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Mathematical Tricks:A must for all competitive exams

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TIPS TO REMEMBER

□ CONGRUENT : Let us start by choosing a base n , then each integer $N = q \cdot n + r$ where q is quotient and r is remainder ($0 \leq r < n$). We write this as $N \equiv r \pmod{n}$ and say that N is congruent to r modulo n . e.g. $78 \equiv 66 \pmod{12}$.

Ex. January 1, 2000 falls on a saturday. What day of the week will January 1, 2020 be ?

Sol. Because there are 20 years in the range 2000 to 2019 of which 5 are leap-years, January 1, 2020 falls on day $20 + 5 \equiv 4 \pmod{7}$, i.e. **Wednesday**.

□ MATHEMATICAL INDUCTION :

(1) **Weak Version :** Let $P(n)$ be a statement such that (i) $P(n_0)$ is true and (ii) $P(k)$ implies $P(k+1)$ for any $k \geq n_0$, then $P(n)$ is true for every $n \geq n_0$.

(2) **Strong Version :** Let $P(n)$ be a statement such that (i) $P(n_0)$ is true and (ii) if $P(n_0), P(n_0+1), \dots, P(k)$ are true for any $k \geq n_0$, then $P(k+1)$ is also true. Then $P(n)$ is true for every $n \geq n_0$.

□ PIGEONHOLE PRINCIPLE : If m pigeons are assigned to n pigeonholes, where $m > n$, then at least two pigeons must occupy the same pigeonhole.

Proof : Let the given conclusion is false, that is, no two pigeons occupy the same pigeonhole. Then every pigeon must occupy a distinct pigeonhole, so $n \geq m$, which is a contradiction. Thus two or more pigeons must occupy some pigeonhole.

□ ROUND-ROBIN TOURNAMENT : In round-robin tournament every team plays every other team exactly once. Suppose there are n teams, labeled 1 through n . Then the tournament can be represented

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by a polygon with n vertices with every pair of vertices connected, every vertex represents a team and every line segment with endpoints i and j represents a game between teams i and j .

Let g_n = number of games by n teams in a tournament. It can be defined recursively :

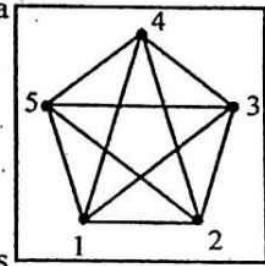
$$g_1 = 0$$

$$g_n = g_{n-1} + (n-1), \text{ where } n \geq 2$$

Solve this recurrence relation, we get

$$g_n = \frac{n(n-1)}{2} = {}^n C_2. \text{ e.g. 5 teams will play 10 games}$$

shows in given figure.



□ FERMAT'S LITTLE THEOREM : Let p be a prime and a is any integer such that $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

Ex. Find the remainder when 24^{1947} is divided by 17.

Sol. $24 \equiv 7 \pmod{17}$. $\therefore 24^{1947} \equiv 7^{1947} \pmod{17}$.

From Fermat's little theorem,

$$7^{16} \equiv 1 \pmod{17}. \therefore 7^{1947} = (7^{16})^{121} \cdot 7^{11} = 1^{121} \cdot 7^{11} \equiv 7^{11} \pmod{17}.$$

$$\text{But } 7^2 \equiv -2 \pmod{17}. \therefore 7^{11} \equiv (7^2)^5 \cdot 7 \equiv (-2)^5 \cdot 7 \equiv -32 \cdot 7$$

$$\equiv 2 \cdot 7 \equiv 14 \pmod{17}.$$

Hence the reqd. remainder is 14.

□ PRINCIPLE OF INCLUSION AND EXCLUSION (P.I.E.) OR SIEVE FORMULA : It is very important principle is a generalization of the sum-rule to sets which need not be disjoint. Venn-diagrams show that $|A \cup B| = |A| + |B| - |A \cap B|$ and $|A \cup B \cup C| =$

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

We generalize to n sets as follows :

$$|A_1 \cup \dots \cup A_n| = \sum_{i=1}^n |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - \dots - (-1)^{n+1} |A_1 \cap \dots \cap A_n|.$$

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e.g. We consider all $n!$ permutations of $1, 2, \dots, n$. If an element i is on place number i , then we say i is a fixed-point of the permutation. Let p_n = number of fixed point free permutations and q_n = number of permutations with at least one fixed point. Then $p_n = n! - q_n$.

Let A_i = number of permutations with i fixed point, then

$$q_n = |A_1 \cup \dots \cup A_n| = \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \dots + (-1)^{n+1} \binom{n}{n}0! =$$

$$= n! \left\{ 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + \frac{(-1)^{n-1}}{n!} \right\}$$

$$\therefore p_n = n! - q_n = n! \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{n-1}}{n!} \right\}.$$

□ DIVISION WITH REMAINDER : For polynomials $f(x)$ and $g(x)$ there exist unique polynomials q and r such that $f(x) = g(x)q(x) + r(x)$, $\deg r(x) < \deg g(x)$ or $r(x) = 0$, where $q(x)$ and $r(x)$ are quotient and remainder respectively on division of $f(x)$ by $g(x)$. If $r(x) = 0$, then we say that $g(x)$ divides $f(x)$ and we write $g(x) | f(x)$.

□ ARITHMETIC-GEOMETRIC-HERMOMIC-MEAN INEQUALITY:

Take any n different positive numbers x_1, x_2, \dots, x_n where n is positive integer, then Arithmetic mean $A = (x_1 + x_2 + \dots + x_n)/n$, Geometric mean $G = (x_1 x_2 \dots x_n)^{1/n}$, Harmonic mean $H =$

$\left\{ \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right) / n \right\}^{-1}$ and Root mean square

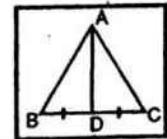
$$[\mu'_2]^{1/2} = \left\{ (x_1^2 + x_2^2 + \dots + x_n^2) \frac{1}{n} \right\}^{1/2}.$$

Verify that whatever the values of n you take and whatever be the positive numbers you choose $[\mu'_2]^{1/2} \geq A \geq G \geq H$, but if x_1, x_2, \dots, x_n are equal, then all these means are also equal.

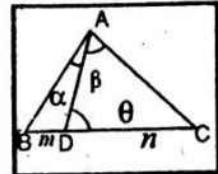
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□ FOLLOWING RESULTS OF HEIGHTS AND DISTANCES:

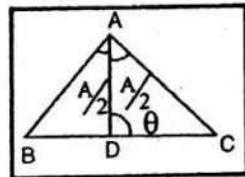
(i) **Appollonius theorem** : If in a $\triangle ABC$, AD is median, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$.



(ii) **m-n theorem** : If $BD : DC = m:n$, then
 $(m+n) \cot \theta = m \cot \alpha - n \cot \beta$
 $= n \cot B - m \cot C$.



(iii) **Angle bisector** : If AD is the angle bisector of $\angle BAC$ $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

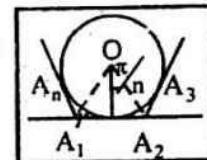


(iv) The exterior angle is equal to sum of interior opposite angles.
(v) If a line is perpendicular to a plane, then its perpendicular to every line in that plane.

□ **REGULAR POLYGON** : Let A_1, A_2, \dots, A_n be a regular polygon of n sides each of length a .

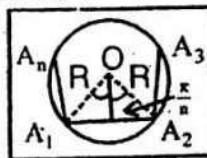
(i) **Inscribed circle of a regular polygon of n sides** :

$$\text{Area} = \pi r^2 \tan \frac{\pi}{n} \text{ and Radius } r = \frac{a}{2} \cos \frac{\pi}{n}.$$



(ii) **Circumscribed circle of a regular polygon of n sides** :

$$\text{Area} = \frac{1}{2} n R^2 \sin \frac{2\pi}{n} \text{ and Radius } R = \frac{a}{2} \cosec \frac{\pi}{n}.$$



□ **MODULUS OR ABSOLUTE VALUE FUNCTION** :

It is defined as $y = f(x) = |x| = \sqrt{x^2} = \begin{cases} x: & x \geq 0 \\ -x: & x < 0 \end{cases}$

$$\therefore f(-x) = x = f(x).$$

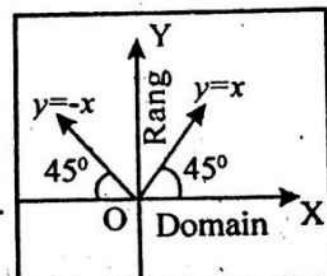
$\therefore f(x)$ is symmetric about y-axis.

Here $|0| = 0$; $|-5| = 5$; $|5| = 5$. Domain : $x \in \mathbb{R}$.

