Conditional Probability

Definition: Conditional probability is the probability of an event occurring when it is already known that another related event has happened. It helps us measure how likely something is under a given condition or extra information. In many real-life situations, we do not look at the whole sample space, but only at a part of it that satisfies the condition.

For example, if we want to know the chance of a student scoring above 80% given that the student studied well, we use conditional probability. It plays an important role in probability theory, statistics, machine learning, and decision-making.

If A and B are two events, then the probability of A happening **given** that B has already happened is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, the probability of B happening **given** that A has already happened is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication Theorem on Probability :

Let E and F be two events associated with a sample space of an experiment. Then
$P(E \cap F) = P(E)\, P(F \mid E),\ P(E) \neq 0$
$= P(F)\, P(E \mid F),\ P(F) \neq 0$
If E, F and G are three events associated with a sample space, then
$P(E \cap F \cap G) = P(E)\, P(F \mid E)\, P(G \mid E \cap F)$

Question 1 : A die is rolled. If the outcome is an even number. What is the probability that it is a prime number?

Solution :

Step-by-Step Solution

Let:

- **Event B** = outcome is **even**
  → $B = \{2, 4, 6\}$
  → $n(B) = 3$
- **Event A** = outcome is **prime**
  → $A = \{2, 3, 5\}$
  → $n(A) = 3$
- **Both prime and even** (intersection)
  → $A \cap B = \{2\}$
  → $n(A \cap B) = 1$

Using Conditional Probability Formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{1}{6}, \quad P(B) = \frac{3}{6}$$

$$P(A|B) = \frac{1/6}{3/6} = \frac{1}{3}$$

✅ Final Answer:

$$\boxed{P(\text{prime}|\text{even}) = \frac{1}{3}}$$

Question 2 : A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Solution :

🔷 Step 1: Define the events

Let:

- **Event B** = Sum of numbers = 6

  Possible outcomes:

$$B = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

  So, $n(B) = 5$

- **Event A** = 4 appears at least once

  From B, choose only those which contain 4:

$$A \cap B = \{(2,4),(4,2)\}$$

  So, $n(A \cap B) = 2$

🔷 Step 2: Apply Conditional Probability Formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{2}{36}, \quad P(B) = \frac{5}{36}$$

$$P(A|B) = \frac{2/36}{5/36} = \frac{2}{5}$$

✅ Final Answer

$$\boxed{P(4 \text{ appears at least once} \mid \text{sum} = 6) = \frac{2}{5}}$$

Question 3 : A coin is tossed three times. Find P(E/F), if the events E and Fore follow.

(i) E: Head on third toss F: heads on first two tosses

(ii) E: At least two heads F: At most two heads

(ii) E: At most two tails F: At least one tail

Solution : Total sample space for 3 tosses

S = 8 outcomes $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(i)

E: Head on **third** toss

$$E = \{HHH, HTH, THH, TTH\} \Rightarrow n(E) = 4$$

F: Heads on **first two** tosses

$$F = \{HHH, HHT\} \Rightarrow n(F) = 2$$

Intersection: both conditions true

$$E \cap F = \{HHH\} \Rightarrow n(E \cap F) = 1$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{2/8} = \frac{1}{2}$$

✔ **Answer (i):** $\boxed{\frac{1}{2}}$

(ii)

E: At least two heads

$$E = \{HHH, HHT, HTH, THH\} \Rightarrow n(E) = 4$$

F: At most two heads

$$F = S \setminus \{HHH\}$$

$$F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\} \Rightarrow n(F) = 7$$

Intersection: outcomes with **2 heads** only

$$E \cap F = \{HHT, HTH, THH\} \Rightarrow n(E \cap F) = 3$$

$$P(E|F) = \frac{3/8}{7/8} = \frac{3}{7}$$

✔ **Answer (ii):** $\boxed{\frac{3}{7}}$

(iii)

E: At most two tails

(i.e., 0, 1 or 2 tails → all except TTT)

$$E = S \setminus \{TTT\} \Rightarrow n(E) = 7$$

F: At least one tail

$$F = S \setminus \{HHH\} \Rightarrow n(F) = 7$$

Intersection:

$$E \cap F = S \setminus \{HHH, TTT\} \Rightarrow n(E \cap F) = 6$$

$$P(E|F) = \frac{6/8}{7/8} = \frac{6}{7}$$

✔ **Answer (iii):** $\boxed{\frac{6}{7}}$

Question 4 : A card is drawn from a well-shuffled deck of 52 cards. If the selected card is a face card (Jack, Queen, King), what is the probability that it is a King?

