Controls and Dynamical Systems

Student Reading Group

Indian Institute of Technology Bombay

Project Name

Author Names

Contents

List of Figures

Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Acknowledgements

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Chapter 1

Introduction

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And after the second paragraph follows the third paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

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Chapter 2

Theory

2.1 Continuous and Discrete Time Signals

In electrical engineering, the fundamental quantity of representing some information is called a signal. It does not matter what the information is i-e: Analog or digital information. In mathematics, a signal is a function that conveys some information. In fact any quantity measurable through time over space or any higher dimension can be taken as a signal. A signal could be of any dimension and could be of any form. Depending on the continuity of the contained information signals can be:

1. Continuous-time A continuous-time signal has a value for all instants in time or space. Example of a continuous-time signal is the voltage of a battery or the position of a pendulum.

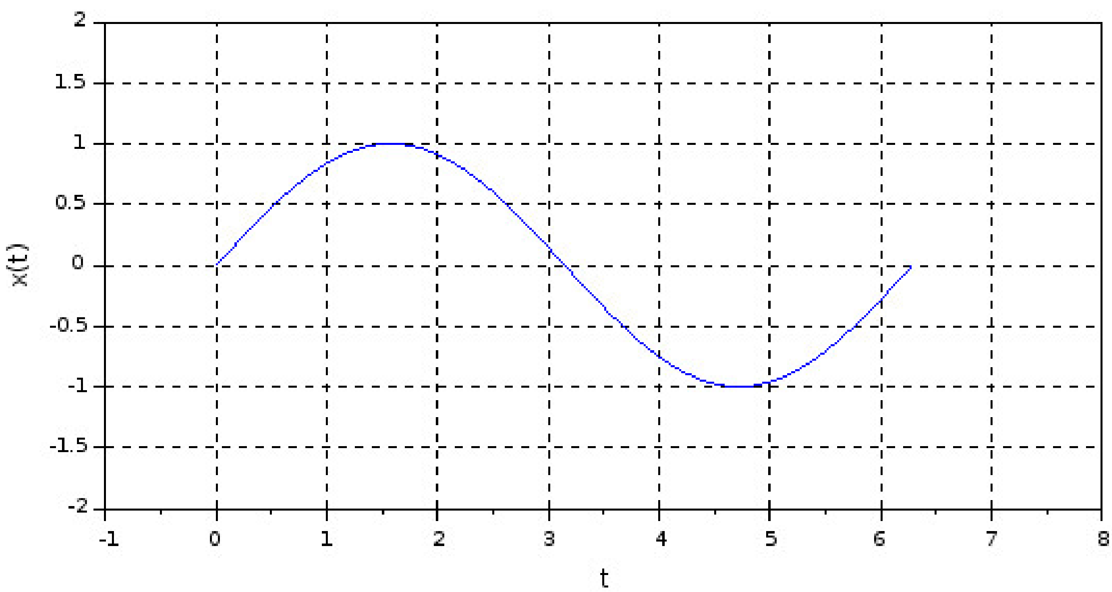


Figure 2.1: Continuous Time Signal

2. Discrete-time A discrete-time signal has a value only at discrete moments in time. Example of a discrete signal is the weight of a human measured early or the daily average temperature measure in a specific area.

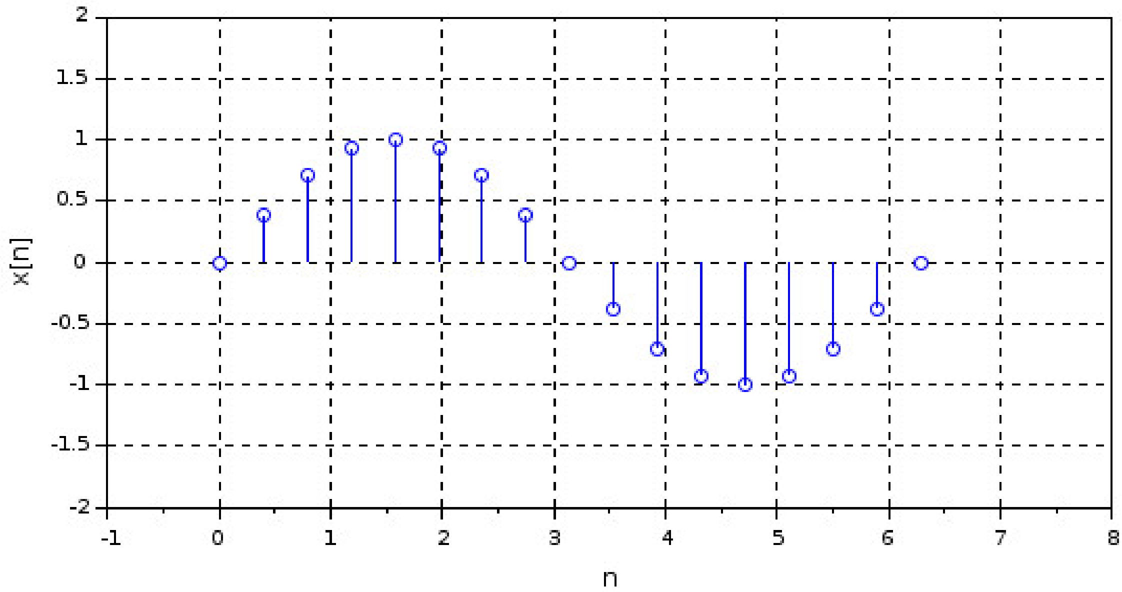


Figure 2.2: Discrete Time Signal

To distinguish between continuous-time and discrete-time signals, we will use the symbol **t** to denote the continuous-time independent variable and **n** to denote the discrete time independent variable.
Continuous Time Signal : x(t)
Discrete Time Signal : x[n]

2.2 Unit Impulse and Unit Step Functions

2.2.1 The Discrete-Time Unit Impulse and Unit Step Sequences

One of the simplest discrete-time signals is the unit impulse (or unit sample), which is defined as

$$\delta[n] = 0, n \neq 0$$

$$\delta[n] = 1, n = 0$$

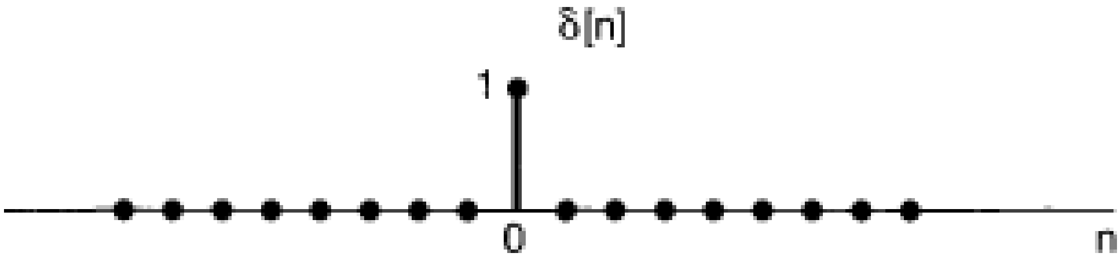


Figure 2.3: Discrete Time Impulse

A second basic discrete-time signal is the discrete-time unit step, denoted by **u[n]** and defined by

$$u[n] = 0, n < 0$$

$$u[n] = 1, n \geq 0$$

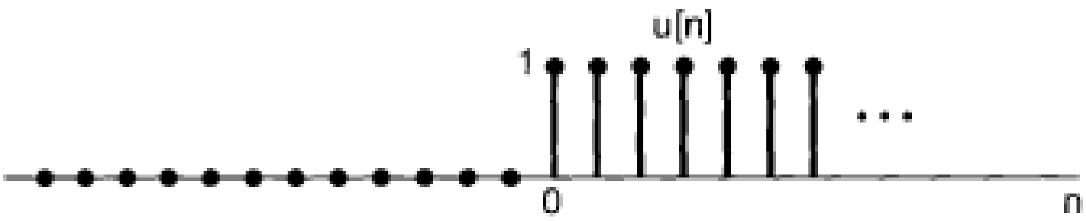


Figure 2.4: Discrete Time Step Function

An important relation between these two :

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

2.2.2 The Continuous-Time Unit Step and Unit Impulse Functions

The continuous-time unit step function u(t) is defined in a manner similar to its discrete time counterpart -

$$u(t) = 0, t < 0$$

$$u(t) = 1, t > 0$$

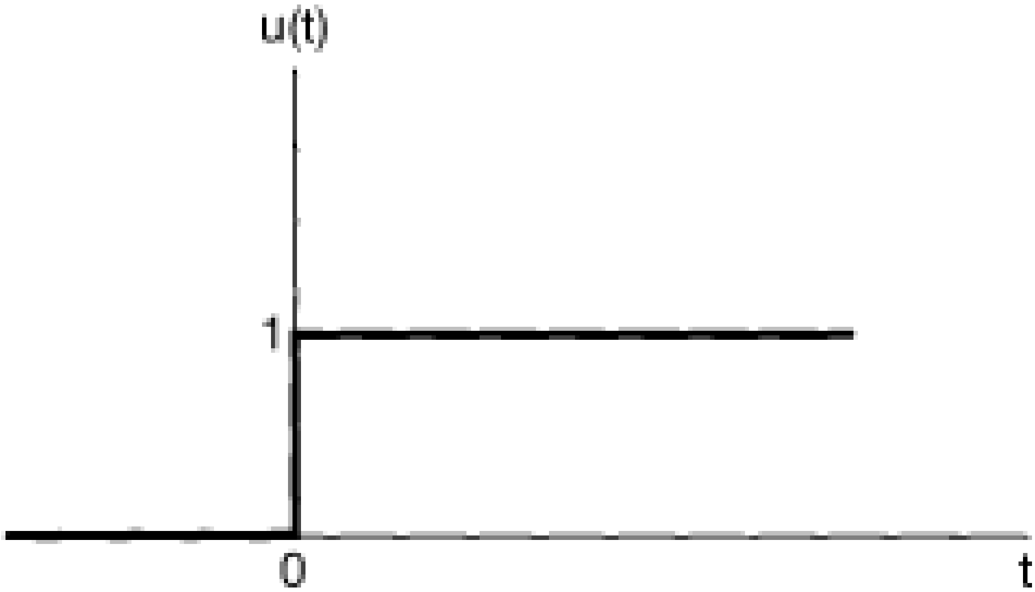


Figure 2.5: Continuous Time Step Function

Note that the unit step is discontinuous at t=0.
The continuous-time unit impulse function is related to the unit step in a manner analogous to the relationship between the discrete-time unit impulse and step functions. In particular,the continuous-time unit step is the running integral of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

The continuous-time unit impulse can be thought of as the first derivative of the continuous-time unit step:

$$\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$$

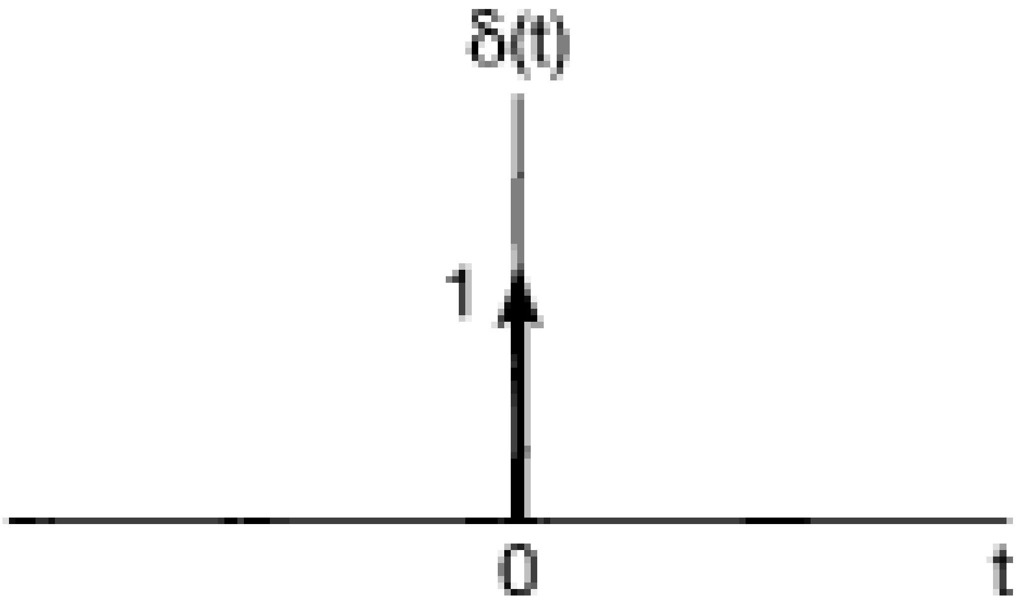


Figure 2.6: Continuous Time Impulse

2.3 Basic System Properties

2.3.1 Systems with and without Memory

A system is said to be memory less if its output for each value of the independent variable at a given time is dependent only on the input at that same time. For example, the system specified by the relationship

$$y[n] = (2x[n] - x^2[n])^2$$

is memory less, as the value of y[n] at any particular time n depends only on the value of x[n] at that time. Similarly, a resistor is a memory less system; with the input x(t) taken as the current and with the voltage taken as the output y(t), the input-output relationship of a resistor is

$$y(t) = Rx(t)$$

where R is the resistance. One particularly simple memoryless system is the identity system, whose output is identical to its input. That is, the input-output relationship for the continuous-time identity system is

$$y(t) = x(t)$$

and the corresponding relationship in discrete time is

$$y[n] = x[n]$$

An example of a discrete-time system with memory is an accumulator or summer

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

and a second example is a delay

$$y[n] = x[n-1]$$

Roughly speaking, the concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about input values at times other than the current time.

2.3.2 Invertibility

A system is said to be invertible if distinct inputs lead to distinct outputs. An example of an invertible continuous-time system is

$$y(t) = 2x(t)$$

for which the inverse system is

$$w(t) = y(t)/2$$

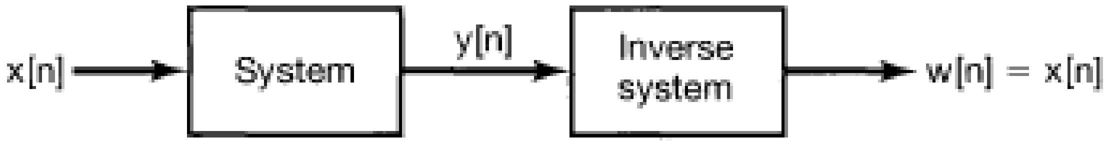


Figure 2.7: Process of inverting a function

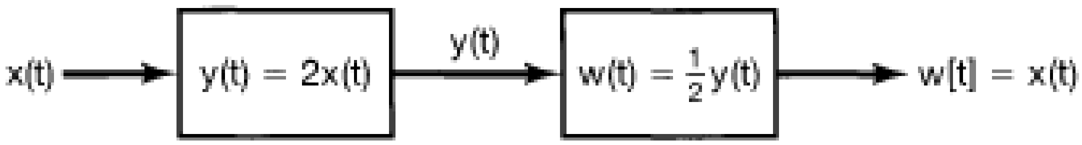


Figure 2.8: Example of inverting a function

Examples of non invertible systems are

$$y[n] = 0$$

that is, the system that produces the zero output sequence for any input sequence, and

$$y(t) = x^2(t)$$

in which case we cannot determine the sign of the input from knowledge of the output.

2.3.3 Causality

A system is said to be causal system if its output depends on present and past inputs only and not on future inputs. Examples of casual system -

$$y(t) = x(t) + x(t-2)$$

$$y(t) = 7x(t-5)$$

Generally all real time systems are causal systems; because in real time applications only present and past samples are present. Since future samples are not present; causal systems are memoryless.
A system whose present response depends on future values of the inputs is called as a non-causal system. Examples of non-casual systems -

$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

All memory less systems are causal, since the output responds only to the current value of the input.
When checking the causality of a system, it is important to look carefully at the inputoutput relation. To illustrate some of the issues involved in doing this, we will check the causality of two particular systems.
The first system is defined by

$$y[n] = x[-n]$$

Note that the output y[n_0] at a positive time n_0 depends only on the value of the input signal x[-n_0] at time (-n_0), which is negative and therefore in the past of n_0. We may be tempted to conclude at this point that the given system is causal. However, we should always be careful to check the input-output relation for all times. In particular, for n less than 0, e.g. n =-4, we see that y[-4] = x[4], so that the output at this time depends on a future value of the input. Hence, the system is not causal.

2.3.4 Stability

Stability is another important system property. Informally, a stable system is one in which small inputs lead to responses that do not diverge.

