Assignment due date: Feb 10, 2017, 11:59pm

Hand-in to be submitted electronically in PDF format with code to the CDF server by the above due date

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I hereby affirm that all the solutions I provide, both in writing and in code, for this assignment are my own. I have properly cited and noted any reference material I used to arrive at my solution and have not share my work with anyone else. I am also aware that should my code be copied from somewhere else, whether found online, from a previous or current student and submitted as my own, it will be reported to the department.

Yijin Wang
Signature

(Note: -3 marks penalty for not completing properly the above section)

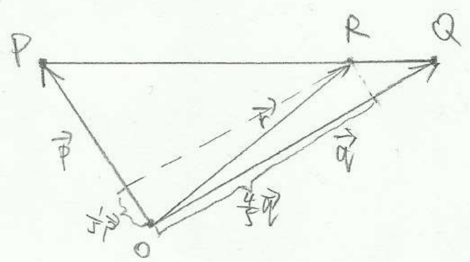
Part 1 total marks: 50
Part 2 total marks: 50
Total: 100

Part 1 - Written work: Be sure to provide clean, legible derivations and not omit any steps which your TA may need to fully understand your work. Use diagrams wherever appropriate.

A) Lines, points, vectors, and dot-products

1) Let a line segment be defined by the points P and Q.

(4 marks) - Let R be the point on the line segment that is four times as far from P as it is from Q. Let $\vec{p} = \overrightarrow{OP}, \vec{q} = \overrightarrow{OQ}, \vec{r} = \overrightarrow{OR}$. Show that $\vec{r} = \frac{1}{5}\vec{p} + \frac{4}{5}\vec{q}$ and *draw* a sketch validating this.



$\overrightarrow{PR} = \vec{r} - \vec{p}$

$\overrightarrow{RQ} = \vec{q} - \vec{r} = \frac{1}{4}\overrightarrow{PR} = \frac{1}{4}(\vec{r} - \vec{p})$

$\vec{q} - \vec{r} = \frac{1}{4}\vec{r} - \frac{1}{4}\vec{p}$

$\frac{5}{4}\vec{r} = \vec{q} + \frac{1}{4}\vec{p}$

$\vec{r} = \frac{1}{5}\vec{p} + \frac{4}{5}\vec{q}$

(3 marks) - Show the parametric form of the line segment formed by P and Q, with parameter t, starting from P in the direction of Q (that is to say, positive values of the parameter will give points along the line towards Q). Be sure to define everything explicitly.

Let R be a point on the line segment formed by P and Q

parametric form: $R = P + t(Q - P)$, $t \in [0, 1]$

(5 marks) Let the points have explicit representations $P(x_0, y_0)$, $Q(x_1, y_1)$ with respect to a coordinate system. Give a simple test to determine if an arbitrary point $R(x, y)$ is on the **line segment** or not.

if ($\min(x_0, x_1) \le x \le \max(x_0, x_1)$ and $\min(y_0, y_1) \le y \le \max(y_0, y_1)$ and $\frac{x_1 - x_0}{x - x_0} = \frac{y_1 - y_0}{y - y_0}$) then $R(x, y)$ is on the line segment.

2)

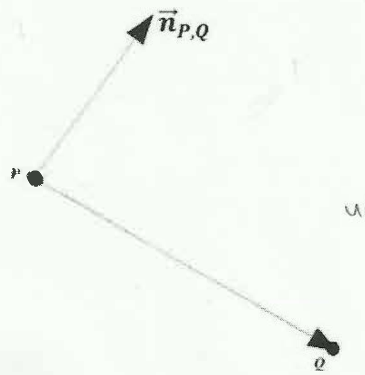
(5 marks) - Derive a **unit** normal vector $\vec{n}$ to the line segment from above, expressed in terms of $\boldsymbol{P}$ and $\boldsymbol{Q}$. $\vec{n}$ should point to the left with respect to the direction of the line segment (the line segment is directed from P to Q).

Let $P = (x_0, y_0)$, $Q = (x_1, y_1)$

normal vector $\vec{N}(\lambda) = \left(-\frac{dy}{d\lambda}(\lambda), \frac{dx}{d\lambda}(\lambda)\right)$

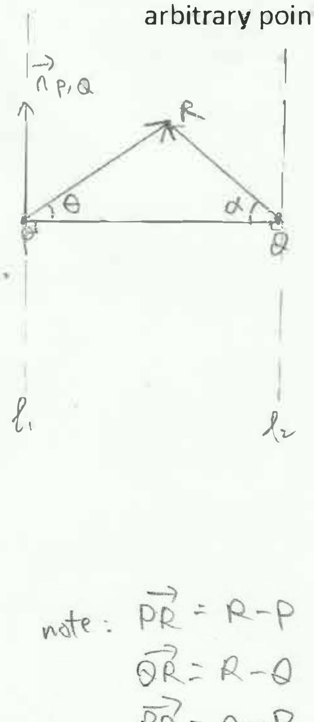
$= (-(y_1 - y_0), x_1 - x_0)$

$= (y_0 - y_1, x_1 - x_0)$



unit normal vector $\vec{n} = \left(\frac{y_0 - y_1}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}}, \frac{x_1 - x_0}{\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}}\right)$

(5 marks) - Using $\vec{n}$ from 2), give an expression that computes the distance between an arbitrary point $R(x,y)$ and the line segment.