Solution :

🔷 Step 1: Define Events

- Event **F**: Card drawn is a **face card**

  Number of face cards (J, Q, K in 4 suits):

$$3 \text{ face cards per suit} \times 4 \text{ suits} = 12 \Rightarrow n(F) = 12$$

- Event **E**: Card drawn is a **King**

  Number of Kings:

$$4 \Rightarrow n(E) = 4$$

- Intersection (King which is also a face card):

$$E \cap F = \{4 \text{ Kings}\} \Rightarrow n(E \cap F) = 4$$

🔷 Step 2: Apply Conditional Probability Formula

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$P(E \cap F) = \frac{4}{52}, \quad P(F) = \frac{12}{52}$$

$$P(E|F) = \frac{4/52}{12/52} = \frac{4}{12} = \frac{1}{3}$$

✅ Final Answer

$$\boxed{P(\text{King} \mid \text{Face Card}) = \frac{1}{3}}$$

Question 5 : Two dice are rolled and the sum of the numbers obtained is 7. What is the conditional probability that one of the dice shows 3?

Solution :

🔷 **Step 1: Define Events**

- Event **B**: Sum of numbers = 7

  Possible outcomes for sum = 7:

$$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \Rightarrow n(B) = 6$$

- Event **A**: One die shows 3

  From B, select outcomes with **3 on either die**:

$$A \cap B = \{(3,4),(4,3)\} \Rightarrow n(A \cap B) = 2$$

🔷 **Step 2: Apply Conditional Probability Formula**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{2}{36}, \quad P(B) = \frac{6}{36}$$

$$P(A|B) = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}$$

✅ **Final Answer**

$$\boxed{P(\text{one die shows 3} \mid \text{sum} = 7) = \frac{1}{3}}$$

Question 6 : A bag contains 5 red balls and 7 blue balls. Three balls are drawn one by one without replacement. It is given that the first ball drawn is blue. What is the probability that exactly two blue balls are drawn in total?

Solution :

🔷 Step 1: Define Events

- Total balls: $5 + 7 = 12$
- Event **B**: First ball is blue (already given)
- Event **A**: Exactly 2 blue balls are drawn in 3 draws

We want $P(A|B)$.

🔷 Step 2: Consider conditional probability

Since the **first ball is blue**, only **2 more draws remain**, and we need **exactly 1 more blue** in these two draws (to make total blue balls = 2).

- Balls remaining after first draw:
  - Blue: $7 - 1 = 6$
  - Red: 5
  - Total remaining: 11
- We need **1 blue + 1 red** in the next 2 draws.

Number of ways to pick 1 blue and 1 red in **any order**:

$$\text{Ways} = \binom{6}{1} \cdot \binom{5}{1} \cdot 2! = 6 \cdot 5 = 30$$

(We multiply by 2 for order of blue and red: BR or RB)

- Total possible outcomes for 2 draws from 11 balls:

$$\binom{11}{2} = 55$$

🔷 Step 3: Compute conditional probability

$$P(A|B) = \frac{\text{Favorable outcomes}}{\text{Total outcomes for 2 draws}} = \frac{30}{55} = \frac{6}{11}$$

✅ Final Answer

$$\boxed{P(\text{exactly 2 blue balls} \mid \text{first is blue}) = \frac{6}{11}}$$

Assignment

Question 1 : Three dice are thrown at the same time. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.

Question 2 : A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.

Question 3 : Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

Question 4 : A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of the second ball being blue?

Question 5 : A box contains 6 red and 4 green balls. Two balls are drawn one by one without replacement. If the first ball drawn is red, what is the probability that the second ball is green?

Answer Sheet

1. 1/10
2. 85/153
3. 7/15
4. 5/9
5. 4/9

Independence of Events

Two events associated with a random experiment are said to be independent if the occurrence of one event does not affect the probability of occurrence of another.

So for two independent events-
P(A/B) = P(A)   P(B/A) = P(B)

Two events A and B associated with the some random experiment are independent if
$P(A \cap B)$ = P(A) P(B)

Note : Let P1, P2, P3 ………….Pr be the Probability of r independent events A1, A2, A3……….Ar then

Probability that none of the events happens = (1 - P1), (1 - P2), (1 - P3)..........(1 - Pr)

Probability that none of the events happens = 1 - (1 - P1), (1 - P2), (1 - P3)..........(1 - Pr)

Example 1 : The probability of student A passing the examination is 3/7 and the of student B passing is 5/7. A pass's and B Passes' as independent, find the probability of

(i) Only A passed the examination.

(ii) Only one of them passed the examination.