Formal definition - If the input to a stable system is bounded (i.e., if its magnitude does not grow without bound), then the output must also be bounded and therefore cannot diverge. And a system that does not diverge is referred to as stable.
If we suspect that a system is unstable, then a useful strategy to verify this is to look for a specific bounded input that leads to an unbounded output. Finding one such example enables us to conclude that the given system is unstable. If such an example does not exist or is difficult to find, we must check for stability by using a method that does not utilize specific examples of input signals. To illustrate this approach, let us check the stability of two systems,

$$S_1 : y(t) = tx(t)$$

$$S_2 : y(t) = e^x(t)$$

In seeking a specific counterexample in order to disprove stability, we might try simple bounded inputs such as a constant or a unit step. For system S_1, a constant input x(t) = 1 yields y(t) = t, which is unbounded, since no matter what finite constant we pick, mod of y(t) will exceed that constant for some t. We conclude that system S_1 is unstable.
For system S_2, which happens to be stable, we would be unable to find a bounded input that results in an unbounded output. So we proceed to verify that all bounded inputs result in bounded outputs. Specifically, let B be an arbitrary positive number, and let x(t) be an arbitrary signal bounded by B; that is, we are making no assumption about x(t), except that

$$\mid x(t) \mid < B$$

or

$$-B < x(t) < B$$

for all t. Using the definition of S_2, y(t) must satisfy

$$e^{-B} <\mid y(t) \mid < e^B$$

We conclude that if any input to S_2 is bounded by an arbitrary positive number B, the corresponding output is guaranteed to be bounded by e^B . Thus, S_2 is stable.

2.3.5 Time Invariance

Conceptually, a system is time invariant if the behavior and characteristics of the system are fixed over time. Specifically, a system is time invariant if a time shift in the input signal results in an identical time shift in the output signal. That is, if y[n] is the output of a discrete-time, time-invariant system when x[n] is the input, then y [n- n_0] is the output when x [n- n_0] is applied. In continuous time with y(t) the output corresponding to the input x(t), a time-invariant system will have y (t- t_0) as the output when x (t - t_0) is the input.
To see how to determine whether a system is time invariant or not, and to gain some

insight into this property, consider the following example -
Consider the continuous-time system defined by

$$y(t) = sin(x(t))$$

To check that this system is time invariant, we must determine whether the time invariance property holds for any input and any time shift t_0 . Thus, let $x_1(t)$ be an arbitrary input to this system, and let

$$y_1(t) = sin(x_1(t))$$

be the corresponding output. Then consider a second input obtained by shifting $x_1(t)$ in time:

$$x_2(t) = x_1(t - t_0)$$

The output corresponding to this input is

$$y_2(t) = sin(x_2(t)) = sin(x_1(t - t_0))$$

Similarly,

$$y_1(t - t_0) = sin[x_1(t - t_0)]$$

We can see that

$$y_1(t - t_0) = y_2(t)$$

Therefore, this system is time invariant.

2.4 Fourier Series Representation of Periodic Signals

We will understand the response of LTI systems to complex polynomials. First we should fathom the importance of the way the response is described when the input is a set of super-positioned signals and a set of basic signals can be used to constitute a broad class of signals. Much of the importance of Fourier analysis results from the fact that both of these properties are provided by the set of complex exponential signals in continuous and discrete-time: i.e., signals of the form e^{st} in continuous-time and z^n in discrete-time, where s and z are complex numbers. As one witnesses the operations carried on certain complex functions by operators which are eigenfunctions of the principal function, yielding an eigenvalue, similarly here we establish the importance of complex exponentials as input to LTI systems referring to the fact that edges over other representations that the response is the same complex exponential with only a change in amplitude. Continuous-time signals:

$$e^{st} \to H(s)e^{st}$$

Discrete-time signals:

$$z^n \to H(z)z^n$$

where the complex amplitude factor H(s) or H(z) will in general be a function of the complex variable s or z. Here the signal is the **eigenfunction** of the system and

the amplitude is referred to as the **eigenvalue** of the system.
Now we need to prove that complex exponentials are indeed eigenfunctions of LTI systems and also understand the nature of the eigenvalues- the functions H(s) and H(z):
Let us consider a continuous-time LTI system with impulse response h(t). For an input x(t), we can determine the output through the use of the convolution integral, so that with $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{\infty} H(\tau)x(t-\tau)d(\tau)$$

$$y(t) = \int_{-\infty}^{\infty} H(\tau)e^{(st-s\tau)}d(\tau)$$

$$y(t) = e^{st}\int_{-\infty}^{\infty} H(\tau)e^{-s\tau}d(\tau)$$

Assuming that the integral is convergent,

$$y(t) = H(s)e^{st}$$

Hence, the complex constant H(s) is defined by:

$$H(s) = \int_{-\infty}^{\infty} H(\tau)e^{-s\tau}d(\tau)$$

Similarly we build the equations necessary for discrete-time LTI systems.That is, suppose that an LTI system with impulse response h[n] has as its input the sequence $x[n] = z^n$ where z is a complex number. Then the output of the system can be determined from the convolution sum as:

$$y[n] = \sum_{k=-\infty}^{\infty} h[n]x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[n]z^{[n-k]}$$

$$y[n] = z^{[n]}\sum_{k=-\infty}^{\infty} h[n]z^{[-k]}$$

Assuming that the sum is convergent,

$$y[n] = H(z)z^{[n]}$$

Hence, the complex constant H(z) is defined by:

$$H(z) = \sum_{k=-\infty}^{\infty} h[n]z^{[-k]}$$

Moving further to take under consideration the otherwise plight condition of signals which can be described hopefully by linear combination; if the input to an LTI system is represented as a linear combination of complex exponentials, then the output can also be represented as a linear combination of the same complex exponential signals.

In other words, if the input to a LTI system (continuous-time and discrete-time) is presented as a linear combination :

$$x(t) = \sum_k a_k e^{s_k t}$$

for a continuous-time LTI system and

$$x[n] = \sum_k a_k z_k^n$$

for a discrete-time LTI system; From the eigenfunction property each component has a separate response and hence the response can be respectively summarised as:

$$y(t) = \sum a_k H(s_k) e^{s_k t}$$

$$y[n] = \sum a_k H(z_k) z_k^n$$

Each coefficient in this representation of the output is obtained as the product of the corresponding coefficient a_k of the input and the system's eigenvalue $H(s_k)$ or $H(z_k)$ associated with the eigenfunction $e^{s_k t}$ or z_k^n respectively.

2.4.1 Fourier Series representation of Continuous-Time Signal

Harmonically related complex exponentials of the form

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(\frac{2\pi}{T_0})t}$$

where each signal is periodic, though not fundamentally periodic, have time period $T_0 = 2\frac{\pi}{\omega_o}$. Which implies that a linear combination of these harmonically related complex exponentials are also periodic with the time period as T. Therefore we exploit the opportunity to introduce the **Fourier Series Representation** :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T_0})t} \quad \rightarrow A$$

As $x(t) = x^*(t)$, we get

$$x(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

and after replacing k with - k which only has the effect of reversing the ends of addition, we get:

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

Comparing it with the first equation, we get $a = a_{-k}^*$ which is equivalent of stating that $a^* = a_{-k}$. Hence we arrive at the decision that all coefficients are real and also $a_k = a_{-k}$. So manipulating the summation as:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

As, $a^* = a_{-k}$,

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{k^*} e^{-jk\omega_0 t}\right]$$

After expressing a_k in its polar form as $a_k = A_k e^{j\theta_k}$ and elaborating further we arrive at:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e(a_k e^{jk\omega_0 t})$$

To further summarise, if a series has Fourier series representation then the coefficients are given by:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(\frac{2\pi}{T_0})t}\,dt \quad \to B$$

This pair of equations A and B establishes the Fourier series of a periodic signal for continuous-time. The set of coefficients a_k are also known as the spectral coefficients of $x(t)$.

2.4.2 Convergence of Fourier series

Some conditions were developed for establishing the convergence of Fourier Series. P. L. Dirichlet had developed an alternate set of conditions which proved suffice to guarantee about any equation (which will be our concern) x(t) equals its Fourier series representation, except at isolated values of t for which x(t) is discontinuous. At these values, the infinite series of the Fourier series representation converges to the average of the values on either side of the discontinuity.

The Dirichlet Conditions:

1. Over any period x(t) must be absolutely integrable $\int_T x(t)\,dt < \infty$
2. In any finite interval of time x(t) is of bounded variation i.e. there are no more than a finite no. of maxima and minima during a single period of the signal.
3. In any finite interval of time there are only a finite no. of discontinuities.Furthermore, each of these discontinuities is finite.

2.4.3 Properties of Continuous Time Fourier Series

1. **Linearity :** Let $x(t)$ and $y(t)$ be two periodic signals having Fourier series coefficients a_k and b_k, then a signal $z(t) = Ax(t) + By(t)$ will have Fourier series coefficients as $Aa_k + Bb_k$.

2. **Time Shifting :** If a signal $x(t)$ having Fourier series coefficients ak is time shifted to $x(t - to)$ then new FS coefficients are $e^{-jk\omega_0 t_0}a_k$.

3. **Time Reversal :** If a signal $x(t)$ having Fourier series coefficients a_k is reversed to $x(-t)$ then new Fourier Series coefficients are a_{-k}.

4. **Time Scaling :** Time scaling is an operation that in general changes the period of the underlying signal. Hence the name scaling.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

5. **Multiplication :** The multiplication of periodic signals $x(t)$, $y(t)$ gives us new coefficients. as

$$h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

.

6. **Conjugation and Conjugate Symmetry:** The Fourier series coefficients will be conjugate symmetric:

$$a^* = a_{-k}$$

and

$$a_k = a_{-k}$$

7. **Parseval's Relation :** The relation thus states that:

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Note that the left-hand side of the equation is the average power (i.e., energy per unit time) in one period of the periodic signal x(t). Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

2.4.4 Fourier Series representation of Discrete-Time Signal

Harmonically related complex exponentials of the form

$$\phi_k[n] = e^{jk\omega_n} = e^{jk(\frac{2\pi}{N})n}$$

where each signal has frequency, though not fundamentally frequency: $N = \frac{2\pi}{\omega_0}$. Which implies that a linear combination of these harmonically related complex exponentials are also periodic with the time period as T. Therefore we exploit the opportunity to introduce the **Fourier Series Representation** :

In discrete-time we have $\phi_k[n] = \phi_{(k+rN)}[n]$, as complex exponentials that differ by frequency 2π are identical. As $\phi_k[n]$ differ over a range of N successive values, we limit the summation as $k =< N >$. Therefore we exploit the opportunity to introduce the **Fourier Series Representation** :

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

is To further summarise, if a series has Fourier series representation then the coefficients are given by: a_k as

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(\frac{2\pi}{N})n}$$

The set of coefficients a_k are also known as the spectral coefficients of $x[n]$.

Properties of Discrete-Time Fourier Series

1. **Linearity :** Let $x[n]$ and $y[n]$ be two periodic signals having Fourier series coefficients a_k and b_k, then a signal $z[n] = Ax[n] + By[n]$ will have Fourier series coefficients as $Aa_k + Bb_k$.

2. **Time Shifting :** If a signal $x[n]$ having Fourier series coefficients a_k is time shifted to $x[n - n_0]$ then new FS coefficients are $e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$.

3. **Time Reversal :** If a signal $x[n]$ having Fourier series coefficients a_k is reversed to $x[-n]$ then new FS coefficients are a_{-k}.

4. **Multiplication :** The multiplication of periodic discrete time signals periodic with frequency N $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$, $y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$ is periodic with period N and gives us new coefficients as

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=<N>} a_l b_{k-l}$$

As the summation can be taken over any set of N consecutive values of l. We refer to this type of operation as a **periodic** convolution between the two periodic sequences of Fourier coefficients. The usual form of the convolution sum (where the summation variable ranges from $-\infty$ to $+\infty$) is sometimes referred to as **aperiodic** convolution, to distinguish it from periodic convolution.

5. **First Difference :** For a signal $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ the first difference coefficients can be denoted by: $x[n] - x[n-1]$

$$x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} D_k(1 - e^{-jk\left(\frac{2\pi}{N}\right)})a_k$$

6. **Parseval's Relation :**

$$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

As in the continuous-time case, the left -hand side of Parseval's relation is the average power in one period for the periodic signal x[n]. Similarly, $|a_k^2|$ is the average power in the k^{th} harmonic component of x[n]. Thus, once again, Parseval's relation states that the average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

2.5 Continuous-Time Fourier Transform

Consider this issue- the time period of a Fourier Series representation of a periodic signal increases infinitely. Thus the harmonically related components become closer in frequency. As the period becomes infinite, the frequency components form a continuum and the Fourier series sum becomes an integral. Now we develop the Fourier series representation for continuous-time periodic signals, and in the sections that follow we build on this foundation as we explore many of the important properties of the continuous-time Fourier transform that form the foundation of frequency-domain methods for continuous-time signals and systems.The resulting spectrum of coefficients in this representation is called the Fourier transform, and the synthesis integral itself, which uses these coefficients to represent the signal as a linear combination of complex exponentials, is called the inverse Fourier transform.

2.5.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Think of an aperiodic signal as a limit of a periodic signal- the period becomes infinitely large. Consider an aperiodic signal $x(t)$ of finite duration i.e. $x(t) = 0$ for $|t| > T_1$ and fundamental time period T. Now we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period(T). As $T \to \infty$ we have $x(t) \to \tilde{x}(t)$.
Writing the synthesis equation again and considering the Fourier Series coefficient integral for the period -T/2 to +T/2 as

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}, dt$$

ω= $2\frac{\pi}{T}$. Also $\tilde{x}(t)$ = x(t) for the interval -T/2 to +T/2 and as x(t) is 0 elsewhere. So we can write:

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}\,dt$$

We then define the envelope Ta_k to be $X(j\omega)$ as:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,dt$$

Thus we get,

$$a_k = \frac{1}{T}X(jk\omega_0)$$

Thus we express $\tilde{x}(t)$ combining all equations and letting $\omega_0 \to 0$ i.e. $(T \to \infty)$ which then implies that $\tilde{x}(t)$ tends to x(t) and the sum converts to an integral:

The Synthesis Equation or the inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{jk\omega t}\,d\omega The Fourier integral :$$

X(jω) $= \int_{-\infty}^{\infty} x(t)e^{-jk\omega t}\,dt These two equations are referred to as the Fourier Transform pair.$

2.5.2 Convergence of Fourier Transform

Let the Synthesis Equation be denoted by:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{jk\omega t}\,d\omega$$

Then we should define conditions so that x(t) satisfies the equation. Just as with periodic signals, there is an alternative set of conditions which are sufficient to ensure that $\hat{x}$(t) is equal to x(t) for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity. These conditions, again referred to as the Dirichlet conditions, require that: 1. Over any period x(t) must be absolutely integrable $\int_T x(t)\,dt < \infty$
2. In any finite interval of time x(t) is of bounded variation i.e. there are no more than a finite no. of maxima and minima during a single period of the signal.
3. In any finite interval of time there are only a finite no. of discontinuities.Furthermore, each of these discontinuities is finite.

2.5.3 Fourier Transform for Periodic Signals

Let us consider a periodic signal $x(t)$ with Fourier Transform

$$X(j\omega) = \sum_{-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

The application of inverse Fourier Transform yields

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

which is the Fourier series representation of periodic signals.

2.5.4 Properties of Continuous Time Fourier Transform

As we are already aware of the common properties that were covered earlier - like linearity, time shifting, time reversal etc. , we would shed light on the other important properties as these properties run parallel and can be understood well by analysing analogy. Also, because of the close relationship between the Fourier series and the Fourier transform, many of the transform properties translate directly into corresponding properties for the continuous-time Fourier series. We would shed light on the important properties which are either different or important:

Convolution Property

As we earlier saw interpretation of the Fourier transform synthesis equation as an expression for x(t) as a linear combination of complex exponentials. We get the following equation by approximating it as limit of sum

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{jk\omega t}\,d\omega = \lim_{\omega_0 \to 0}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

The response of a linear system with impulse response $h(t)$ to a complex exponential is $H(jk\omega_0)e^{jk\omega_0}$ where

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(t)e^{-jk\omega_0}\,dt$$

The response to the linear system $x(t)$ is

$$y(t) = \int_{-\infty}^{\infty} X(j\omega)H(j\omega)e^{jk\omega t}\,d\omega = \lim_{\omega_0 \to 0}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}H(j\omega_0)e^{jk\omega_0 t}\omega_0$$

On comparing with the fourier transform of y(t) we get $Y(j\omega) = X(j\omega)H(j\omega)$ and $y(t) = x(t) * h(t)$ Which we sum up on together as:

$$y(t) = x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = X(j\omega)H(j\omega)$$

Multiplication property

The convolution property states that convolution in the time domain corresponds to multiplication in the frequency domain. Because of duality in time and frequency we expect a dual property also to hold.

$$r(t) = s(t)p(t) \overset{R}{\longleftrightarrow} (j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is known as amplitude modulation. Hence, this equation is also referred as **modulation property**.
The following table summarises the properties of Continuous Time Fourier Transform: Source: Alan V. Oppenheim, Alan S. Willsky, with S. Hamid-Signals and Systems-Prentice Hall (1996)

Property	Aperiodic signal	Fourier transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Figure 2.9: Table of Continuous Time Fourier Transform Properties

2.5.5 Some Common Fourier Transforms

Source: Alan V. Oppenheim, Alan S. Willsky, with S. Hamid-Signals and Systems-Prentice Hall (1996)

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Figure 2.10: Table of Common Continuous Time Fourier Transforms

2.5.6 Linear Constant-Coefficients Differential Equation Systems

An important class of LTI systems, we will determine the frequency response of the these systems characterised by the equations:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

and by the convolution property

$$Y(j\omega) = X(j\omega)H(j\omega)$$

or

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Taking FT on both sides :

$$\mathcal{F}\sum_{k=0}^{k=N} a_k \frac{d^k y(t)}{dt^k} = \mathcal{F}\sum_{k=0}^{k=M} b_k \frac{d^k y(t)}{dt^k}$$

From the linearity and differentiation property we have:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

Finally we can now determine the frequency response of these systems by examining the rational polynomial $H(j\omega)$ given by:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

2.6 Discrete-Time Fourier Transform

2.6.1 Development of the Discrete-Time Fourier Transform

To develop the Fourier Transform representation for for aperiodic sequences discrete time signals we build the concepts in parallel analogy with that done in case of continuous-time: by extending the Fourier series description of periodic signals in order to develop a Fourier transform representation for discrete-time aperiodic signals Consider a signal x[n] that is of finite duration i.e. for some positive integers N_1 and N_2 such that $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$. From this aperiodic signal, we can construct a periodic signal $\tilde{x}[n]$ for which $x[n]$ is one period. Now we choose $N \to \infty$ for $x[n] = \tilde{x}[n]$ for any finite value of n. We can write this as

$$x[n] = \tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \frac{1}{N} \sum_{k=<N>} \tilde{x}[n] e^{-jk(\frac{2\pi}{N})n}$$

Since x[n] = $\tilde{x}$[n] over a period that includes the interval $-N_1 \leq n \leq N_2$, it's convenient to choose the interval of summation to include this interval, so that$\tilde{x}$[n] can be replaced by x[n] in the summation.

