Combinatorics

→ Permutations
→ Combinations
→ Formulas & derivations
→ Properties
→ 1 Problem

Combinatorics deals with selection and arrangement of objects according to some pattern ⇒ counting the no. of ways it can be done

Question: there is an exam with 3 T/F questions

Q1: T/F 2 ways
Q2: T/F 2 ways
Q3: T/F 2 ways

$2 \times 2 \times 2 = \boxed{8}$

$\boxed{8}$
6
9

If a question can be left unanswered,

Q1: T/F/_ ⇒ 3 ways ⇒ $(1+1+1=3)$
Q2: T/F/_ ⇒ 3
Q3: T/F/_ ⇒ 3

$3 \times 3 \times 3 = \boxed{27}$

$3+3+3$

$3 \times 3 \times 3 = 27$

Multiplicative Principle

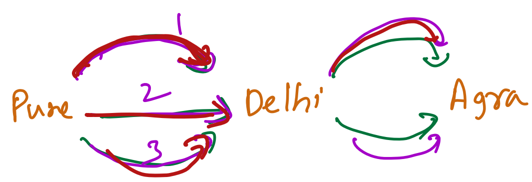
Task A : n ways

Task B : m ways

No. of ways of doing Task A & Task B

is $n \times m$

→ Travel Pune to Agra (via Delhi)



$2 + 2 + 2$

$= 2 \times 3$

Task A : Travel from Pune to Delhi ⇒ 3

Task B : Travel from Delhi to Agra ⇒ 2

Task A & Task B : Travel from Pune to Agra via Delhi

$= \text{num}(\text{Task A}) \times \text{num}(\text{Task B})$

$3 \times 2 = 6$

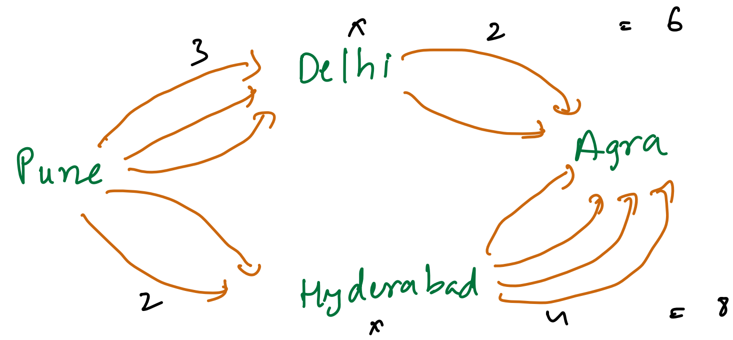
Addition Principle

Task A : n ways

Task B : m ways

No. of ways of doing Task A or Task B

is $n + m$



6 ← Task A : Travel from Pune to Agra (via Delhi)

8 ← Task B : Travel from Pune to Agra (via Hyd)

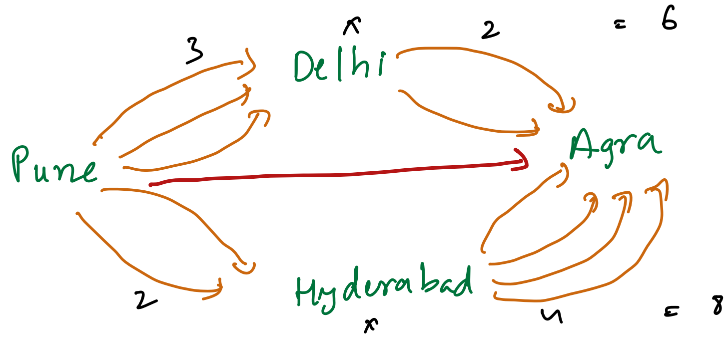
Task A or Task B : Travel from Pune to Agra

(via Delhi / Hyd)

$= \text{num}(\text{Task A}) + \text{num}(\text{Task B})$

$= 6 + 8$

$= 14$



6 ⇐ Task A: Travel from Pune to Agra (via Delhi)

8 ⇐ Task B: Travel from Pune to Agra (via Hyd)

1 ⇐ Task C: Trave from Pune to Agra directly

$6 + 8 + 1 = 15$

AND ⟹ Multiplication

OR ⟹ Addition

Q1: T / F / _

Q2: T / F / _

Q3: T / F / _

Permutations

Ordered arrangement of objects

Str = "abc"

a b c
a c b
b a c
b c a
c a b
c b a

6

abc != acb

(abc != cab)

