CSC411 - Project #2

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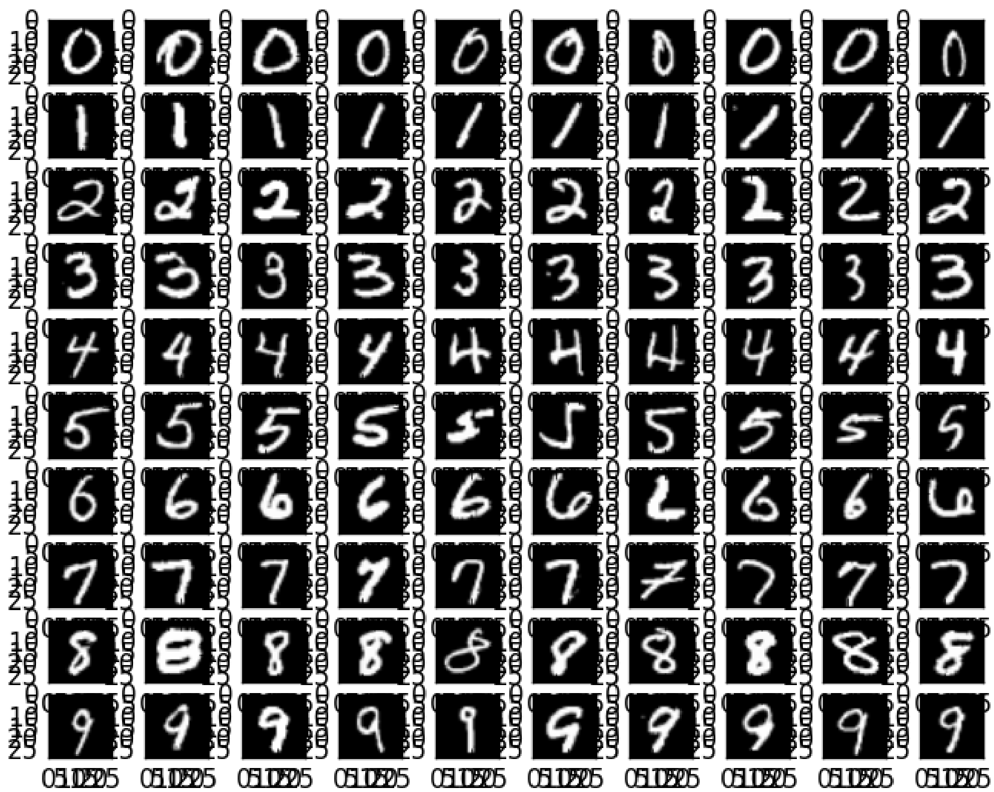
Part 1

Question Describe the dataset of digits. In your report, include 10 images of each of the digits. You may find matplotlibs subplot useful.

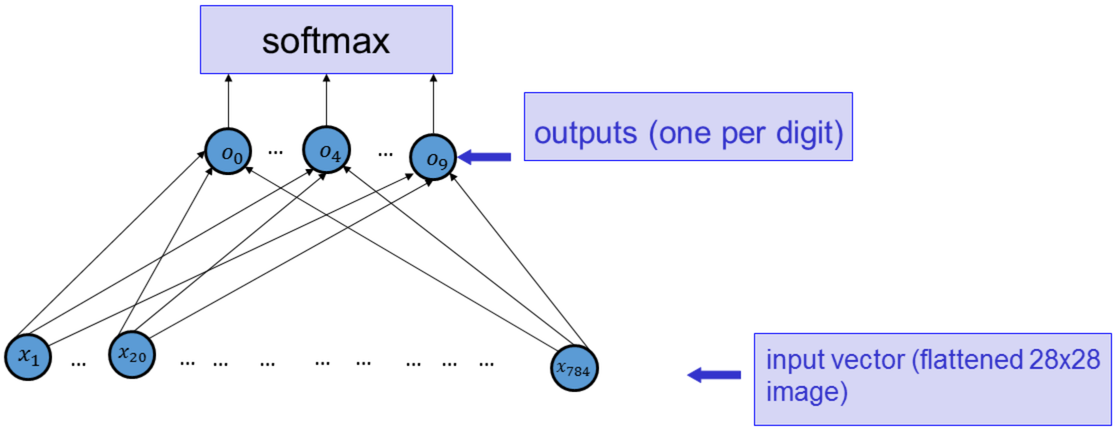
Answer Please see the images of each digits below. For the same digit, some images have thicker lines (like “8” in the second column) and some have thiner lines (like “8” in the third column). Some digit images are straight (like “1” in the first column) and some digits are rotated (like “1” in the ninth column). Most digits are clear to be identified manually but some digits are not clear (like the “6” in the seventh column).



10 images of digits from 0 to 9

Part 2

Question Implement a function that computes the network below. The o's here should simply be



linear combinations of the x's (that is, the activation function in the output layer is the identity). Supecifically, use $o_i = \sum_j w_{ji}x_j + b_i$. Include the listing of your implementation in your report for this Part.

Answer Please see the source code of the function below:

```
def softmax(y):
    '''Return the output of the softmax function for the matrix of output y. y
    is an NxM matrix where N is the number of outputs for a single case, and M
    is the number of cases'''
    return exp(y)/tile(sum(exp(y),0), (len(y),1))


def lin_combin(w, b, x):
    o = (dot(w.T, x) + b)
    return softmax(o)
```

Part 3

We would like to use the sum of the negative log-probabilities of all the training cases as the cost function.