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(\frac{2\pi}{N})n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(\frac{2\pi}{N})n}$$

Now we define Na_k to be $X(j\omega)$ as follows:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Thus we get

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

Now we express $\tilde{x}[n]$ combining all equations and letting $\omega_0 \to 0$ i.e. $(T \to \infty)$ which then implies that $\tilde{x}[n]$ tends to x[n] and the sum converts to an integral:
The Synthesis Equation:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

The analyst equation or the discrete-time Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

These two equations are referred to as Fourier Transform pair in Discrete time.

2.6.2 Fourier Transform for Periodic Signals

Let us consider a periodic signal $x[n]$ with Fourier Transform

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$

The application of inverse Fourier Transform yields

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{N})n}$$

which is the Fourier series representation of periodic signals.

2.6.3 Properties Of Discrete Time Fourier Transform

As we are already aware of the common properties that were covered earlier - like linearity, time shifting, time reversal etc. , we would shed light on the other important properties as these properties run parallel and can be understood well by analysing analogy.Also, because of the close relationship between the Fourier series and the Fourier transform, many of the transform properties translate directly into corresponding properties for the discrete-time Fourier series. We would shed light on the important properties which are either different or important:

Periodicity

The discrete time FT is always periodic in ω with period of 2π

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

This is in contrast with the continuous time FT which is in general not periodic.

Time Expansion

In properties of continuous time FT we can see that

$$x(\alpha t) \xrightarrow{\mathcal{DTFT}} \frac{1}{\alpha} X\left(\frac{j\omega}{\alpha}\right)$$

However, we face difficulties in defining $x[\alpha n]$ as α may not be an integer. Thus we define a signal $x_{(k)}[n] = x[\frac{n}{k}]$ if n is a mutiple of k otherwise $x_{(k)}[n] = 0$. Here we let $n = rk$ where r is any integer.
Thus the FT of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x_{(k)}[rk] e^{-j\omega rk}$$

Furthermore since $x_{(k)}[rk] = x[r]$ we get

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$$

That is

$$x_{(k)}[n] \xrightarrow{\mathcal{DTFT}} (e^{jk\omega})$$

The Convolution Property

In representing and analyzing discrete-time LTI systems, the Convolution property takes credit as it takes in the case of Continuous Time Fourier Transform. So instead of more praising we proceed to know why this property gets it. If x[n], h[n], and y[n] are essentially the input, impulse response, and output, respectively, of an LTI system, so that

$$y[n] = x[n] * h[n]$$

then,

$$Y(e^{jw}) = X(e^{jw}) * H(e^{jw})$$

where X,Y,H are the Fourier Transform in Discrete-Time. We also learn that the frequency response of a discrete-time LTI system is the Fourier transform of the impulse response of the system.

The Multiplication Property

An analogous property exists for discrete-time signals and plays a similar role in applications as in the case of Continuous Time Signals. The Multiplication Property proves to be beneficial in Sampling. Consider y[n] equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms.

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_1(e^{(j(\omega-\theta))})d\theta$$

The following table summarises the properties of Discrete Time Fourier Transform: Source: Alan V. Oppenheim, Alan S. Willsky, with S. Hamid-Signals and Systems-Prentice Hall (1996)

Property	Aperiodic Signal	Fourier Transform
	$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Figure 2.11: Table of Discrete Time Fourier Transform Properties

2.6.4 Some common Fourier Transforms

Source: Alan V. Oppenheim, Alan S. Willsky, with S. Hamid-Signals and Systems-Prentice Hall (1996)

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Figure 2.12: Table of Common Discrete-Time Fourier Transforms

2.7 The Laplace Transform

Owing to the fact that the eigenfunction property introduced earlier and many of its consequences apply as well for arbitrary real values and not only those values that are purely imaginary. Thus we exploit this issue to generalise the continuous-time Fourier transform, known as the Laplace transform. Earlier we witnessed the response to LTI systems where the response to an exponential signal e^{st} was h(t):

$$y(t) = H(s)e^{st}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where $s \in \mathbb{C}$. In case of an imaginary s, the integral is called the Fourier transform of h(t). When s is of the form $\sigma+j\omega$ it's termed as Laplace Transform. The definition of Laplace Transform of a signal h(t) is :

$$X(s) =^{\delta} \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where $s = \sigma + j\omega$ Some important denotations: 1. The Laplace Operator $\mathcal{L}$ denotes the relation between x(t) and its Laplace Transform X(s):

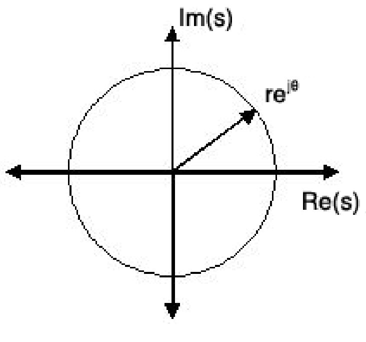
$$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$$

when s=$j\omega$ we arrive at the Fourier Transform:

$$X(s)|_{s=j\omega} = \mathcal{F}x(t)$$

2.7.1 Region Of Convergence

Owing to the fact that two signals can have identical algebraic expressions for X(s), we need some tool to distinguish between their Laplace Transform. Here comes the definition of ROC- Region Of Convergence. Let's explore some specific constraints on the ROC for various classes of signals. As we will see, an understanding of these constraints often permits us to specify implicitly or to reconstruct the ROC from knowledge of only the algebraic expression for X(s) and certain general characteristics of x(t) in the time domain.



Properties of ROC:

Property 1:
The ROC of X(s) consists of strips parallel to the jω-axis in the s-plane.

Property 2:
For rational Laplace transforms, the ROC does not contain any poles.

Property 3:
If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

Property 4:
If x(t) is right sided, and if the line $\mathcal{R}e(s) = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e(s) > \sigma_0$ will also be in the ROC.A right-sided signal is a signal for which x(t) = 0 prior to some finite time T.

Property 5:
If x(t) is left sided, and if the line $\mathcal{R}e(s) = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e(s) < \sigma_0$ will also be in the ROC.

Property 6:
If x(t) is two sided, and if the line $\mathcal{R}e(s) = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\mathcal{R}e(s) = \sigma_0$.

Property 7:
If the Laplace transform X(s) of x(t) is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of X(s) are contained in the ROC.

Property 8:
If the Laplace transform X(s) of x(t) is rational, then if x(t) is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If x(t) is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

2.7.2 The Inverse Laplace Transform

As we know already,

$$X(\sigma + j\omega) = \mathcal{F}(x(t)e^{-\sigma t}) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}e^{-\sigma t}dt$$

Using our knowledge of Inverse Fourier Transform, we can write x(t) as:

$$x(t) = \mathcal{F}^{-1}(X(\sigma + j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-(j\omega+\sigma)t}d\omega$$

This equation states that x(t) can be represented as a weighted integral of complex exponentials ans we can recover x(t) using $s = \sigma + j\omega$.

2.7.3 Properties of Laplace Transform

The following table summarises the properties of Laplace Transform: Source: Alan V. Oppenheim, Alan S. Willsky, with S. Hamid-Signals and Systems-Prentice Hall (1996)

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$$

Figure 2.13: Table of Common Laplace Transform Properties

2.7.4 Some common Laplace Transforms

Source: Alan V. Oppenheim, Alan S. Willsky, with S. Hamid-Signals and Systems-Prentice Hall (1996)

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$

Figure 2.14: Table of Common Laplace Transforms

2.8 Filtering

The concept of Filtering rises from the convolution property. We must understand that filtering is a consequence-for linear,time-invariant systems the output is frequency response times the Fourier transform. The process of changing the relative amplitude or eliminating some frequency components of the signal is known as Filtering. Frequency-selective filters attempt to exactly pass some bands of frequencies and exactly reject others. Frequency-shaping filters more generally attempt to reshape the signal spectrum by multiplying the input spectrum by some specified shaping.

Frequency Shaping Filters : LTI systems that change the shape of the spectrum are often referred to as frequency shaping filters. One major application of this filter is audio systems where we modify the relative amounts of low-frequency bands (bass) and high-frequency bands (treble). Also in high fidelity audio systems(high quality reproduction of time) an equalizing filter is often included in the pre-amplifier to compensate for the frequency response characteristics of the speaker.

Another important class of frequency shaping filters is the differentiating filters $y(t) = \frac{dx(t)}{dt}$. In differentiating filters, for input $x(t)$ we get output as $y(t) = j\omega x(t)$. Thus $H(j\omega) = j\omega$ i.e. a complex exponential signal will receive greater amplification for larger values of ω. Thus, one main use of these filters is enhancing edges in image processing.

Frequency Selective Filters are designed to pass some frequencies essentially un-distorted and significantly attenuate or eliminate others are known as frequency selective filters.Ideal frequency-selective filters, such as lowpass, highpass, and bandpass filters, are useful abstractions mathematically but arenot exactly implementable. Furthermore, even if they were implementable, in practical situations they may not be desirable. They are used in variety of situations for example noise in the background can be removed, communications systems is also an important application.

The Low Pass Filter – It allows low frequency signals from 0Hz to its cut-off frequency, c point to pass and blocks higher frequencies.

The High Pass Filter - It allows high frequency signals from its cut-off frequency, c point and higher to infinity to pass and blocks lower frequencies.

The Band Pass Filter – It allows signals falling within a certain frequency band to pass through and blocks other frequencies.

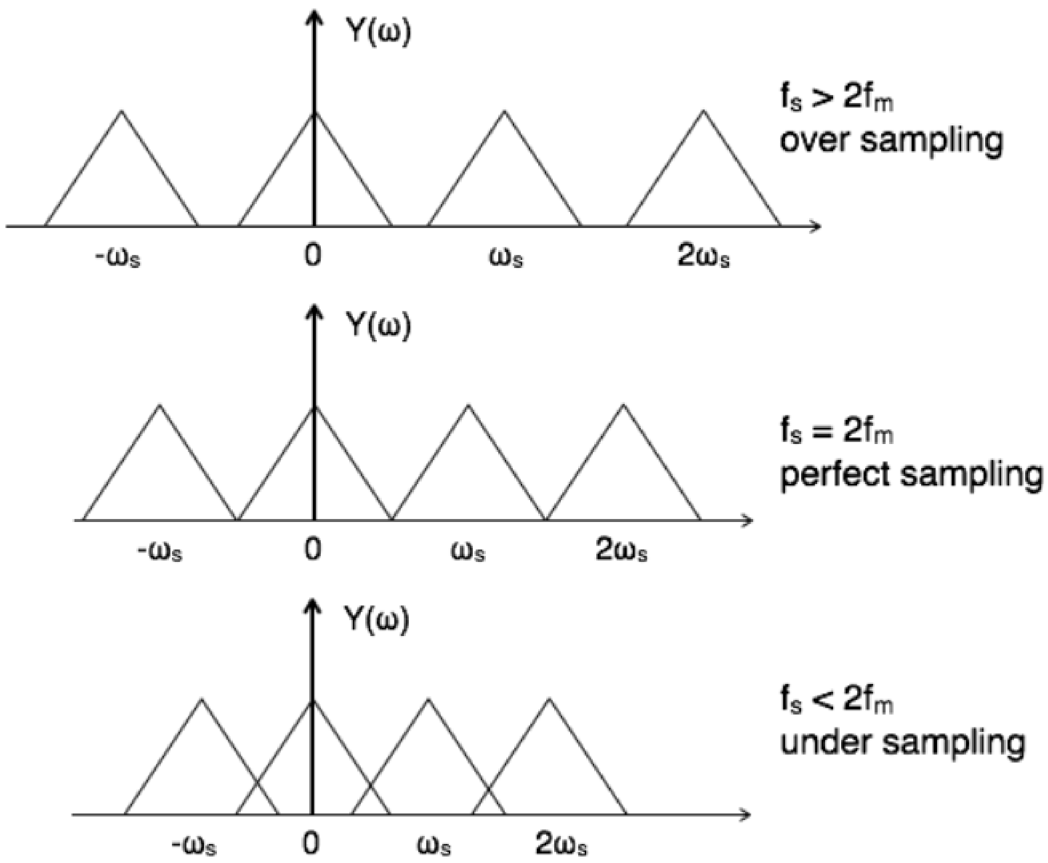
2.9 Sampling

2.9.1 Sampling Theorem

A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal. i. e.

$$f_s \geq 2f_m$$

Possibility of sampled frequency spectrum with different conditions is given by the following diagrams:



2.9.2 Aliasing Effect

The overlapped region in case of under sampling represents aliasing effect, which can be removed by
1. Considering f_s greater than $2f_m$
2. By using anti aliasing filters

2.9.3 Signals Sampling Techniques

There are three types of sampling techniques:
1. Impulse sampling
2. Natural sampling
3. Flat Top sampling

2.9.4 Impulse Sampling

Impulse sampling can be performed by multiplying input signal x(t) with impulse train

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

of period 'T'. Here, the amplitude of impulse changes with respect to amplitude of input signal x(t). The output of sampler is given by

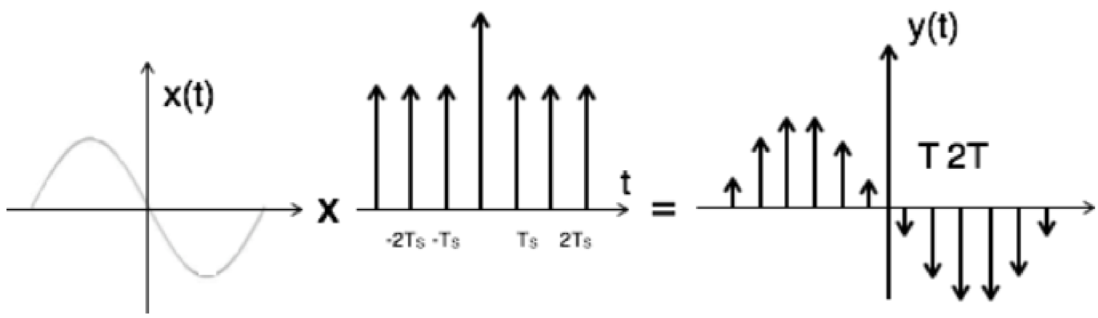


Figure 2.15: Output of Sampler

$$y(t) = x(t) \times ImpulseTrain$$

$$y(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$y(t) = y_\delta(t) = \sum_{n=-\infty}^{\infty} x(nt)\delta(t - nT)\text{.........}(1)$$

To get the spectrum of sampled signal, consider Fourier transform of equation 1 on both sides

$$Y(\omega) = 1/T \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

This is called ideal sampling or impulse sampling. You cannot use this practically because pulse width cannot be zero and the generation of impulse train is not possible practically.

