GCD $\Rightarrow$ Greatest Common Divisor
(HCF)
↳ Highest common factor.

$$GCD(a,b) = x$$
$$a \% x = 0$$
$$b \% x = 0$$

GCD is the highest no. which divides both a & b.

Ex

$gcd(15, 25) = 5$

15	25
1	1
3	5
5	25
15	

$gcd(12, 30) = 6$

12	30
1	1
2	2
3	3
4	5
6	6
12	10
	15
	30

Ex

$gcd(10, -25) = 5$

10	-25
1	-25
2	-5
5	-1
10	1
	5
	25

Ex $\gcd(0, 8) = 8$

$$\begin{bmatrix} -\infty \\ \vdots \\ 0\times \\ \vdots \\ +\infty \end{bmatrix}$$

1
2
4
8

* Every no. (except 0) is a factor of 0.

Ex $\gcd(0, -10) = \boxed{10}$

$0\times$

$$\begin{bmatrix} -\infty \\ \vdots \\ 10 \\ +\infty \end{bmatrix}$$

−10
−5
−2
−1
1
2
5
$\boxed{10}$

Ex $\gcd(-16, -24) = 8$

1	1
2	2
4	3
8	4
16	6
	8
	12
	24

Note

$$\boxed{\gcd(a, b) \geq 1}$$

Properties of GCD

1) $\gcd(a, b) = \gcd(b, a) \Rightarrow$ Commutative

2) $\gcd(a, b) = \gcd(|a|, |b|)$

3) $\gcd(0, x) = |x|$

4) $\gcd(0, 0) \Rightarrow$ undefined

$\begin{bmatrix} -\infty \\ | \\ +\infty \end{bmatrix}$ $\begin{bmatrix} -\infty \\ | \\ -\infty \end{bmatrix}$

5) $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$

Associative

$\gcd(\gcd(b, c), a)$

$\gcd(\gcd(a, c), b)$

Ex $A: \{24, 12, 6, 3, 18\} \Rightarrow \gcd(A) = 3$

$\underbrace{24, 12}_{g=12}$

$g = 12$ → $g = 6$ → $g = 3$ → $g = 3$

6) If $\gcd(a, b, c) = x$

$\gcd(\underbrace{a, b, c}_{x}, d) \leq x$

Ex $\gcd(24, 12, 18) = 6$

$$\left.\begin{aligned} \gcd(24, 12, 18, 30) &= 6 \\ \gcd(24, 12, 18, 10) &= 2 \\ \gcd(24, 12, 18, 1) &= 1 \end{aligned}\right\} \le 6$$

$\Rightarrow$ Adding a no. to a list of no's can never increase the GCD. GCD either remains same or decreases.

Ex $\gcd(10, 5) = 5$

$\gcd(6, 13) = 1$

$\gcd(5, 7) = 1$

5	7
1	1
5	7

$\Rightarrow$ If $\gcd(a, b) = 1$: Coprime no's.

$\gcd(8, 9) = 1 \Rightarrow 8, 9$

8	9
1	1
2	3
4	9
8	

Q. Given 2 no's A, B. A, B > 0
find the gcd(A, B)

Sol^n ① :-

$$\boxed{gcd(a,b) \leq min(a,b)}$$

⇒ find the factors of min(a,b) & check if it is a factor of both a & b.

```
g = 1
for ( i = 1 ; i*i <= min(a,b) ; i++) {
    fact1 = i
    fact2 = min(a,b)/i
    if ( a % fact1 == 0 && b % fact1 == 0)
        g = max(g, fact1);

    if ( a % fact2 == 0 && b % fact2 == 0)
        g = max(g, fact2);

}
return g;
```

$i_{max} = \sqrt{min(A,B)}$

TC: O(sqrt(min(A,B)))

SC: O(1)

$\Rightarrow$ If $\gcd(A, B) = g$

① — $A = g \times K_1$
② — $B = g \times K_2$

K_1 & K_2 will be co-prime no's.
$\gcd(K_1, K_2) = 1$

$\gcd(15, 25) = 5$

$15 = 5 \times 3 \, K_1$
$25 = 5 \times 5 \, K_2$

$K_1 = 3$, $K_2 = 5$ } co-prime no's

$\Rightarrow$ K_1 & K_2 will have NO common factors.