Part 3(a)

Question Compute the gradient of the cost function with respect to the weight w_{ij}. Justify every step. You may refer to Slide 7 of the One-Hot Encoding lecture , but note that you need to justify every step there, and that your cost function is the sum over all the training examples.

Answer When sample size is one, we could calculate $\frac{\partial p_j}{\partial o_i}$ in two cases:

1. If $j = i$,

$$\begin{aligned}\frac{\partial p_i}{\partial o_i} &= \frac{e^{o_i}}{\sum_k e^{o_k}} - \frac{e^{o_i}}{\sum_k e^{o_k}} \cdot \frac{e^{o_i}}{\sum_k e^{o_k}} \\ &= p_i(1 - p_i)\end{aligned}$$

2. If $j \neq i$,

$$\begin{aligned}\frac{\partial p_j}{\partial o_i} &= -\frac{e^{o_j}}{\sum_k e^{o_k}} \cdot \frac{e^{o_i}}{\sum_k e^{o_k}} \\ &= -p_j p_i\end{aligned}$$

In summary,

$$\frac{\partial p_j}{\partial o_i} = \begin{cases} p_i(1 - p_i) & \text{, if } j = i \\ -p_j p_i & \text{, if } j \neq i \end{cases} \tag{1}$$

$$\frac{\partial C}{\partial p_j} = -\frac{y_j}{p_j}$$

$$\begin{aligned}\frac{\partial C}{\partial o_i} &= \sum_j \frac{\partial C}{\partial p_j}\frac{\partial p_j}{\partial o_i} \\ &= \sum_{j, j\neq i} \frac{\partial C}{\partial p_j}\frac{\partial p_j}{\partial o_i} + \frac{\partial C}{\partial p_i}\frac{\partial p_i}{\partial o_i} \\ &= \sum_{j, j\neq i} (-\frac{y_j}{p_j}) \cdot (-p_j p_i) - \frac{y_i}{p_i} \cdot p_i(1 - p_i) \\ &= \sum_{j, j\neq i} y_j p_i - y_i + y i p_i \\ &= \sum_j y_j p_i - y_i\end{aligned}$$

Because $\sum_j y_j = 1$, $\sum_j y_j p_i = p_i$. Then,

$$\frac{\partial C}{\partial o_i} = p_i - y_i$$

For the following equation, let j refers to the input size (number of x per sample) and i refers to the output size (number of o per sample), and k refers to the training set size (number of samples).

X is the input matrix with dimension $n \times m$, where n is the number x (related to j) and m is the sample size (related to k). P is the calculated result matrix after using softmax with dimension $d \times m$, where d is the output size (number of outputs per sample, related to i). Y is the expected output with the same dimension as P.

$$\frac{\partial C}{\partial o_{ik}} = p_{ik} - y_{ik}$$

$$\begin{aligned}\frac{\partial o_{ik}}{\partial w_{ij}} &= \frac{\partial}{\partial w_{ij}}(w_{i1}x_{1k} + \cdots + w_{ij}x_{jk} + \cdots + w_{in}x_{nk} + b_{ik}) \\ &= x_{jk}\end{aligned}$$

$$\begin{aligned}\frac{\partial C}{\partial w_{ij}} &= \sum_k \frac{\partial C}{\partial o_{ik}} \cdot \frac{\partial o_{ik}}{\partial w_{ij}} \\ &= \sum_k x_{jk}(p_{ik} - y_{ik}) \\ &= X(P-Y)^T\end{aligned}$$

Part 3(b)

Question Write vectorized code that computes the gradient with respect to the cost function. Check that the gradient was computed correctly by approximating the gradient at several coordinates using finite differences. Include the code for computing the gradient in vectorized form in your report.

```
def part3b(M):
    """Test gradient for part 3b."""

    x1 = M["train8"][0]
    x2 = M["train8"][1]
    x = vstack((x1, x2)).T
    y = array([[1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0, 0, 0]]).T
    w = zeros((x.shape[0], 10))
    b = zeros((10, 2))
    test_gradient(x, y, b, w)


def test_gradient(x, y, b, w):
    """
    Compare the result of my gradient function NLL_gradient and the gradient get
    from finite differences method. Test three times on different w_ij.
    """
```

```python
    j_list = [5, 8, 8]
    i_list = [300, 601, 600]

    h = 10e-5

    for k in range(0, 3):
        i = i_list[k]
        j = j_list[k]
        g_finite = finite_diff_gradient(x, y, w, b, i, j, h)
        g_mine = NLL_gradient(x, y, w, b)[i, j]
        diff = abs(g_mine - g_finite)
        print("Test "+str(k)+":")
        print("gradient_finite_differences = "+str(g_finite))
        print("gradient_my_function = "+str(g_mine))
        print("difference = "+str(diff))
        print("--------------------------------")


def finite_diff_gradient(x, y, w, b, i, j, h):
    """
    Use finite difference to calculate the gradient on w_ij with h.
    x: input
    y: expected output
    w: weight
    b: bias
    """

    c = NLL(x, y, w, b)
    new_w = w
    new_w[i, j] += h
    new_c = NLL(x, y, new_w, b)
    return (new_c - c)/h


def NLL(x, y, w, b):
    p = lin_combin(w, b, x)
    return -sum(y*log(p))


def NLL_gradient(x, y, w, b):
    p = lin_combin(w, b, x)
    return dot(x, (p - y).T)
```