INTERNATIONAL CENTRE for THEORETICAL SCIENCES
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Cutoff for the Ising Model on Finite Regular Graphs via Information Percolation

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Mixing Times and Total Variation Distance

- A finite, irreducible, and aperiodic Markov chain has a unique stationary distribution π.
- The distribution of X_t converges to π, regardless of the starting state X_0.

Total Variation Distance:

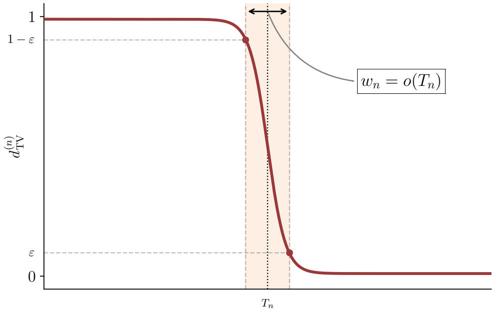
$$\|\mu - \nu\|_{\mathrm{TV}} := \max_{A \subset \Omega} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{\sigma \in \Omega} |\mu(\sigma) - \nu(\sigma)|$$

Mixing Time Definitions:

$$d_{\mathrm{TV}}(t) := \max_{x_0 \in \Omega} \left\| \mathbb{P}^t(x_0, \cdot) - \pi(\cdot) \right\|_{\mathrm{TV}}$$

$$t_{\mathrm{MIX}}(\varepsilon) := \inf\{t \ : \ d_{\mathrm{TV}}(t) \leq \varepsilon\}$$

The Cutoff Phenomenon



A sequence of Markov chains indexed by $n = 1, 2, 3, \ldots$, exhibits a **cutoff** at time T_n with constant window if:

$$T_n \to \infty \quad \text{and} \quad \forall\, 0 < \varepsilon < 1,\ \exists\, t_\varepsilon :$$

$$t_{\mathrm{MIX}}^{(n)}(\varepsilon) \le T_n + t_\varepsilon \quad \text{and} \quad t_{\mathrm{MIX}}^{(n)}(1-\varepsilon) \ge T_n - t_\varepsilon$$

The Ising Model

- Let $G = (V, E)$ be a finite graph, and $\Omega = \{+1, -1\}^V$.
- The *Ising measure* on Ω is defined by:

$$\pi(\sigma) = \frac{1}{Z(\beta)} \exp\left(\beta \sum_{u \sim v} \sigma(u)\sigma(v)\right)$$

- $Z(\beta)$ is the normalizing constant (partition function), and $\beta > 0$ is the inverse temperature.

Glauber Dynamics for the Ising Model

- Given the current configuration σ, pick a vertex $v \in V$ uniformly at random.
- Compute the neighborhood sum:

$$S(\sigma, v) := \sum_{u \in N_G(v)} \sigma(u)$$

- Resample the spin at v. Assign $+1$ with probability:

$$p(\sigma, v) = \frac{e^{\beta S(\sigma,v)}}{e^{\beta S(\sigma,v)} + e^{-\beta S(\sigma,v)}} = \frac{1 + \tanh(\beta S(\sigma, v))}{2}$$

- This update rule satisfies detailed balance with respect to the Ising measure, hence ensuring stationarity.

Continuous-Time Glauber Dynamics

- Consider a finite, d-regular graph $G = (V, E)$, with $|V| = n$.
- Assign an independent Poisson clock with rate 1 (i.i.d. $\mathrm{Exp}(1)$) to each vertex of the graph.
- When the clock at a vertex v rings, update its spin according to the law given by the Glauber dynamics:

$$\mathbb{P}[\text{new spin at } v = +1] = \frac{1 + \tanh\left(\beta \sum_{u \sim v} X(u)\right)}{2}$$

- This defines a continuous-time version of Glauber dynamics (a.k.a. the *heat-bath* dynamics).

The Main Theorem: Lubetzky and Sly (2014)

Theorem. For any $d \geq 2$, there exists $\beta_0 = \beta_0(d) > 0$ such that the following holds:

