

# Projected Gradient Descent Efficiently Solves the Trust Region Subproblem

Mark Nishimura

Reese Pathak

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## Abstract

We show that projected gradient descent asymptotically converges to a global minimizer of the trust region subproblem. We then show that iterates shortly hit the boundary, after which consecutive iterates remain on the boundary. Conditional on a single conjectured inequality from empirical evidence we are able to show that projected gradient descent achieves the typical  $O(\log(1/\varepsilon))$  rate enjoyed by smooth convex functions.

## 1 Introduction

Trust region methods are sequential programming procedures in which heuristics are used to approximately solve a general optimization problem through multiple constrained quadratic programs. As a subroutine, these methods formulate and solve many instances of the following *trust region subproblem*

$$\begin{aligned} &\text{minimize} && (1/2)x^T A x + b^T x \\ &\text{subject to} && \|x\| \leq R \end{aligned} \tag{1}$$

with variable  $x \in \mathbf{R}^n$ . The problem data are a symmetric matrix  $A \in \mathbf{R}^{n \times n}$ , a vector  $b \in \mathbf{R}^n$ , and a radius parameter  $R > 0$ . Crucially, the matrix  $A$  is possibly indefinite.

### 1.1 Previous works

The trust region subproblem is well-studied, and thus there many previous works worth mentioning. In earlier papers, the problem was solved either via subspace methods such as Steihaug-Toint (where no global convergence guarantees have been proven, to our knowledge), or using fast eigenvector and eigenvalue computation procedures like the Lanczos method [CGT00, EG09, GLRT99, GRT10]. More recently, however, some authors have provided convergence guarantees for this problem. For example, by reducing the trust region subproblem to a sequence of approximate eigenvector computations, Hazan and Koren [HK16] demonstrate that  $\tilde{O}(1/\sqrt{\varepsilon})^1$  matrix-vector multiplies are enough to guarantee an  $\varepsilon$ -suboptimal point. In [HK17], Nguyen and Kiling-Karzan reduce the trust region problem to a convex QCQP using eigenvector calculations, where first-order methods apply.

However, perhaps the most obvious algorithm to solve (1), is the *projected gradient method*, which we study in this paper. To our knowledge, the only previous work that analyzes the convergence properties of this procedure on (1) is [TA98], where Tao and An augment this procedure

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<sup>1</sup>We use the  $\tilde{O}(\cdot)$  notation to hide logarithmic factors.

by a restarting scheme, requiring possibly  $O(d)$  restarts, which could scale poorly for large-scale problems. We also mention a recent work by Carmon and Duchi [CD16], studying the closely related problem

$$\text{minimize } (1/2)x^T A x + b^T x + (\rho/3)\|x\|_2^3, \quad (2)$$

in variable  $x \in \mathbf{R}^n$ , again with  $A$  symmetric, possibly indefinite, and parameter  $\rho > 0$ . The authors analyze gradient descent, proving that  $\tilde{O}(1/\varepsilon)$  gradient steps are enough to output an  $\varepsilon$ -suboptimal point.

In this paper we demonstrate that the projected gradient method on (1) asymptotically converges to a global minimizer on the trust region subproblem.

## 1.2 Notation and classical results

In the sequel, we refer to the objective function as  $f : \mathbf{R}^n \rightarrow \mathbf{R}$ , given by  $f(x) = (1/2)x^T A x + b^T x$ . Additionally, the constraint set is the closed ball  $\mathcal{B}(R) \triangleq \{x \in \mathbf{R}^n \mid \|x\| \leq R\}$ , where  $\|\cdot\|$  denotes the Euclidean norm. We use the notation  $x^*$  to denote the global minimum of  $f$  when it is unique, so that  $x^* = \operatorname{argmin}_{x \in \mathcal{B}(R)} f(x)$ . We use  $f^*$  to denote the optimal value of  $f$ , so that  $f^* = \inf_{x \in \mathcal{B}(R)} f(x)$ . Hence, when  $x^*$  exists,  $f^* = f(x^*)$ .