Range : $y \in [0, \infty)$. Continuous everywhere.

Non differentiable at $x = 0$, elsewhere

differentiable. Even as well as Many-one function. Monotonically



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increasing and decreasing for all $x > 0$ and $x < 0$ respectively.

Note : $|x|$ is read modulus or mod x . $|x| = 2 \Leftrightarrow x = \pm 2$; $|x| < 2 \Leftrightarrow -2 < x < 2$ and $|x| > 2 \Leftrightarrow x < -2$ or $x > 2$.

Properties :

(i) $|x+y| = |x| + |y| \Leftrightarrow x, y \geq 0$ or $x, y \leq 0$.

(ii) $|x-y| = |x| - |y| \Leftrightarrow x \geq 0, |x| \geq |y|$ or $x \leq 0, y \leq 0$ and $|x| \leq |y|$.

(iii) $|x \pm y| \leq |x| + |y|$.

(vi) $|x \pm y| \geq |x| - |y|$.

□ SIGNUM FUNCTION : It is defined as $y = f(x) =$

$$\text{Sgn}(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|}; x \neq 0 \\ 0; x = 0 \end{cases} = \begin{cases} -1; x < 0 \\ 0; x = 0 \\ 1; x > 0 \end{cases}$$

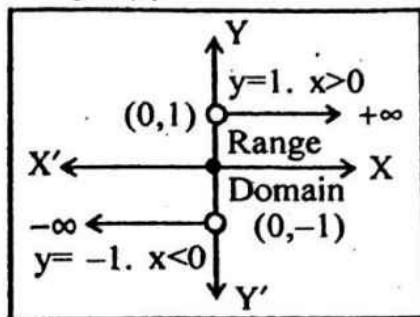
Domain : $x \in \mathbb{R}$. Range : $y \in \{-1, 0, 1\}$.

Continuous everywhere except

at $x=0$ and discontinuous at $x=0$. Odd as well as many-one function.

Neither monotonically increasing nor decreasing.

Note : O indicates that the points $(0, 1)$ and $(0, -1)$ are not included and denote the point $(0, 0)$ is included in the graph.



□ GREATEST INTEGER OR FLOOR OR STEP FUNCTION:

It is defined as $y = f(x) = [x] = \text{gint}(x) = n$ where $n \leq x < n+1$.

e.g. $[\pi] = 3, [\log_{10} 3] = 0, [-3.5] = -4$, etc and $[0.7] = 0$ etc.

\therefore Real number = Integral Part (I) + fractional part (f), where $I = [x]$ and

$0 \leq f < 1$. $\therefore y = f(x) = [x]$

For $-3 \leq x < -2$; $y = -3$

,, $-2 \leq x < -1$; $y = -2$

,, $-1 \leq x < 0$; $y = -1$

,, $0 \leq x < 1$; $y = 0$

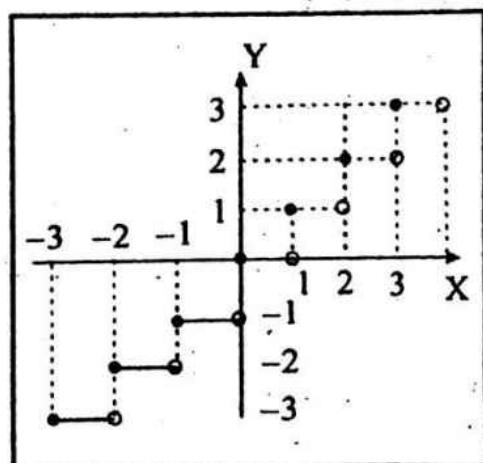
,, $1 \leq x < 2$; $y = 1$ and so on.

• Domain : $x \in \mathbb{R}$, Range : $y \in \mathbb{Z}$

• Continuous and differentiable

everywhere except at $x = n$, $n \in \mathbb{Z}$

• Many one and into function for co-domain as all real numbers.



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Properties :

(i) $[x+n] = n + [x]$, $n \in \mathbb{I}$.

(ii) $[-x] = -[x] \forall n \in \mathbb{I}$ and $[-x] = -[x] - 1$, $x \notin \mathbb{I}$.

(iii) $x = [x] + \{x\}$, $\{\}$ denote fractional part of x .

(iv) $[x_1 + x_2] \geq [x_1] + [x_2]$.

(v) $\left[\frac{[x]}{n} \right] = \left[\frac{x}{n} \right] \forall n \in \mathbb{N}$.

(vi) If $[f(x)] \geq I$, then $f(x) \geq I$ and for $[f(x)] \leq I$, then $f(x) < I+1$.

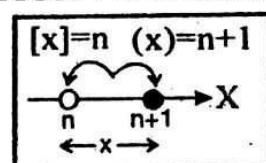
(vii) $[p + [q + [r + [s + [t]]]]] = [p] + [q] + [r] + [s] + [t]$.

(viii) $[x] + \left[x + \frac{1}{n} \right] + \left[x + \frac{2}{n} \right] + \dots + \left[x + \frac{n-1}{n} \right] = [nx]$.

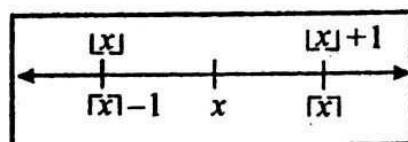
□ LEAST INTEGER OR CEILING FUNCTION : It is defined.

as $y = f(x) = (x) = [x] = \text{int}(x) = n+1$,

where $n < x \leq n+1$.



e.g. $(\pi) = 4$, $(\log_{10} 3) = 1$, $(-3.5) = -3$ etc.

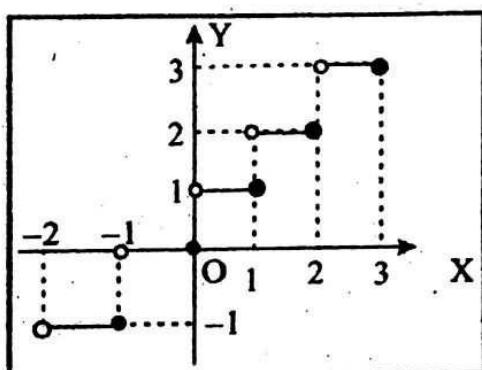


$\therefore y = (x) = 0$ if $-1 < x \leq 0$
 $= 1$ if $0 < x \leq 1$
 $= 2$ if $1 < x \leq 2$
 $= 3$ if $2 < x \leq 3$ and so on.

• Domain : $x \in \mathbb{R}$ • Range : $y \in \mathbb{I}$

• Many one into function for co-domain of all real numbers.

Discontinuous at all integral values of x .



Properties :

(i) $(x+n) = (x)+n$, $n \in \mathbb{I}$

(ii) $(-x) = -(x)$, $x \in \mathbb{I}$ and $(-x) = -(x)+1$, $x \notin \mathbb{I}$.

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(iii) $x = (x) + \{x\} - 1$, $\{x\}$ denotes the fractional part of x .

(iv) $(x_1 + x_2) \leq (x_1) + (x_2)$.

(v) $\left(\frac{(x)}{n} \right) = \left(\frac{x}{n} \right)$, $n \in \mathbb{N}$.

(vi) $(x) + \left(x + \frac{1}{n} \right) + \left(x + \frac{2}{n} \right) + \dots + \left(x + \frac{n-1}{n} \right) = (nx) + n - 1$, $n \in \mathbb{N}$.

□ FRACTIONAL PART FUNCTION : It is defined as $y = f(x) = \{x\} = x - [x]$ (where $0 \leq f < 1$) $= x - n$, $n \leq x < n+1$.

e.g. $\{1.6\} = 0.6$, $\{1\} = 0$, $\{-3.6\} = 0.4$

$\therefore y = \{x\} = x$ if $0 \leq x < 1$

$= x-1$ if $1 \leq x < 2$

$= x-2$ if $2 \leq x < 3$ and so on.

• Domain : $x \in \mathbb{R}$, • Range : $y \in [0, 1)$,

• Discontinuous and Non-differentiable at all integers, • Monotonically

increasing throughout • Many-one and

into function for co-domain of all real numbers.

Properties :

(i) If $0 \leq x < 1$, then $\{x\} = x$.

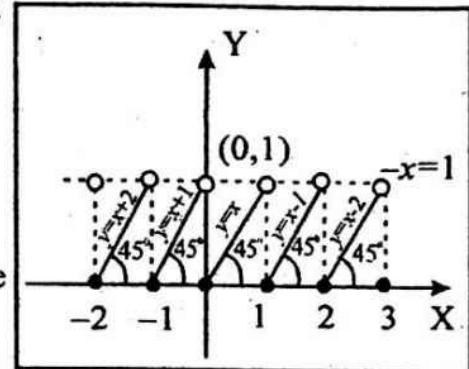
(ii) If $x \in \mathbb{Z}$, then $\{x\} = 0$.