2.9.5 Natural Sampling

Natural sampling is similar to impulse sampling, except the impulse train is replaced by pulse train of period T. i.e. you multiply input signal x(t) to pulse train

$$\sum_{n=-\infty}^{\infty} P(t-nT)$$

as shown below

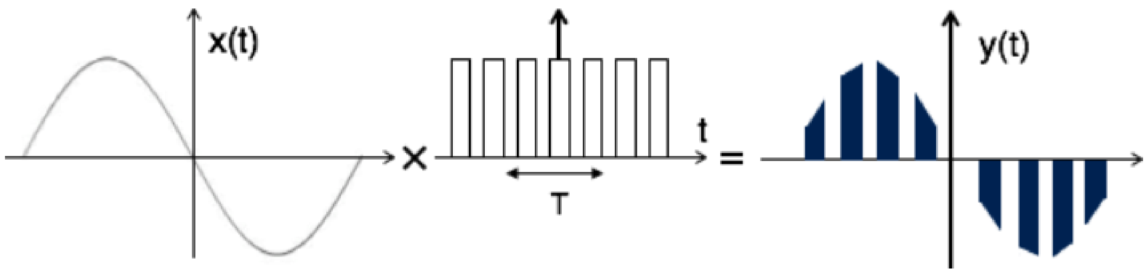


Figure 2.16: Natural Sampling of an input signal

The output of sampler is

$$y(t) = x(t) \times PulseTrain$$

$$y(t) = x(t) \times p(t)$$

$$y(t) = x(t) \times \sum_{n=-\infty}^{\infty} P(t-nT)..........1$$

The exponential Fourier series representation of p(t) can be given as

$$p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_s t}..........2$$

$$p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn2\pi f_s t}$$

Where

$$1/T \int_{-T/2}^{T/2} p(t)e^{-jn\omega_s t}dt$$

$$F_n = n\omega_s/TP$$

Substitute F_n value in equation 2

$$p(t) = 1/T \sum_{n=-\infty}^{\infty} P(n\omega_s)e^{jn\omega_s t}$$

Substitute p(t) in equation 1

$$y(t) = 1/T \sum_{n=-\infty}^{\infty} P(n\omega_s)x(t)e^{jn\omega_s t}$$

To get the spectrum of sampled signal, consider the Fourier transform on both sides

$$\boldsymbol{F.T\,[y(t)] = F.T[\tfrac{1}{T}\Sigma_{n=-\infty}^{\infty}P(n\omega_s)\,x(t)\,e^{jn\omega_s t}]}$$

$$\boldsymbol{= \tfrac{1}{T}\Sigma_{n=-\infty}^{\infty}P(n\omega_s)\,F.T\,[x(t)\,e^{jn\omega_s t}]}$$

Figure 2.17: Fourier Transform performed on both sides

According to frequency shifting property

$$\boldsymbol{F.T\,[x(t)\,e^{jn\omega_s t}] = X[\omega - n\omega_s]}$$

$$\boldsymbol{\therefore\ Y[\omega] = \tfrac{1}{T}\Sigma_{n=-\infty}^{\infty}P(n\omega_s)\,X[\omega - n\omega_s]}$$

Figure 2.18: Shifting Property Applied

Refer [1]

Chapter 3

Assignment Problems

3.1 Question 1

A low pass RC filter has been designed with R=100KΩ and C= 4μF .What is the transfer function and corner frequency (cut-off frequency) of this filter. Sketch the bode plot of the filter and suggest a situation where you would use this filter.

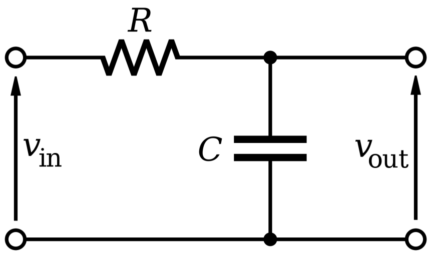


Figure 3.1: RC low pass filter

Authors

(i) Saketika Chekuri

Solution

Consider the Kirchoff's Voltage Law equation for the above circuit in the time domain:

$$V_{in}(t) - \frac{dq}{dt}R - \frac{q}{C} = 0$$

We can write $V_{out}(t)$ as follows, from the diagram:

$$V_{out}(t) = \frac{q}{C}$$

Taking the Laplace Transform of the above equations, while assuming that the capacitor is initially uncharged, we get:

$$V_{in}(s) - sQ(s)R - \frac{Q(s)}{C} = 0$$

$$V_{out}(s) = \frac{Q(s)}{C}$$

Combining both the equations, we get:

$$V_{in}(s) - sQ(s)R - V_{out}(s) = 0$$

which can be rearranged as:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1}$$

Plugging in R=100KΩ and C= 4μF yields the transfer function

$$H(s) = \frac{1}{0.4s + 1}$$

The theoretical cut-off for this filter is given by

$$f_c = \frac{1}{2\pi RC} \approx 0.398\ Hz$$

The bode plots are:

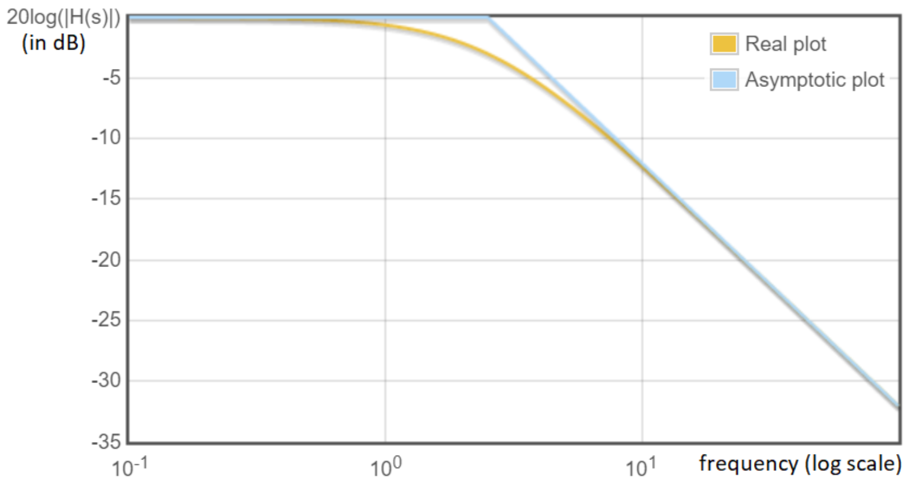


Figure 3.2: Bode plot- Magnitude

As we can see, the given setup acts as a passive(since it has only passive components like resistors, capacitors) low pass filter as it passes only those signals with

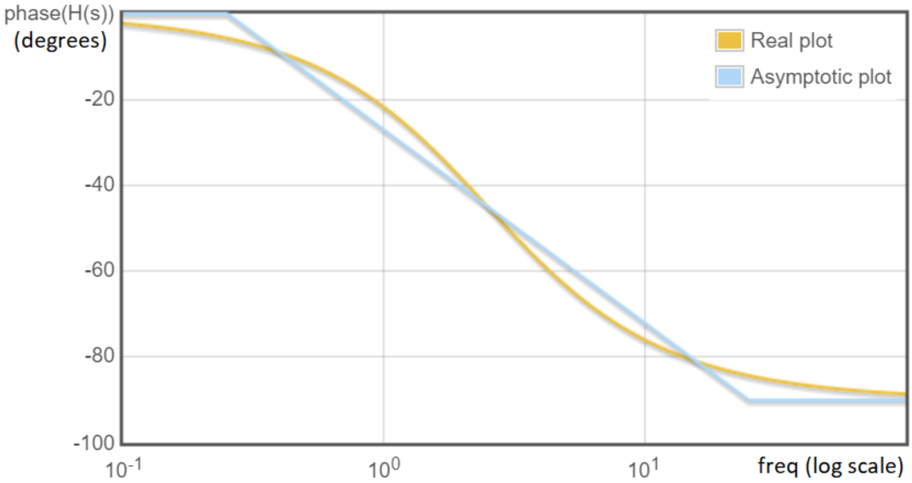


Figure 3.3: Bode plot- Phase

a frequency lower than a selected cutoff frequency and attenuates signals with frequencies higher than the cutoff frequency.

For the bode plots, we take H(jw). Substituting jω in place of s in the transfer function for frequency response, we get $H(jw) = \frac{1}{0.4jw+1}$. From this relation, it is easy to that the bode plot for phase is consistent (for example, as $\omega \to 0$, $\angle H(jw)$ is also near 0 degrees and as $\omega \to \infty$, $\angle H(jw) \to$ -90 degrees).

Arguably, the most frequent use of such low-pass filters are to reduce electrical noise in circuits. By choosing a capacitor of suitable capacitance and connecting it directly across the terminals of the load, we can prevent the high frequency noise interfering with our system (Such a capacitor is also known as a 'decoupling' capacitor).

3.2 Question 2

Samples of signal x(t) are shown in the following figure. Draw reconstruction of this signal using:

(A) Shannon's reconstruction theorem

(B) Zero Order Hold

Authors

(i) K Sudheeradh

Solution

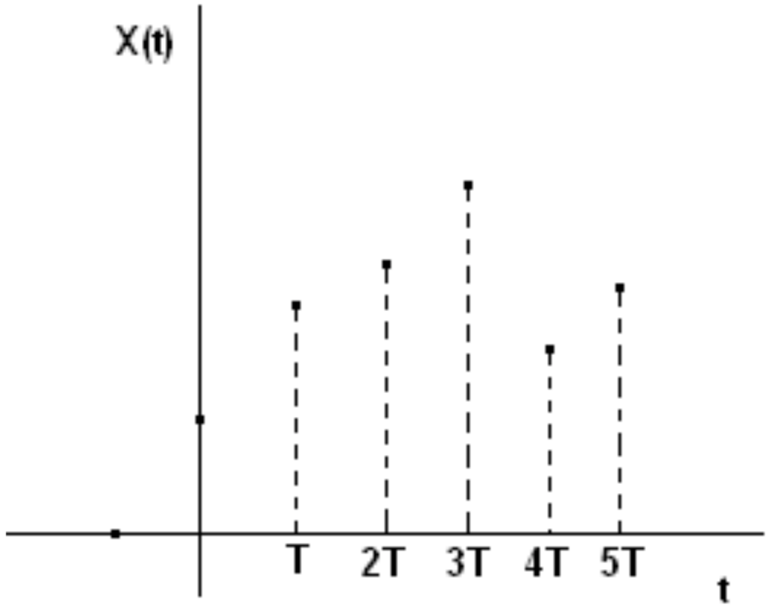


Figure 3.4: Sampled Signal

(A) Shannon's reconstruction theorem
Sampling theorem **signalsandsystems**:
Let $x(t)$ be a band-limited signal with $X(\omega) = 0$ for $|\omega_s| > 2\omega_M$, where $X(\omega)$ is the fourier transform of the signal $x(t)$, ω_s is the sampling frequency and ω_M is the frequency message signal. Then x(t) is uniquely determined by its samples x(nT), n = 0,± 1, ± 2, ... , if

$$\omega_s > 2\omega_M,$$

where

$$\frac{2\pi}{T}$$

Given these samples, we can reconstruct x(t) by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than ω_s - ω_M. The resulting output signal will 1 exactly equal x(t).

To retrieve the original signal from the sampled signal of time period T_S, the sampled signal is passed through Low-Pass filter of cutoff frequency ω_c and gain T_S where,

$$\omega_M \leq \omega_c \leq \omega_s - \omega_M$$

Without loss of generality we can chose $\omega_c = \omega_M$.

Low-pass filter is defined as follows:

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_M \\ 0 & \text{otherwise.} \end{cases}$$

The impulse response of ideal Low-Pass filter given above which is required for reconstruction of the signal is as follows:

$$h(t) = \frac{T_s \cdot \sin(\omega_m t)}{\pi \cdot t}$$

To recover the message signal, we must pass the sampled signal $x_s(t)$ through this Low-Pass filter. If the reconstructed signal is represented by $x_R(t)$ **signalsandsystems**, then,

$$x_R(t) = x_s(t) * h(t)$$

$$x_R(t) = x_s(t) * \frac{\sin(w_m t) \cdot T_s}{\pi \cdot t}$$

from the Nyquist rate, we have $\omega_s = 2\omega_M$. On substituting this value in above equation, we get,

$$x_R(t) = x_s(t) * \frac{\sin\left(\frac{w_s}{2} t\right) \cdot T_s}{\pi \cdot t}$$

On convolving, we get the final reconstructed signal as follows,

$$x_R(t) = T_s \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left(\frac{w_s}{2}\left(t - nT_s\right)\right)}{\pi\left(t - nT_s\right)}$$

This is equivalent to interpolating using sinc function in time domain. On plotting the sinc functions at the discrete time intervals scaled by the sampled signal, we get the following plots,

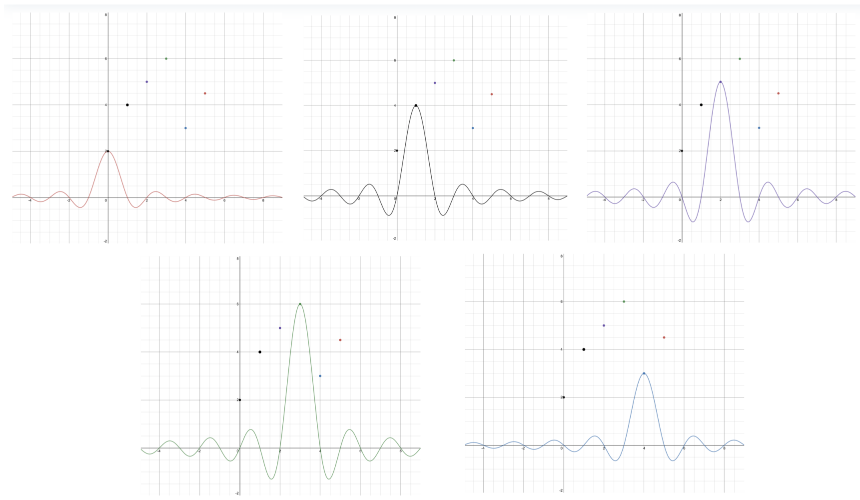


Figure 3.5: sinc functions

On summing up all the sinc functions from the sampled signal, we get the recovered message signal,

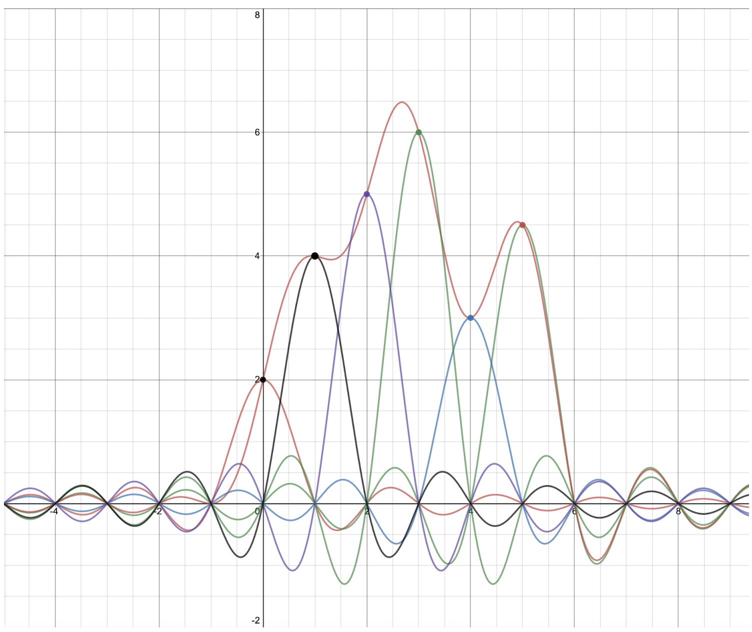


Figure 3.6: Reconstructed signal

Hence, the reconstructed signal is as follows,

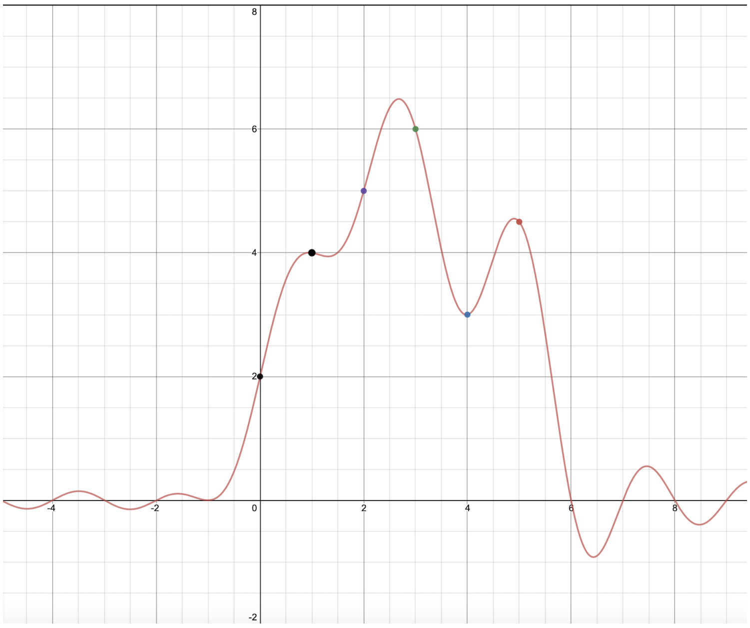


Figure 3.7: Reconstructed signal

(B) Zero Order Hold **signalsandsystems**

A zero-order hold reconstructs the continuous-time waveform from a sample sequence x[n], assuming one sample per time interval T i.e., constant value of x[n] is held for T seconds. The zero order hold reconstructed signal can be modeled as the output of a linear time-invariant filter with impulse response equal to a rect function. The rect function is as follows:

$$h_o(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

The reconstructed signal $x_R(t)$ is derived as follows,

$$x_R(t) = x_s(t) * h_o(t)$$

$$x_R(t) = \sum_{n=-\infty}^{\infty} x_s[n]\delta(t - nT) * h_o(t)$$

$$x_R(t) = \sum_{n=-\infty}^{\infty} x_s[n] \cdot h_o(t - nT)$$

The reconstructed signal obtained when the sampled signal is passed through the Zero Order Hold system(ZOH) is shown in the next page.