We fix the eigendecomposition of  $A = U D U^T$ , where  $D = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$ , and  $U$  has orthonormal columns  $u_i$ . We impose without loss that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . By  $\|\cdot\|_{\text{op}}$ , we denote the  $\ell_2$ -operator norm  $\|M\|_{\text{op}} = \sup_{\|x\|=1} \|Mx\|$ , for any  $M \in \mathbf{R}^{n \times n}$ . A useful identity is that  $\|M\|_{\text{op}} = \max_i |\lambda_i(M)|$  when  $M$  is a symmetric  $n \times n$  matrix. We will put  $\beta \triangleq \|A\|_{\text{op}}$ .

Additionally, say a differentiable function  $g : \mathbf{R}^n \rightarrow \mathbf{R}$  is  $L$ -smooth on convex set  $C \subset \mathbf{R}^n$ , provided that

$$\|\nabla g(x) - \nabla g(y)\| \leq L\|x - y\| \quad \text{for any } x, y \in C.$$

It is well known that this implies

$$g(x) - g(y) \leq \nabla g(y)^T (x - y) + \frac{L}{2} \|x - y\|^2 \quad \text{for any } x, y \in C. \quad (3)$$

Equivalently,  $\|g(x)\|_{\text{op}} \leq L$ , for Lebesgue almost every  $x \in C$ . For nonempty, closed, convex sets  $C \subset \mathbf{R}^n$ , associate the projection operator  $\Pi_C : \mathbf{R}^n \rightarrow C$  given by

$$\Pi_C(x) = \operatorname{argmin}_{y \in C} \left( \frac{1}{2} \|x - y\|^2 \right),$$

for any  $x \in \mathbf{R}^n$ . In the sequel we denote by  $I : \mathbf{R}^n \rightarrow \mathbf{R}^n$  the identity operator on  $\mathbf{R}^n$ .

## 2 Asymptotic convergence to a global minimizer

### 2.1 Projected gradient descent

Projected gradient descent (PGD) begins at an initialization  $x^{(0)} \in \mathbf{R}^n$  and generates iterates

$$y^{(k+1)} = x^{(k)} - \eta \nabla f(x^{(k)}) \quad (4)$$

$$x^{(k+1)} = \Pi_{\mathcal{B}(R)}(y^{(k+1)}), \quad (5)$$

for nonnegative integer  $k$  and step size  $\eta$ . We make the following assumptions about this procedure.

**Assumption 2.1.** In (4), the step size  $\eta$  satisfies  $0 < \eta < \frac{1}{\beta}$ .

**Assumption 2.2.** The initial point satisfies  $x^{(0)} = 0$ .

## 2.2 Asymptotic convergence to a global minimizer

We begin by providing a few results, which characterize the iterates of projected gradient descent.

**Lemma 2.3.** *Let Assumptions 2.1 and 2.2 hold. Then the iterates of gradient descent satisfy  $(u_i^T x^{(k)})(u_i^T b) \leq 0$  for all  $i = 1, \dots, n$  and every  $k \geq 0$ . 0*

*Proof.* Evidently, the claim holds due to Assumption 2.2 when  $k = 0$ . Thus, inductively assume that for some  $k$

$$(u_i^T x^{(k)})(u_i^T b) \leq 0 \quad \text{for all } i = 1, \dots, n. \quad (6)$$

By definition,  $x^{(k+1)} = cy^{(k+1)}$  for some  $c \in (0, 1]$ , so it suffices to ensure  $(u_i^T y^{(k+1)})(u_i^T b) \leq 0$ . Using (6) along with Assumption 2.1,

$$(u_i^T y^{(k+1)})(u_i^T b) = (1 - \eta\lambda_i)(u_i^T x^{(k)})(u_i^T b) - \eta(u_i^T b)^2 \leq 0,$$

since  $\eta < \beta^{-1} \leq \lambda_i^{-1}$ , for all  $i = 1, \dots, n$ . This proves the result.  $\square$

The following result shows projected gradient descent is a descent method for (1).

