Part - 2 - Kalman Filter :

- Let us assume the state of the system as a 2×1 vector i.e Initial State $X_0 = \begin{bmatrix} GDP_0 \\ Rate_0 \end{bmatrix}$

- We know that $\boxed{GDP = GDP_0 + Rate_0 \, \Delta t}$

Δt = time step (unit)

- ① Transition Model $(X_{t+1} | X_t)$:

$$X_{t+1_p} = A X_t + W_a$$

$$X_{t+1_p} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} GDP_0 \\ Rate_0 \end{bmatrix} + \text{Noise}$$

X_{t+1_p} = Predicted GDP at time $t+1$

- This is the estimate of GDP in the absence of any sensor information.

② Process Covariance Matrix :

⇓

Defines how the errors/Noise in the states are correlated.

$$\cancel{P_{kp}} \cancel{\approx} \cancel{A} \quad P_{t+1_p} = A \, P_{t_p} \, A^T$$

$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{GDP}^2 & \sigma_{GDP}\,\sigma_{Rate} \\ \sigma_{GDP}\,\sigma_{Rate} & \sigma_{Rate}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}$$

③ Sensor Model (E) - Evidence :

We are considering two sensors that will affect GDP.

$$e_{t+1} = C\, x_t + z$$

$$\begin{bmatrix} e_{t+1\,1} \\ e_{t+1\,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GDP_t \\ Rate_t \end{bmatrix} + \text{Sensor Noise}$$

e_1 & e_2 are the two sensor readings/observations that will affect only the GDP.

④ Kalman Gain (K) :

$$K = \frac{P_{t+1_p} H^T}{H P_{t+1_p} H^T + R}$$

R - Sensor Noise Covariance Matrix

$$R = \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1}\sigma_{e_2} \\ \sigma_{e_1}\sigma_{e_2} & \sigma_{e_2}^2 \end{bmatrix}$$

H - Transformative matrix to maintain the dimensions

$P_{t+1_p} H^T$ is very similar to $\sigma_t^2 + \sigma_x^2$.

P represents the error in GDP estimate and GDP transition model.

R - represents the error in sensor reading.

(5) <u>Update</u> ($\underline{X_{t+1}} | E_{t+1}$):

$$X_{t+1} = X_{t+1_p} + K [E - H X_{t+1_p}]$$

X_{t+1_p} - Predicted state from transition model.

K - Kalman Gain

E = Sensor model state

This is the estimate of GDP in the presence of 2 sensors.

(6) ~~Upattad~~ Updating Process Covariance Matrix :-

$$P_{t+1} = (I - KH) P_{t+1_p}$$

P_{t+1} - New ~~Predi~~ Process Covariance Matrix

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Kalman Filter Example.

- Initial State assumed, $X_0 = \begin{bmatrix} GDP_0 \\ Rate_0 \end{bmatrix}$

$$X_0 = \begin{bmatrix} 15 \\ 0.255 \end{bmatrix} \text{ (in trillions)}$$

$$\Delta t = 1$$

① Transition Model $(X_{t+1} | X_t)$:

$$X_{t+1_p} = A X_t + \text{Noise.}$$

$$X_{t+1_p} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} GDP_0 \\ Rate_0 \end{bmatrix} + \text{Noise}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 0.255 \end{bmatrix} + \text{Noise}$$

$$X_{t+1_p} = \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} + \underbrace{\text{Noise}}_{\sim N(\mu, \sigma^2)}$$

Here, $\mu = 0$.

- This is the estimate of GDP in the absence of any sensor information, i.e 15.255 trillions

② Process Covariance Matrix:

$$P_{t+1_p} = A P_{t_p} A^T$$

$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{GDP}^2 & \sigma_{Rate}\,\sigma_{GDP} \\ \sigma_{GDP}\,\sigma_{Rate} & \sigma_{Rate}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}$$

We don't know the relation between the variance of Rate and GDP, so we are considering $\sigma_{GDP}\,\sigma_{Rate} = 0$, it will automatically converge to the right values.

$$P_{t+1_p} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0 \\ 0 & 0.005 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\sigma_{GDP}^2 = 0.6$ & $\sigma_{Rate}^2 = 0.005 \Rightarrow$ Assumed.

$$P_{t+1_p} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.005 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$P_{t+1_p} = \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

③ <u>Sensor Model (E)</u>:

$$e_{t+1} = C X_t + z$$

$$\begin{bmatrix} e_{t+1_1} \\ e_{t+1_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GDP_t \\ Rate_t \end{bmatrix} + \text{Sensor Noise.}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 15 \\ 0.255 \end{bmatrix} + \text{Sensor Noise}$$

We considered two sensors for GDP only and not for growth rate.

Consider Sensor 1 (e_{t+1_1}) as Consumption and sensor 2 (e_{t+1_2}) as imports. in the sensor model.

$$\begin{bmatrix} e_{t+1_1} \\ e_{t+1_2} \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix} + \text{Sensor Noise}$$

R - sensor covariance matrix.

$$R - \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1}\sigma_{e_2} \\ \sigma_{e_1}\sigma_{e_2} & \sigma_{e_2}^2 \end{bmatrix}$$

We don't know the relation between the variance of Sensor 1 and Sensor 2, so we are considering $\sigma_{e_1}\sigma_{e_2} = 0$, it will automatically converge to right values.