① Test if R is between l_1 and l_2:

if $\cos\theta$ and $\cos\alpha$ are both greater than 0, R is between l_1 and l_2:

$\therefore$ if $\frac{\overrightarrow{PR} \cdot \overrightarrow{PQ}}{\|\overrightarrow{PR}\| \cdot \|\overrightarrow{PQ}\|} > 0$ and $\frac{\overrightarrow{QR} \cdot \overrightarrow{QP}}{\|\overrightarrow{QR}\| \cdot \|\overrightarrow{PQ}\|} > 0$: ($\cos\theta > 0$ and $\cos\alpha > 0$)

distance $= \vec{n} \cdot \overrightarrow{PR}$

else if $\frac{\overrightarrow{PR} \cdot \overrightarrow{PQ}}{\|\overrightarrow{PR}\| \cdot \|\overrightarrow{PQ}\|} > 0$: ($\cos\theta > 0$ and $\cos\alpha < 0$)

distance $= \|\overrightarrow{QR}\|$

else: ($\cos\theta < 0$ and $\cos\alpha > 0$)

distance $= \|\overrightarrow{PR}\|$

note: $\overrightarrow{PR} = R - P$

$\overrightarrow{QR} = R - Q$

$\overrightarrow{PQ} = Q - P$

$\overrightarrow{QP} = P - Q$

B: Transformations and transformation properties

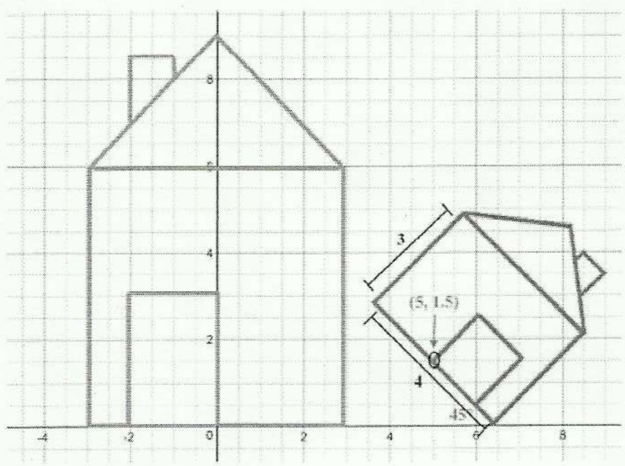
(3 marks each) - Prove whether or not the following pairs of transformations commute. The transformations are general, 2D affine transforms (*provide solutions on separate sheet*)

1) Rotations and Translations
2) Two rotations
3) Translation, Reflection
4) Shear wrt to x axis an uniform scaling
5) Rotation, non-uniform scaling

If not-commutative, a simple counter example will suffice. If commutative, algebraic proof is required. Please see the paper attached at the end.

C: Affine transformation properties

1) (5 marks) - Give the sequence of 2D affine transformations that maps the object in the left figure to the object in the right figure. You can use R, T, S, Sh, Re, to express rotation, translation, scaling, shear, and reflection operations respectively.



Transformation matrix = TRSRe where

$$T=\begin{bmatrix}1&0&5\\0&1&1.5\\0&0&1\end{bmatrix}\quad R=\begin{bmatrix}\cos45^\circ&\sin45^\circ&0\\-\sin45^\circ&\cos45^\circ&0\\0&0&1\end{bmatrix}\quad S=\begin{bmatrix}\frac{2}{3}&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\quad Re=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

2) (5 marks) A triangle can be expressed in parametric form as:

$$T(\alpha,\beta) = \overrightarrow{p_1} + \alpha\overrightarrow{d_1} + \beta\overrightarrow{d_2}$$

provided that $\alpha + \beta \leq 1$ and $\alpha, \beta \geq 0$ Prove that under an invertible affine transform, the triangle remains a triangle.

Proof: $T'(\alpha,\beta) = A(\vec{p_1} + \alpha\vec{d_1} + \beta\vec{d_2})$, where A is an affine transform matrix.

$T'(\alpha,\beta) = A\vec{p_1} + \alpha A\vec{d_1} + \beta A\vec{d_2}$

let $\vec{p_1}' = A\vec{p_1}$, $\vec{d_1}' = A\vec{d_1}$, $\vec{d_2}' = A\vec{d_2}$

$T'(\alpha,\beta) = \vec{p_1}' + \alpha\vec{d_1}' + \beta\vec{d_2}'$

$\therefore T'(\alpha,\beta)$ is still a triangle.

$\because \vec{p_1}'$ has the same dimension as $\vec{p_1}$,

$\vec{d_1}'$ has the same dimension as $\vec{d_1}$,

$\vec{d_2}'$ has the same dimension as $\vec{d_2}$.

3) (3 marks) - Can affine transformations be represented with a formula where points can be represented in Cartesian coordinates instead of homogeneous coordinates? What is the main advantage of representing affine transformations in the same matrix form as general homographies?

Yes, affine transformations can be represented as $p' = Ap + \vec{t}$ for point p in Cartesian coordinates, where A is the 2×2 transformation matrix and $\vec{t}$ is the 1×2 translation vector.

The main advantage of representing affine transformations in the same matrix form as general homographies is that if we have a series of mixed transformations and translations, in homographies we could just do a series of matrix multiplications while in Cartesian form we have additions mixed with multiplication, which is more complex, and hard to inverse.

CSC418 A1 Part B:

1) Rotations and Translations : does NOT commute

Let Rotation $R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Translation $T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\left(R = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$

$$RT = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TR = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \neq RT$$

2) Two rotations : Commute

Let Rotation $R_1 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$R_1R_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta & 0 \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2R_1 = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta & 0 \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_1R_2$$

3) Translation, Reflection : does NOT commute

Let Translation $T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, Reflection $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$TR = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \neq TR$$

4) shear wrt to x axis, an uniform scaling: Commute

Let shearing $Sh = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, Scaling $Sc = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$ShSc = \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & ab & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_c S_h = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & ab & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} = S_h S_c$$

5) Rotation, non-uniform scaling: does NOT commute

Let Rotation $R = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, non-uniform Scaling $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$RS = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$SR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq RS$$