(iii) If $x \notin \mathbb{Z}$ and $x > 0$, then $\{-x\} = 1 - \{x\}$.

(iv) Domain and Range of $\frac{1}{\{x\}}$ are $\mathbb{R} - \mathbb{Z}$ and $(0, 1)$ respectively.

Note : $f(x+T) = f(x)$ ($x \in \mathbb{R}$), where T be a (+)ve real number and $f(x) = \{x\}$. $\Rightarrow x+T - [x+T] = x - [x] \quad \forall x \in \mathbb{R}$, $\Rightarrow [x+T] - [x] = T \quad \forall x \in \mathbb{R}$, $\Rightarrow T = 1, 2, 3, \dots$. Thus there exist $T > 0$ such that $f(x+T) = f(x) \quad \forall x \in \mathbb{R}$, so $f(x)$ is periodic and the smallest value of T satisfies $f(x+T) = f(x) \quad \forall x \in \mathbb{R}$ is 1.

□ DIRICHLET FUNCTION : Suppose c and d be real numbers (usually taken as $c=1$ and $d=0$), then the **Dirichlet function** is defined



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by $y=f(x)=\begin{cases} c \text{ for } x \text{ rational} \\ d \text{ for } x \text{ irrational} \end{cases}$ and is discontinuous everywhere.

Analytically : $f(x) = \lim_{m \rightarrow \infty} \left\{ \lim_{n \rightarrow \infty} \cos^{2n}(m!\pi x) \right\}$

- Domain : $x \in \mathbb{R}$
- Range : $y \in \{c, d\} = \{0, 1\}$
- Neither increasing nor decreasing
- Not a periodic function.
- Discontinuous everywhere
- Many one and into function for co-domain of all real numbers.

WORKED OUT EXAMPLES

Ex.1. Let $\max(x, y)$ denote the maximum of x, y and $\min(x, y)$ their minimum, where x and y are any integer, prove that $\max(x, y) - \min(x, y) = |x - y|$.

Sol. Case 1. Let $x \geq y$, then $|x - y| = x - y$, $\max(x, y) = x$ and $\min(x, y) = y$. Thus $\max(x, y) - \min(x, y) = x - y = |x - y|$.

Case 2. Let $x < y$, then $|x - y| = y - x$, $\max(x, y) = y$ and $\min(x, y) = x$. Thus $\max(x, y) - \min(x, y) = y - x = |x - y|$.

Ex.2. Solve : (i) $|x+1|^{\log(x+1)} = (x-3)|x|$
(ii) $|x-1| + |y-1| = 4$, $x - |y-1| = 2$.

Sol. (i) Given : $|x+1|^{\log(x+1)} = (x-3)|x| \dots\dots\dots(1)$
 $\because 3+2x-x^2 > 0$ (for $\log x, x > 0$)
 $\therefore -1 < x < 3 \dots(2)$.

Let $x+1>1$, then $x>0 \dots(3)$.

\therefore From (1) : $(x+1)^{\log(x+1)(3+2x-x^2)} = (x-3)x$

$$\Rightarrow 3+2x-x^2 = x^2 - 3x \Rightarrow 2x^2 - 5x - 3 = 0 \therefore x = -\frac{1}{2}, 3.$$

Now, from (2) and (3) : $x \neq -\frac{1}{2}, 3$.

Hence it has no solution.

(ii) Given : $|x-1| + 3y = 4 \Rightarrow \begin{cases} x+3y=5, x \geq 1 \dots(1) \\ -x+3y=3, x < 1 \dots(2) \end{cases}$ and also given : $x - |y-1| = 2$

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$$\Rightarrow \begin{cases} x-y=1, y \geq 1 \dots (3) \\ x+y=3, x < 1 \dots (4) \end{cases} \text{ Solving (1), (3): } x=2, y=1 \text{ and solving (1), (4): } x=2, y=1.$$

\therefore No solution as $x \geq 1, y < 1$.

Solving (2), (3) : $x=3, y=2$, i.e. no solution as $x < 1, y \geq 1$ and solving

$$(2), (4) : x = \frac{5}{2}, y = \frac{3}{2}, \text{ i.e. no solution as } x < 1, y < 1.$$

Hence the reqd. solution is unique, i.e. $x=2, y=1$.

Ex. 3 If $f(x)=3[x]+5$ and $f(x)=5[x-2]+7$, find the value of $[x+f(x)]$.

Sol. given : $3[x]+5 = 5[x-2]+7 \Rightarrow 3[x]+5 = 5[x]-10+7 \Rightarrow 2[x]=8$,
 $\Rightarrow [x]=4 \therefore 4 \leq x < 5$, i.e. $x=4+\text{fractional part}$.

$$\therefore f(x)=3[x]+5=3.4+5=17. \therefore [x+f(x)]=[17+\text{fractional part}+4]=21.$$

Ex. 4. Solve : $\{x+1\}+2x=4[x+1]-6$ where $\{.\}=f.p$; $[.] = G.I.$

$$\text{Sol. } x+1-[x+1]+2x=4[x+1]-6 \quad (\because \{x\}=x-[x]), \Rightarrow 3x+1=5[x+1]-6 \\ =5([x]+1)-6, \Rightarrow 3x=5[x]-2 \dots (i), \Rightarrow 3(\{x\}+[x])=5[x]-2.$$

$$\therefore 3\{x\}=2[x]-2 \dots (ii).$$

$$\text{Now, } 0 \leq \{x\} < 1, \Rightarrow 0 \leq 3\{x\} < 3, \Rightarrow 0 \leq 2[x]-2 < 3 \text{ [using (ii)]},$$

$$\Rightarrow 2 \leq 2[x] < 5, \Rightarrow 1 \leq [x] < \frac{5}{2}, \Rightarrow [x]=1, 2.$$

$$\therefore \text{From (i): } [x]=1, \text{ i.e. } x=1 \text{ and } [x]=2, \text{ i.e. } x=\frac{8}{3}.$$

Hence reqd. solutions are $1, \frac{8}{3}$.

Ex. 5. Solve : $(x)^2 + (x+2)^2 = 20$, where $(.) = \text{least integer}$.

Sol. Let $x=l+f$, then from the given equation, we have $(l+f)^2 + (l+f+2)^2$

$$= 20, \Rightarrow (l+1)^2 + (l+3)^2 = 20, \Rightarrow l^2 + 2l + 1 + l^2 + 6l + 9 = 20,$$

$$\Rightarrow l^2 + 4l - 5 = 0, \Rightarrow l = 1, -5.$$

$$\therefore x = l+f, -5+f, \Rightarrow 0 < f < 1, \Rightarrow 1 < l+f < 2 \text{ and } -5 < -5+f < -4.$$

Further $x = l$.

$$\therefore (x)^2 + (x+2)^2 = 20, \text{ i.e. } x \in (-5, -4] \cup (1, 2].$$

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Ex. 6. Solve : $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and $[y + [y]] = 2\cos x$.

Sol. we have

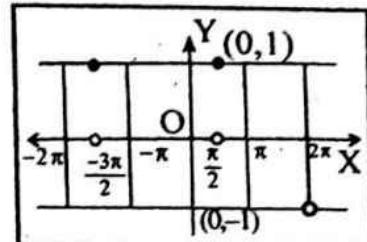
$y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]] = \frac{1}{3}[\sin x + [+\sin x] + [\sin x]]$ ($\because [I+x] = I + [x]$ for any integer I) $= \frac{1}{3}([\sin x] + [\sin x] + [\sin x]) \dots \dots (1)$

and $[y + [y]] = 2\cos x$, $\Rightarrow 2[y] = 2\cos x$, $\Rightarrow [y] = \cos x \dots \dots (2)$.

\therefore From (1) and (2) : $[\sin x] = \cos x$,
 $\Rightarrow [\sin x] = \cos x$.

Now, plotting the curves $[\sin x]$ and $\cos x$ on the same frame, we see that the two curves have no intersection points.

Hence the given equation has no solution.



Ex. 7. How many integers between 1 and 300 (inclusive) are (i) divisible by at least one of 3, 5 and 7? (ii) divisible by 3 and by 5, but not by 7? (iii) divisible by 5 but by neither 3 nor 7?

