

# Projected Gradient Descent Efficiently\* Solves the Trust Region Subproblem

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## Abstract

We show that projected gradient descent asymptotically converges to a global minimizer of the trust region subproblem. We remark on next steps in this project at the end.

## 1 Introduction

Trust region methods are sequential programming procedures in which heuristics are used to approximately solve a general optimization problem through multiple constrained quadratic programs. As a subroutine, these methods formulate and solve many instances of the following *trust region subproblem*

$$\begin{aligned} & \text{minimize} && (1/2)x^T A x + b^T x \\ & \text{subject to} && \|x\| \leq R \end{aligned} \tag{1}$$

with variable  $x \in \mathbf{R}^n$ . The problem data are a symmetric matrix  $A \in \mathbf{R}^{n \times n}$ , a vector  $b \in \mathbf{R}^n$ , and a radius parameter  $R > 0$ . Crucially, the matrix  $A$  is possibly indefinite.

### 1.1 Previous works

The trust region subproblem is well-studied, and thus there many previous works worth mentioning. In earlier papers, the problem was solved either via subspace methods such as Steihaug-Toint (where no global convergence guarantees have been proven, to our knowledge), or using fast eigenvector and eigenvalue computation procedures like the Lanczos method [CGT00, EG09, GLRT99, GRT10]. More recently, however, some authors have provided convergence guarantees for this problem. For example, by reducing the trust region subproblem to a sequence of approximate eigenvector computations, Hazan and Koren [HK16] demonstrate that  $\tilde{O}(1/\sqrt{\varepsilon})^1$  matrix-vector multiplies are enough to guarantee an  $\varepsilon$ -suboptimal point. In [HK17], Nguyen and Kiling-Karzan reduce the trust region problem to a convex QCQP using eigenvector calculations, where first-order methods apply.

However, perhaps the most obvious algorithm to solve (1), is the *projected gradient method*, which we study in this paper. To our knowledge, the only previous work that analyzes the convergence properties of this procedure on (1) is [TA98], where Tao and An augment this procedure by a restarting scheme, requiring possibly  $O(d)$  restarts, which could scale poorly for large-scale

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\*Technically, a conjecture.

<sup>1</sup>We use the  $\tilde{O}(\cdot)$  notation to hide logarithmic factors.

problems. We also mention a recent work by Carmon and Duchi [CD16], studying the closely related problem

$$\text{minimize } (1/2)x^T Ax + b^T x + (\rho/3)\|x\|_2^3, \quad (2)$$

in variable  $x \in \mathbf{R}^n$ , again with  $A$  symmetric, possibly indefinite, and parameter  $\rho > 0$ . The authors analyze gradient descent, proving that  $\tilde{O}(1/\varepsilon)$  gradient steps are enough to output an  $\varepsilon$ -suboptimal point.

In this paper we demonstrate that the projected gradient method on (1) asymptotically converges to a global minimizer on the trust region subproblem.

## 1.2 Notation and classical results

In the sequel, we refer to the objective function as  $f : \mathbf{R}^n \rightarrow \mathbf{R}$ , given by  $f(x) = (1/2)x^T Ax + 2b^T x$ . Additionally, the constraint set is the closed ball  $\mathcal{B}(R) \triangleq \{x \in \mathbf{R}^n \mid \|x\| \leq R\}$ , where  $\|\cdot\|$  denotes the Euclidean norm. We use the notation  $x^*$  to denote the global minimum of  $f$  when it is unique, so that  $x^* = \operatorname{argmin}_{x \in \mathcal{B}(R)} f(x)$ . We use  $f^*$  to denote the optimal value of  $f$ , so that  $f^* = \inf_{x \in \mathcal{B}(R)} f(x)$ . Hence, when  $x^*$  exists,  $f^* = f(x^*)$ .

