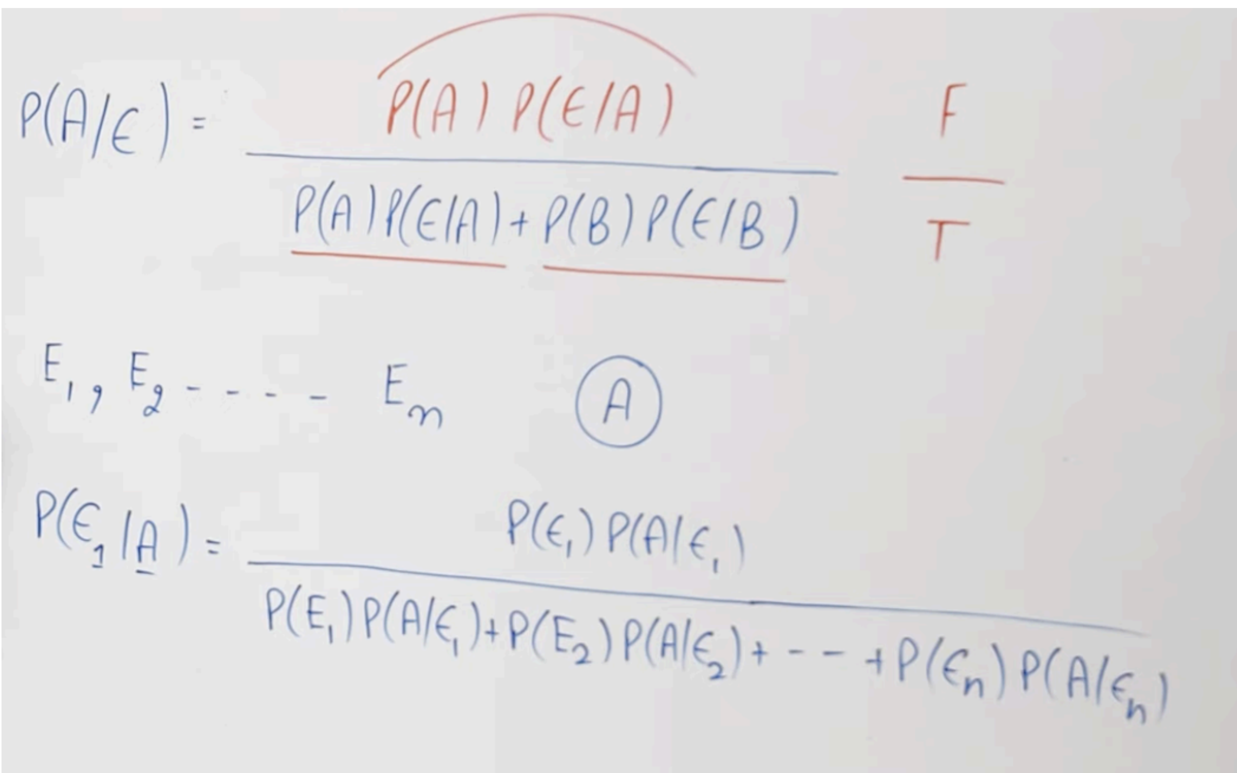
🎯 Bayes’ Theorem

Statement (Bayes' Theorem).
If E1, E2,..... En are n mutually exclusive and exhaustive events, Such that P(Ei) > 0 for each i and A is an arbitrary event for which P(A) ≠ 0, then for each 1≤ i ≤n, the conditional probability of occurrence of Ei, given that A has occurred is :

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{k=1}^{n} P(E_k)P(A|E_k)}, i = 1, 2, 3, \ldots, n$$



Bayes Theorem Formula for Events

The formula for events derived from the definition of conditional probability is:
P (A | B) = P (B | A) . P (A) / P (B) ,   P (B) ≠ 0

Example 1 - Three urns contain 2 red, 3 black; 3 red, 2 black and 4 red, 1 black balls respectively. One of the urns is selected at random and a ball is drawn. If the ball drawn is red, find the probability that it is drawn from the second urn.

Solution :

Let

- U_1: first urn (2R, 3B)
- U_2: second urn (3R, 2B)
- U_3: third urn (4R, 1B)

  Each urn is chosen **at random**, so

$$P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$$

Step 1: Probabilities of drawing a red ball from each urn

$$P(R|U_1) = \frac{2}{5}, \quad P(R|U_2) = \frac{3}{5}, \quad P(R|U_3) = \frac{4}{5}$$

Step 2: Total probability of drawing a red ball

$$P(R) = P(U_1)P(R|U_1) + P(U_2)P(R|U_2) + P(U_3)P(R|U_3)$$
$$= \frac{1}{3}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{3}{5} + \frac{1}{3}\cdot\frac{4}{5} = \frac{1}{3}\cdot\frac{9}{5} = \frac{3}{5}$$