Sol. Let A, B and C be the set of those integers between 1 and 300 which are divisible by 3, 5, 7 respectively.

$$A = \{n \mid 1 \leq n \leq 300, 3 \mid n\}; B = \{n \mid 1 \leq n \leq 300, 5 \mid n\};$$

$$C = \{n \mid 1 \leq n \leq 300, 7 \mid n\}.$$

To be divisible by 3 or 5 or 7 is to be at least in one of A, B, or C.

$$\text{Now, } |A| = \left[\frac{300}{3} \right], \text{ [.] is the greatest integer} = 100.$$

$$\text{Similarly, } |B| = \left[\frac{300}{5} \right] = 60 \text{ and } |C| = \left[\frac{300}{7} \right] = 42, \text{ where } |A|, |B|, |C| \text{ being}$$

the cardinalities of sets A, B, C. Again, $A \cap B$ is the set of integers between 1 and 300, which are divisible by both 3 and 5, since 3 and 5 are relatively prime, any number divisible by them must be divisible by product of them.

$\therefore A \cap B$ is the set of integers divisible by 15.

Similarly $B \cap C$, $A \cap C$, $A \cap B \cap C$ are the sets of integers between 1 and 300 which are divisible by 35, 21 and 105.

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$$|A \cap B| = \left[\frac{300}{15} \right] = 20, |A \cap C| = \left[\frac{300}{21} \right] = 14, |B \cap C| = \left[\frac{300}{35} \right] = 8,$$

$$|A \cap B \cap C| = \left[\frac{300}{105} \right] = 2.$$

(i) $|A \cap B \cap C| = |A| + |B| + |C| - |AB| - |BC| - |CA| + |ABC| = 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162.$

(ii) The numbers divisible by 5 and 3, but not by 7, are precisely those numbers in $(ABC)/C$, whose cardinality is $|A \cap B| - |A \cap B \cap C| = 20 - 2 = 18.$

(iii) The numbers divisible by 5 but by neither 3 nor 7 are those in $B \setminus (A \cup C)$, having cardinality $|B| - |B \cap (A \cup C)|$. Since $|B \cap (A \cup C)| = |B \cap A| + |B \cap C| - |B \cap A \cap C| = |B \cap A| + |B \cap C| - |B \cap A \cap C| = 20 + 8 - 2 = 26$ and the number we are looking for is $|B| - |B \cap (A \cup C)| = 60 - 26 = 34.$

Ex. 8. Let n be a positive integer. Suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n=1 \\ L(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1 \end{cases} \quad (\lfloor \cdot \rfloor \text{ is the floor function}).$$

Find $L(25)$.

Sol. $L(25)$ is found recursively as follows:

$$L(25) = (12) + 1 \dots \text{(i)}; L(12) = L(6) + 1 \dots \text{(ii)}$$

$$L(6) = L(3) + 1 \dots \text{(iii)}; L(3) = L(1) + 1 \dots \text{(iv)}$$

Back calculating $L(6) = L(3) + 1 = 1 + 1 = 2$; $L(12) = L(6) + 1 = 2 + 1 = 3$.

$$\therefore L(25) = L(12) + 1 = 3 + 1 = 4.$$

Ex. 9. If $f(x) = x^3 - 9x^2 + 24x + c$ has three real and distinct roots α, β, γ then find the possible values of c . Hence, otherwise show that $|\alpha| + |\beta| + |\gamma|$ can take only two values and determine these values, $|\cdot|$ denotes the greatest integer function.

Sol. Suppose $y = x^3 - 9x^2 + 24x$.

$$\therefore y_1 = 3(x-2)(x-4). \therefore \text{For turning point } y_1 = 0.$$

$\therefore x = 2, 4$ are turning points.

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Thus $y(2)=8-36+48=20$ and $y(4)=64-144+96=16$.

From the graph we can see that x-axis will cut the graph 3 times, if it shifted downward by 16 to 20 units, i.e.

$c \in (-20, -16)$. For

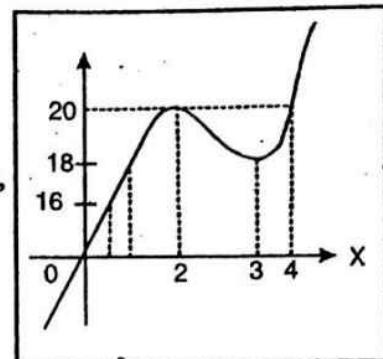
$[\alpha]+[\beta]+[\gamma]$, if $c \in (-20, -16)$, then $\alpha \in (1, 2)$, $\beta \in (3, 4)$, $\gamma \in (4, 5)$.

$\therefore [\alpha]+[\beta]+[\gamma]=1+3+4=8$. Again, if

$c \in (-20, -18)$, then $\alpha \in (1, 2)$,

$\beta \in (2, 3)$, $\gamma \in (4, 5)$.

$\therefore [\alpha]+[\beta]+[\gamma]=1+2+4=7$.



Finally, We can say that $[\alpha]+[\beta]+[\gamma]=\begin{cases} 8 & \text{if } -18 < c < -16 \\ 7 & \text{if } -20 < c < -18. \end{cases}$

Ex. 10. If a, b, c are the cube root of p , ($p < 0$) then for any permissible

value of x, y, z which is given by $\left| \frac{xa+yb+zc}{xb+yc+za} \right| + (a^2_1 - 2b^2_1) \omega + \omega^2 ([x] + [y] + [z]) = 0$, where ω is cube root of unity and a_1, b_1 , are real positive numbers, b_1 is a prime, find the value of $[x+a_1] + [y+b_1] + [z]$ ($[.] = \text{G.I.F.}$).

Sol. Given : $1 + \omega + \omega^2 = 0 \dots \text{(i)}$ and $\left| \frac{xa+yb+zc}{xb+yc+za} \right| + (a^2_1 - 2b^2_1) \omega + ([x] + [y] + [z])\omega^2 = 0 \dots \text{(ii)}$.

From (i) and (ii) : $\left| \frac{xa+yb+zc}{xb+yc+za} \right| = (a^2_1 - 2b^2_1) = [x] + [y] + [z] \dots \text{(iii)}$.

Since a, b, c are the roots of $(p)^{1/3}$ and let $a = t$, $b = t\omega$ and $c = t\omega^2$.

$$\therefore \left| \frac{xa+yb+zc}{xb+yc+za} \right| = \left| \frac{xt+y+\omega+t\omega^2.z}{x.t\omega+y.t\omega^2+x.t} \right| = \left| \frac{x+y\omega+z\omega^2}{x\omega+y\omega^2+z} \right| = \left| \frac{1}{\omega} \right| = 1.$$

\therefore From (iii) : $a^2_1 - 2b^2_1 = 1$, i.e. $a^2_1 = 2b^2_1 + 1$ (odd).....(iv)

Thus a_1 can be written in the form $(2n+1)$, $\Rightarrow (2n+1)^2 = 1 + 2b^2_1 \Rightarrow 4n^2 + 4n + 1 = 2b^2_1 = 2n(n+1) \Rightarrow b^2_1 = 2n(2n+1)$, an even number and given prime, so b^2_1 is also prime. $\therefore b_1 = 2$ ($\because 2$ is only even prime)

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$\therefore a_1^2 = 9, \Rightarrow a_1 = 3$. From (iii) : $[x] + [y] + [z] = a_1^2 - 2b_1^2 = 1$.

$\therefore [x+a_1] + [y+b_1] + [z] = [x+3] + [y+2] + [z] = [x] + [y] + [z] + 5 = 1 + 5 = 6$.

Ex. 11. Find the number of solutions of $4\{x\} = x + [x]$, where $\{\cdot\}$, $[\cdot]$ denotes the fractional part, greatest integer function respectively.

Sol. Given : $4\{x\} = x + [x]$, $\Rightarrow 4(x - [x]) = x + [x]$ ($\because x = [x] + \{x\}$), $\Rightarrow 3x = 5[x]$.

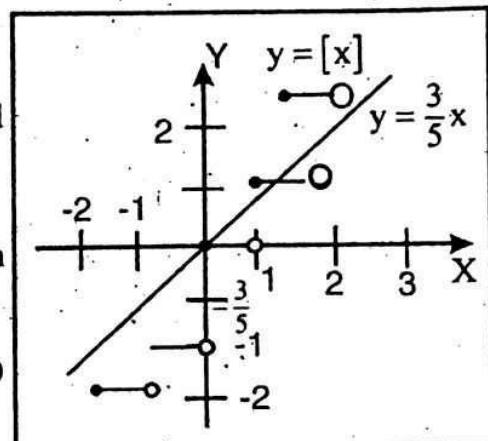
$$\therefore [x] = \frac{3}{5}x \dots \text{(i)}$$

\therefore To plot the graph of both $y = [x]$ and $y = \frac{3}{5}x$.