We fix the eigendecomposition of  $A = UDU^T$ , where  $D = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$ , and  $U$  has orthonormal columns  $u_i$ . We impose without loss that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . By  $\|\cdot\|_{\text{op}}$ , we denote the  $\ell_2$ -operator norm  $\|M\|_{\text{op}} = \sup_{\|x\|=1} \|Mx\|$ , for any  $M \in \mathbf{R}^{n \times n}$ . A useful identity is that  $\|M\|_{\text{op}} = \max_i |\lambda_i(M)|$  when  $M$  is a symmetric  $n \times n$  matrix. We will put  $\beta \triangleq \|A\|_{\text{op}}$ .

Additionally, say a differentiable function  $g : \mathbf{R}^n \rightarrow \mathbf{R}$  is  $L$ -smooth on convex set  $C \subset \mathbf{R}^n$ , provided that

$$\|\nabla g(x) - \nabla g(y)\| \leq L\|x - y\| \quad \text{for any } x, y \in C.$$

It is well known that this implies

$$g(x) - g(y) \leq \nabla g(y)^T (x - y) + \frac{L}{2}\|x - y\|^2 \quad \text{for any } x, y \in C. \quad (3)$$

Equivalently,  $\|g(x)\|_{\text{op}} \leq L$ , for Lebesgue almost every  $x \in C$ . For nonempty, closed, convex sets  $C \subset \mathbf{R}^n$ , associate the projection operator  $\Pi_C : \mathbf{R}^n \rightarrow C$  given by

$$\Pi_C(x) = \operatorname{argmin}_{y \in C} \left( \frac{1}{2}\|x - y\|^2 \right),$$

for any  $x \in \mathbf{R}^n$ . In the sequel we denote by  $I : \mathbf{R}^n \rightarrow \mathbf{R}^n$  the identity operator on  $\mathbf{R}^n$ .

## 2 Asymptotic convergence to a global minimizer

### 2.1 Projected gradient descent

Projected gradient descent (PGD) begins at an initialization  $x^{(0)} \in \mathbf{R}^n$  and generates iterates

$$y^{(k+1)} = x^{(k)} - \eta \nabla f(x^{(k)}) \quad (4)$$

$$x^{(k+1)} = \Pi_{\mathcal{B}(R)}(y^{(k+1)}), \quad (5)$$

for nonnegative integer  $k$  and step size  $\eta$ . We make the following assumptions about this procedure.

**Assumption 2.1.** In (4), the step size  $\eta$  satisfies  $0 < \eta < \frac{1}{\beta}$ .

**Assumption 2.2.** The initial point satisfies  $x^{(0)} = 0$ .

## 2.2 Asymptotic convergence to a global minimizer

We begin by providing a few results, which characterize the iterates of projected gradient descent.

**Lemma 2.3.** *Let Assumptions 2.1 and 2.2 hold. Then the iterates of gradient descent satisfy  $(u_i^T x^{(k)})(u_i^T b) \leq 0$  for all  $i = 1, \dots, n$  and every  $k \geq 0$ . 0*

*Proof.* Evidently, the claim holds due to Assumption 2.2 when  $k = 0$ . Thus, inductively assume that for some  $k$

$$(u_i^T x^{(k)})(u_i^T b) \leq 0 \quad \text{for all } i = 1, \dots, n. \quad (6)$$

By definition,  $x^{(k+1)} = cy^{(k+1)}$  for some  $c \in (0, 1]$ , so it suffices to ensure  $(u_i^T y^{(k+1)})(u_i^T b) \leq 0$ . Using (6) along with Assumption 2.1,

$$(u_i^T y^{(k+1)})(u_i^T b) = (1 - \eta\lambda_i)(u_i^T x^{(k)})(u_i^T b) - \eta(u_i^T b)^2 \leq 0,$$

since  $\eta < \beta^{-1} \leq \lambda_i^{-1}$ , for all  $i = 1, \dots, n$ . This proves the result.  $\square$

The following result shows projected gradient descent is a descent method for (1).