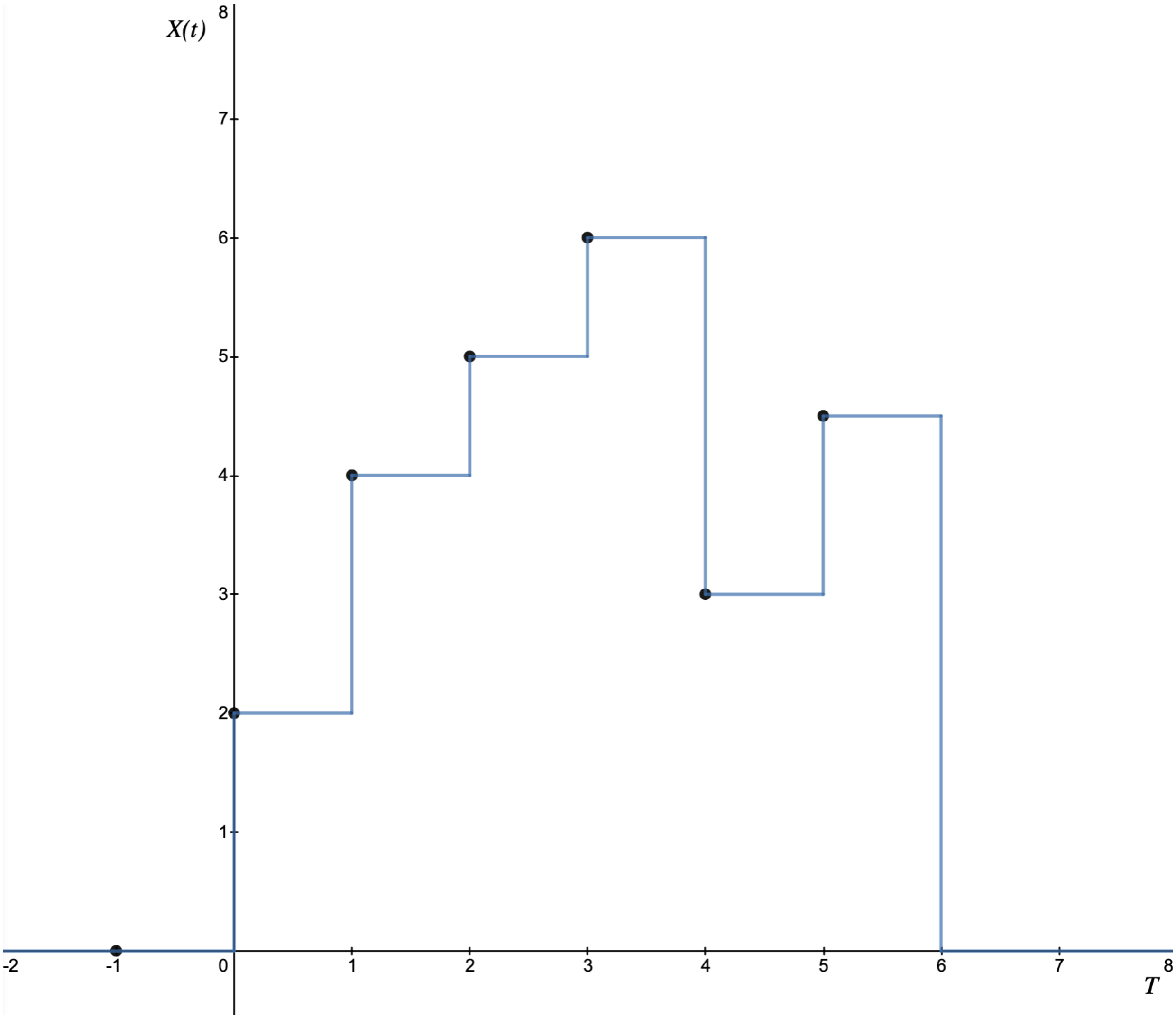


Figure 3.8: ZOH Reconstructed signal

3.3 Question 3

A moving-average filter operates by averaging a number of points from the input signal to produce each point in the output signal. It is usually of the form:

$$y[i] = \frac{1}{M}\sum_{k=0}^{M-1} x[i-k]$$

Eg: For M=2,

$$y[i] = \frac{x[i] + x[i-1]}{2}$$

(i) Show that the moving average filter is basically a convolution of the input signal with a rectangular pulse of area 1.

The moving average filter is often the first thing tried when faced with a problem - as it is very good for many applications, and being optimal for a common problem, by reducing random white noise while keeping the sharpest step response

(ii) Compare the frequency response of an ideal Low-pass filter with that of a moving average function - one being of a lower order (3-point), and another of a higher-order (11-point). Compare the roll-off and stop-band attenuation, and show that a moving average filter while being a good smoothing filter, is a relatively bad low-pass filter

(iii) Implement a recursive moving average filter on MATLAB.

For a given stock market data to identify the trend lines using a 3-point and 11-point filter.

For a given rectangular pulse hidden in noise use a 3-point and 11-point filter to identify it

Author

Avish Wagde

Solution

(i) The convolution of two discrete time function is given by ,

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Now if we analyse $u[k] - u[k-M]$ where $u[n]$ is the unit step response. We see that it's a rectangular pulse with area M. Therefore, it's evident that

$$\frac{1}{M}\big(u[k] - u[k-M]\big)$$

has area 1, and it's a Rectangular pulse.

Now we can say as $u[k]$ is zero for all negative values of k,

$$\sum_{k=-\infty}^{\infty} x[n-k]u[k] = \sum_{k=0}^{\infty} x[n-k]$$

And similarly,

$$\sum_{k=-\infty}^{\infty} x[n-k]u[k-M] = \sum_{k=M}^{\infty} x[n-k]$$

Subtracting the above two equations, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} x[n-k]\big(u[k] - u[k-M]\big) &= \sum_{k=0}^{\infty} x[n-k] - \sum_{k=M}^{\infty} x[n-k] \\ &= \sum_{k=0}^{M-1} x[n-k]\end{aligned}$$

Dividing the above equation throughout by M we get the moving average filter,

$$\begin{aligned}\frac{1}{M}\sum_{k=-\infty}^{\infty} x[n-k]\big(u[k] - u[k-M]\big) &= \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] \\ \sum_{k=-\infty}^{\infty} x[n-k]\big(\frac{u[k] - u[k-M]}{M}\big) &= \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\end{aligned}$$

By the above equation it's precisely visible that the moving average filter is nothing but the convolution of input function x and a rectangular pulse function h with area 1 , where h is given by

$$h[n] = \frac{u[n] - u[n-M]}{M}$$

Hence Proved.

(ii) The frequency response of Ideal Low pass filter is given by

$$\begin{aligned} H(f) &= a \qquad \forall f \in (-B, B) \\ &= 0 \qquad otherwise \end{aligned}$$

here B is called the *Bandwidth* of the response, below plotted is one such response for bandwidth 1 and $a = 1$.

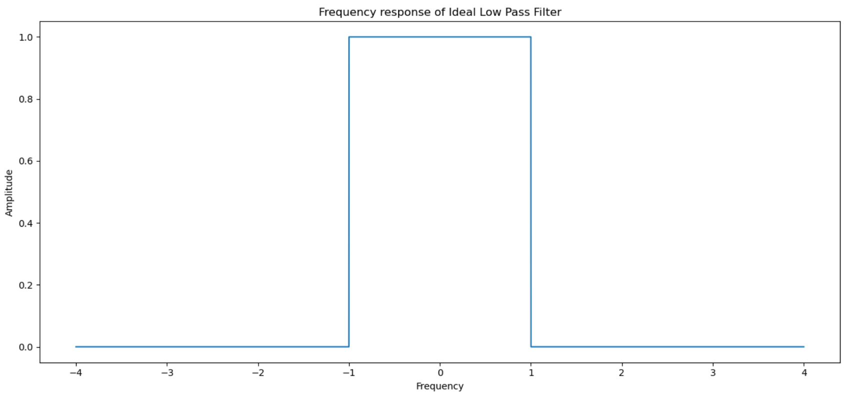


Figure 3.9: Frequency response of Ideal Low Pass filter

And the frequency response of a Moving Average Filter with M averaging order is given by

$$H(f) = \frac{1}{M}\frac{(1 - e^{-j2\pi fM})}{(1 - e^{-j2\pi f})}$$

where the amplitude of each frequency is the absolute value of $H(f)$ i.e. $|H(f)|$

$$|H(f)| = \frac{1}{M}\frac{\sin \pi fM}{\sin \pi f}$$

below plotted are such responses for $M = 3$ (blue) and $M = 11$ (orange).

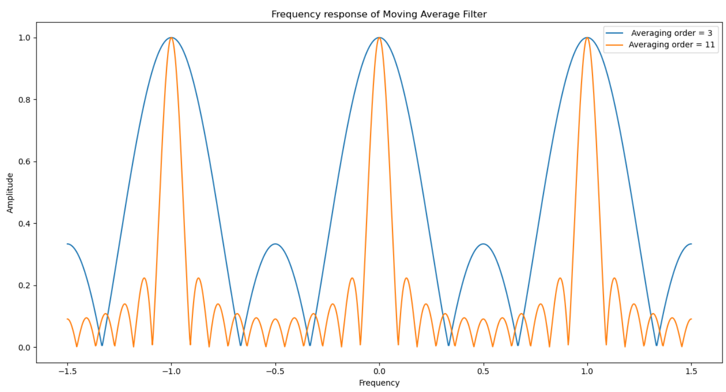


Figure 3.10: Frequency response of Moving Average filter

From the frequency response it can be asserted that the roll-off is very slow and the stop band attenuation is not good. Given this stop band attenuation,clearly, the moving average filter cannot separate one band of frequencies from another. But of all the possible linear filters that could be used, the moving average produces the lowest noise for a given edge sharpness. To justify our fact lets imagine we want to design a filter with a fixed edge sharpness. Let's assume we fix the edge sharpness by specifying that there are eleven points in the rise of the step response. This requires that the filter kernel have eleven points. Now the optimization question is : how do we choose the eleven values in the filter kernel to minimize the noise on the output signal? Since the noise we are trying to reduce is random, none of the input points is special; each is just as noisy as its neighbor. Therefore, it is useless to give preferential treatment to any one of the input points by assigning it a larger coefficient in the filter kernel. The lowest noise is obtained when all the input samples are treated equally, i.e., the **moving average filter** .

As the filter length increases (the parameter M) the smoothness of the output increases, whereas the sharp transitions in the data are made increasingly blunt. This implies that this filter has excellent time domain response but a poor frequency response. As visible from its frequency response the Moving average filter with length 11 has better roll off and stop band attenuation than of length 3, but still the roll off and stop band attenuation is very bad in comparison to Ideal Low pass filter.

The figure on the next page shows how powerful Moving average filter can be for smoothing. On the first plot, we have the input that is going into the moving average filter. The input is noisy and our objective is to reduce the noise. The next figure is the output response of a 3-point Moving Average filter. It can be deduced from the figure that the 3-point Moving Average filter has not done much in filtering out the noise. We increase the filter taps to 51-points and we can see that the noise in the output has reduced a lot, which is depicted in next figure. We increase the taps further to 101 and 501 and we can observe that even-though the noise is almost zero, the transitions are blunted out drastically (observe the slope on the either side of the signal and compare them with the ideal brick wall transition in our input).

As the filter length increases (the parameter M) the smoothness of the output increases, whereas the sharp transitions in the data are made increasingly blunt. This implies that this filter has excellent time domain response but a poor frequency response.Remember, good performance in the time domain results in poor performance in the frequency domain, and vice versa. In short, the moving average is an exceptionally good smoothing filter (the action in the time domain), but an exceptionally bad low-pass filter (the action in the frequency domain).

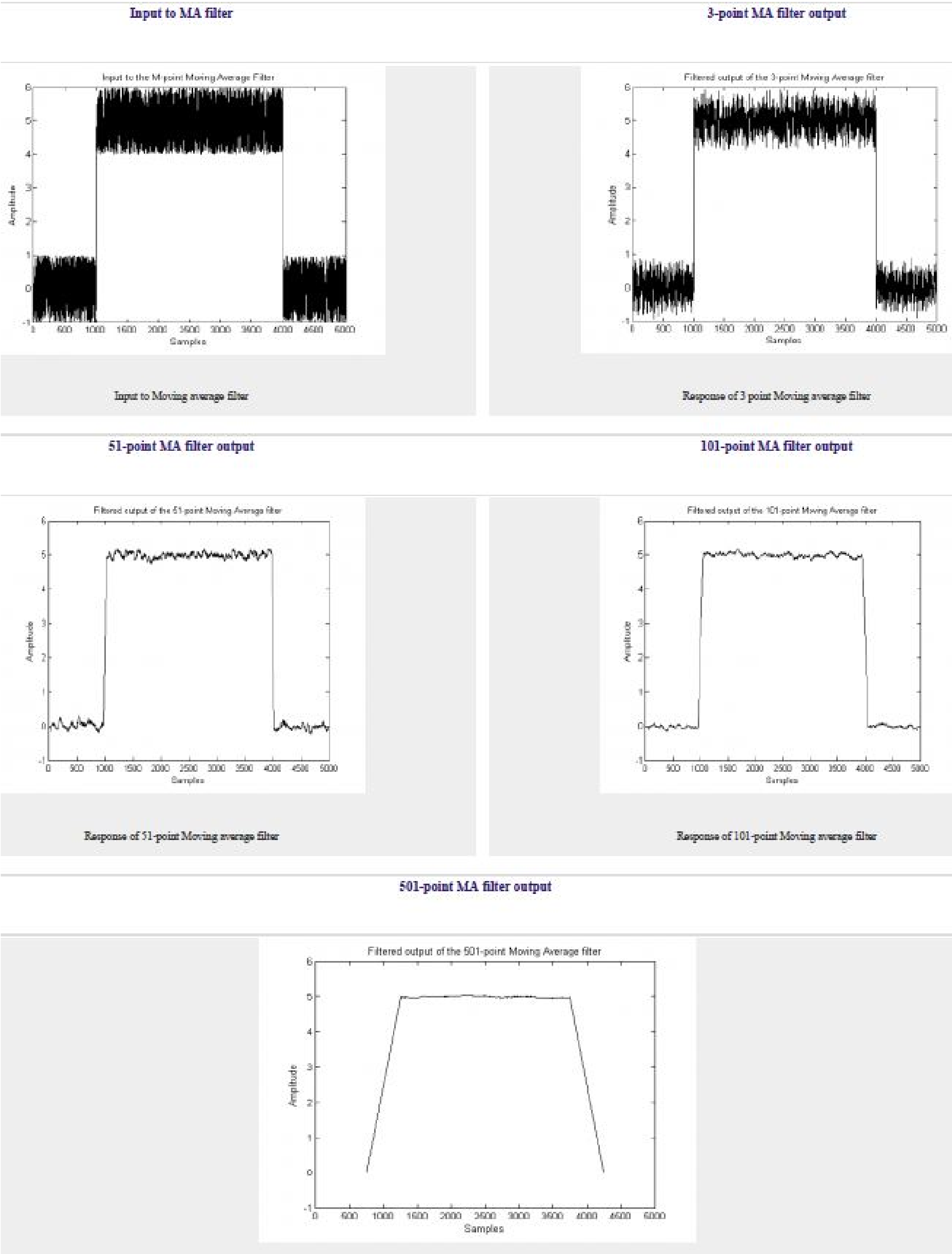


Figure 3.11: Smoothing action of Moving Average filter. This is a case study by Mathuranathan Viswanathan. (https://www.gaussianwaves.com/2010/11/moving-average-filter-ma-filter-2/)

(iii) MATLAB offers a excellent and easy way to implement a recursive moving average filter by the use of a inbuilt function called **movmean**.

The assignment $M = movmean(A, k)$ returns an array of local k-point mean values, where each mean is calculated over a sliding window of length k across neighboring elements of A. When k is odd, the window is centered about the element in the current position. When k is even, the window is centered about the current and previous elements. The window size is automatically truncated at the endpoints when there are not enough elements to fill the window. When the window is truncated, the average is taken over only the elements that fill the window. M is the same size as A.

Given below is a plot of Highest price of stocks over a day from 2012 - 2016. Each day is represented by a number from 1 to 1227 in succession. the red line plot is the actual data plot and the green, black and blue dashed line plots are the moving average trend lines for lengths 3,11 and 51 respectively.