Thus the two curves intersects when $[x] = 0$ and $[x] = 1$(ii).

$$\therefore \text{From (i) and (ii)} : x = \frac{5}{3}[x], \text{ i.e. } x = \frac{5}{3} \cdot 0$$

$$\text{and } x = \frac{5}{3} \cdot 1.$$



Hence $x = 0, \frac{5}{3}$, i.e. the only two solutions.

Ex. 12. Draw the graph of $[y] = \sin^{-1}\left(\frac{x}{2}\right)$, where $[\cdot] = \text{G.I.F.}$

Sol. First draw the graph of

$$y = \sin^{-1}\left(\frac{x}{2}\right), \Rightarrow -1 \leq \frac{x}{2} \leq 1, \text{ i.e. } -2 \leq x \leq 2$$

$$\text{and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

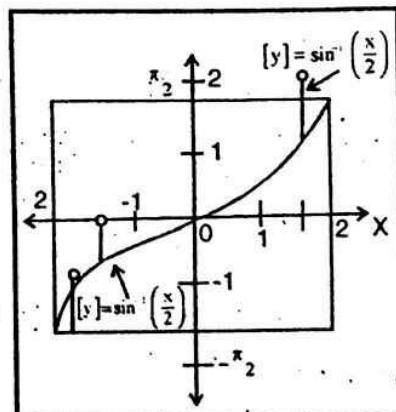
$$[y] = -2, -1, 0, 1. \therefore \sin^{-1}\left(\frac{x}{2}\right) = -2, -1, 0, 1. \text{ or}$$

$$\frac{x}{2} = -\sin 2, -\sin 1, 0, \sin 1,$$

$$\Rightarrow x = -2 \sin 2, -2 \sin 1, 0, 2 \sin 1.$$

Now when $x = -2 \sin 2, -2 \leq y < -1$.

$$\therefore x = -2 \sin 1, -1 \leq y < 0; x = 0, 0 \leq y < 1; x = 2 \sin 1, 1 \leq y < 2.$$



Ex. 13. Find the domain and range of $f(x) = \frac{e^x}{1+x} \forall x \in \mathbb{R}$.

$g(x)$

Sol. Let $f(x) = \frac{e^x}{F(x)}$, where $g(x) = e^x$ and $F(x) = 1+[x]$, now domain of $g(x) = e^x$ is set of all values of x as it is defined

$\forall x \in \mathbb{R}$.

$\therefore D_f$ of e^x = Set of all real numbers.

Also, domain of $f(x) = 1+[x]$ is set of all values of $x \in \mathbb{R}$. But

$$1+[x]=0 \quad \forall -1 \leq x < 0.$$

\therefore Domain of $f(x)$ will be $\mathbb{R} - \{-1 \leq x < 0\} = (-\infty, -1) \cup [0, \infty)$.

For range $x \geq 0$, $1+[x] \geq x, \forall x \in [0, \infty)$ and $e^x \geq 1, \forall x \in [0, \infty)$.

For $x \in [0, 1)$; $f(x) = e^x$ and for $x \in [1, 2)$, $f(x) = \frac{e^x}{2}$ and so on.

Thus all values of y or $e \geq 1$. \therefore Range $f(x) \in [1, \infty)$.

Ex. 14. Evaluate : $\lim_{n \rightarrow \infty} \frac{\{x\} + \{2x\} + \{3x\} + \dots + \{nx\}}{n^2}$ where $\{x\}$ is fractional part of x .

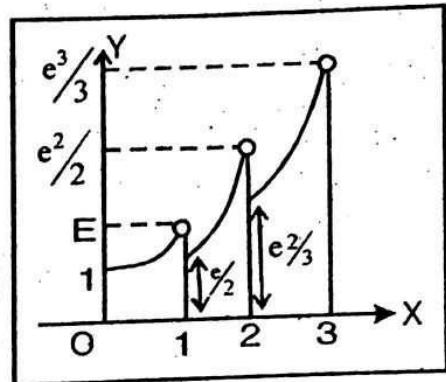
Sol. $\because 0 \leq \{rx\} < 1$ where $r = 1, 2, 3, \dots, n$.

$$\therefore 0 \leq \sum_{r=1}^n \{rx\} < \sum_{r=1}^n (1), \Rightarrow 0 \leq \sum_{r=1}^n \{rx\} < n.$$

Dividing throughout by n^2 , we get

$$\frac{0}{n^2} \leq \frac{\sum_{r=1}^n \{rx\}}{n^2} < \frac{1}{n}, \Rightarrow \lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} \frac{\sum_{r=1}^n \{rx\}}{n^2} < \lim_{n \rightarrow \infty} \frac{1}{n},$$

$$\Rightarrow 0 \leq \lim_{n \rightarrow \infty} \frac{\sum_{r=1}^n \{rx\}}{n^2} < 0.$$



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$$\therefore 0 \leq \lim_{n \rightarrow \infty} \frac{\{x\} + \{2x\} + \dots + \{nx\}}{n^2} < 0.$$

According to Sandwich theorem or Squeeze principle

$$\lim_{n \rightarrow \infty} \frac{\{x\} + \{2x\} + \dots + \{nx\}}{n^2} = 0.$$

Ex. 15. If $[x]$ denotes the integral part of x and

$$f(x) = [x] \left\{ \frac{\sin \frac{\pi}{x+1} + \sin \pi x + 1}{1+x} \right\}, \text{ then show that } f(x) \text{ is}$$

discontinuous at all integral points.

Sol. $\sin \pi [x+1] = 0$ and $[x+1] = [x] + 1$.

$$\therefore f(x) = \frac{[x]}{1+[x]} \sin \frac{\pi}{[x]+1}.$$

At $x = n$, $n \in I$, then $f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}$. For $n < x < n+1$, $n \in I$,

$$f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}. \therefore \lim_{x \rightarrow n+0} f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}.$$

$$\text{For } n-1 < x < n, [x] = n-1. \therefore f(x) = \frac{n-1}{n} \sin \frac{\pi}{n}.$$

$$\therefore \lim_{x \rightarrow n-0} f(x) = \frac{n-1}{n} \sin \frac{\pi}{n} \text{ and } f(x) = \frac{n}{1+n} \sin \frac{\pi}{n+1}.$$

Hence $f(x)$ is discontinuous at all $n \in I$.

Ex. 16. Let $[x]$ stands for the g.i.f. Find the derivative of $f(x) = (x + [x^3 + 1])^{x^2 + \sin x}$, wherever it exists in $(1, 3/2)$. Indicate the points where it does not exist.

Sol. We see that $[x^3 + 1]$ takes a jump from 2 to 3 at $2^{1/3}$ and again from 3 to 4 at $3^{1/3}$ in $(1, 3/2)$.

\therefore It is discontinuous at $x=2^{1/3}$ and $x=3^{1/3}$, i.e. it is not differentiable at

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$$x=2^{1/3} \text{ and } x=3^{1/3}.$$

$$\text{Now, } f(x) = (x + [x^3 + 1])^{x^2 + \sin x} = \begin{cases} (x+2)^{x^2 + \sin x} & \text{if } 1 < x < 2^{1/3} \\ (x+3)^{x^2 + \sin x} & \text{if } 2^{1/3} < x < 3^{1/3} \\ (x+4)^{x^2 + \sin x} & \text{if } 3^{1/3} < x < \frac{3}{2} \end{cases}$$

$$f'(x) = \begin{cases} (x+2)^{x^2 + \sin x} \left\{ (2x + \cos x) \log \left((x+2) + \frac{x^2 + \sin x}{x+2} \right) \right\} & \text{if } 1 < x < 2^{1/3} \\ (x+3)^{x^2 + \sin x} \left\{ (2x + \cos x) \log \left((x+3) + \frac{x^2 + \sin x}{x+3} \right) \right\} & \text{if } 2^{1/3} < x < 3^{1/3} \\ (x+4)^{x^2 + \sin x} \left\{ (2x + \cos x) \log \left((x+4) + \frac{x^2 + \sin x}{x+4} \right) \right\} & \text{if } 3^{1/3} < x < \frac{3}{2} \end{cases}$$