**Lemma 2.4.** *Let Assumption 2.1 hold. Then for any  $k > 0$ ,*

$$f(x^{(k+1)}) - f(x^{(k)}) \leq \left( \frac{\beta}{2} - \frac{1}{2\eta} \right) \|x^{(k+1)} - x^{(k)}\|^2.$$

*Proof.* Basic manipulations imply

$$\nabla f(x^{(k)})^T (x - x^{(k)}) + \frac{1}{2\eta} \|x - x^{(k)}\|^2 = \frac{1}{2\eta} \|x - (x^{(k)} - \eta \nabla f(x^{(k)}))\|^2 - \frac{\eta}{2} \|\nabla f(x^{(k)})\|^2.$$

Thus, as  $\eta > 0$  it follows that

$$\operatorname{argmin}_{x \in \mathcal{B}(R)} \left( \nabla f(x^{(k)})^T (x - x^{(k)}) + \frac{1}{2\eta} \|x - x^{(k)}\|^2 \right) = \operatorname{argmin}_{x \in \mathcal{B}(R)} \left( \frac{1}{2} \|x - (x^{(k)} - \eta \nabla f(x^{(k)}))\|^2 \right).$$

Comparing the display above to (4), (5), and the definition of  $\Pi_{\mathcal{B}(R)}$ ,

$$x^{(k+1)} = \operatorname{argmin}_{x \in \mathcal{B}(R)} \left( \nabla f(x^{(k)})^T (x - x^{(k)}) + \frac{1}{2\eta} \|x - x^{(k)}\|^2 \right). \quad (7)$$

Appealing to the  $\beta$ -smoothness of  $f$  and evaluating (7) at  $x^{(k)} \in \mathcal{B}(R)$ ,

$$f(x^{(k+1)}) - f(x^{(k)}) \leq \nabla f(x^{(k)})^T (x^{(k+1)} - x^{(k)}) + \frac{\beta}{2} \|x^{(k+1)} - x^{(k)}\|^2 \leq \left( \frac{\beta}{2} - \frac{1}{2\eta} \right) \|x^{(k+1)} - x^{(k)}\|^2.$$

$\square$

The following result provides a useful optimality criterion for the trust region subproblem (1).

**Theorem 2.5** ([CGT00], Corollary 7.2.2.). *A point  $x \in \mathcal{B}(R)$  is a global minimizer of  $f$  subject to  $\|x\| \leq R$  if and only if for some  $z \geq 0$ ,*

$$(A + zI)x = -b \quad A + zI \succeq 0 \quad z(\|x\| - R) = 0.$$

*Furthermore,  $x$  is unique if and only if  $A + zI \succ 0$ . In this case, we write  $x = x^*$ .*

An important special case from Theorem 2.5 is that when  $\|x^*\| < R$ , then  $\nabla f(x^*) = 0$ . Furthermore, with a simplifying assumption, we can provide a set of simpler optimality criterion.

**Corollary 2.6.** *Suppose that  $b^T u_1 \neq 0$ . Then if for some  $\tilde{x} \in \mathcal{B}(R)$  and  $z \geq 0$ , it holds that*

$$(A + zI)\tilde{x} = -b \quad z(\|\tilde{x}\| - R) = 0 \quad (u_1^T \tilde{x})(u_1^T b) \leq 0 \quad (8)$$

*then  $\tilde{x}$  is the unique global minimizer to  $f$  over  $\mathcal{B}(R)$ , i.e.,  $\tilde{x} = x^*$ .*

*Proof.* Focusing on the first condition,  $b^T u_1 = -(z + \lambda_1)(u_1^T \tilde{x})$ . Thus,  $b^T u_1 \neq 0$  implies that  $(u_1^T \tilde{x}) \neq 0$  and  $z + \lambda_1 \neq 0$ , strengthening the third condition to  $(u_1^T \tilde{x})(u_1^T b) < 0$ . But this implies that  $z + \lambda_1 = -(u_1^T b)(u_1^T \tilde{x}) / (u_1^T \tilde{x})^2 > 0$ , which implies that  $z > \lambda_i$  for all  $i$ , whence  $A + zI \succ 0$ , establishing the result.  $\square$

The assumptions along with Corollary 2.6 and Lemmas 2.3 and 2.4 give us our desired asymptotic convergence guarantee.