**Lemma 2.4.** *Let Assumption 2.1 hold. Then for any  $k > 0$ ,*

$$f(x^{(k+1)}) - f(x^{(k)}) \leq \left( \frac{\beta}{2} - \frac{1}{2\eta} \right) \|x^{(k+1)} - x^{(k)}\|^2.$$

*Proof.* Basic manipulations imply

$$\nabla f(x^{(k)})^T (x - x^{(k)}) + \frac{1}{2\eta} \|x - x^{(k)}\|^2 = \frac{1}{2\eta} \|x - (x^{(k)} - \eta \nabla f(x^{(k)}))\|^2 - \frac{\eta}{2} \|\nabla f(x^{(k)})\|^2.$$

Thus, as  $\eta > 0$  it follows that

$$\operatorname{argmin}_{x \in \mathcal{B}(R)} \left( \nabla f(x^{(k)})^T (x - x^{(k)}) + \frac{1}{2\eta} \|x - x^{(k)}\|^2 \right) = \operatorname{argmin}_{x \in \mathcal{B}(R)} \left( \frac{1}{2} \|x - (x^{(k)} - \eta \nabla f(x^{(k)}))\|^2 \right).$$

Comparing the display above to (4), (5), and the definition of  $\Pi_{\mathcal{B}(R)}$ ,

$$x^{(k+1)} = \operatorname{argmin}_{x \in \mathcal{B}(R)} \left( \nabla f(x^{(k)})^T (x - x^{(k)}) + \frac{1}{2\eta} \|x - x^{(k)}\|^2 \right). \quad (7)$$

Appealing to the  $\beta$ -smoothness of  $f$  and evaluating (7) at  $x^{(k)} \in \mathcal{B}(R)$ ,

$$f(x^{(k+1)}) - f(x^{(k)}) \leq \nabla f(x^{(k)})^T (x^{(k+1)} - x^{(k)}) + \frac{\beta}{2} \|x^{(k+1)} - x^{(k)}\|^2 \leq \left( \frac{\beta}{2} - \frac{1}{2\eta} \right) \|x^{(k+1)} - x^{(k)}\|^2.$$

$\square$

The following result provides a useful optimality criterion for the trust region subproblem (1).

**Theorem 2.5** ([CGT00], Corollary 7.2.2.). *A point  $x \in \mathcal{B}(R)$  is a global minimizer of  $f$  subject to  $\|x\| \leq R$  if and only if for some  $z \geq 0$ ,*

$$(A + zI)x = -b \quad A + zI \succeq 0 \quad z(\|x\| - R) = 0.$$

*Furthermore,  $x$  is unique if and only if  $A + zI \succ 0$ . In this case, we write  $x = x^*$ .*

An important special case from Theorem 2.5 is that when  $\|x^*\| < R$ , then  $\nabla f(x^*) = 0$ . Furthermore, with a simplifying assumption, we can provide a set of simpler optimality criterion.

**Corollary 2.6.** *Suppose that  $b^T u_1 \neq 0$ . Then if for some  $\tilde{x} \in \mathcal{B}(R)$  and  $z \geq 0$ , it holds that*

$$(A + zI)\tilde{x} = -b \quad z(\|\tilde{x}\| - R) = 0 \quad (u_1^T \tilde{x})(u_1^T b) \leq 0 \quad (8)$$

*then  $\tilde{x}$  is the unique global minimizer to  $f$  over  $\mathcal{B}(R)$ , i.e.,  $\tilde{x} = x^*$ .*

*Proof.* Focusing on the first condition,  $b^T u_1 = -(z + \lambda_1)(u_1^T \tilde{x})$ . Thus,  $b^T u_1 \neq 0$  implies that  $(u_1^T \tilde{x}) \neq 0$  and  $z + \lambda_1 \neq 0$ , strengthening the third condition to  $(u_1^T \tilde{x})(u_1^T b) < 0$ . But this implies that  $z + \lambda_1 = -(u_1^T b)(u_1^T \tilde{x}) / (u_1^T \tilde{x})^2 > 0$ , which implies that  $z > \lambda_i$  for all  $i$ , whence  $A + zI \succ 0$ , establishing the result.  $\square$

The assumptions along with Corollary 2.6 and Lemmas 2.3 and 2.4 give us our desired asymptotic convergence guarantee.