Ex. 17. Evaluate :

$$(i) \frac{\int_0^n [x] dx}{\int_0^n \{x\}}$$

$$(ii) \int_0^{[x]} \left(\int_0^{[x]} - [x - \frac{1}{2}] dx \right) dx$$

$$(iii) \int_0^{n\pi} \sin \left[\frac{2x}{n} \right] dx$$

$$(iv) \int_{-10}^0 \frac{2[x]}{3x - [x]} dx$$

$$(v) \int_{-1}^1 \frac{\sin^2 x}{\left[\frac{x}{\sqrt{2}} \right] + \frac{1}{2}} dx$$

$$(vi) \int_0^\infty [ne^{-x}] dx, \text{ where } [x] = \text{G.I.F.}; \quad \{x\} = \text{F.P. and } n \text{ is natural numbers.}$$

$$\int_0^n x dx = \int_0^1 x dx + \int_1^2 x dx + \int_2^3 x dx + \dots + \int_{n-1}^n x dx$$

$$17. (i) \text{ Let } I = \frac{\int_0^n \{x\} dx}{n \int_0^n \{x\} dx}$$

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$$= \frac{0+1 \int_1^2 dx + 2 \int_2^3 dx + \dots + (n-1) \int_{n-1}^n dx}{n \int_0^n x dx} = \frac{(n-1)n/2}{n/2} = (n-1).$$

$$(ii) \int_0^{[x]} \left(\int_0^{[x]} \left(x - \left[x + f\left(-\frac{1}{2}\right) \right] \right) dx \right) dx \quad (\text{let } x = [x] + f, 0 \leq f < 1)$$

$$= \int_0^{[x]} \left(\int_0^{[x]} -\left[f\left(-\frac{1}{2}\right) \right] dx \right) dx. \quad \because 0 \leq f < 1 \therefore -\frac{1}{2} \leq f\left(-\frac{1}{2}\right) < \frac{1}{2}, \text{i.e. } \left[f\left(-\frac{1}{2}\right) \right] = -1, 0.$$

$$\text{Case 1. If } \left[f\left(-\frac{1}{2}\right) \right] = 0, \text{ then } \int_0^{[x]} \left(\int_0^{[x]} 0 \cdot dx \right) dx = 0.$$

$$\text{Case 2. If } \left[f\left(-\frac{1}{2}\right) \right] = -1, \text{ then } \int_0^{[x]} \left(\int_0^{[x]} 1 \cdot dx \right) dx = 0$$

$$\int_0^{[x]} [x] dx = \int_0^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx + \dots + \int_{[x]-1}^{[x]} [x] dx = 0 + 1 + 2 + \dots + ([x]-1)[x] = -\frac{([x]-1)[x]}{2}.$$

$$(iii) \int_0^{\frac{\pi}{2}} \sin \left[\frac{2x}{\pi} \right] dx = \int_0^{\frac{\pi}{2}} \sin 0 dx + \int_{\frac{\pi}{2}}^{\pi} \sin 1 dx + \int_{\pi}^{\frac{3\pi}{2}} \sin 2 dx + \dots$$

$$+ \dots + \int_{\frac{n\pi}{4}}^{\frac{n\pi}{2}} \sin (2n-2) dx + \int_{\frac{n\pi}{2}}^{\frac{n\pi}{2}} \sin (2n-1) dx =$$

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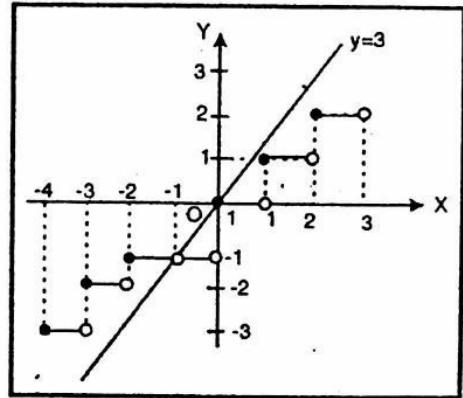
$$\frac{\pi}{2} \{ \sin 1 + \sin 2 + \sin 3 + \dots + \sin (2n-2) + \sin (2n-1) \} =$$

$$\frac{\pi}{2} \left[\frac{\sin \left(a + \frac{(2n-2)h}{2} \right) h}{\sin \frac{h}{2}} \times \sin \left\{ \frac{(2n-1)h}{2} \right\} \right] \text{ where } a = 1, h = 1,$$

$$= \frac{\pi}{2} \left[\frac{\sin \left\{ 1 + \frac{2(n-1)}{2} \right\} \cdot \sin \frac{(2n-1)}{2}}{\sin \frac{1}{2}} \right] = \frac{\pi}{2} \cdot \frac{\sin n}{\sin \frac{1}{2}} \sin \frac{(2n-1)}{2}$$

(iv) Let $f(x) = \frac{2[x]}{3x-[x]}$.

Clearly f is not defined if $x=0$ and when $3x=[x]$.



\therefore In $(-10, 0)$, f is not defined at $x = -\frac{1}{3}$.

Case 1. $x \in \left(-10, -\frac{1}{3}\right)$, $[x] < 0$ and $3x-[x] < 0$

$$\therefore \frac{[x]}{3x-[x]} > 0, \text{ i.e. } f(x) = 1.$$

Case 2. $x \in \left(-\frac{1}{3}, 0\right)$, $[x] < 0$ and $3x-[x] > 0$, i.e. $f[x] = -1$.

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$$\text{Now, } \int_{-10}^0 f(x)dx = \int_{-10}^{-1/3} dx + \int_{-1/3}^0 (-1)dx = (x)_{-10}^{-1/3} - (x)_{-1/3}^0 = \left(-\frac{1}{3} + 10\right)$$

$$-\left(0 + \frac{1}{3}\right) = \frac{28}{3}$$

$$(v) \text{ Let } I = \int_{-1}^1 f(x)dx, \text{ where } f(x) = \frac{\sin^2 x}{[x/\sqrt{2}] + 1/2}$$

$$\therefore f(-x) = \frac{\sin^2 x}{[-x/\sqrt{2}] + 1/2} = \frac{\sin^2 x}{-1 - [x/\sqrt{2}] + 1/2} \quad (\because [x] - [-x] = -1 \forall x \in I)$$

$\therefore x/\sqrt{2}$ is not an integer in $(-1, 0)$ and $(0, 1)$.

$$\therefore f(-x) = \frac{\sin^2 x}{[x/\sqrt{2}] - 1/2} = -f(x), \text{ i.e. } f(x) \text{ is an odd function in } x$$

$$\therefore I = \int_{-1}^1 f(x)dx = 0.$$

$$(vi) \text{ Let } f(x) = n.e^{-x} \quad \forall n \in \mathbb{N}, x \in \mathbb{R}^+.$$

$$\therefore f'(x) = -ne^{-x} < 0 \quad \forall x \in \mathbb{R}.$$

This is decreasing 0 to ∞ and in such type of problems we always break the internal for $x > \log n$, we have $[ne^{-x}] = 0$.

$$\therefore \int_0^\infty [xe^{-x}]dx = \int_0^{\log \frac{n}{n-1}} (x-1)dx + \int_{\log \frac{n}{n-1}}^{\log \frac{n}{n-2}} (x-2)dx + \int_{\log \frac{n}{n-2}}^{\log \frac{n}{n-3}} (x-3)dx + \dots +$$

$$\int_{\log \frac{n}{2}}^{\log n} 1dx + \int_{\log n}^{\infty} 0dx = (n-1) \left(\log \frac{n}{n-1} - 0 \right) + (n-2) \left(\log \frac{n}{n-2} - \log \frac{n}{n-1} \right)$$

$$+ \dots + 1 \cdot \left(\log \frac{n}{1} - \log \frac{n}{2} \right) + \dots + 0(\infty - \log n).$$

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Now writing the terms in reversed order, we have

$$\begin{aligned}
 \int_0^{\infty} n e^{-x} dx &= 1 \left(\log \frac{n}{1} - \log \frac{n}{2} \right) + 2 \left(\log \frac{n}{2} - \log \frac{n}{3} \right) + 3 \left(\log \frac{n}{3} - \log \frac{n}{4} \right) + \\
 &\dots + (n-2) \left(\log \frac{n}{n-2} - \log \frac{n}{n-1} \right) + (n-1) \log \frac{n}{n-1} \\
 &= \log n + (2-1) \log \frac{n}{2} + (3-2) \log \frac{n}{3} + \dots + [(n-1)-(n-2)] \log \frac{n}{n-1} \\
 &= \log n + \log \frac{n}{2} + \log \frac{n}{3} + \dots + \log \frac{n}{n-1} = \log \left(\frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{n-1} \right) \\
 &= \log \left\{ \frac{n^{n-1}}{(n-1)!} \right\} = \log \left\{ \frac{n^n \cdot n^{-1}}{(n-1)!} \right\} = \log \left(\frac{n^n}{n!} \right)
 \end{aligned}$$

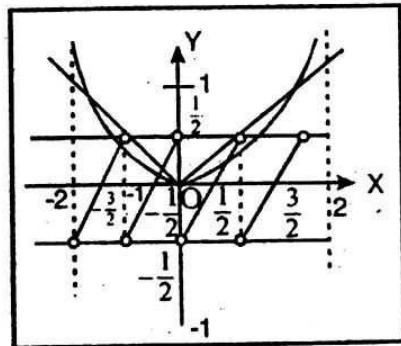
Ex. 18. Let $f(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \notin \mathbb{I} \\ 0 & \text{if } x \in \mathbb{I} \end{cases}$ where $[.]$ denotes the G.I.F. If

$g(x) = \max \{x^2, f(x), |x|\}$, $\forall x \in [-10, 10]$, then find the value

of $\int_{-2}^2 g(x) dx$.