$\sigma_{e_1}^2 = 0.4$ (Consumption) & $\sigma_{e_2}^2 = 0.2$ (Imports)

(Assumed)

(4), Kalman Gain:-

$$K = \frac{P_{t+1_p} H^T}{H P_{t+1_p} H^T + R}$$

$$= \frac{\begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 0.005 & 0.2 \end{bmatrix}}$$

$$= \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.0250 & 5.00 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

(5) <u>Update</u> $(\underline{X_{t+1}} \mid \underline{E_{t+1}})$

$$X_{t+1} = X_{t+1_p} + K[E - H X_{t+1_p}]$$

$$= \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} + \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix} \left(\begin{bmatrix} 15 \\ 15 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} + \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix} \begin{bmatrix} -0.255 \\ 14.745 \end{bmatrix}$$

$$= \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} + \begin{bmatrix} -0.153 \\ -0.001275 \end{bmatrix}$$

$$\boxed{X_{t+1} = \begin{bmatrix} 15.102 \\ 0.253725 \end{bmatrix}} \text{ (in trillions)}$$

This is the estimate of the GDP given that we have the information from the 2 sensors.

(6) Updating Process Covariance Matrix:

$$P_{t+1} = (I - KH)\, P_{t+1p}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix} \right) \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0 \\ -0.005 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

$$P_{t+1} = \begin{bmatrix} 0.24 & 0 \\ +0.002 & 0 \end{bmatrix}$$

This is the new process covariance matrix i.e $\sigma_t^2 + \sigma_x^2$, that will affect the Kalman Gain in the following iteration

Extra Credit Part:

In this case, steps 1 & 2 i.e Transition Model and Process Covariance Matrix step will be the same and all the assumptions are same.

Now, 3 Sensor Model (E):-

$$e_{t+1} = C X_t + Z$$

We are considering two sensors that will affect the GDP such as consumptions (e_{t+1_1}), imports (e_{t+1_2}) and the third sensor affects the rate of GDP (e_{t+1_3})

$$\therefore \begin{bmatrix} e_{t+1_1} \\ e_{t+1_2} \\ e_{t+1_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} GDP_t \\ Rate_t \end{bmatrix} + \text{Sensor Noise.}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 0.255 \end{bmatrix} + \text{Sensor Noise}$$

$$= \begin{bmatrix} 15 \\ 15 \\ 0.255 \end{bmatrix} + \text{Sensor Noise}$$

R - Sensor Covariance Matrix

$$R = \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1}\sigma_{e_2} & \sigma_{e_1}\sigma_{e_3} \\ \sigma_{e_2}\sigma_{e_1} & \sigma_{e_2}^2 & \sigma_{e_2}\sigma_{e_3} \\ \sigma_{e_3}\sigma_{e_1} & \sigma_{e_3}\sigma_{e_2} & \sigma_{e_3}^2 \end{bmatrix}$$

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We don't know the relation between the variance of sensor 1, sensor 2 and sensor 3, so we are considering

$$\sigma_{e_1}\sigma_{e_2} = \sigma_{e_2}\sigma_{e_3} = \sigma_{e_1}\sigma_{e_3} = 0.$$

$$\sigma_{e_1}^2 = 0.4 \quad , \quad \sigma_{e_2}^2 = 0.2 \quad , \quad \sigma_{e_3}^2 = 0.0001 \Rightarrow \text{Assumed.}$$

(4) Kalman Gain :

$$K = \frac{P_{t+1_p} H^T}{H P_{t+1_p} H^T + R}$$

$$= \frac{\begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.0001 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 0.6 & 0.6 & 0 \\ 0.005 & 0.005 & 0 \end{bmatrix}}{\begin{bmatrix} 0.6 & 0 \\ 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.0001 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 0.6 & 0.6 & 0 \\ 0.005 & 0.005 & 0 \end{bmatrix}}{\begin{bmatrix} 1.0 & 0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0.005 & 0.005 & 0.0001 \end{bmatrix}}$$

$$K = \begin{bmatrix} 0.6 & 0.6 & 0 \\ 0.005 & 0.005 & 0 \end{bmatrix} \begin{bmatrix} 2.0 & -1.0 & 0 \\ -1.0 & 2.0 & 0 \\ -23 & -45 & 10000 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.2727 & 0.5455 & 0 \\ 0.0023 & 0.0045 & 0 \end{bmatrix}$$

(5) Update $(X_{t+1} \mid E_{t+1})$:

$$X_{t+1} = X_{t+1_p} + K\left[E - H X_{t+1_p}\right]$$

$$= \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} + \begin{bmatrix} 0.2727 & 0.5455 & 0 \\ 0.0023 & 0.0045 & 0 \end{bmatrix} \left(\begin{bmatrix} 15 \\ 15 \\ 0.255 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} + \begin{bmatrix} 0.2727 & 0.5455 & 0 \\ 0.0023 & 0.0045 & 0 \end{bmatrix} \begin{bmatrix} -0.255 \\ -0.255 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 15.255 \\ 0.255 \end{bmatrix} + \begin{bmatrix} -0.2086 \\ -0.0017 \end{bmatrix}$$

$$\boxed{X_{t+1} = \begin{bmatrix} 15.0464 \\ 0.2533 \end{bmatrix}}$$ (in trillions)

This is the estimate of GDP and Growth Rate in the presence of 3 sensors.

⑥ Updating Process Covariance Matrix:

$$P_{t+1} = (I - KH)\, P_{t+1_p}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2727 & 0.5455 & 0 \\ 0.0023 & 0.0045 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8182 & 0 \\ 0.0068 & 0 \end{bmatrix} \right) \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1818 & 0 \\ -0.0068 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0 \end{bmatrix}$$

$$P_{t+1} = \begin{bmatrix} 0.1091 & 0 \\ -0.00092 & 0 \end{bmatrix}$$

This is the new process covariance matrix i.e $\sigma_t^2 + \sigma_x^2$ that will affect the Kalman Gain in the following iteration