Order of chars/ objects matters

Str = "abc"   N = 3

(3) (2) (1)
1st 2nd 3rd

TaskA : No. of ways of placing char in Pos1 => 3

TaskB : No. of ways of placing char in Pos2 => 2

TaskC : No. of ways of placing char in Pos3 => 1

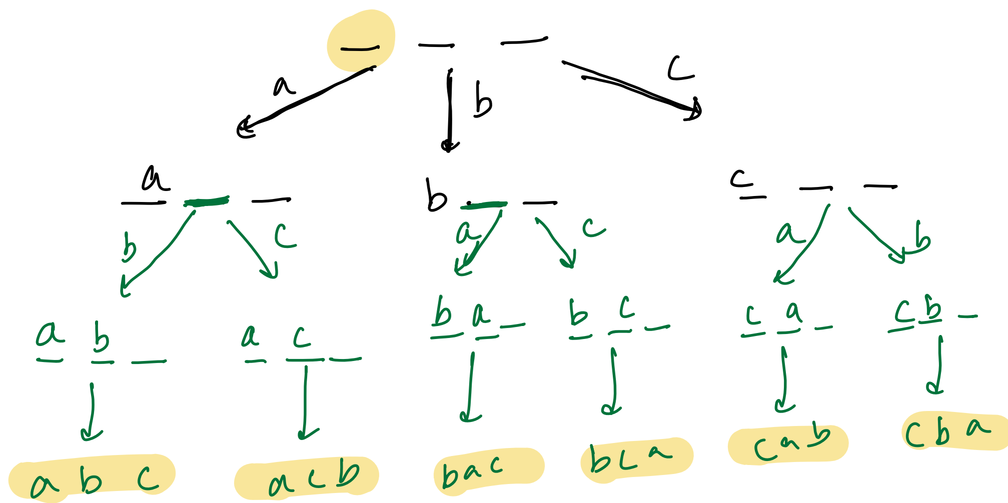
$3 \times 2 \times 1 = 6$

N chars

$$\underset{N}{\_}\ \underset{(N-1)}{\_}\ \underset{(N-2)}{\_}\ \_\ \cdots\ \underset{3}{\_}\ \underset{2}{\_}\ \underset{1}{\_}$$

$$N \times (N-1)(N-2) \cdots 3 \cdot 2 \cdot 1 = N!$$

S: "a b c"



Question: No. of ways to arrange any R characters among N character

S = "abcde"

N = 5

$\boxed{R = 2}$

$\frac{a \times x}{x}$

$\underline{(5)}\ \underline{(4)} = 5 \times 4 = \boxed{20}$

$= \boxed{5 \times 4} = \rightarrow \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$

$= 20$

$= \boxed{\frac{5!}{3!}}$

S = "abcde"

N = 5

$\boxed{R = 3}$

$\underline{5}\ \underline{4}\ \underline{3} = 5 \times 4 \times 3 = 60$

$= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!}$

$N, R \Rightarrow \frac{N!}{(N-R)!}$

No. of ways of arranging R chars among N chars $= \frac{N!}{(N-R)!} = {}^{N}P_{R}$

$$^{N}P_{R} = \frac{N!}{(N-R)!}$$

Permutations with repetitions

S = "aba"

a_1 a_2

N = 3

$\boxed{3! = 6}$

aab
aba
baa

$\Rightarrow$ 3 permutations

$a_1 b a_2 \Rightarrow aba$

$a_1 a_2 b \Rightarrow aab$

$a_2 a_1 b \Rightarrow aab$

$b a_1 a_2 \Rightarrow baa$

$b a_2 a_1 \Rightarrow baa$

$a_2 b a_1 \Rightarrow aba$

$\Rightarrow \boxed{3}$

S = "abbbcd"