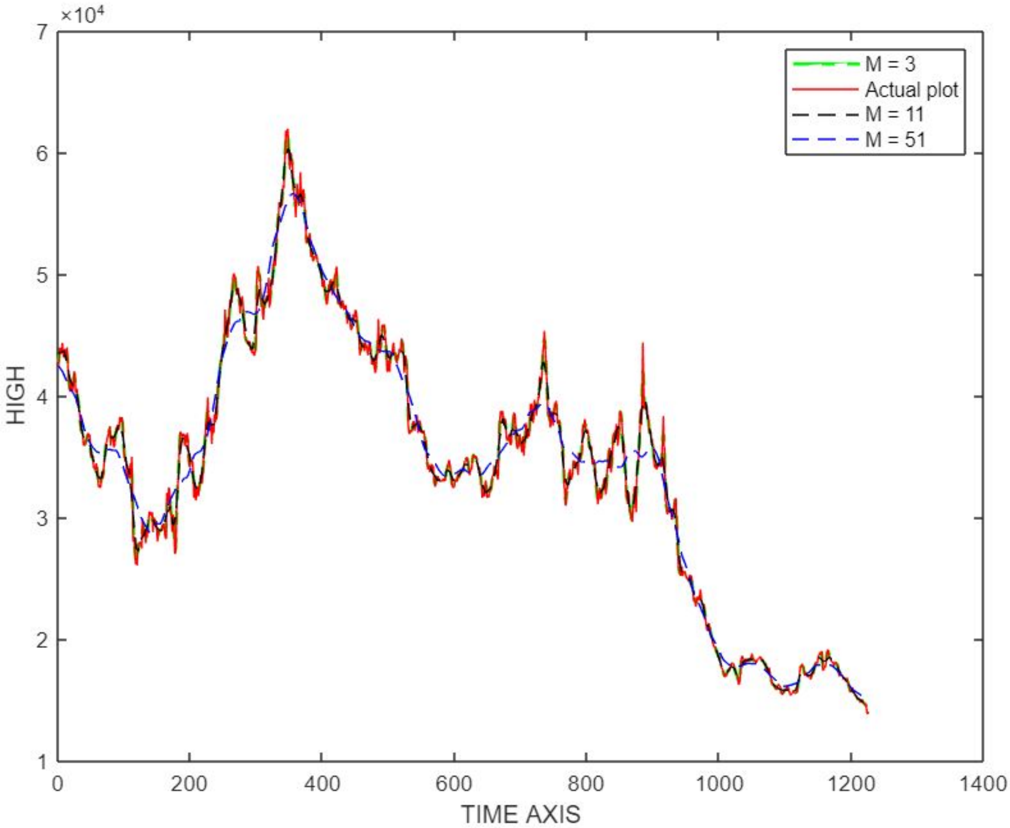


Figure 3.12: Trendlines using Moving Average filter.

as the plot above might not be visible a zoomed view is given to see the trends.

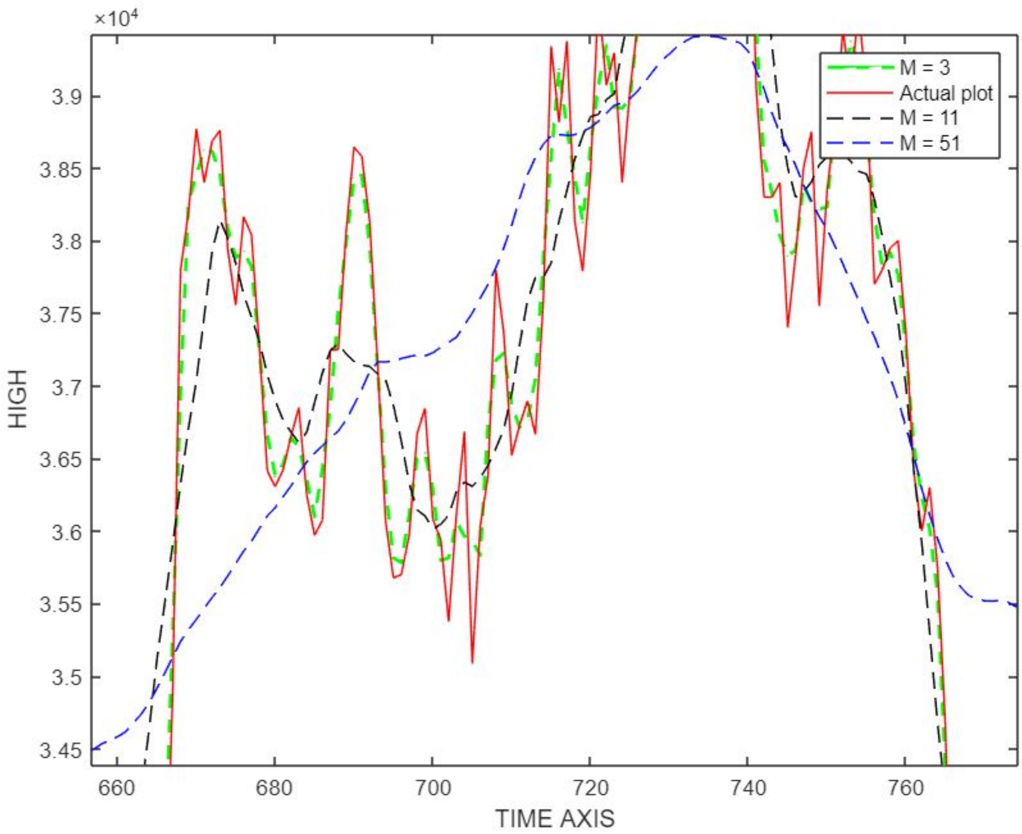


Figure 3.13: Zoomed view.

The above was implemented using the following program in MATLAB.

https://github.com/avish-meliodas/Moving-average-Filter.git

3.4 Question 4

Frequency domain analysis need not be restricted to just signals in the time domain. We can very well have signals in the spatial domain as well, such as those of images. Simple processes such as blurring an image, or sharpening an image can be performed easily in the frequency domain - use the convolution theorem to show this is valid.

For a given grayscale image - the following can be implemented on MATLAB:

1. Take the Fourier Transform of an image.
    (a) Add noise to the image data, and then take its Fourier Transform, comparing it to the previous
    (b) Implement a Low-Pass / High-Pass filter
    (c) Take the Inverse Fourier Transform to get an image

2. Create an image with periodical patterns with different orientation, which represent varying frequency. Take the Inverse Fourier Transform of this to investigate the effects of adding/removing patterns in the frequency domain on the spatial domain picture

Authors

(i) Madhav Joshi

Solution

The edges in the image correspond to the high frequency regions as there is sudden change of intensities in that region. Therefore:

$$\text{Cancelling High Frequency} \rightarrow \text{Blurring}$$

$$\text{Cancelling Low Frequency} \rightarrow \text{Sharpening}$$

The Fourier Transform (abbreviated later as FT) of an image represents the image of spacial domain in frequency domain i.e. how much contribution does each frequency have in making of the image.

The convolution theorem for continuous domain in $\Im^2$ is:

$$\mathrm{y}(x_1, x_2) = \mathrm{x}(x_1, x_2) * \mathrm{h}(x_1, x_2) = \int_{\Re}\int_{\Re} \mathrm{x}(\tau_1, \tau_2)\,\mathrm{h}(x_1 - \tau_1, x_2 - \tau_2) d\tau_1 d\tau_2$$

where, x_1, x_2 may represent the 2D space coordinates, $\mathrm{y}(x_1, x_2)$ is the output signal/image of the input signal/image $\mathrm{x}(x_1, x_2)$, $\mathrm{h}(x_1, x_2)$ is the impulse response of the system which can also be referred to as filter of the image as $\mathrm{h}(x_1, x_2)$ scans through the input $\mathrm{x}(x_1, x_2)$ and makes changes accordingly to give the output $\mathrm{y}(x_1, x_2)$ and * represents convolution between $\mathrm{x}(x_1, x_2)$ and $\mathrm{h}(x_1, x_2)$.

By the convolution property of FT this becomes simple multiplication of $\mathfrak{F}\{\mathrm{x}(x_1, x_2)\}$ (say $= \mathrm{X}(\omega_1, \omega_2)$) and $\mathfrak{F}\{\mathrm{h}(x_1, x_2)\}$ (say $\mathrm{H}(\omega_1, \omega_2)$) in the frequency domain to give $\mathfrak{F}\{\mathrm{y}(x_1, x_2)\}$ (say $\mathrm{Y}(\omega_1, \omega_2)$):

$$\mathrm{Y}(\omega_1, \omega_2) = \mathrm{X}(\omega_1, \omega_2) \times \mathrm{H}(\omega_1, \omega_2)$$

Hence to cancel the high frequency i.e. blurring an image, we choose $\mathrm{H}(\omega_1, \omega_2)$ as:

$$\|\,\mathrm{H}(\omega_1, \omega_2)\| = 1,\ if\ |\omega| \leqslant \omega_c\ and\ 0\ otherwise$$

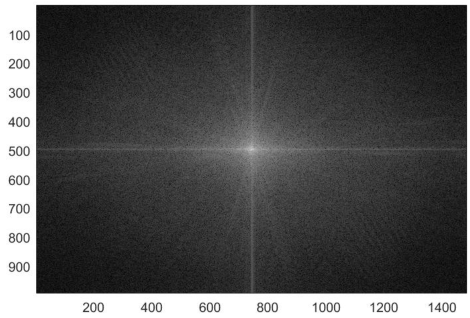
to allow only those frequencies which are $\leqslant$ critical frequency ω_c.
And to cancel low frequency i.e. sharpening an image, we choose $\mathrm{H}(\omega_1, \omega_2)$ as:

$$\|\,\mathrm{H}(\omega_1, \omega_2)\| = 1,\ if\ |\omega| \geqslant \omega_c\ and\ 0\ otherwise$$

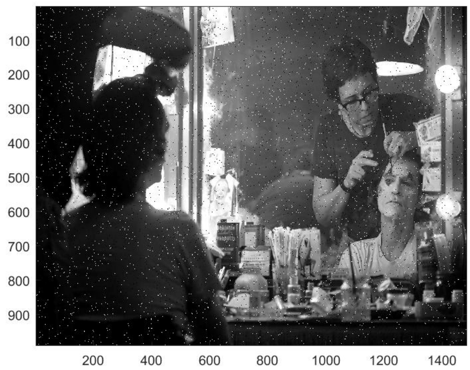
to allow only those frequencies which are $\geqslant$ critical frequency ω_c. To get the output image $\mathrm{y}(x_1, x_2)$ we just do inverse FT i.e. $\mathfrak{F}^{-1}\{\mathrm{y}(x_1, x_2)\}$

Now applying the above mentioned procedure on an image:

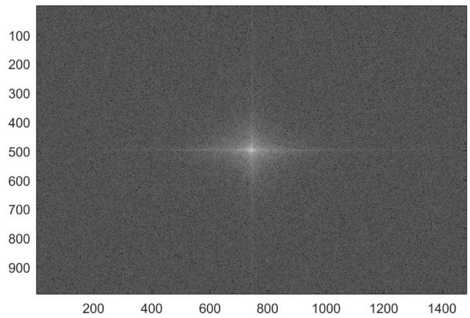
(a) Sample image



(b) FT of left image



(c) Noise in image



(d) FT of left image

Figure 3.14: Image in grey scale and adding noise

1. Taking FT of a normal grey scale image and then adding noise to the image.

2. Applying a low pass filter first with high then with medium and finally with small critical frequency value to the noised image.

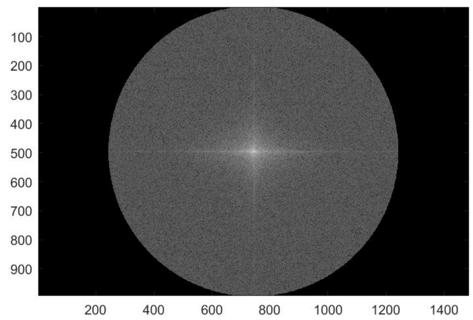
3. Now applying high pass filter with extremely low critical frequency value and then higher and then even higher.

After this we take images with periodic nature and observe the orientation of their inverse FT.

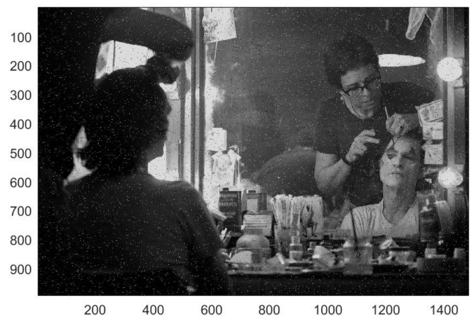
1. We take an image of horizontal dotted line with each point of one pixel size and then take its inverse FT.

2. Then we tilt the line by 45 deg anticlockwise and take its inverse FT. Notice that by keeping the number of points constant, the distance between them has increased, so the frequency.

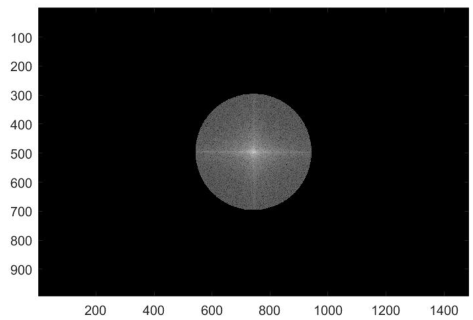
3. Then we again tilt it to make the line vertical.