**Proposition 2.7** (Asymptotic convergence). *Let Assumptions 2.1 and 2.2 hold, and suppose  $b^T u_1 \neq 0$ . Then as  $k \rightarrow \infty$ , the iterates of projected gradient descent satisfy  $x^{(k)} \rightarrow x^*$  and  $f(x^{(k)}) \downarrow f(x^*)$ .*

*Proof.* It suffices to demonstrate that  $x^{(k)} \rightarrow x^*$ , because then the conclusion follows via continuity of  $f$  and To that end, Lemma 2.4. Lemma 2.4 and Assumption 2.1 yield the following bound for any integer  $T \geq 1$ ,

$$\left(\frac{1}{2\eta} - \frac{\beta}{2}\right) \sum_{k=0}^{T-1} \|x^{(k+1)} - x^{(k)}\|^2 \leq f(x^{(0)}) - f(x^{(T)}) \leq f(x^{(0)}) - f^*. \quad (9)$$

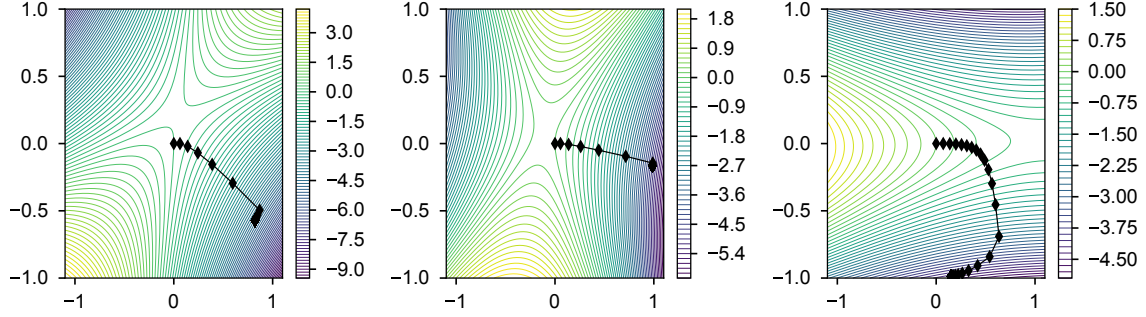
Now, define  $\phi : \mathcal{B}(R) \rightarrow \mathbf{R}^n$  by  $\phi(x) = \Pi_{\mathcal{B}(R)}(x - \eta \nabla f(x)) - x$ , for points  $x \in \mathcal{B}(R)$ . The bound in (9) implies that the displayed series is convergent as  $T \rightarrow \infty$  and thus  $\phi(x^{(k)}) \rightarrow 0$ . Note also that the map  $\phi$  is evidently continuous, as  $\nabla f$  is  $\beta$ -Lipschitz and  $\Pi_{\mathcal{B}(R)}$  is non-expansive, thus 1-Lipschitz.

Suppose now that  $\tilde{x} \in \mathcal{B}(R)$  is a subsequential limit of  $(x^{(k)})$  (indeed, one exists since this sequence is bounded), and observe by continuity  $\phi(\tilde{x}) = 0$ . To show that  $\tilde{x} = x^*$ , by Corollary 2.6, it suffices to establish the first two conditions of (8), as the third immediately holds by Lemma 2.3. Observe first that  $\phi(\tilde{x}) = 0$  implies that for some  $c \geq 1$ ,

$$\tilde{x} - \eta \nabla f(\tilde{x}) = \tilde{x} - \eta(A\tilde{x} - b) = c\tilde{x}. \quad (10)$$

Indeed, setting  $z = (c - 1)\eta^{-1}$ , this implies that  $(A + zI)\tilde{x} = -b$ . If  $\tilde{x}$  lies on the boundary of  $\mathcal{B}(R)$ , so that  $\|\tilde{x}\| = R$ , then as  $z \geq 0$ , this establishes (8) and hence  $\tilde{x} = x^*$ . On the other hand, if  $\tilde{x}$  is in the interior of  $\mathcal{B}(R)$ , so that  $\|\tilde{x}\| < R$ , then  $\phi(\tilde{x}) = 0$  implies that  $c = 1$  in (10), and thus  $z = 0$ , once again establishing (8), hence also that  $\tilde{x} = x^*$ . As this analysis applies to any such subsequential limit  $\tilde{x}$  of the bounded sequence  $(x^{(k)})$ , the claim is now proven (since the iterates lie in  $\mathcal{B}(R)$ , which is compact).  $\square$

We provide some numerical evidence demonstrating the effect of Proposition 2.7 in Figure 1.