Solution :
Given :
$P(A) = 3/7$ P(A)=3/7 → Probability student A passes
$P(B) = 5/7$ P(B)=5/7 → Probability student B passes
Events A and B are independent

Step 1: Compute complementary probabilities

- P(A′)=1−P(A)=1−3/7=4/7
- P(B′)=1−P(B)=1−5/7=2/7

(i) Probability that only A passes

Only A passes → A passes and B fails

$$P(\text{only A}) = P(A \cap B') = P(A) \cdot P(B') \quad \text{(independent events)}$$

$$P(\text{only A}) = \frac{3}{7} \cdot \frac{2}{7} = \frac{6}{49}$$

(ii) Probability that only one of them passes

Only one passes → either A passes & B fails **or** B passes & A fails

$$P(\text{only one}) = P(A \cap B') + P(A' \cap B)$$

$$P(A' \cap B) = P(A') \cdot P(B) = \frac{4}{7} \cdot \frac{5}{7} = \frac{20}{49}$$

$$P(\text{only one}) = \frac{6}{49} + \frac{20}{49} = \frac{26}{49}$$

✅ **Final Answers**

1. **Only A passes:** $\frac{6}{49}$
2. **Only one passes:** $\frac{26}{49}$

Example 2 : A student takes examinations in four subjects A, B, C, D. His chance of passing in A are 4/5, in B are 3/4 , in Core 5/6 and in D are 2/3.To qualify he must pass in A and at-least in two other subjects. What is the probability that he qualifies ?

Solution : **Given:**

A student takes 4 subjects: **A, B, C, D**

- P(A)=4/5
- P(B)=3/4
- P(C)=5/6
- P(D)=2/3

To **qualify**, the student must:

- Pass in **A** AND
- Pass in **at least two of the other three subjects (B, C, D)**
  We assume all subjects are **independent events**.

Step 1: Pass A

$$P(A) = 4/5$$

Step 2: Pass at least 2 of B, C, D

Let's denote:

- Pass exactly 2 subjects:

$$P(\text{exactly 2 pass}) = P(B \cap C \cap D') + P(B \cap C' \cap D) + P(B' \cap C \cap D)$$

- Pass all 3 subjects:

$$P(B \cap C \cap D)$$

Step 2a: Pass exactly 2 subjects

$$P(B \cap C \cap D') = P(B) \cdot P(C) \cdot P(D') = \frac{3}{4} \cdot \frac{5}{6} \cdot \left(1 - \frac{2}{3}\right) = \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{3} = \frac{15}{72} = \frac{5}{24}$$

$$P(B \cap C' \cap D) = P(B) \cdot P(C') \cdot P(D) = \frac{3}{4} \cdot \left(1 - \frac{5}{6}\right) \cdot \frac{2}{3} = \frac{3}{4} \cdot \frac{1}{6} \cdot \frac{2}{3} = \frac{6}{72} = \frac{1}{12}$$

$$P(B' \cap C \cap D) = P(B') \cdot P(C) \cdot P(D) = \left(1 - \frac{3}{4}\right) \cdot \frac{5}{6} \cdot \frac{2}{3} = \frac{1}{4} \cdot \frac{5}{6} \cdot \frac{2}{3} = \frac{10}{72} = \frac{5}{36}$$

$$P(\text{exactly 2 pass}) = \frac{5}{24} + \frac{1}{12} + \frac{5}{36} = \frac{15 + 6 + 10}{72} = \frac{31}{72}$$

Step 2b: Pass all 3 subjects

$$P(B \cap C \cap D) = \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{2}{3} = \frac{30}{72} = \frac{5}{12}$$

Step 2c: Pass at least 2 subjects (B, C, D)

$$P(\text{at least 2}) = P(\text{exactly 2}) + P(\text{all 3}) = \frac{31}{72} + \frac{30}{72} = \frac{61}{72}$$

Step 3: Pass A and at least 2 others

$$P(\text{qualify}) = P(A) \cdot P(\text{at least 2 of B, C, D}) = \frac{4}{5} \cdot \frac{61}{72} = \frac{244}{360} = \frac{61}{90}$$

✅ **Final Answer**

$$\boxed{P(\text{qualify}) = \frac{61}{90}}$$

Example 3 : A bag has 4 red balls and 5 black balls; and the second bag has 3 red and 7 black balls. One ball is drawn from the first and two from the second. Find the probability that out of three balls 2 are black and one is red.

