Projected Gradient Descent Efficiently Solves the Trust Region Problem

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EE364b: Convex Optimization II Class Project

Introduction

Trust region methods are sequential programming procedures which formulate and solve many instances of the following **trust region problem**

$$\begin{array}{ll} \text{minimize} & (1/2)x^TAx + b^Tx \\ \text{subject to} & \|x\| \le R \end{array} \tag{1}$$

with variable x. The matrix A may be indefinite, in which case the resulting problem is not convex.

Projected Gradient Descent

We investigate the behavior of **projected gradient descent** (PGD) which consists of iterating the following two steps:

$$\begin{aligned} y^{(k+1)} &= x^{(k)} - \eta \nabla f(x^{(k)}) \\ x^{(k+1)} &= \Pi_{\mathcal{B}(R)}(y^{(k+1)}). \end{aligned} \tag{2}$$

Structured group lasso

Another approach is the *structured group lasso*:

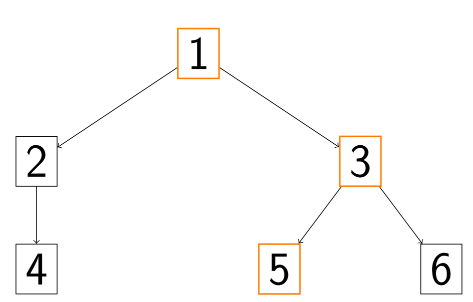
$$\text{minimize} \;\; f(x) + \textstyle\sum_{i=1}^N \lambda_i \|x_{g_i}\|_2$$

where $g_i \subseteq [n]$ and $\mathcal{G} = \{g_1, \ldots, g_N\}$

- like group lasso, but the groups can overlap arbitrarily
- particular choices of groups can impose 'structured' sparsity
- *e.g.*, topic models, selecting interaction terms for (graphical) models, tree structure of gene networks, fMRI data
- generalizes to the **composite absolute penalties family**:

$$r(x) = \|(\|x_{g_1}\|_{p_1}, \ldots, \|x_{g_N}\|_{p_N})\|_{p_0}$$

Hierarchical selection



- $\mathcal{G} = \{\{4\}, \{5\}, \{6\}, \{2,4\}, \{3,5,6\}, \{1,2,3,4,5,6\}\}$
- nonzero variables form a rooted and connected subtree
  - if node is selected, so are its ancestors
  - if node is not selected, neither are its descendants

Algorithm

We solve this problem using an ADMM lasso implementation:

```
prox_f = @(v,rho) (rho/(1 + rho))*(v - b) + b;
prox_g = @(v,rho) (max(0, v - 1/rho) - max(0, -v - 1/rho));

AA = A*A';
L  = chol(eye(m) + AA);

for iter = 1:MAX_ITER
    xx = prox_g(xz - xt, rho);
    yx = prox_f(yz - yt, rho);

    yz = L \ (L' \ (A*(xx + xt) + AA*(yx + yt)));
    xz = xx + xt + A'*(yx + yt - yz);

    xt = xt + xx - xz;
    yt = yt + yx - yz;
end
```

Line search

If L is not known (usually the case), can use the following line search:

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given x^k, λ^{k-1}, and parameter $\beta \in (0,1)$.
Let $\lambda := \lambda^{k-1}$.
repeat
1. Let $z := \mathbf{prox}_{\lambda g}(x^k - \lambda \nabla f(x^k))$.
2. **break if** $f(z) \le \hat{f}_\lambda(z, x^k)$.
3. Update $\lambda := \beta\lambda$.

return $\lambda^k := \lambda$, $x^{k+1} := z$.

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typical value of β is $1/2$, and

$$\hat{f}_\lambda(x,y) = f(y) + \nabla f(y)^T(x-y) + (1/2\lambda)\|x-y\|_2^2$$

Convergence proof

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Numerical example

Consider a numerical example with $f(x) = \|Ax - b\|_2^2$ with $A \in \mathbf{R}^{10\times 100}$ and $b \in \mathbf{R}^{10}$. Entries of A and b are generated as independent samples from a standard normal distribution. Here, we have chosen λ using cross validation.

Results

On this numerical example, the ADMM method converges quickly. We give two realizations corresponding to different parameters A and b.

Conclusion

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