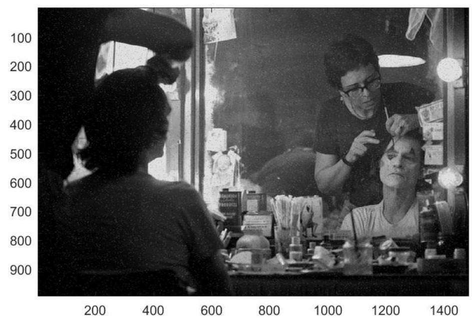
(a) High ω_c



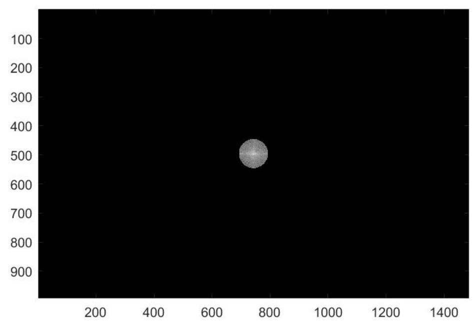
(b) Inverse FT of left



(c) Medium ω_c



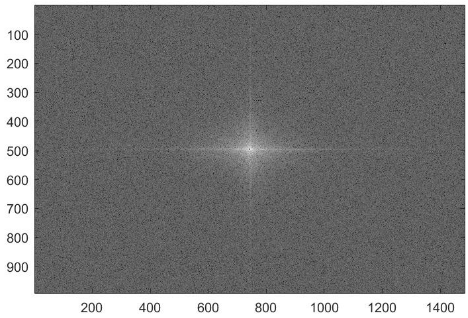
(d) Inverse FT of left



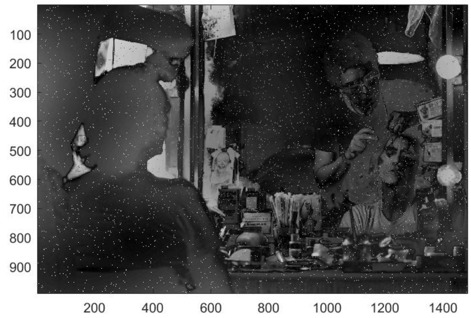
(e) Low ω_c

(f) Inverse FT of left

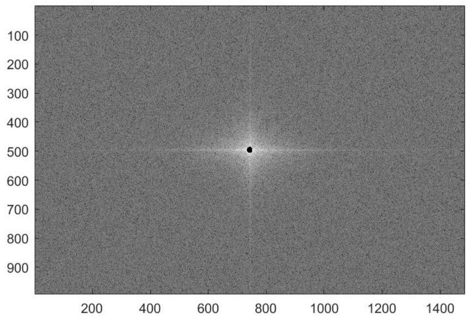
Figure 3.15: Applying Low pass filters



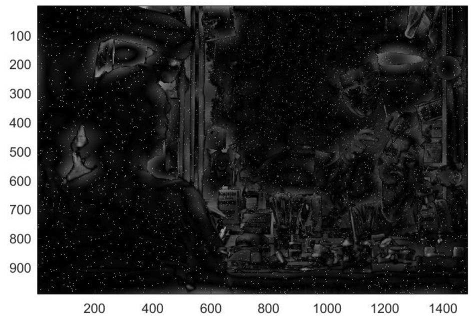
(a) Low ω_c



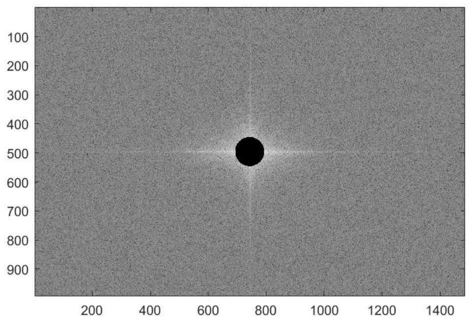
(b) Inverse FT of left



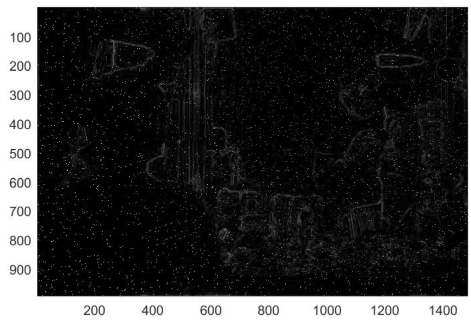
(c) Medium ω_c



(d) Inverse FT of left

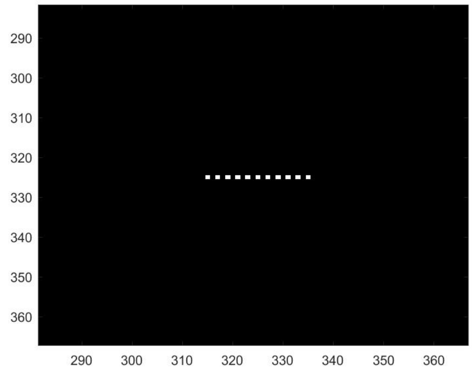


(e) High ω_c

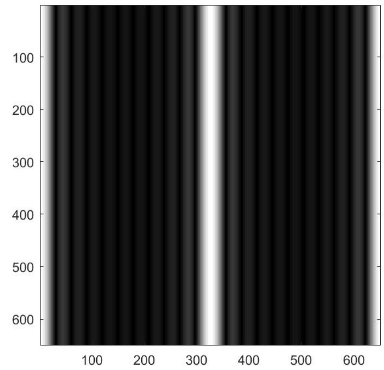


(f) Inverse FT of left

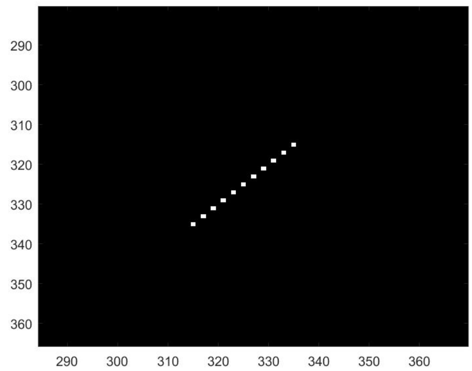
Figure 3.16: Applying High pass filters



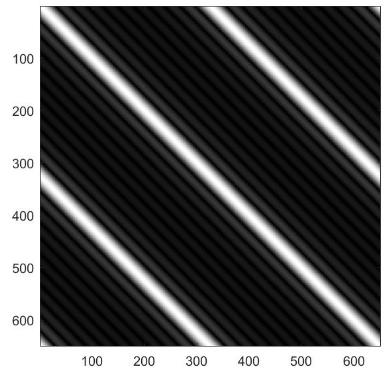
(a) Horizontal Line



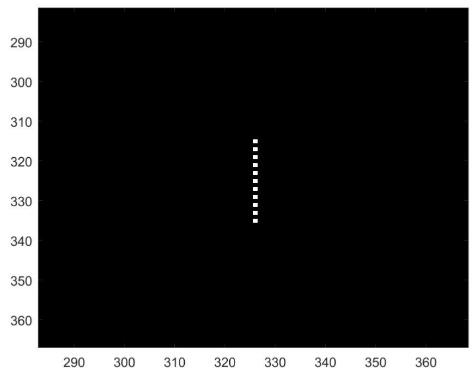
(b) Inverse FT of left



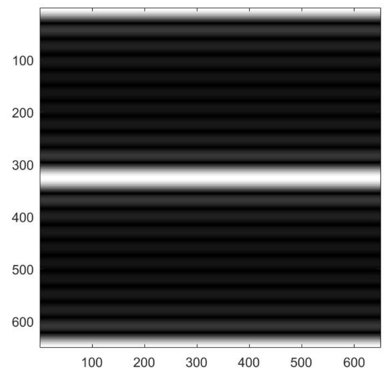
(c) Diagonal Line



(d) Inverse FT of left



(e) Vertical Line

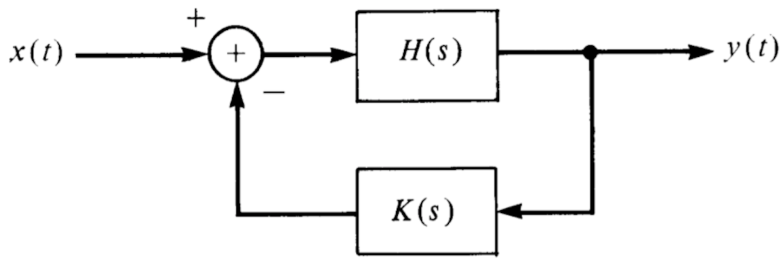


(f) Inverse FT of left

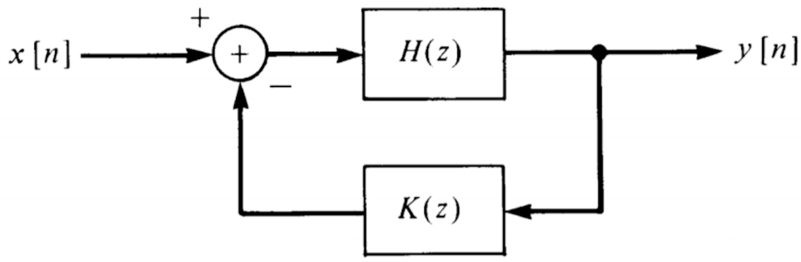
Figure 3.17: Applying inverse FT to a dotted line

3.5 Question 5

Consider the two feedback systems:



(a) Continuous-time System



(b) Discrete-time System

Figure 3.18: Feedback Systems

1. Express the overall system functions
   (a) $Q(s) = \frac{Y(s)}{X(s)}$
   (b) $Q(z) = \frac{Y(z)}{X(z)}$, in terms of the system functions in the forward and feedback paths - H, K

   Assume $K(s), K(z)$ are both constant values K and that:
   $H(s) = \frac{2}{s-2}$, $H(z) = \frac{2}{z-2}$
2. Plot the pole-zero pattern of $Q(s)$ and $Q(z)$ for $K = 0, 1, -1$
3. Indicate the locus in the s-plane of the poles of $Q(s)$ as K increase from $K = 0$ towards large positive values and as K decreases from $K = 0$ towards large negative values
4. Specify the range of values of K for which each of the systems is stable.

Authors

(i) Ilindra Shreya

Solution

1. System functions :

   (a) Continuous-time Feedback System :

   Considering $e(t)$ as the input for the control element and $z(t)$ as the output of the feedback element or simply the feedback signal, we have :

   from the Convolution property of Laplace transform (reference [1])

$$Y(s) = H(s)E(s) \tag{3.1}$$

   where

$$E(s) = X(s) - Z(s) \tag{3.2}$$

   and

$$Z(s) = K(s)Y(s) \tag{3.3}$$

   from which we get

$$Y(s) = H(s)[X(s) - K(s)Y(s)] \tag{3.4}$$

   or

$$Q(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + H(s)K(s)} \tag{3.5}$$

   (b) Discrete-time Feedback System :

   Considering $e[n]$ as the input for the control element and $z[n]$ as the output of the feedback element or simply the feedback signal, we have :

   from the Convolution property of z - transform

$$Y(z) = H(z)E(z) \tag{3.6}$$

   where

$$E(z) = X(z) - Z(z) \tag{3.7}$$

   and

$$Z(z) = K(z)Y(z) \tag{3.8}$$

   from which we get

$$Y(z) = H(z)[X(z) - K(z)Y(z)] \tag{3.9}$$

   or

$$Q(z) = \frac{Y(z)}{X(z)} = \frac{H(z)}{1 + H(z)K(z)} \tag{3.10}$$

2. Pole-zero patterns of Q(s) :

Substituting $H(s) = \frac{2}{s-2}$, $K(s) = K$ in equation 3.5, we get

$$Q(s) = \frac{\frac{2}{s-2}}{1 + (\frac{2}{s-2})K} \tag{3.11}$$
$$= \frac{2}{s - 2(1-K)}, s \neq 2$$

For rational Laplace transform - $X(s)$, the roots of the numerator polynomial are referred as the zeros of $X(s)$ and the roots of the denominator polynomial are referred as the poles of $X(s)$. The poles and zeroes of $X(s)$ in the finite s-plane completely characterize the algebraic expression for $X(s)$ to within a scale factor.
Therefore, the zeroes of $Q(s)$ are roots of

$$f(s) = 2 = 0 \tag{3.12}$$

i.e. $Q(s)$ does not have zeroes
and poles are roots of

$$g(s) = s - 2(1-K) = 0 \tag{3.13}$$

i.e. $s = 2(1-K)$ are the poles of $Q(s)$
Since $s \neq 2$, for $K = 0$ poles do not exist

Pole-zero patterns of $Q(z)$:

For $H(z) = \frac{2}{z-2}$, $K(z) = K$ $Q(z)$ would be similar to that of the above continuous-time system with z replacing s

$$Q(z) = \frac{2}{z - 2(1-K)}, z \neq 2 \tag{3.14}$$

Poles and and zeroes of z-transform are defined in the same fashion as that of Laplace transform. Therefore similar to the case of continuous-system we have
poles at $z = 2(1-K)$
Since $z \neq 2$, for $K = 0$ poles do not exist
zeroes does not exist

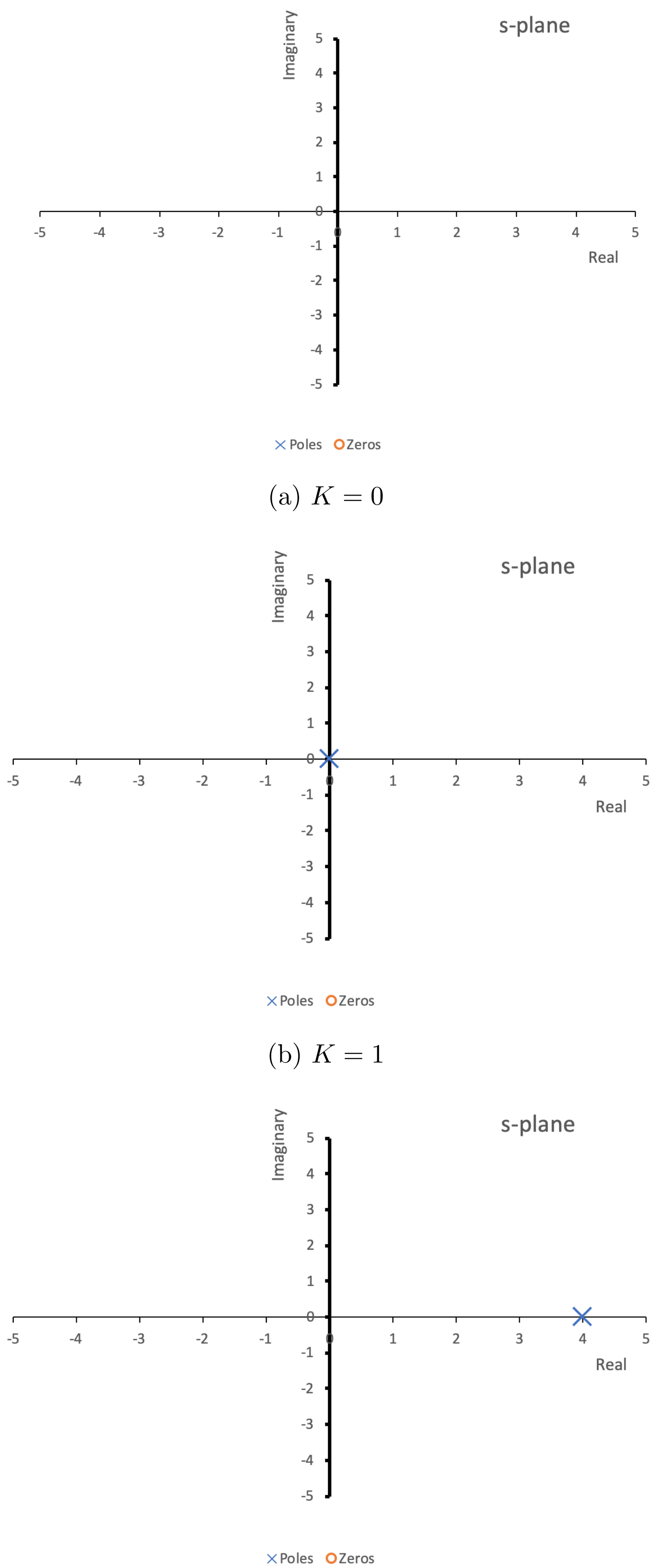
3. Poles of $Q(s)$ and $Q(z)$ for different values of k are given by

$$s = 2(1-K) \tag{3.15}$$

and

$$z = 2(1-K) \tag{3.16}$$

respectively
For K increasing from $K = 0$ to large positive values, the locus of poles of $Q(s)$ and $Q(z)$ are same except the change of plane which is a ray on Real-axis with starting point (2,0) and directed towards negative real-axis. While the locus for K decreasing from $K = 0$ towards large negative values is also a ray on Real-axis with staring point (2,0) but directed towards negative real-axis. The loci are shown in figures 3.21 and 3.22

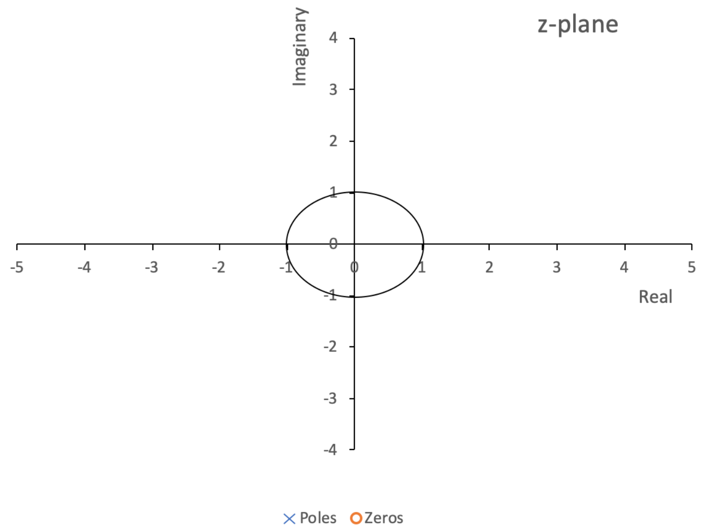


(a) $K = 0$

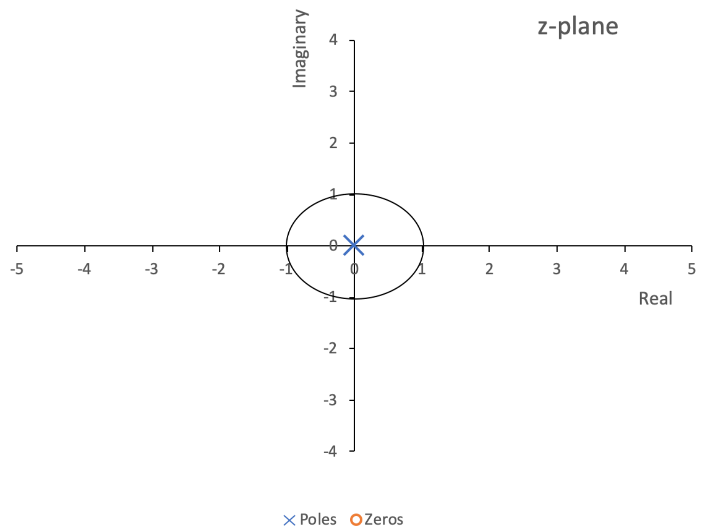
(b) $K = 1$

(c) $K = -1$

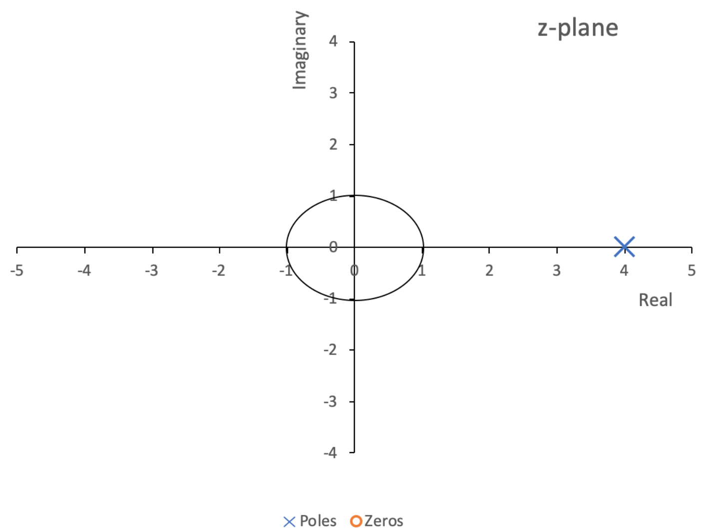
Figure 3.19: Pole-zero plots of Q(s) for different values of k



