Conditional Probability

Definition: Conditional probability is the probability of an event occurring when it is already known that another related event has happened. It helps us measure how likely something is under a given condition or extra information. In many real-life situations, we do not look at the whole sample space, but only at a part of it that satisfies the condition.

For example, if we want to know the chance of a student scoring above 80% given that the student studied well, we use conditional probability. It plays an important role in probability theory, statistics, machine learning, and decision-making.

If A and B are two events, then the probability of A happening **given** that B has already happened is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, the probability of B happening **given** that A has already happened is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication Theorem on Probability :

Let E and F be two events associated with a sample space of an experiment. Then
$P(E \cap F) = P(E)\,P(F \mid E),\ P(E) \neq 0$
$= P(F)\,P(E \mid F),\ P(F) \neq 0$
If E, F and G are three events associated with a sample space, then
$P(E \cap F \cap G) = P(E)\,P(F \mid E)\,P(G \mid E \cap F)$

Question 1 : A die is rolled. If the outcome is an even number. What is the probability that it is a prime number?

Solution :

Step-by-Step Solution

Let:

- **Event B** = outcome is **even**
  → $B = \{2, 4, 6\}$
  → $n(B) = 3$
- **Event A** = outcome is **prime**
  → $A = \{2, 3, 5\}$
  → $n(A) = 3$
- **Both prime and even** (intersection)
  → $A \cap B = \{2\}$
  → $n(A \cap B) = 1$

Using Conditional Probability Formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{1}{6}, \quad P(B) = \frac{3}{6}$$

$$P(A|B) = \frac{1/6}{3/6} = \frac{1}{3}$$

✅ **Final Answer:**

$$\boxed{P(\text{prime}|\text{even}) = \frac{1}{3}}$$

Question 2 : A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Solution :

🔷 Step 1: Define the events

Let:

- **Event B** = Sum of numbers = 6

  Possible outcomes:

$$B = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

  So, $n(B) = 5$

- **Event A** = 4 appears at least once

  From B, choose only those which contain 4:

$$A \cap B = \{(2,4),(4,2)\}$$

  So, $n(A \cap B) = 2$

🔷 Step 2: Apply Conditional Probability Formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{2}{36}, \quad P(B) = \frac{5}{36}$$

$$P(A|B) = \frac{2/36}{5/36} = \frac{2}{5}$$

✅ Final Answer

$$\boxed{P(4 \text{ appears at least once} \mid \text{sum} = 6) = \frac{2}{5}}$$

Question 3 : A coin is tossed three times. Find P(E/F), if the events E and Fore follow.

(i) E: Head on third toss F: heads on first two tosses

(ii) E: At least two heads F: At most two heads

(ii) E: At most two tails F: At least one tail

Solution : Total sample space for 3 tosses

S = 8 outcomes $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(i)

E: Head on **third** toss

$$E = \{HHH, HTH, THH, TTH\} \Rightarrow n(E) = 4$$

F: Heads on **first two** tosses

$$F = \{HHH, HHT\} \Rightarrow n(F) = 2$$

Intersection: both conditions true

$$E \cap F = \{HHH\} \Rightarrow n(E \cap F) = 1$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{2/8} = \frac{1}{2}$$

✔ **Answer (i):** $\boxed{\frac{1}{2}}$

(ii)

E: At least two heads

$$E = \{HHH, HHT, HTH, THH\} \Rightarrow n(E) = 4$$

F: At most two heads

$$F = S \setminus \{HHH\}$$

$$F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\} \Rightarrow n(F) = 7$$

Intersection: outcomes with **2 heads** only

$$E \cap F = \{HHT, HTH, THH\} \Rightarrow n(E \cap F) = 3$$

$$P(E|F) = \frac{3/8}{7/8} = \frac{3}{7}$$

✔ **Answer (ii):** $\boxed{\frac{3}{7}}$

(iii)

E: At most two tails
(i.e., 0, 1 or 2 tails → all except TTT)

$$E = S \setminus \{TTT\} \Rightarrow n(E) = 7$$

F: At least one tail

$$F = S \setminus \{HHH\} \Rightarrow n(F) = 7$$

Intersection:

$$E \cap F = S \setminus \{HHH, TTT\} \Rightarrow n(E \cap F) = 6$$

$$P(E|F) = \frac{6/8}{7/8} = \frac{6}{7}$$

✔ **Answer (iii):** $\boxed{\frac{6}{7}}$

Question 4 : A card is drawn from a well-shuffled deck of 52 cards. If the selected card is a face card (Jack, Queen, King), what is the probability that it is a King?