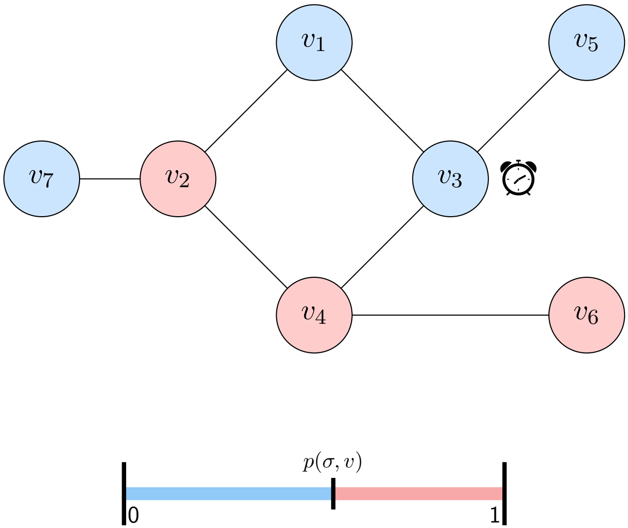
Let G be a d-regular transitive graph on n vertices. For any fixed $0 < \varepsilon < 1$, the continuous-time Glauber dynamics for the Ising model on G at inverse-temperature $0 \leq \beta \leq \beta_0$ satisfies:

$$t_{\text{MIX}}(\varepsilon) = T_n \pm O_\varepsilon(1).$$

where,

$$T_n := \inf\left\{t > 0 : \mathfrak{m}_t := \mathbb{E}[X_t^+] \leq \frac{1}{\sqrt{n}}\right\}$$

Update Sequence Representation

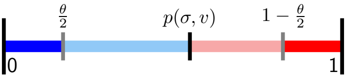


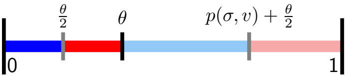
Oblivious Updates

- The local field satisfies $-d < S(\sigma, v) < d$, so for $\theta := 1 - \tanh(\beta d)$, the update probability lies in

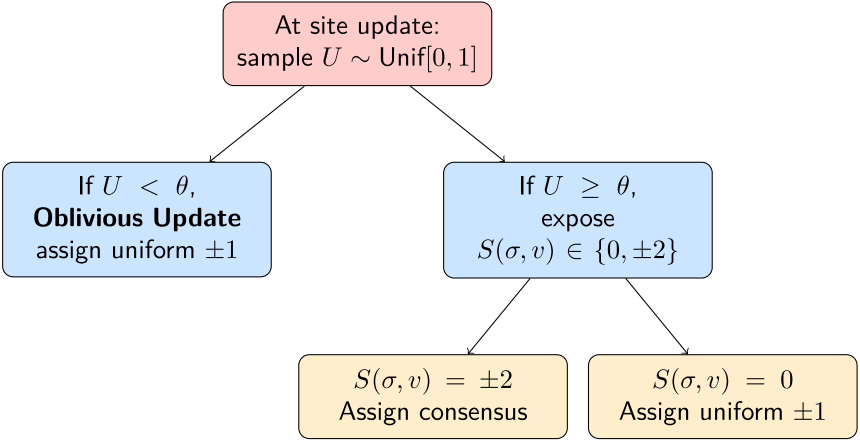
$$p(\sigma, v) = \frac{1 + \tanh(\beta S(\sigma, v))}{2} \in \left[\frac{\theta}{2}, 1 - \frac{\theta}{2}\right]$$

- An update is called *oblivious* if $U < \theta$: in this case, the spin is sampled uniformly from $\{+1, -1\}$, independent of the local field or past state.





An Example: $\mathbb{Z}_n$



An Example: $\mathbb{Z}_n$ (contd.)

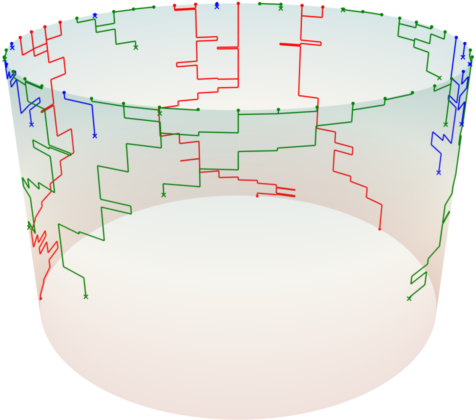
At site update:
sample $U \sim \mathsf{Unif}[0, 1]$

If $U < \theta$,
Oblivious Update
assign uniform ± 1

If $U \geq \theta$,
copy one of the
neighbors randomly

This gives rise to a system of **coalescing random walks** on $\mathbb{Z}_n$, percolating backwards in time, each *killed* at rate θ.

A Simulation



The Information Percolation Clusters

- If the system run long enough that all the clusters die out before hitting time zero, then the configuration at this time follows the stationary measure.
- If the system hasn't run for very long, a lot of red clusters remain and the effect of the initial state is significant.
- The balancing point is at the **critical mixing time** which in this case is

$$T_n \asymp \log(n)$$

Thank You & Acknowledgements

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Questions and Discussions Welcome.