**Figure 1:** Three random indefinite instances of the trust region subproblem (1), with  $R = 1$ ,  $\eta = 1/(2\|A\|_{\text{op}})$  and  $x^{(0)} = 0$ . From left to right, the eigenvalues are  $\lambda = (-8, 3)$ ,  $\lambda = (-9, 3)$ , and  $\lambda = (-7, 1)$ . The dots indicate iterates of projected gradient descent and the lines indicate the process  $\dot{x} = -\nabla f(x)$ .

### 3 Non-asymptotic convergence guarantees

In this section, we use the notation  $\pi^{(k)} \in (0, 1]$  to denote a constant such that  $x^{(k)} = \pi^{(k)} y^{(k)}$ . Additionally we tacitly assume that Assumptions 2.1 and 2.2 hold.

We first prove a technical result about the signs of an iterative process.

**Lemma 3.1.** *Let  $\kappa \in \mathbf{R}^n$  satisfy  $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n \leq 1$ , and let  $c(t)$  denote a non-negative sequence. If  $z_i^{(0)} = 0$  for all  $i$ , and*

$$z_i^{(k+1)} = c^{(k)}(1 - \kappa_i)z_i^{(k)} + 1$$

*for all  $k$ , then the following three statements hold*

1. *If  $z_j^{(k)} \leq c^{(k-1)}z_j^{(k-1)}$  then  $z_j^{(k')} \leq c^{(k'-1)}z_j^{(k'-1)}$  for all  $k' > k$ .*
2. *If  $z_i^{(k+1)} \leq c^{(k)}z_i^{(k)}$  then for all  $j \geq i$ ,  $z_j^{(k+1)} \leq c^{(k)}z_j^{(k)}$ .*

*Proof.* To see (a), it suffices to show (by induction) that the claim holds for  $k' = k + 1$ . Thus,

$$z_j^{(k+1)} - c^{(k)}z_j^{(k)} = c^{(k)}(1 - \kappa_j) \left( z_j^{(k)} - c^{(k-1)}z_j^{(k-1)} \right) \leq 0,$$

by assumption and  $c^{(k)} \geq 0$ ,  $1 - \kappa_j \geq 0$ .

To see (b), fix  $j \geq i$ . Note that evidently  $z_i^{(k)} \geq 0$  for all  $i, k$  thus

$$\frac{c^{(k)}z_j^{(k)}}{z_j^{(k+1)}} - \frac{c^{(k)}z_i^{(k)}}{z_i^{(k+1)}} = \frac{(c^{(k)})^2(\kappa_j - \kappa_i)z_j^{(k)}z_i^{(k)}}{z_i^{(k+1)}z_j^{(k+1)}} \geq 0.$$

Above, we use that  $\kappa_j \geq \kappa_i$  when  $j \geq i$ . Hence,  $c^{(k)}z_j^{(k)}/z_j^{(k+1)} \geq c^{(k)}z_i^{(k)}/z_i^{(k+1)} \geq 1$ . The claim follows.  $\square$

**Lemma 3.2.** *Fix  $k \in \mathbf{N}$  such that  $\nabla f(x^{(k)})^T x^{(k)} \leq 0$ . Then  $x^{(k)T} A \nabla f(x^{(k)}) \geq \beta x^{(k)T} \nabla f(x^{(k)})$ .*

*Proof.* Note first that  $x^{(k')} = \pi^{(k')}y^{(k')}$ , thus, for all  $k' \leq k$ ,

$$\sum_{i=1}^n (u_i^T y^{(k')})(u_i^T (x^{(k')} - y^{(k'+1)})) \leq 0.$$

Define the following sets for  $k' \leq k$

$$\begin{aligned} I_+^{(k')} &\triangleq \{i \in [n] : (u_i^T y^{(k')})(u_i^T (x^{(k')} - y^{(k'+1)})) \geq 0\} \\ I_-^{(k')} &\triangleq \{i \in [n] : (u_i^T y^{(k')})(u_i^T (x^{(k')} - y^{(k'+1)})) \leq 0\}. \end{aligned}$$