Solution :

🔷 Step 1: Define Events

- Event **F**: Card drawn is a **face card**

  Number of face cards (J, Q, K in 4 suits):

$$3 \text{ face cards per suit} \times 4 \text{ suits} = 12 \Rightarrow n(F) = 12$$

- Event **E**: Card drawn is a **King**

  Number of Kings:

$$4 \Rightarrow n(E) = 4$$

- Intersection (King which is also a face card):

$$E \cap F = \{4 \text{ Kings}\} \Rightarrow n(E \cap F) = 4$$

🔷 Step 2: Apply Conditional Probability Formula

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$P(E \cap F) = \frac{4}{52}, \quad P(F) = \frac{12}{52}$$

$$P(E|F) = \frac{4/52}{12/52} = \frac{4}{12} = \frac{1}{3}$$

✅ Final Answer

$$\boxed{P(\text{King} \mid \text{Face Card}) = \frac{1}{3}}$$

Question 5 : Two dice are rolled and the sum of the numbers obtained is 7. What is the conditional probability that one of the dice shows 3?

Solution :

🔷 **Step 1: Define Events**

- Event **B**: Sum of numbers = 7

  Possible outcomes for sum = 7:

$$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \Rightarrow n(B) = 6$$

- Event **A**: One die shows 3

  From B, select outcomes with **3 on either die**:

$$A \cap B = \{(3,4),(4,3)\} \Rightarrow n(A \cap B) = 2$$

🔷 **Step 2: Apply Conditional Probability Formula**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{2}{36}, \quad P(B) = \frac{6}{36}$$

$$P(A|B) = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}$$

✅ **Final Answer**

$$\boxed{P(\text{one die shows 3} \mid \text{sum} = 7) = \frac{1}{3}}$$

Question 6 : A bag contains 5 red balls and 7 blue balls. Three balls are drawn one by one without replacement. It is given that the first ball drawn is blue. What is the probability that exactly two blue balls are drawn in total?

Solution :

🔷 Step 1: Define Events

- Total balls: $5 + 7 = 12$
- Event **B**: First ball is blue (already given)
- Event **A**: Exactly 2 blue balls are drawn in 3 draws

We want $P(A|B)$.

🔷 Step 2: Consider conditional probability

Since the **first ball is blue**, only **2 more draws remain**, and we need **exactly 1 more blue** in these two draws (to make total blue balls = 2).

- Balls remaining after first draw:
  - Blue: $7 - 1 = 6$
  - Red: 5
  - Total remaining: 11
- We need **1 blue + 1 red** in the next 2 draws.

Number of ways to pick 1 blue and 1 red in **any order**:

$$\text{Ways} = \binom{6}{1} \cdot \binom{5}{1} \cdot 2! = 6 \cdot 5 = 30$$

(We multiply by 2 for order of blue and red: BR or RB)

- Total possible outcomes for 2 draws from 11 balls:

$$\binom{11}{2} = 55$$

🔷 Step 3: Compute conditional probability

$$P(A|B) = \frac{\text{Favorable outcomes}}{\text{Total outcomes for 2 draws}} = \frac{30}{55} = \frac{6}{11}$$

✅ Final Answer

$$\boxed{P(\text{exactly 2 blue balls} \mid \text{first is blue}) = \frac{6}{11}}$$

Assignment

Question 1 : Three dice are thrown at the same time. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.

Question 2 : A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.

Question 3 : Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

Question 4 : A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of the second ball being blue?

Question 5 : A box contains 6 red and 4 green balls. Two balls are drawn one by one without replacement. If the first ball drawn is red, what is the probability that the second ball is green?

Answer Sheet

1. 1/10
2. 85/153
3. 7/15
4. 5/9
5. 4/9