b_1 b_2 b_3

num('b') = 3

N = 6

$$\Rightarrow \frac{6!}{3!} = 6 \times 5 \times 4 = \boxed{120}$$

$\rightarrow$ a b_1 c b_2 b_3 d

$\rightarrow$ a b_2 c b_1 b_3 d

a b_1 c b_3 b_2 d

abcbbd

abcbbd

Same Permutation

$a \quad b_2 \quad c \quad b_3 \quad b_1 \quad d$

$a \quad b_3 \quad c \quad b_1 \quad b_2 \quad d$

$a \quad b_3 \quad c \quad b_2 \quad b_1 \quad d$

Given N characters = $\boxed{N!}$

$b_1 \quad c \quad b_2 \quad a \quad b_3 \quad d$

$b_1 \quad c \quad b_3 \quad a \quad b_2 \quad d$

$b_2 \quad c \quad b_1 \quad a \quad b_3 \quad d$

$b_2 \quad c \quad b_3 \quad a \quad b_1 \quad d$

$b_3 \quad c \quad b_1 \quad a \quad b_2 \quad d$

$b_3 \quad c \quad b_2 \quad a \quad b_1 \quad d$

6 times

$b_1 \quad b_2 \quad b_3$

$b_1 \quad b_2 \quad b_3$

$b_1 \quad b_3 \quad b_2$

$b_2 \quad b_1 \quad b_3$

$b_2 \quad b_3 \quad b_1$

$b_3 \quad b_1 \quad b_2$

$b_3 \quad b_2 \quad b_1$

$3! = \boxed{6}$

$\Rightarrow$ $d\ b_1\ a\ b_2\ b_3\ c$

$d\ b_1\ a\ b_3\ b_2\ c$

$d\ b_2\ a\ b_1\ b_3\ c$

$d\ b_2\ a\ b_3\ b_1\ c$

$d\ b_3\ a\ b_1\ b_2\ c$

$d\ b_3\ a\ b_2\ b_1\ c$

Any permutation would occur 6 times.

Unique Permutation: $\dfrac{\text{Total Permutation}}{3!}$

$\underbrace{P_1\ P_1\ P_1\ P_1\ P_1\ P_1}_{6}$

$\underbrace{P_2\ P_2\ P_2\ P_2\ P_2\ P_3}_{6}$

$\underbrace{P_3\ P_3\ P_3\ P_3\ P_3\ P_3}_{6}$

#permutation = 4 $\{P_1, P_2, P_3, P_4\}$ 6

$P_1 = 6$

$P_2 = 6$

$P_3 = 6$

$P_4 = 6$

$\dfrac{\boxed{4 \times 6}}{6}$ $\Rightarrow$ Total Perm $\Rightarrow$

$\dfrac{6 + \cancel{6}\,6 + 6}{6}$:

$\dfrac{18}{6} = 3$

$S =$ "a $\underbrace{bbb}_{\text{3! arrangement of b's}}$ c" $\Rightarrow \dfrac{5!}{3!}$

(Freq of b)

$N = 6$

$S =$ "aa bbb c" $\qquad \dfrac{6!}{2! \times 3! \; 1!}$

b a_1 b a_2 b c

b a_2 b a_1 b c

a_1 c b b a_2 b

a_2 c b b a_1 b

$a_1 \; a_2$

$a_2 \; a_1$

$\overline{}$

$2!$

Every Permutation occurs 2 times

(If we consider)

=

Permutations :

$$\frac{N!}{r_1! \cdot r_2! \cdot r_3! \cdot r_4! \ldots r_k!}$$

r_i = no. of times i^{th} character would occur

S = "accbaa"

$N = 6$

$$\frac{6!}{3! \cdot 2! \cdot 1!}$$

$$= \frac{6 \times 5 \times \cancel{4}^{2} \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1} \times \cancel{2}}$$

$= 6 \times 5 \times 2 = 60$

Combinations

(10: 20 PM)

Selection of objects (unordered)

abc == cba ≡ cab ≡ acb ≡ bac = bca

S : "abcde"

{abcd} {abce} {abde} {acde} {bcde}

$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{5 \text{ ways}}$

N_{C_r} ⇒ No. of ways of choosing 'r' characters from N characters

$$= \frac{N!}{r!\,(N-r)!}$$

Ex2 : S = "abcd" N = 4 r = 2

{ab, ac, ad, bc, bd, cd} ⇒

ba ca da cb db dc

6 ways

$$N_{C_r} = \frac{N!}{r!\,(N-r)!} = \frac{4!}{2!\cdot 2!}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2 \times 2}$$

$$= \boxed{6}$$

Derivation:

Given N characters, No. of ways of arranging r characters

$$^{N}P_{R} = \frac{N!}{(N-R)!}$$

S = "abcde"　　　N = 5 , R = 3

N = 5

R = 3

$$^{5}C_{3} = \frac{5!}{3!.2!} = \frac{5 \times \cancel{4}^{2} \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2} \times \cancel{2}} = \boxed{10}$$

$$\#\text{permutations with R characters} = \frac{\;\frac{N!}{(N-R)!}\;}{R!}$$

cdb
cbd
bcd
bdc
dbc
dcb

⇒ 1 Combination

R = 3

$\boxed{R!}$

3 chars ⇒ $\boxed{3!}$

ace
aec
cae
cea
eac
eca

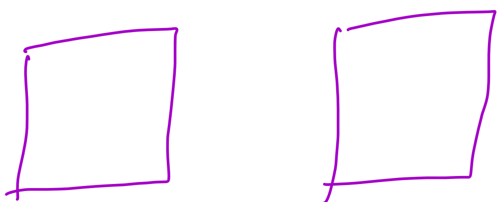
⇒ 1 Combination

$\boxed{3!}$

$$ {}^{N}C_{R} = \frac{\left[\frac{N!}{(N-R)!}\right]}{R!} = \frac{N!}{R!\,(N-R)!} $$

Property :

1) There are N Balls



Bag1 Bag2

⇒ No. of ways of placing R balls in Bag1 and remaining balls in bag.

$$ \boxed{{}^{N}C_{R}} \Rightarrow \; - \; 1 \qquad \frac{N!}{R!\,(N-R)!} $$

⇒ No. of ways of placing $(N-R)$ balls in Bag2

$$ \boxed{{}^{N}C_{N-R}} \; - \; 2 $$

$$ \Rightarrow \frac{N!}{(N-R)!\,(N-(N-R))!} $$

$N-N+R$

$$ = \frac{N!}{(N-R)! \times R!} $$

$$ 1 == 2 $$

$$\boxed{{}^{N}C_{R} = {}^{N}C_{N-R}}$$

2) $${}^{N}C_{0} = \frac{N!}{0!\,(N-0)!} = \frac{N!}{N!} = \boxed{1}$$

$0! = 1$

3) $${}^{N}C_{1} = \frac{N!}{1!\,(N-1)!} = \frac{N \times \cancel{(N-1)}!}{\cancel{(N-1)}!} = \boxed{N}$$

$S = abcde$

$\{a, b, c, d, e\}$

4) $${}^{N}C_{N} = \frac{N!}{N!\,(N-0)!} = \frac{N!}{N!} = \boxed{1}$$

$${}^{N}C_{N} = {}^{N}C_{N-N} = {}^{N}C_{0} \qquad ({}^{n}C_{r} = {}^{n}C_{n-r})$$

5) Find no of ways of selecting r items from N items

$\Downarrow$

$^{N}C_{r}$

$\underbrace{a_1 \quad a_2 \quad a_3 \quad a_4 \quad \dots \quad a_{N-1}}_{N-1} \quad a_N$

Pick a_N $\rightarrow$ $^{N-1}C_{R-1}$ $+$ Don't pick a_N $\rightarrow$ $^{N-1}C_{R}$

Choices for n^{th} item:

r1) Select a_N $\Rightarrow$

OR

r2) Don't select a_N $\Rightarrow$

LHS RHS

$$\boxed{^{N}C_{R} = {}^{N-1}C_{R-1} + {}^{N-1}C_{R}}$$

LHS: $\dfrac{N!}{R!\,(N-R)!}$

RHS: $\left[\dfrac{(N-1)!}{(R-1)!\,(N-R)!} + \dfrac{(N-1)!}{R!\,(N-1-R)!}\right]$