Let $B > A$

$B - A = g(K_2 - K_1)$ — ③

$\Rightarrow$ $(B-A)$ will be divisible by g

$A = g \times K_1$
$B - A = g(K_2 - K_1)$

☆ $\Rightarrow$ $\boxed{\gcd(A, B-A) = g}$

Proof *

* $\gcd(a, b) = g$
* $\gcd(a, b-a) = x$

To prove $\boxed{x = g}$

* $a \% g = 0$ — (1)

  $b \% g = 0$ — (2)

$\Rightarrow a = g \times k_1$

$b = g \times k_2$

$b - a = g(k_2 - k_1)$

$\Rightarrow$ g is a factor of a, b & $b-a$

$a \% x = 0$

$(b-a) \% x = 0$

$a = x \times t_1$ — (3)

$b - a = x \times t_2$ — (4)

(3) + (4)

$b = x(t_1 + t_2)$

* x is factor of a, b & $b-a$

$\gcd(a, b) = g$

$\Rightarrow$ x is a factor of a, b

$\boxed{x \leq g}$ — (5)

$\gcd(a, b-a) = x$

$\Rightarrow$ g is a factor of a & $b-a$

$$\boxed{g <= x} \quad \text{--- (6)}$$

from (5) & (6)

$$\boxed{g = x}$$

* Hence Proved

$\Rightarrow$ # $\boxed{\gcd(A, B) = \gcd(A, B-A)}$ $\quad B > A$

Ex $\quad \gcd(\underset{A}{6}, \underset{B}{8}) \quad$ $B > A$

$\gcd(6, 8) = \gcd(6, 8-6) = \gcd(\underset{A}{6}, \underset{B}{2})$ $\quad B \ngtr A$

$= \gcd(2, 6)$

$= \gcd(2, 4)$

$= \gcd(2, 2)$

$= \gcd(2, 2-2)$

$= \gcd(\overset{A}{2}, \overset{B}{0}) = \gcd(\underset{A}{0}, \underset{B}{2})$

$= 2$

$gcd(0, x) = |x|$

Code :-

```
int gcd(a, b) {
    if (a == 0)
        return b;
    if (a > b)
        swap(a, b)

    return gcd(a, b-a)
}
```

$\Rightarrow \underset{A\ \ B}{gcd(5, 0)} \rightarrow a = 0$
$b = 5$
$gcd(0, 5) \Rightarrow 5$

Time Complexity

$\underbrace{gcd(1, 20) \rightarrow gcd(1, 19) \rightarrow gcd(1, 18) \rightarrow gcd(1, 17) \cdots}_{20 \text{ iterations}}$

$O(\max(A, B))$

$\Rightarrow$ 20 iterations :- Repeated subtraction

% Modulo

Ex $gcd(2,5) \rightarrow gcd(2,3)$

$\downarrow$

$gcd(2,1)$

$\downarrow$

$gcd(1,2)$

$\downarrow$

$gcd(1,1)$

$\downarrow$

$gcd(1,0)$
A B

$\downarrow$

$gcd(0,1)$

$\downarrow$

$gcd(0,1-0)$

* Base case

$\Rightarrow 1$

$\Rightarrow gcd(60,200) \rightarrow gcd(60, 200-60)$

$\rightarrow gcd(60, 200-120)$

$\rightarrow gcd(60, \underbrace{200-180}_{20})$

$20 \rightarrow 200 - 3*60$

$\rightarrow gcd(60, 20)$

$\boxed{200 \% 60} \Rightarrow$

Remainder when 200 is divided by 60.

$$\boxed{D = d*q + r}$$

$$r = D - d*q$$

Code

$$gcd(a,b) = gcd(a, b-a)$$
$$= gcd(a, \underbrace{b \% a}_{<a})$$
$$= gcd(b \% a, a)$$

```
int gcd(a, b) {
    if(a == 0) return b;
    return gcd(b % a, a);
}
```