Associated to these sets, define  $\lambda_+^{(k')} = \lambda_i$  and  $\lambda_-^{(k')} = \lambda_j$  for  $i = \min I_+^{(k')}$ , and  $j = \max I_-^{(k')}$ . Then now observe that, expanding  $y^{(k)T} A \nabla f(x^{(k)})$  in the eigenbasis of  $A$ ,

$$\begin{aligned} y^{(k)T} A \nabla f(x^{(k)}) &= \frac{1}{\eta} \left( \sum_{i \in I_+^{(k)}} \lambda_i (u_i^T y^{(k)})(u_i^T (x^{(k)} - y^{(k+1)})) + \sum_{i \in I_-^{(k)}} \lambda_i (u_i^T y^{(k)})(u_i^T (x^{(k)} - y^{(k+1)})) \right) \\ &\geq \frac{1}{\eta} \left( \lambda_+^{(k)} \sum_{i \in I_+^{(k)}} (u_i^T y^{(k)})(u_i^T (x^{(k)} - y^{(k+1)})) + \lambda_-^{(k)} \sum_{i \in I_-^{(k)}} (u_i^T y^{(k)})(u_i^T (x^{(k)} - y^{(k+1)})) \right) \\ &\geq \lambda_-^{(k)} y^{(k)T} \nabla f(x^{(k)}) \geq \beta y^{(k)T} \nabla f(x^{(k)}). \end{aligned}$$

Recalling that  $x^{(k)} = \pi^{(k)}y^{(k)}$  with  $\pi^{(k)} \in (0, 1]$ , this proves the claim. The last inequality was obtained by assumption that  $x^{(k)T} \nabla f(x^{(k)}) \leq 0$ , and that  $\lambda_i \leq \lambda_n \leq \beta$  for all  $i \leq n$ . The penultimate inequality is due to  $\lambda_-^{(k)} \leq \lambda_+^{(k)}$ . To see this, it suffices to show that

$$(u_i^T y^{(k)})(u_i^T (x^{(k)} - y^{(k+1)})) \geq 0 \quad \text{implies} \quad (u_j^T y^{(k)})(u_j^T (x^{(k)} - y^{(k+1)})) \geq 0 \quad \text{for all } j \geq i \quad (11)$$

We prove this using Lemma 3.1. Indeed,  $z_j^{(k)} = (u_j^T y^{(k)})/(-\eta u_j^T b)$  for all  $k \in \mathbf{N}$ . Then

$$z_j^{(k+1)} = \frac{u_j^T ((I - \eta A)x^{(k)} - \eta b)}{-\eta u_j^T b} = \underbrace{\pi^{(k)}}_{\triangleq c^{(k)}} (1 - \underbrace{\eta \lambda_i}_{\kappa_i}) z_j^{(k)} + 1.$$

Note that  $z_j^{(0)} = 0$ , and additionally  $\eta \lambda_i = \kappa_i$  is non-decreasing in  $i$ , and bounded above by 1 since  $\eta \leq 1/\beta$ . Additionally, by assumption we have  $c^{(k')} = \pi^{(k')} = R/\|y^{(k')}\|$ , which is evidently non-negative. Thus (2.) of Lemma 3.1 implies that

$$\pi^{(k)} z_i^{(k)} - z_i^{(k+1)} \geq 0 \quad \text{implies} \quad \pi^{(k)} z_i^{(k)} - z_i^{(k+1)} \geq 0 \quad \text{for all } j \geq i.$$

Note that as  $z_i^{(k)} \geq 0$ , this is equivalent to the display in (11). The result is now proven.  $\square$

**Lemma 3.3.** *For all  $k \geq 0$ , the iterates of projected gradient descent satisfy  $b^T x^{(k)} \leq 0$ .*

*Proof.* We inductively establish the following, stronger, result.

$$\text{for all } i \in [n], \quad (u_i^T x^{(k)})(u_i^T b) \leq 0 \quad \text{for all } k \in \mathbf{N}. \quad (12)$$

Claim (12) evidently holds when  $k = 0$ , so now suppose it holds for  $k' \leq k$ . Fix  $i \in [n]$ . Note

$$\begin{aligned} \text{sign}(u_i^T x^{(k+1)}) &= \text{sign}\left(\pi^{(k+1)}((1 - \eta\lambda_i)u_i^T x^{(k)} - \eta u_i^T b)\right) \\ &= \text{sign}((1 - \eta\lambda_i)u_i^T x^{(k)} + \eta(-u_i^T b)) = -\text{sign}(u_i^T b) \end{aligned}$$