$(N-R)\,(N-1-R)!$ $\quad$ $R \times (R-1)!$

$$\frac{(N-1)!}{(R-1)!\,(N-R-1)!}\left[\frac{1}{N-R} + \frac{1}{R}\right]$$

$$= \frac{(N-1)!}{(R-1)!\,(N-R-1)!}\left[\frac{\cancel{R} + N - \cancel{R}}{(N-R)\,(R)}\right]$$

$$= \frac{(N-1)!}{(R-1)!\,(N-R-1)!} \times \left[\frac{N}{R \times (N-R)}\right]$$

$$= \frac{N!}{R!\,(N-R)!} = {}^{N}C_{R}$$

Question: Given N, r, M

compute $({}^{N}C_{r}) \% M$

$N, r, M \le 10^9$

Ex1: $N = 4 \quad r = 2 \quad M = 10$

Ans = 6

$({}^{4}C_{2}) \% 10$

$\frac{4 \times 3 \times \cancel{2} \times \cancel{1}}{2 \times \cancel{2}} = 6$

$6 \% 10 = 6$

$N = 10^9$

$r = 10^3$

$(10^9)!$

$\left(\frac{N!}{r!\,(N-r)!}\right) \% m$

$\left(\frac{a}{b}\right) \% m \neq \left(\frac{a \% m}{b \% m}\right) \% m$

LHS: RHS:

$a = 7, \quad b = 2, \quad M = 6$

LHS: $\left(\frac{7}{2}\right) \% 6 = 3 \% 6 = \boxed{3}$

RHS: $\left(\frac{7 \% 6}{2 \% 6}\right) \% 6 = \left(\frac{1}{2}\right) \% 6 = 0 \% 6 = \boxed{0}$

$$\left(\frac{a}{b}\right)\%m = \left(a\%m \times (b^{-1})\%m\right)\%m$$

Fermat's theorem

$$b^{-1}\%M = b^{M-2}\%M$$

→ M has to be Prime

→ b and M has to be co-prime $\gcd(b, M) = 1$

$${}^{N}C_{R}\%M = \left(\frac{N!}{R!\,(N-R)!}\right)\%M$$

$$= \left(N!\%M \times (R!)^{-1}\%M \times ((N-R)!)^{-1}\%M\right)\%M$$

R! & M are co-prime ?

$\gcd(M, R!) = 1$

$M = 11$

$R = 12$

$\gcd(11, 12!)$

$\gcd(11, 22) = 11$

$\gcd(11, 12 \times 11 \times 10 \times 9 \ldots\ldots 1)$

11

$N, r < M$ && M is prime

$\gcd(R!, M) = 1$ $\gcd((N-R)!, M) = 1$

11

$N, r < M$ $\gcd(10!, 11) =$

$N-r < M$

9!

8!

7!

Conditions to apply Inverse modulo

1) M has to be prime

2) $N, R < M$

Solution:

Property:

$$^{N}C_{R} = {}^{N-1}C_{R} + \underbrace{{}^{N-1}C_{R-1}}$$

$N = 5$ $R = 2$ $M = 10$ $({}^5C_2)\% 10$

$^{N}C_{0} = 1$, $^{N}C_{1} = N$, $^{N}C_{N} = 1$

$^{5}C_{2} = {}^{4}C_{2} + {}^{4}C_{1} \Rightarrow 4 \Rightarrow 6+4 = 10$

$^{4}C_{2} = 3\ {}^{3}C_{2} + {}^{3}C_{1} \Rightarrow 3 = 6$

$^{3}C_{2} = 1\ {}^{2}C_{2} + {}^{2}C_{1} \Rightarrow 2 = 2+1 = 3$

$$\left({}^{N}C_{R}\right)\%M = \left(\left({}^{N-1}C_{R}\right)\%M + \left({}^{N-1}C_{R-1}\right)\%M\right)\%M$$

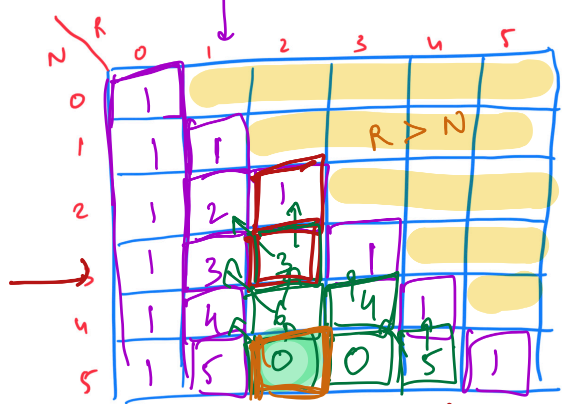
$$(a+b)\%M = (a\%M + b\%M)\%M$$

${}^{N}C_{0} = 1$

${}^{N}C_{0} = 1,\quad {}^{N}C_{1} = N,\quad {}^{N}C_{N} = 1$

$\left({}^{5}C_{2}\right)\%10$

A =



Pascal Triangle.