Sol. It is clear from the graph

$$g(x) = \begin{cases} x^2 & \text{if } -2 \leq x \leq -1 \\ -x & \text{if } -1 \leq x \leq -\frac{1}{4} \\ x + \frac{1}{2} & \text{if } -\frac{1}{4} \leq x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ x^2 & \text{if } 1 \leq x \leq 2 \end{cases}$$



$$\therefore \int_{-2}^2 f(x) dx = \int_{-2}^{-1} x^2 dx + \int_{-1}^{-1/4} (-x) dx + \int_{-1/4}^0 \left(x + \frac{1}{2} \right) dx + \int_0^1 x dx = \frac{275}{98}$$

Ex. 19. Let the curves $C_1: y^2 = 4[\sqrt{y}]x$ and $C_2: x^2 = 4[\sqrt{x}]y$ where $[.]$ denotes the G.I.F., find the area of the region enclosed by these two curves with in the square formed by the lines, $x=1, y=1, x=4, y=4$.

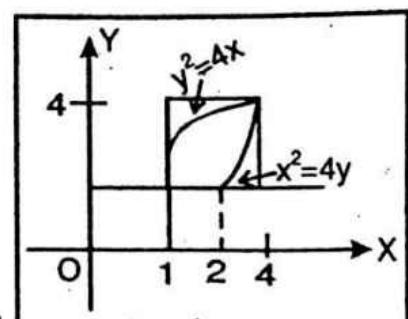
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Sol. Since $1 < x < 4$, so $1 < \sqrt{x} < 2$. $\therefore [\sqrt{x}] = 1$ and similarly $[\sqrt{y}] = 1$.

Now, $c_1: y^2 = 4x$ and $c_2: x^2 = 4y$.

$$\therefore \text{Reqd. area} = \int_1^4 2\sqrt{x} dx - (2-1) \cdot 1 - \int_2^4 \frac{x^2}{4} dx$$

$$= \frac{4}{3} x^{3/2} \Big|_1^4 - 1 - \frac{1}{4} x^3 \Big|_2^4 = \frac{11}{3} \text{ sq. units.}$$



Ex. 20. If x is a real number in $[0,1]$, then

show that the value of $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \{1 + \cot^{2m}(n! \pi x)\}$ is given by 2 or 1 according as x is rational or irrational.

Sol. Case 1. If $x \in \mathbb{Q}$, then $n! \pi x$ will be an integral multiple of π for large values of n .

$\therefore \cos(n! \pi x)$ will be either 1 or -1, i.e. $\cos^{2m}(n! \pi x) = 1$

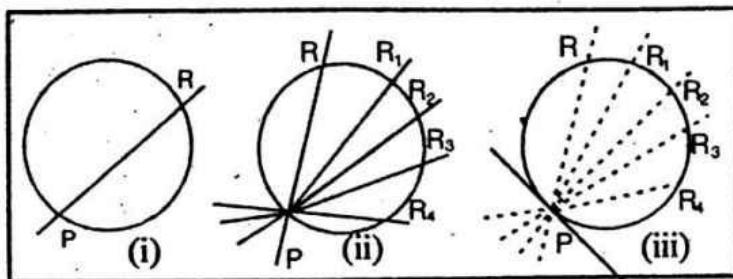
$$\therefore \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \{1 + \cos^{2m}(n! \pi x)\} = 1+1=2.$$

Case 2. If $x \in \mathbb{Q}$, then $x! \pi x$ will not be an integral multiple of π .

$\therefore \cos(x! \pi x)$ will be between -1 or 1, i.e. $\cos^{2m}(x! \pi x) = 0$.

$$\text{Hence, } \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \{1 + \cos^{2m}(n! \pi x)\} = 1+0=1.$$

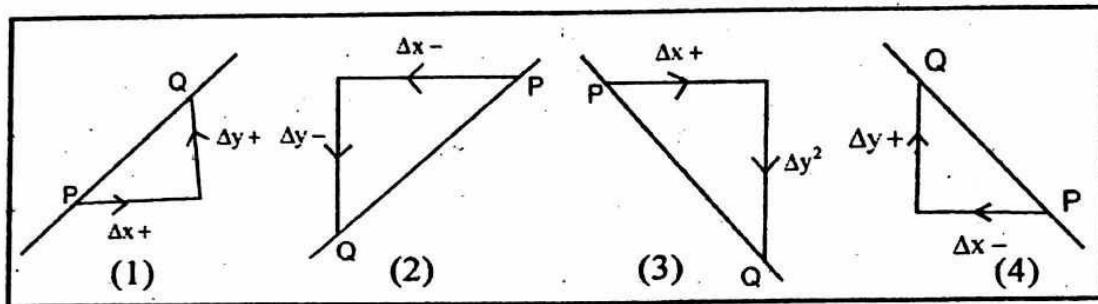
□ LIMING POSITION OF A SECANT : A secant (Latin *seco*, I cut) is a straight line that cuts a curve at two points. If the secant in fig (i) moves in such a way that P remains fixed in position, but R approaching P more and more closely, successively occupies the position R_1, R_2, R_3 and R_4 , shown in fig.(ii). It will be seen that the length of chord PR and the length arc PR both becomes successively less and less as R approaches P , while all the time the secant of which PR is part approaches closer and closer to its **limiting position**, which is that of the tangent to the curve at point P as shown in fig(iii).



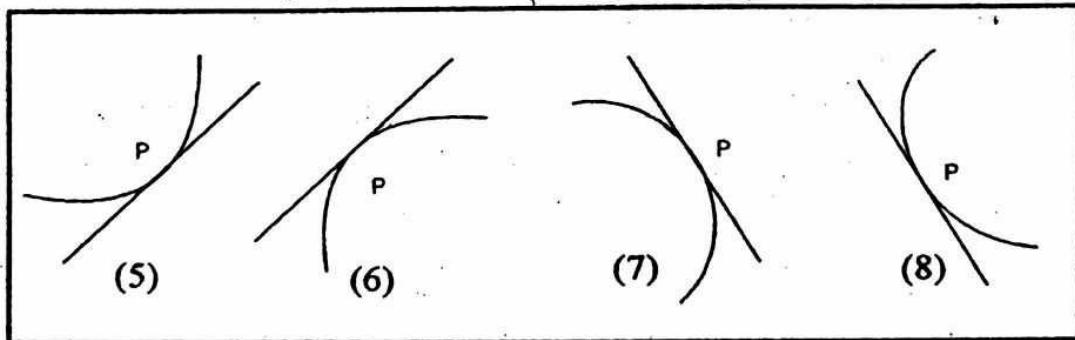
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□ **SIGN OF $\frac{dy}{dx}$** : Let $y=f(x)$. If $\frac{\Delta y}{\Delta x}$ is (+)ve Δx and Δy have the same sign, i.e. an increase in x produces an increase in y and a decrease in x produces a decrease in y .

If $y=f(x)$ be drawn, P being the point (x, y) and $Q (x + \Delta x, y + \Delta y)$. Then if Δx and Δy both (+)ve P and Q will be placed as in (1), if both negative as in (2). In each case the chord PQ has a (+)ve gradient. But if Δx be (+)ve and Δy (-)ve, P and Q will be placed as in (3), if Δx (-)ve, and Δy (+)ve, as in (4).



So, if $\frac{dy}{dx}$ is (+)ve for a given value of x , x and y are both increasing or both decreasing, but if $\frac{dy}{dx}$ is (-)ve, x is increasing and y decreasing or vice-versa. In the graph if $\frac{dy}{dx}$ is (+)ve at the point (x, y) , the gradient of



the tangent is (+)ve and the curve in the neighbourhood of P is shaped like (5) or (6), but if $\frac{dy}{dx}$ is (-)ve, the shape is like (7) or (8).

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□ MAXIMA AND MINIMA :

At A and D, the gradient is changing from (-)ve to (+)ve and these are called **minimum points**, FA and HD are **minimum (or minima) values** of y . At B and E, the gradient is changing from (+)ve to (-)ve and these are called **maximum (or maxima) points**, GB and JE are **maximum values** of y .