The final equality holds since  $\pi^{(k+1)} \in (0, 1]$ , and the final inequality holds due to the inductive assumption. This proves Claim (12), and the result follows by summing these inequalities:  $b^T x^{(k)} = \sum_{i=1}^n (x^{(k)})^T u_i (u_i^T b) \leq 0$ .  $\square$

**Lemma 3.4.** *For all  $k \geq 0$ , the iterates of projected gradient descent satisfy  $x^{(k)T} \nabla f(x^{(k)}) \leq 0$ . Furthermore, for all  $k$ ,  $\|y^{(k+1)}\| \geq \|x^{(k)}\|$ .*

*Proof.* By definition of the projected gradient descent iteration, we have

$$\|y^{(k+1)}\|^2 = \|x^{(k)}\|^2 + \eta \|\nabla f(x^{(k)})\|^2 - 2\eta \nabla f(x^{(k)})^T x^{(k)}.$$

Thus, to prove the claim it would be sufficient to show inductively that  $x^{(k)T} \nabla f(x^{(k)}) \leq 0$ . The basis of induction is clear as the statement trivially holds at  $x^{(0)} = 0$ . Suppose the claim holds for  $k$ , and note it suffices to demonstrate  $y^{(k+1)T} \nabla f(x^{(k+1)}) \leq 0$ . For all  $k$ , denote  $0 < \pi^{(k)} \leq 1$  such that  $\pi^{(k)} y^{(k)} = x^{(k)}$ . Lemma 3.3 implies

$$\begin{aligned} y^{(k+1)T} \nabla f(x^{(k+1)}) &\leq \pi^{(k)} \left( x^{(k)T} \nabla f(x^{(k)}) - \eta x^{(k)T} A \nabla f(x^{(k)}) - \eta \|\nabla f(x^{(k)})\|^2 + \eta^2 \nabla f(x^{(k)})^T A \nabla f(x^{(k)}) \right) \\ &\leq -\pi^{(k)}(\eta - \eta^2 \beta) \|\nabla f(x^{(k)})\|^2 + \pi^{(k)}(1 - \eta\beta) x^{(k)T} \nabla f(x^{(k)}) \leq 0 \end{aligned}$$

The penultimate inequality is due Lemma 3.2, and the final inequality is because  $\eta \leq 1/\beta$ .  $\square$

Lemma 3.4 immediately implies the following claim.

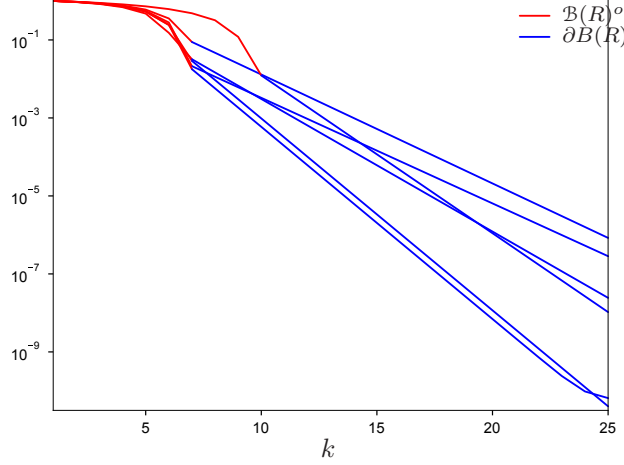
**Corollary 3.5.** *Suppose an iterate of projected gradient descent satisfies  $x^{(\tau)} \in \partial\mathcal{B}(R)$  for some  $\tau \in \mathbf{N}$ . Then  $x^{(t)} \in \partial\mathcal{B}(R)$  for all  $t \geq \tau$ .*

*Proof.* Lemma 3.4 implies  $\|y^{(\tau+1)}\|^2 \geq \|x^{(\tau)}\|^2 \geq R^2$ , thus  $x^{(\tau+1)} \in \partial B(R)$ . The claim now follows via induction.  $\square$

In words, once an iterate hits the boundary, all subsequent iterates remain on the boundary. We believe this result is relevant. Figure 2 demonstrates the effect of iterates remaining on the boundary; empirically we observe exponential convergence, characteristic of gradient descent on smooth, convex functions.