**Proposition 2.7** (Asymptotic convergence). *Let Assumptions 2.1 and 2.2 hold, and suppose  $b^T u_1 \neq 0$ . Then as  $k \rightarrow \infty$ , the iterates of projected gradient descent satisfy  $x^{(k)} \rightarrow x^*$  and  $f(x^{(k)}) \downarrow f(x^*)$ .*

*Proof.* It suffices to demonstrate that  $x^{(k)} \rightarrow x^*$ , because then the conclusion follows via continuity of  $f$  and to that end, Lemma 2.4. Lemma 2.4 and Assumption 2.1 yield the following bound for any integer  $T \geq 1$ ,

$$\left(\frac{1}{2\eta} - \frac{\beta}{2}\right) \sum_{k=0}^{T-1} \|x^{(k+1)} - x^{(k)}\|^2 \leq f(x^{(0)}) - f(x^{(T)}) \leq f(x^{(0)}) - f^*. \quad (9)$$

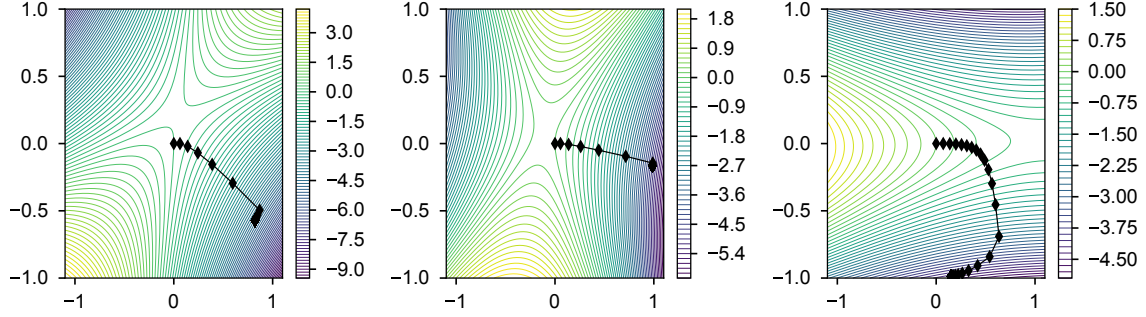
Now, define  $\phi : \mathcal{B}(R) \rightarrow \mathbf{R}^n$  by  $\phi(x) = \Pi_{\mathcal{B}(R)}(x - \eta \nabla f(x)) - x$ , for points  $x \in \mathcal{B}(R)$ . The bound in (9) implies that the displayed series is convergent as  $T \rightarrow \infty$  and thus  $\phi(x^{(k)}) \rightarrow 0$ . Note also that the map  $\phi$  is evidently continuous, as  $\nabla f$  is  $\beta$ -Lipschitz and  $\Pi_{\mathcal{B}(R)}$  is non-expansive, thus 1-Lipschitz.

Suppose now that  $\tilde{x} \in \mathcal{B}(R)$  is a subsequential limit of  $(x^{(k)})$  (indeed, one exists since this sequence is bounded), and observe by continuity  $\phi(\tilde{x}) = 0$ . To show that  $\tilde{x} = x^*$ , by Corollary 2.6, it suffices to establish the first two conditions of (8), as the third immediately holds by Lemma 2.3. Observe first that  $\phi(\tilde{x}) = 0$  implies that for some  $c \geq 1$ ,

$$\tilde{x} - \eta \nabla f(\tilde{x}) = \tilde{x} - \eta(A\tilde{x} - b) = c\tilde{x}. \quad (10)$$