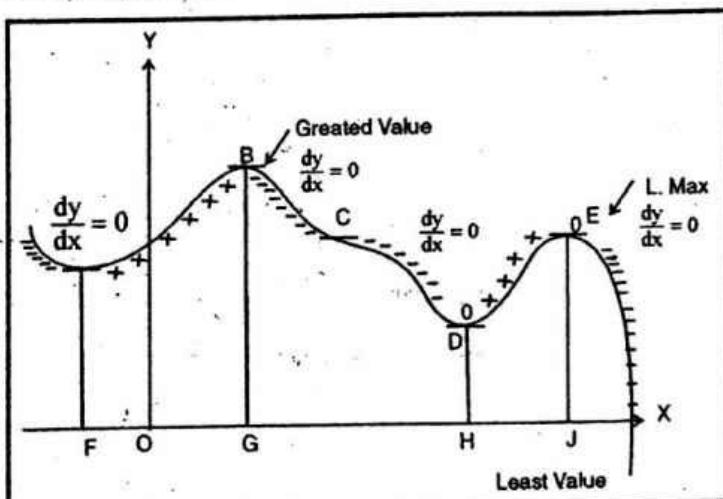
The words max. and min. are used in the sense of greatest and least only in the immediate vicinity of the point, this local meaning is brought out clearly in this curve, since a maximum value, JE as in fact less than a minimum value FA.

At C the gradient is zero, but is not changing sign, this point is a **point of inflexion**, which may be linked to the point on an S-bend at which a road stops turning left and begins to turn right or vice-versa. The gradient of a curve at a point of inflexion need not be zero (the reader should be able to spot four more in given fig.), however at this stage we are concerned only with searching for maximum and minimum. We need to

bear in mind points of inflexion only as a third possibility at points where the gradient of a curve is zero, At any point where the gradient of a curve is zero, y is said to be a **stationary value**.

Any max or min point is called a **turning point** and, y

is said to be a **turning value** there.



Note: While a function has at most only one **greatest value** or **absolute max.** or **global max** and at most only one **least value** or **absolute min.** or **global min.**. It may have several **max. value** or **local max** and several **min value** or **local min**.

□ DISTINGUISHING BETWEEN MAX AND MIN POINTS :

At a max. point $\frac{dy}{dx} = 0$. Just before the point $\frac{dy}{dx}$ is (+)ve and just after if $\frac{dy}{dx}$ is (-)ve. Thus in passing from one side of the point to the other $\frac{dy}{dx}$ decrease. If a graph of $\frac{dy}{dx}$ against x is plotted, it has a downward slope

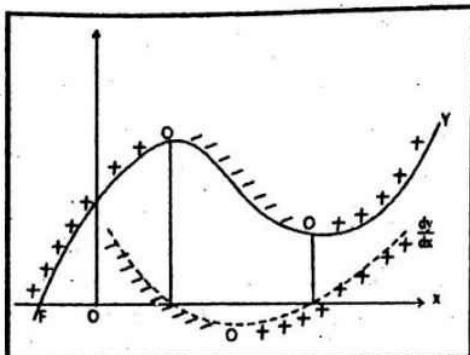
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in the region of the point under consideration. Now the slope of the

$\left(\frac{dy}{dx} - x\right)$ graph is given by $\frac{d}{dx}\left(\frac{dy}{dx}\right) =$

$\frac{d^2y}{dx^2}$. So, at the max. point $\frac{d^2y}{dx^2}$ is (-)

ve (except as below). In fig. the graph of y is represented by a continuous



line and the graph of $\frac{dy}{dx}$ by a dotted line. In a similar way it can be seen

that at a min. point $\frac{d^2y}{dx^2}$ will be (+)ve (except as below) This provides a

method of investigating the nature of a turning point.

If, at a point, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, i.e. max. point.

If, at a point, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, i.e. min. point.

Ex. O is a point on a straight line. A particle moves along the line so that it is s ft. from O, t sec. after a certain instant, where $s = t(t-2)^2$. Describe the motion before and after $t=0$.

Sol. Here $s = t^3 - 4t^2 + 4t$.

$$\therefore \frac{ds}{dt} = 3t^2 - 8t + 4 = (3t-2)(t-2) = v.$$

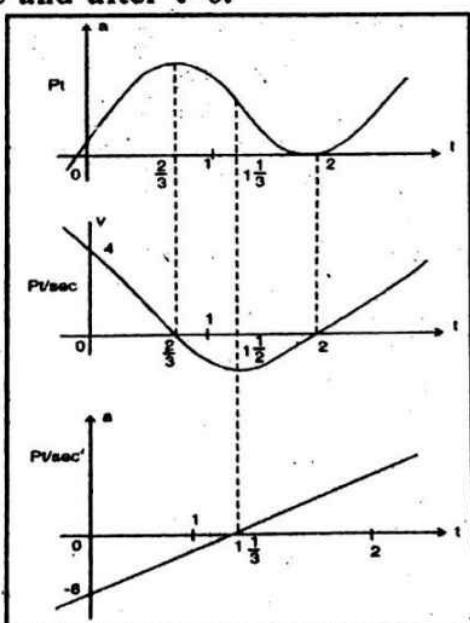
This graph has a min. point $\left(1\frac{1}{3}, -1\frac{1}{3}\right)$

and passes through $\left(\frac{2}{3}, 0\right)$, $(2, 0)$

and $(0, 4)$, it is the middle sketch and

upper sketch the max pt. $\left(\frac{2}{3}, \frac{32}{27}\right)$

and min. point $(2, 0)$.



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Again, $\frac{dv}{dt} = a = 6t - 8$ and is the bottom sketch.

Notice that the max. and min. values of s occur when $v = \frac{ds}{dt} = 0$ and that

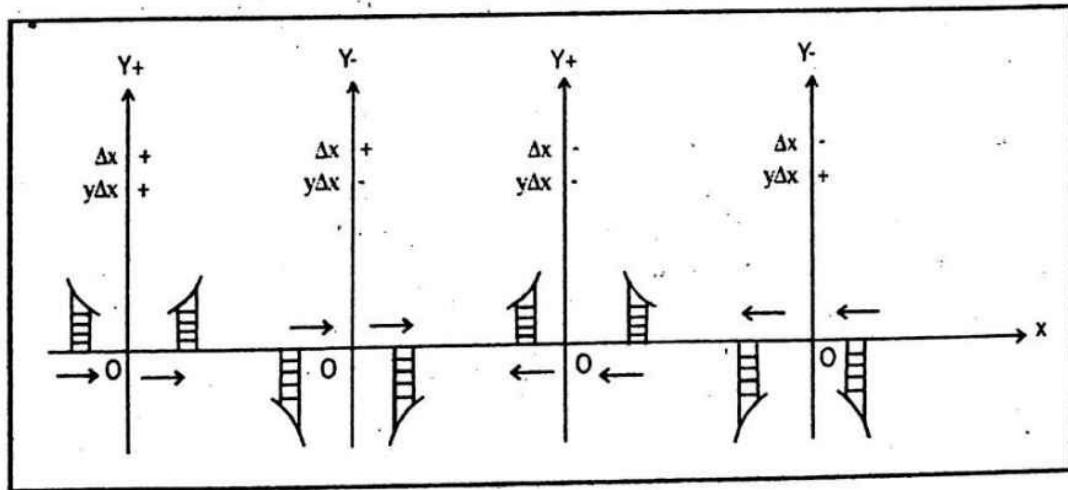
the min. value of v occurs when $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 0$.

Before $t=0$, the particle is approaching 0 from the (-)ve side, at $t=0$ it is passing through 0 with velocity 4ft./Sec. and acceleration - 8ft/Sec².

Hence its speed is decreasing and it comes momentarily to rest $\frac{32}{27}$ ft.

from 0 (on the (+)ve side), when $t = \frac{2}{3}$, it returns to 0, where it is momentarily at rest when $t=2$, and thereafter it moves away from 0 in the (+)ve direction.

□ SIGN OF AREA : If we advance from left to right, i.e. if Δx is (+)ve, $y\Delta x$ representing the area of our typical strip is (+)ve if y is



(+)ve, i.e. if the strip lies above the x-axis and (-)ve if the strip lies below the x-axis is given fig.

If we advance from right to left, i.e. if Δx is (-)ve, $y\Delta x$ is (+)ve if the strip lies below and (-)ve if the strip lies above the x-axis.

□ DIVISION BY ZERO : Values like $\frac{0}{0}, \frac{3}{0}, \frac{a}{0}$ are inadmissible. There

have no meaning. Some students are under the impression that $\frac{5}{0}$ is