* Euclidean's Algorithm

Ex

$$gcd(\underset{A}{5}, \underset{B}{2}) \Rightarrow gcd(2 \% 5, 5)$$
$$\Rightarrow gcd(\underset{a}{2}, \underset{b}{5})$$
$$\Rightarrow gcd(5 \% 2, 2)$$
$$\Rightarrow gcd(1, 2)$$
$$\Rightarrow gcd(2 \% 1, 1)$$
$$\Rightarrow gcd(0, 1)$$
$$\Rightarrow 1$$

Ex

$$\gcd(\underset{A}{23}, \underset{B}{15}) = 1$$

$\hookrightarrow \gcd(15 \% 23, 23) = \gcd(15, 23)$

$\downarrow$

$\gcd(8, 15)$

$\downarrow$

$\gcd(7, 8)$

$\downarrow$

$\gcd(1, 7)$

$\downarrow$

$1 \Leftarrow \gcd(0, 1)$

Time comlexity

$N \to N-1 \to N-2 \to N-3 \to - \text{———} \; 1 \Rightarrow O(N)$

$N \to \frac{N}{2} \to \frac{N}{4} \to \frac{N}{8} \to \dots \text{---} \; 1 \Rightarrow O(\log N)$

$\gcd(a, b)$

$\downarrow \quad \downarrow$

$b \% a$

$a < b/2$

$b \% a$

$\underbrace{b - \boxed{x * a} \le b}_{\ge 0}$

$b \% a \in [0, a-1]$

$\in [0, \frac{b}{2} - 1]$

$\boxed{a = \frac{b}{2}}$

$b = 8$

$a = 4$

$8 \% 4 = 0$

$\boxed{b \% a = 0}$

$b = 9$

$a = 4$

$\boxed{b \% a = 1}$

Single step.

$\boxed{a > \frac{b}{2}}$

$b \% a \in [0, a-1]$

$\boxed{a > \frac{b}{2}}$

$\boxed{2a > b} \Leftarrow$

$b \% a \Rightarrow b - 1 * a$

$\Rightarrow b - a$

$\Rightarrow \boxed{b - \left(> \frac{b}{2}\right)} < \frac{b}{2}$

⇒ Upper Bound

$\frac{b}{2}$

⇒ Problem of size (N) is getting reduced to problem of size $\frac{N}{2}$ [Upper Bound]

$$\boxed{TC: O(\log(\max(A, B)))}$$

Subsequences :-

↳ Sequence generated by deleting 0 or more elements from the Array.

{ 3 , 4 , 6 , 8 , 9 , 12 }

1) { 3 , 8 , 9 } ✓

2) { 4 } ✓

3) { } ✓

4) { 9, 4, 6, 8 } × ⇒ Order matters

[No. of subsequences ⇒ 2^n

Q. Given an array, Return true if there exists a subsequence with GCD = 1

A : {4, 6, 3, 8} → true

A : {2, 4, 6, 8} → false

A : {3, 6, 9} → false

* If there's a subsequence with gcd = 1,
⇒ GCD of the entire Array will be 1

1) find the gcd of entire array (g)

2) If (g == 1) return true
else return false;

```
g = gcd(a[0], a[1])
for( i = 2; i < n; i++ )        ⇒ N
    g = gcd(g, a[i])            ⇒ log(max(A[i]))
if ( g == 1)
    return true;
return false;
```

$\boxed{TC: O(N * \log(\max(A[i])))}$

Q. Given an Array A, Delete minimum no. of elements such that gcd of A becomes 1.
If it's possible return true.
If not, return false.

A : { 6, 10, 15, 25, 24 , 18 } ⇒ 1 ✓

min no. of elements = 0

⇒ True

A : { 6, 18, 24, 36 } ⇒ 6

⇒ false

a	b	c	d
x_1	x_5	x_7	x_9
x_2	x_2	x_8	x_{10}
x_3	x_4	x_2	x_2
x_4	x_6	x_4	x_4

→ $\boxed{x_4}$

$x_4 > x_2$

⇒ If the gcd of entire Array == 1
return true

else return false;

Doubts