Indeed, setting  $z = (c - 1)\eta^{-1}$ , this implies that  $(A + zI)\tilde{x} = -b$ . If  $\tilde{x}$  lies on the boundary of  $\mathcal{B}(R)$ , so that  $\|\tilde{x}\| = R$ , then as  $z \geq 0$ , this establishes (8) and hence  $\tilde{x} = x^*$ . On the other hand, if  $\tilde{x}$  is in the interior of  $\mathcal{B}(R)$ , so that  $\|\tilde{x}\| < R$ , then  $\phi(\tilde{x}) = 0$  implies that  $c = 1$  in (10), and thus  $z = 0$ , once again establishing (8), hence also that  $\tilde{x} = x^*$ . As this analysis applies to any such subsequential limit  $\tilde{x}$  of the bounded sequence  $(x^{(k)})$ , the claim is now proven (since the iterates lie in  $\mathcal{B}(R)$ , which is compact).  $\square$

We provide some numerical evidence demonstrating the effect of Proposition 2.7 in Figure 1.



**Figure 1:** Three random indefinite instances of the trust region subproblem (1), with  $R = 1$ ,  $\eta = 1/(2\|A\|_{\text{op}})$  and  $x^{(0)} = 0$ . From left to right, the eigenvalues are  $\lambda = (-8, 3)$ ,  $\lambda = (-9, 3)$ , and  $\lambda = (-7, 1)$ . The dots indicate iterates of projected gradient descent and the lines indicate the process  $\dot{x} = -\nabla f(x)$ .

### 3 Non-asymptotic convergence guarantees

We plan to continue this work by providing convergence rates for this problem. In particular, our current goal is to obtain a non-asymptotic convergence rate, similar to Theorem 3.1, [CD16].

**Lemma 3.1** (Boundary). *Suppose that  $t$  is the first time that  $\|y^{(t)}\|_2^2 \geq R^2$ . Then  $\|y^{(t+1)}\|_2^2 \geq R^2$ .*

*Proof.* By assumption we have that with  $r_j \triangleq 1 - \eta d_j$ , that

$$y^{(t)} = -\eta \sum_{k=0}^{t-1} (I - \eta A)^k b = -\sum_{j=1}^n (1 - r_j^t) \frac{b_j}{d_j} e_j.$$

Hence we have  $R^2 = \sum_{j=1}^n (b_j/d_j)^2 (1 - r_j^t)^2$ . Now, write

$$y^{(t+1)} = \frac{R}{\|y^{(t)}\|} \left( (I - \eta A)y^{(t)} - \eta b \right) - \left( 1 - \frac{R}{\|y^{(t)}\|} \right) \eta b = \frac{R}{\|y^{(t)}\|} g^{(t+1)} - \left( 1 - \frac{R}{\|y^{(t)}\|} \right) \eta b$$

where  $g^{(t+1)} = -\eta \sum_{k=0}^t (I - \eta A)^k b$ . Expanding the norm,

$$\begin{aligned} \|y^{(t+1)}\|_2^2 &= \frac{R^2}{\|y^{(t)}\|_2^2} \sum_{j=1}^n \frac{b_j^2}{d_j^2} (1 - r_j^{t+1})^2 + \left( 1 - \frac{R}{\|y^{(t)}\|} \right)^2 \eta^2 \|b\|^2 + 2\eta \left( 1 - \frac{R}{\|y^{(t)}\|} \right) \left( \frac{R}{\|y^{(t)}\|} \right) \sum_{j=1}^n (1 - r_j^{t+1}) \frac{b_j^2}{d_j} \\ &= \frac{R^2}{\|y^{(t)}\|_2^2} \left[ \sum_{j=1}^n b_j^2 \left( \frac{1 - r_j^{t+1}}{d_j} - \eta \right)^2 \right] + \frac{2R\eta}{\|y^{(t)}\|} \left[ \sum_{j=1}^n b_j^2 \left( \frac{1 - r_j^{t+1}}{d_j} - \eta \right) \right] + \eta^2 \|b\|_2^2 \\ &= \sum_{j=1}^n b_j^2 \left( \eta + \frac{R}{\|y^{(t)}\|} \alpha_j \right)^2, \end{aligned}$$

where  $\alpha_j = (1 - r_j^{t+1})d_j^{-1} - \eta$ . □

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