(a) $K = 0$

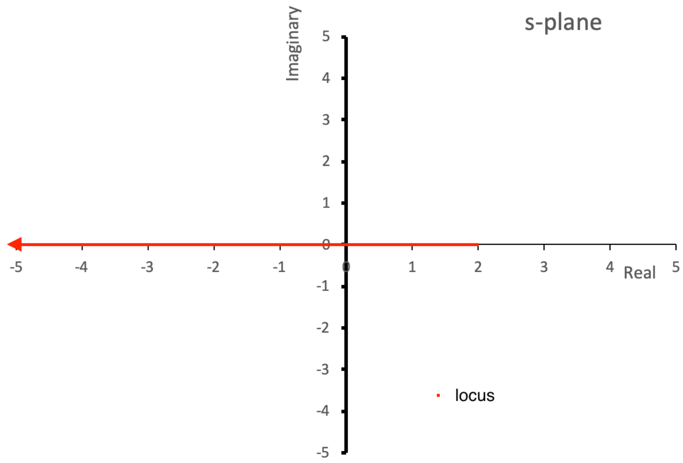


(b) $K = 1$

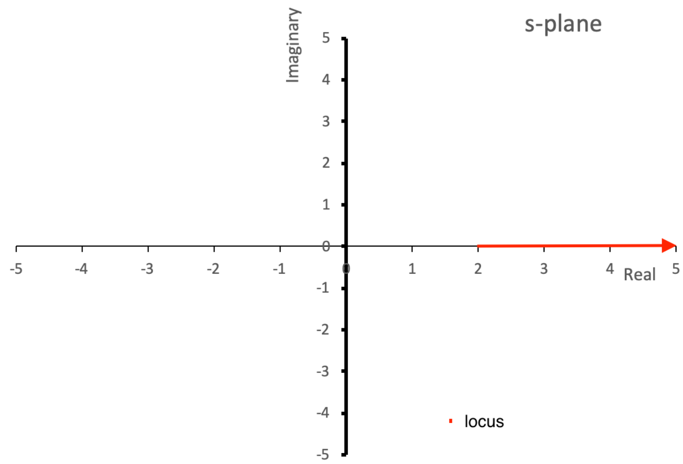


(c) $K = -1$

Figure 3.20: Pole-zero plots of $Q(z)$ for different values of k

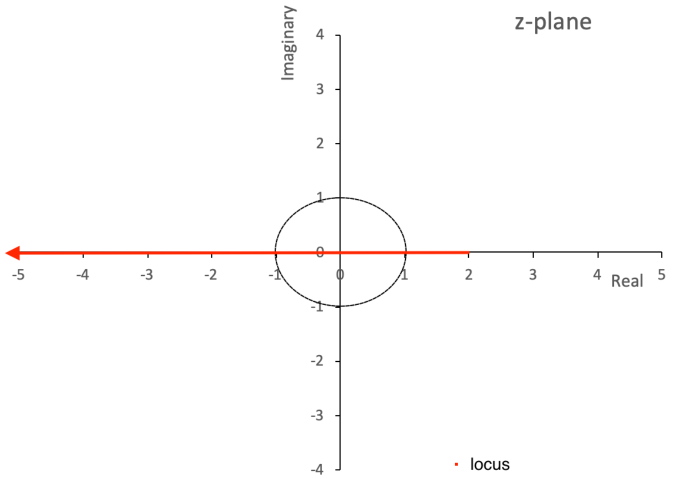


(a) K increasing to large positive values starting with 0

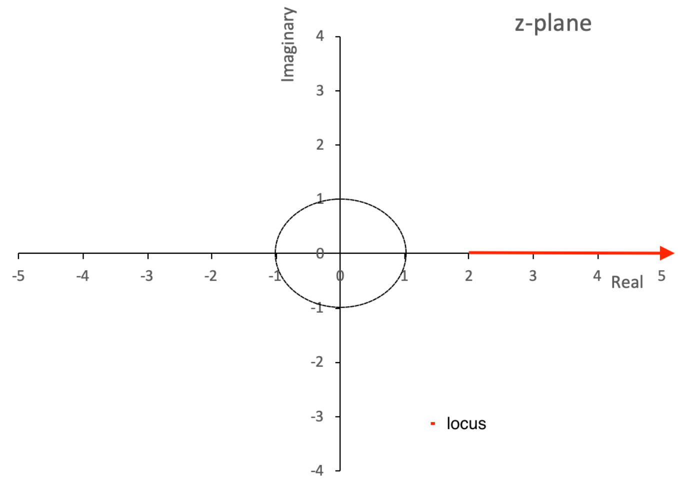


(b) K decreasing to large positive values starting with 0

Figure 3.21: locus of poles of $Q(s)$



(a) K increasing to large positive values starting with 0



(b) K decreasing to large positive values starting with 0

Figure 3.22: locus of poles of $Q(z)$

4. Stability is a system property that any bounded input yields a bounded output.

Accordingly, for a continuous LTI system to be stable, whenever the input $x(t)$ is bounded the output $y(t)$ should be bounded.
Consider an input $x(t)$ that is bounded in magnitude :

$$| x(t) | < B \text{ for all t} \tag{3.17}$$

Applying this input to an LTI system with unit impulse response $h(t)$, we have

$$\begin{aligned} | y(t) | &= | \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau | \\ &\leq \int_{-\infty}^{\infty} | h(\tau) || x(t-\tau) | d\tau \\ &\leq B \int_{-\infty}^{\infty} | h(\tau) | d\tau \end{aligned} \tag{3.18}$$

Therefore, a system is stable if its impulse response is absolutely integrable, i.e. if,

$$\int_{-\infty}^{\infty} | h(\tau) | d\tau < \infty \tag{3.19}$$

This implies the Fourier transform of the impulse response converges. Since the Fourier transform of a signal equals the Laplace transform evaluated along the jw-axis, we have :
An LTI system is stable if and only if the ROC of its system function includes the entire jw-axis.

Because of the typical applications in which feedback is utilized, it can be reasonably assumed that the given feedback systems are causal.
The output of a causal system is is a right-sided signal
Therefore, from the properties of ROC for Laplace Transform :

(a) ROC for a Laplace Transform is entire s-plane, right-half plane, left-half plane, a strip in s-plane parallel to imaginary axis for signal of finite duration, right-sided signal, left-sided signal, two-sided signal respectively. A two-sided signal may not have ROC at all.

(b) If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

(c) If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

The system function $Q(s)$ in eq (3.11) being rational with one pole at $2(1-K)$, ROC of $Q(s)$ is the region in the s-plane to the right of the rightmost pole(here it is the only pole of $Q(s)$). For the system to be stable this right-half plane should contain the imaginary axis, requiring the pole should have negative real part, i.e.,

$$Re(2(1-K)) < 0 \implies Re(K) > 1 \tag{3.20}$$

Therefore the given continuous-time feedback system is stable for $Re(K) \in (1, \infty)$ and $Im(K) \in (-\infty, \infty)$

For discrete-time systems, we obtain an analogous characterisation of stability in terms of impulse response of an LTI system. Specifically, if $\mid x[n] \mid < B$ for all n, then in analogy with eqs (3.17)-(3.20), it follows that the system is stable if the impulse response is absolutely summable, i.e. if,

$$\sum_{k=-\infty}^{\infty} \mid h[k] \mid < \infty \tag{3.21}$$

In this case the Fourier transform of h[n] converges and consequently, the ROC of the system function must include the unit circle.
Since the system is causal, the system function $Q(z)$ in (3.14) is z-transform of a right-sided signal.
From the properties of ROC for z-transform which are analogous to those mentioned for Laplace transform, ROC of $Q(z)$ is the region in the z-plane outside the outermost pole(here it is the only pole)-i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore, system being causal ROC also includes $z = \infty$. Therefore, for this system to be stable ROC should include the unit circle requiring the magnitude of the pole to be less than 1, i.e.,

$$\mid 2(1-K) \mid < 1 \implies \mid (1-K) \mid < 1/2 \tag{3.22}$$

$$\sqrt{(1-x)^2 + y^2} < 1/2 \tag{3.23}$$

where x and y are real and imaginary parts of K respectively

Therefore, the given discrete-time feedback system is stable for

$Im(K) \in (-1/2, 1/2)$ and $Re(K) \in (1 - \sqrt{\frac{1}{4} - y^2}, 1 + \sqrt{\frac{1}{4} - y^2})$
where y is the imaginary part of K

3.6 Question 6

Simulate a room with echo - suppose this speech signal is played in a large room which results in significant echo. Model the echo as being composed of a direct path which has no delay and two reflected paths, one corresponding to a delay of τ seconds and another corresponding to a delay of 2τ seconds. Let the reflection coefficient for these two paths be 0.9 and 0.81 respectively. Simulate the received signal y[n] from an input signal x[n]. Investigate this for different values of and, determine for which value of one can hear the echo, and also, some distortion in the signal. Additional: If you know the values of delays and reflection coefficient, implement a filter to recover the original signal x[n] from y[n]

Author

(i) Nishant Mittal

Solution

Simulating an echo from an audio signal and vice versa can be easily done via the 'z' domain. As this is a discrete system, we first have to calculate the delay in terms of sampling periods from the time delay τ. Considering we know the frequency of the audio clip i.e. sampling frequency of the system as Fs, the delay(N_0) is simply given by $N_0 = \tau * Fs$.

Writing the equation in the time domain:

$$y[n] = x[n] + 0.9x[n - N_0] + 0.81x[n - 2N_0]$$

Taking the Laplace Transform i.e. transform into the 'z' domain:

$$Y(z) = X(z) + 0.9X(z)e^{-N_0} + 0.81X(z)e^{-2N_0}$$

$$\frac{Y(z)}{X(z)} = 1 + \frac{0.9}{e^{N_0}} + \frac{0.81}{e^{2N_0}}$$

$$H(z) = \frac{e^{2N_0} + 0.9e^{N_0} + 0.81}{e^{2N_0}}$$

This is our transfer function to add the echo. I tested an audio clip using various values of τ, the results of which can be seen below as audio wave forms. The audio clips can be heard here

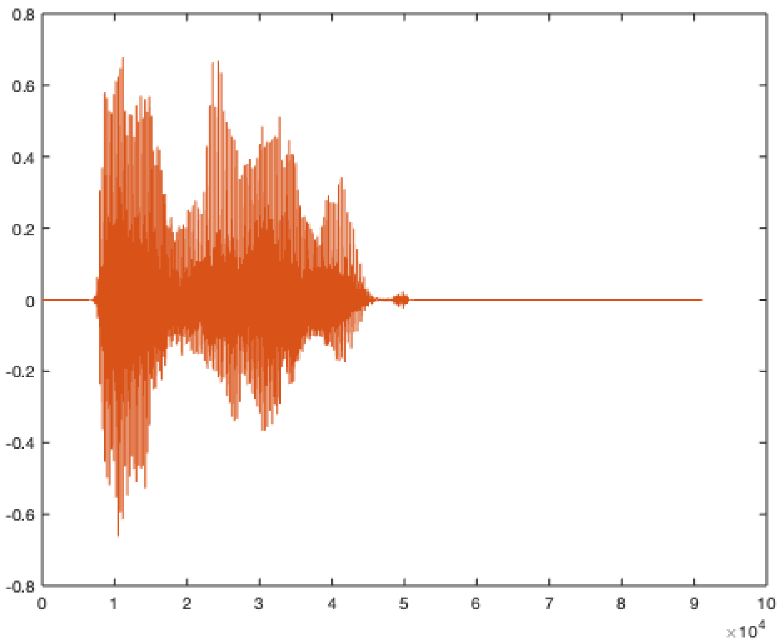


Figure 3.23: Original Audio Wave Form of First Clip

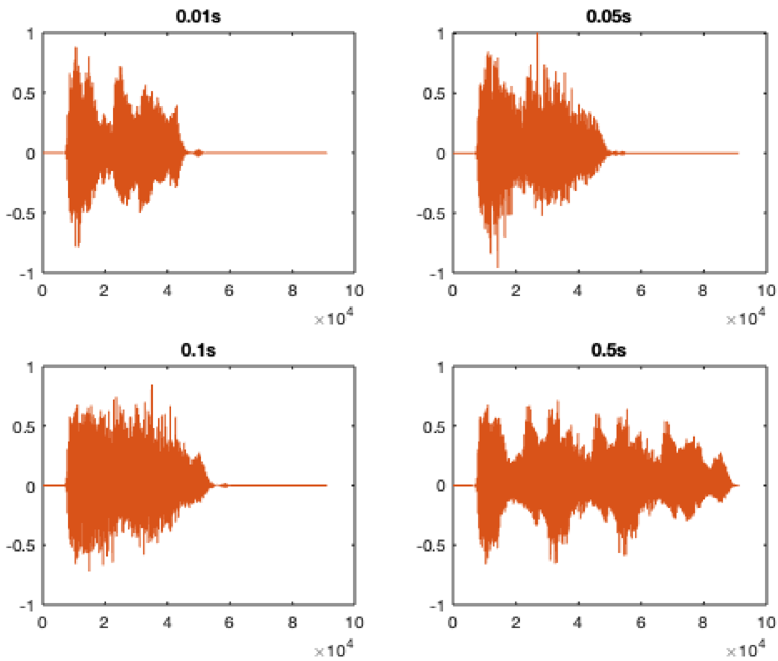


Figure 3.24: Audio Wave Forms of First Clip at Different Delays

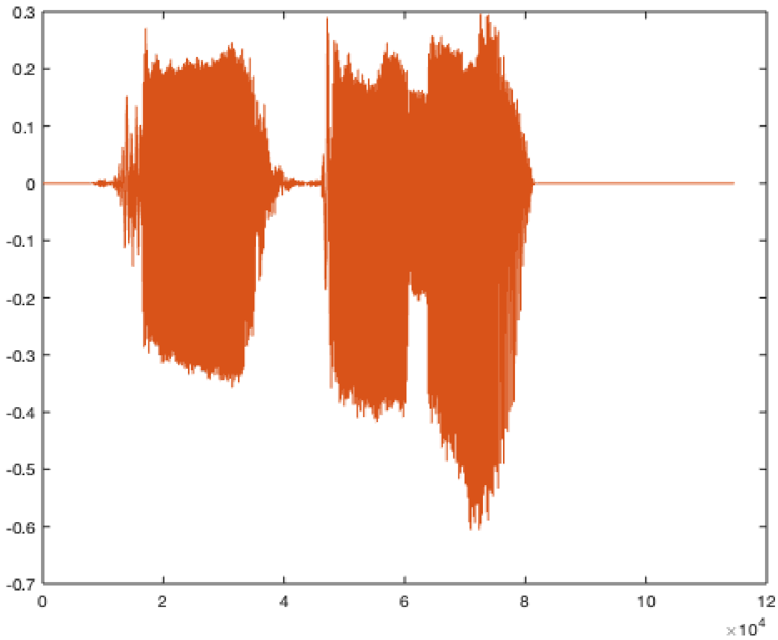


Figure 3.25: Original Audio Wave Form of Second Clip

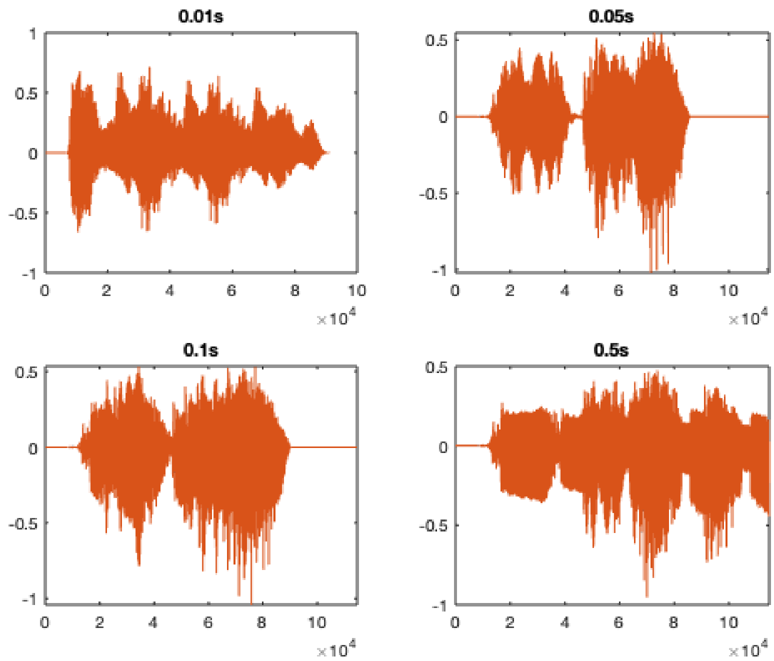


Figure 3.26: Audio Wave Forms of Second Clip at Different Delays

After listening to various clips the echo was first heard at about $\tau = 0.05s$. Obtaining the original audio from the echo is a fairly simple process. The transfer function for the inverse process is simply the inverse of the one above.

$$G(z) = \frac{1}{H(z)} = \frac{e^{2N_0}}{e^{2N_0} + 0.9e^{N_0} + 0.81}$$

The Matlab functions for the same are given below:

```
function [y_echo,Hz] = echoz(y,Fs,tau)
%This Function simulates echo

N0 = tau*Fs;
numerator = [1,zeros(1,N0 - 1),0.9,zeros(1,N0 - 1),0.81];
denominator = [1,zeros(1,2*N0)];

Hz = tf(numerator,denominator,1/Fs);

y_echo = filter(numerator,denominator,y);

end
```

(a) Adding Echo

```
function [y,Gz] = de_echoz(y_echo,Fs,tau)
%This Function recovers original
% signal from signal with echo

N0 = tau*Fs;
numerator = [1,zeros(1,2*N0)];
denominator = [1,zeros(1,N0 - 1),0.9,zeros(1,N0 - 1),0.81];

Gz = tf(numerator,denominator,1/Fs);

y = filter(numerator,denominator,y_echo);

end
```

(b) Removing Echo

Figure 3.27: Matlab Functions

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Bibliography

[1] A. S. W. Alan V. Oppenheim and S. H. Nawab, *Signals Systems*. Prentice Hall, 1997.