⟹ Dynamic Programming

(Tabulation)

$N = 5$
$R = 2$
$M = 10$

$(4+6)\%10$

ans = A[N][R]

$A[3][2] = \left({}^{3}C_{2}\right)\%10$

$A[x][y] = \left({}^{x}C_{y}\right)\%M$

$R > N$

A[4][2]

A[3][2] =

A[4][2] =

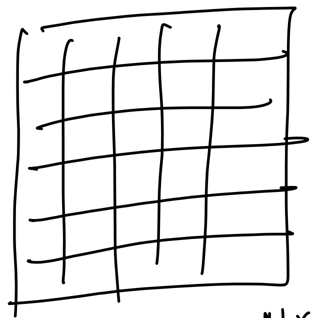
${}^{3}C_{2} = {}^{2}C_{2} + {}^{2}C_{1}$

$(2+1)\%10$

A[4][2] = A[3][2] + A[3][1]

A[i][j] = A[i-1][j] + A[i-1][j-1]

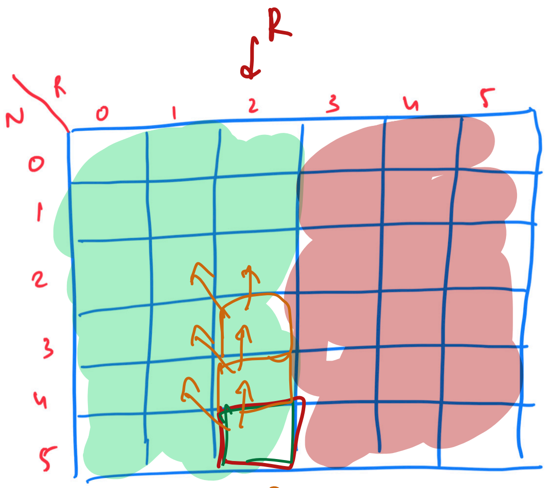
```
int   mat[N+1][N+1];
  // 1st   column   :   1
  for(i=0;  i<=N; i++){
        mat[i][0] = 1
  }
  // 2nd   column   :  i
  for(i=0;  i<=N; i++){
        mat[i][1] = i;
  }

   // Diagonal :  1
   for(i=0;  i<=N; i++){
        mat[i][i] = 1
   }
                                        { r <= N
  for(i=3;  i<=N; i++){
        for(j=2;  j<i;  j++){
           mat[i][j] =  (mat[i-1][j] + mat[i-1][j-1])%m

  }     }
  return   mat[N][R]


     T.C :
```



N x N

T.C: $O(N^2)$

S.C: $O(N^2)$



R columns

$N = 5$

$R = 2$

T.C: $O(N \cdot R)$

S.C: $O(N \cdot R)$

$$^8C_4 \Rightarrow F(^7C_4) + F(^7C_3)$$

$$^8C_5 = {^7C_5} + {^7C_5}$$

$\boxed{N, R < M}$

$R > M$

$M = 11, R = 12$

$(R!)^{-1} \% M$

$\gcd(R!, M) = 1$

$\gcd(12!, 11)$

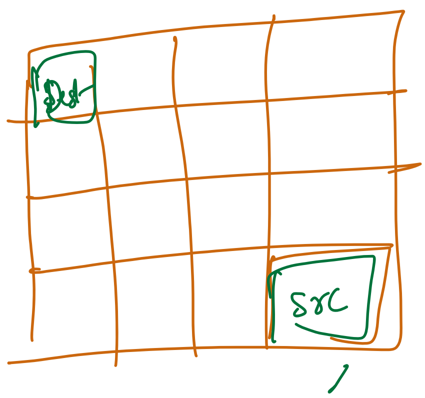
$\gcd(12 \times 11 \times 10 \ldots \ldots 3 \times 2 \times 1, 11) = \boxed{11}$

$R < M$

$R = 9 \quad M = 11$

$\gcd(9!, 11)$ → $1, 11$

$\gcd(9.8.7.6 \ldots 1, 11) = \boxed{1}$



$\boxed{D.P}$