The following result bounds the time to the boundary.

**Claim 3.6.** *Suppose that  $\lambda_1 < 0$ , and  $b^T u_1 \neq 0$ . Then after  $O\left(\frac{R|\lambda_1|}{|b^T u_1| \log(1 - \eta\lambda_1)}\right)$  iterations, the iterates of projected gradient descent lie on the boundary.*



**Figure 2:** Two regimes of convergence, before and after hitting the boundary.

*Proof.* Suppose that for  $k$  iterations,  $\|y^{(k)}\|^2 < R^2$ . Then, as  $x^{(0)} = 0$ , we have

$$y^{(k)} = -\eta \sum_{t=0}^{k-1} (I - \eta A)^t b = -\eta \sum_{i=1}^n \left( \sum_{t=0}^{k-1} (1 - \eta \lambda_i)^t \right) b^T u_i u_i$$

Hence,

$$\|y^{(k)}\|^2 = \sum_{i=1}^n \frac{(b^T u_i)^2}{\lambda_i^2} (1 - (1 - \eta \lambda_i)^k)^2 \geq \frac{(b^T u_1)^2}{\lambda_1^2} (1 - (1 - \eta \lambda_1)^k)^2.$$

Setting the right-hand side to  $R^2$ , one obtains that if  $k \geq 1 + \frac{R|\lambda_1|}{|b^T u_1|(\log(1 - \eta \lambda_1))}$ , then the display above is larger than  $R^2$ . Thus, the claim now follows via Corollary 3.5.  $\square$

**Lemma 3.7.** *Let  $\tau^{\text{bd}} = \min\{k : x^{(k)} \in \partial\mathcal{B}(R)\}$ . If for some  $\delta > 0$ , and for all  $k \geq \tau^{\text{bd}}$ ,  $(x^{(k+1)} - \eta\delta x^{(k)})^T x^* \geq (1 - \eta\delta)R^2$ , then  $f(x^{(t)})$  is  $\varepsilon > 0$ -suboptimal provided that  $t \geq \tau^{\text{bd}} + \frac{1}{\eta\delta} \log\left(\frac{2R^2(\beta+z)}{\varepsilon}\right)$ .*

*Proof.* By hypothesis, we have  $\|x^{(k+1)} - x^*\|^2 \leq (1 - \eta\delta)^2 \|x^{(k)} - x^*\|^2$ , provided  $k \geq \tau^{\text{bd}}$ . Thus, as  $\delta < 1/\eta$ , and with  $\tau := \tau^{\text{bd}}$ ,

$$\|x^{(k+\tau)} - x^*\|^2 \leq (1 - \eta\delta)^{2k} \|x^\tau - x^*\|^2 \leq 4R^2(1 - \eta\delta)^k \leq 4R^2 e^{-2\eta\delta k},$$

as  $1 + \alpha \leq e^\alpha$  for all  $\alpha \in \mathbf{R}$ . Now appealing to the  $\beta$ -smoothness of  $f$ ,

$$f(x^{(k+\tau)}) - f^* \leq 2(\beta + z)R^2 e^{-\eta\delta k} \leq \varepsilon,$$

provided that  $k \geq \frac{1}{\eta\delta} \log((2R^2(\beta + z))/\varepsilon)$ . Hence if

$$t \geq \tau^{\text{bd}} + \frac{1}{\eta\delta} \log\left(\frac{2R^2(\beta + z)}{\varepsilon}\right),$$

then  $f(x^{(t)}) - f^* \leq \varepsilon$ .  $\square$



**Conjecture 3.8.** *In the notation of Lemma 3.7, for all  $k \geq \tau^{\text{bd}}$ ,  $(x^{(k+1)} - \eta\delta x^{(k)})^T x^* \geq (1 - \eta\delta)R^2$  holds with  $\delta = 1/(z + \lambda_1)$ .*

Conditional on Conjecture 3.8, this implies the  $\tilde{O}(\log(1/\varepsilon))$  convergence rate enjoyed by smooth, convex functions under the gradient descent iteration. Of course, the most obvious open problem is to prove (or disprove) Conjecture 3.8.

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