Solution : **Given:**

- Bag 1 → 4 Red, 5 Black → Total = 9 balls
- Bag 2 → 3 Red, 7 Black → Total = 10 balls

You draw:

- 1 ball from Bag 1
- 2 balls from Bag 2

You want: **Exactly 2 black balls and 1 red ball in total**

There are two possible cases:

Case 1:

- Red from Bag 1
- 2 Black from Bag 2

$$P(\text{Red from Bag1}) = \frac{4}{9}$$

$$P(\text{2 Black from Bag2}) = \frac{7}{10} \cdot \frac{6}{9} = \frac{42}{90} = \frac{14}{30}$$

$$P(\text{Case 1}) = \frac{4}{9} \cdot \frac{42}{90} = \frac{168}{810} = \frac{28}{135}$$

Case 2:

- Black from Bag 1
- 1 Red & 1 Black from Bag 2

$$P(\text{Black from Bag1}) = \frac{5}{9}$$

Now from Bag 2 we have **two ways**:

- Red first, Black second
- Black first, Red second

$$P(RB) = \frac{3}{10} \cdot \frac{7}{9} = \frac{21}{90}$$

$$P(BR) = \frac{7}{10} \cdot \frac{3}{9} = \frac{21}{90}$$

So,

$$P(\text{1 Red, 1 Black from Bag2}) = \frac{21}{90} + \frac{21}{90} = \frac{42}{90}$$

$$= \frac{14}{30}$$

$$P(\text{Case 2}) = \frac{5}{9} \cdot \frac{42}{90} = \frac{210}{810} = \frac{35}{135}$$

Total Probability

$$P(\text{Exactly 2B + 1R}) = \frac{28}{135} + \frac{35}{135} = \frac{63}{135} = \frac{7}{15}$$

✅ Final Answer

$$\boxed{P = \frac{7}{15}}$$

Example 4 : A, B and C in order to toss a coin. The first one to throw a head wins. What are their respective chances of winning, assuming that the game continues indefinitely ?

Solution : A, B, and C toss a coin **in order** and the **first one to get a Head wins**.
Assume the game continues until someone wins.

Each toss:

- P(H)=1/2P(H) = 1/2P(H)=1/2
- P(T)=1/2P(T) = 1/2P(T)=1/2

We need their **probability of winning the entire game**.

✓ Probability A wins

A can win:

- on 1st toss → $\frac{1}{2}$
- OR all three get tails first, then A gets head → $(TTT)(H)$
- OR tails in first 6 tosses... and so on.

So:

$$P(A) = \frac{1}{2} + \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^6 \cdot \frac{1}{2} + \cdots$$

This is a geometric series:

$$P(A) = \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots\right]$$

Common ratio $r = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$$P(A) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{1}{2} \cdot \frac{1}{\frac{7}{8}} = \frac{1}{2} \cdot \frac{8}{7} = \frac{4}{7}$$

✔️ Probability B wins

B can only win if:

- A fails (T), and
- B gets H

Similar repeated cycles:

$$P(B) = \frac{1}{2} \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^4 \cdot \frac{1}{2} + \cdots$$

$$P(B) = \frac{1}{4} \left[1 + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \cdots\right]$$

$$P(B) = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{1}{4} \cdot \frac{8}{7} = \frac{2}{7}$$

✔️ Probability C wins

C wins if:

- A gets T
- B gets T
- C gets H

$$P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \left[1 + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \cdots\right]$$

$$P(C) = \frac{1}{8} \cdot \frac{8}{7} = \frac{1}{7}$$

🔍 Verification:

$$P(A) + P(B) + P(C) = \frac{4}{7} + \frac{2}{7} + \frac{1}{7} = \frac{7}{7} = 1$$

✔️ Total = 1 → Correct

Assignment

Question 1 :

A box contains 6 red balls and 4 blue balls. Two balls are drawn with replacement.
Find the probability that:
(a) Both are red
(b) First is red and second is blue

Question 2 :

The probability that student A knows the correct answer to a question is 2/3 and the probability that student B knows the correct answer is 3/5.
If both students answer independently, find the probability that:
(a) Both answer correctly
(b) At least one answers correctly

Question 3 :

Two coins are tossed simultaneously three times. What is the probability that the head appears in all three tosses assuming independence of tosses?

Question 4 :

The probability that machine A works properly on a day is 0.85 and for machine B is 0.9.
Assuming independence, find:
(a) Probability that both work properly
(b) Probability that at least one stops working

Question 5 :

A die is rolled once. Event A: Getting an even number.
Event B: Getting a number greater than 3.
Find whether events A and B are independent.

Answer Sheet

Q1 (a)	(9/25)
Q1 (b)	(6/25)
Q2 (a)	(2/5)
Q2 (b)	(13/15)
Q3	(27/64)
Q4 (a)	(0.765 = 153/200)
Q4 (b)	(0.235 = 47/200)
Q5	**Not independent** (since (P(A)P(B)=1/4 \ne P(A\cap B)=1/3))