CSER 2207_8: Numerical Analysis-I

Lecture-6
Solution of equation in single variable

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Error Analysis

Order of Convergence

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ **converges to p of order α, with asymptotic error constant λ.** ■

An iterative technique of the form $p_n = g(p_{n-1})$ is said to be of *order* α if the sequence $\{p_n\}_{n=0}^{\infty}$ converges to the solution $p = g(p)$ of order α.

In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order. The asymptotic constant affects the speed of convergence but not to the extent of the order. Two cases of order are given special attention.

(i) If $\alpha = 1$ (and $\lambda < 1$), the sequence is **linearly convergent**.

(ii) If $\alpha = 2$, the sequence is **quadratically convergent**.

The next illustration compares a linearly convergent sequence to one that is quadratically convergent. It shows why we try to find methods that produce higher-order convergent sequences.

Illustration

Suppose that $\{p_n\}_{n=0}^{\infty}$ is linearly convergent to 0 with

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = 0.5$$

and that $\{\tilde{p}_n\}_{n=0}^{\infty}$ is quadratically convergent to 0 with the same asymptotic error constant,

$$\lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} = 0.5.$$

For simplicity we assume that for each n we have

$$\frac{|p_{n+1}|}{|p_n|} \approx 0.5 \quad \text{and} \quad \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} \approx 0.5.$$

For the linearly convergent scheme, this means that

$$|p_n - 0| = |p_n| \approx 0.5|p_{n-1}| \approx (0.5)^2|p_{n-2}| \approx \cdots \approx (0.5)^n|p_0|,$$

whereas the quadratically convergent procedure has

$$\begin{aligned} |\tilde{p}_n - 0| = |\tilde{p}_n| &\approx 0.5|\tilde{p}_{n-1}|^2 \approx (0.5)[0.5|\tilde{p}_{n-2}|^2]^2 = (0.5)^3|\tilde{p}_{n-2}|^4 \\ &\approx (0.5)^3[(0.5)|\tilde{p}_{n-3}|^2]^4 = (0.5)^7|\tilde{p}_{n-3}|^8 \\ &\approx \cdots \approx (0.5)^{2^n-1}|\tilde{p}_0|^{2^n}. \end{aligned}$$

Cont…

Table 2.7 illustrates the relative speed of convergence of the sequences to 0 if $|p_0| = |\tilde{p}_0| = 1$.

Table 2.7

n	Linear Convergence Sequence $\{p_n\}_{n=0}^{\infty}$ $(0.5)^n$	Quadratic Convergence Sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ $(0.5)^{2^n-1}$
1	5.0000×10^{-1}	5.0000×10^{-1}
2	2.5000×10^{-1}	1.2500×10^{-1}
3	1.2500×10^{-1}	7.8125×10^{-3}
4	6.2500×10^{-2}	3.0518×10^{-5}
5	3.1250×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

The quadratically convergent sequence is within 10^{-38} of 0 by the seventh term. At least 126 terms are needed to ensure this accuracy for the linearly convergent sequence. □

Linear Convergence

Theorem 2.8 Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x \in [a,b]$. Suppose, in addition, that g' is continuous on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a,b).$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \ge 1,$$

converges only linearly to the unique fixed point p in $[a,b]$. ■

Proof We know from the Fixed-Point Theorem 2.4 in Section 2.2 that the sequence converges to p. Since g' exists on (a,b), we can apply the Mean Value Theorem to g to show that for any n,

$$p_{n+1} - p = g(p_n) - g(p) = g'(\xi_n)(p_n - p),$$

where ξ_n is between p_n and p. Since $\{p_n\}_{n=0}^{\infty}$ converges to p, we also have $\{\xi_n\}_{n=0}^{\infty}$ converging to p. Since g' is continuous on (a,b), we have

$$\lim_{n\to\infty} g'(\xi_n) = g'(p).$$

Thus

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} = \lim_{n\to\infty} g'(\xi_n) = g'(p) \quad \text{and} \quad \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|.$$

Hence, if $g'(p) \neq 0$, fixed-point iteration exhibits linear convergence with asymptotic error constant $|g'(p)|$. ■ ■ ■

Quadratic Convergence of Newton-Raphson method

Successive approximations of *Newton–Raphson formula* are

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.24}$$

To obtain the rate of convergence of the method, we note that $f(\xi) = 0$ so that Taylor's expansion gives

$$f(x_n) + (\xi - x_n) f'(x_n) + \frac{1}{2}(\xi - x_n)^2 f''(x_n) + \cdots = 0,$$

from which we obtain

$$-\frac{f(x_n)}{f'(x_n)} = (\xi - x_n) + \frac{1}{2}(\xi - x_n)^2 \frac{f''(x_n)}{f'(x_n)} \tag{2.27}$$

From (2.24) and (2.27), we have

$$x_{n+1} - \xi = \frac{1}{2}(x_n - \xi)^2 \frac{f''(x_n)}{f'(x_n)} \tag{2.28}$$

Setting

$$\varepsilon_n = x_n - \xi, \tag{2.29}$$

Equation (2.28) gives

$$\varepsilon_{n+1} \approx \frac{1}{2}\varepsilon_n^2 \frac{f''(\xi)}{f'(\xi)}, \tag{2.30}$$

so that the Newton–Raphson process has a second-order or quadratic convergence.

Thank You