Step 3: Using Bayes' Theorem

$$P(U_2|R) = \frac{P(U_2)\,P(R|U_2)}{P(R)} = \frac{\frac{1}{3}\cdot\frac{3}{5}}{\frac{3}{5}} = \frac{1}{3}$$

✅ **Final Answer:**

$$\boxed{\frac{1}{3}}$$

Example 2 - Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins; in box II, both are silver coins and in box III, there is one gold and one silver coin. A person chooses a box at random and takes a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Solution :

Let the three boxes be:

- **Box I (GG)**: 2 Gold coins
- **Box II (SS)**: 2 Silver coins
- **Box III (GS)**: 1 Gold, 1 Silver

Each box is chosen **at random**, so

$$P(I) = P(II) = P(III) = \frac{1}{3}$$

Step 1: Probability of drawing a gold coin from each box

$$P(G|I) = 1, \quad P(G|II) = 0, \quad P(G|III) = \frac{1}{2}$$

Step 2: Total probability of drawing a gold coin

Using Total Probability Theorem:

$$P(G) = P(I)P(G|I) + P(II)P(G|II) + P(III)P(G|III)$$

$$P(G) = \frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{3}\left(\frac{1}{2}\right)$$

$$P(G) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Step 3: Required probability (Bayes' Theorem)

We need:

$$P(\text{other coin is gold} \mid \text{drawn coin is gold}) = P(I|G)$$

$$P(I|G) = \frac{P(I)P(G|I)}{P(G)} = \frac{\frac{1}{3} \cdot 1}{\frac{1}{2}} = \frac{2}{3}$$

Example - 3 A doctor is to visit a patient. From past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively 3/10, 1/5, 1/10 and 2/5. The probabilities that he will be late are 1/4, 1/3, 1/12 if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Solution :

Let the modes of transport be:

- T: Train
- B: Bus
- S: Scooter
- O: Other means
- L: Event that the doctor is **late**

Given:

$$P(T) = \frac{3}{10}, \quad P(B) = \frac{1}{5}, \quad P(S) = \frac{1}{10}, \quad P(O) = \frac{2}{5}$$

Probabilities of being late:

$$P(L|T) = \frac{1}{4}, \quad P(L|B) = \frac{1}{3}, \quad P(L|S) = \frac{1}{12}, \quad P(L|O) = 0$$

Step 1: Find the total probability that the doctor is late

Using Total Probability Theorem:

$$P(L) = P(T)P(L|T) + P(B)P(L|B) + P(S)P(L|S) + P(O)P(L|O)$$

$$P(L) = \frac{3}{10} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{3} + \frac{1}{10} \cdot \frac{1}{12} + \frac{2}{5} \cdot 0$$

$$P(L) = \frac{3}{40} + \frac{1}{15} + \frac{1}{120}$$

Take LCM = 120:

$$P(L) = \frac{9}{120} + \frac{8}{120} + \frac{1}{120} = \frac{18}{120} = \frac{3}{20}$$

Step 2: Apply Bayes' Theorem

We need:

$$P(T|L) = \frac{P(T)P(L|T)}{P(L)}$$

$$P(T|L) = \frac{\frac{3}{10} \cdot \frac{1}{4}}{\frac{3}{20}} = \frac{\frac{3}{40}}{\frac{3}{20}} = \frac{3}{40} \cdot \frac{20}{3} = \frac{1}{2}$$

Example - 4 There are three coins. One is a two headed coin (having heads on both faces), another is a biased coin that tails 25% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin?

Solution :

Let the three coins be:

- C_1: **Two-headed coin**
- C_2: **Biased coin** (Tail = 25% ⇒ Head = 75% = $\frac{3}{4}$)
- C_3: **Unbiased coin** (Head = $\frac{1}{2}$)

Each coin is chosen **at random**, so:

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$$

Let H = event that **Head appears.**

Step 1: Probabilities of getting Head from each coin

$$P(H|C_1) = 1, \quad P(H|C_2) = \frac{3}{4}, \quad P(H|C_3) = \frac{1}{2}$$

Step 2: Total probability of getting Head

$$P(H) = P(C_1)P(H|C_1) + P(C_2)P(H|C_2) + P(C_3)P(H|C_3)$$

$$P(H) = \frac{1}{3}(1) + \frac{1}{3}\left(\frac{3}{4}\right) + \frac{1}{3}\left(\frac{1}{2}\right)$$

$$P(H) = \frac{1}{3} + \frac{1}{4} + \frac{1}{6}$$

Take LCM = 12:

$$P(H) = \frac{4}{12} + \frac{3}{12} + \frac{2}{12} = \frac{9}{12} = \frac{3}{4}$$

Step 3: Apply Bayes' Theorem

We need:

$$P(C_1|H) = \frac{P(C_1)P(H|C_1)}{P(H)}$$

$$P(C_1|H) = \frac{\frac{1}{3}\cdot 1}{\frac{3}{4}} = \frac{1}{3}\cdot\frac{4}{3} = \frac{4}{9}$$

Example 5 - An engineer visits sites by Car, Bus, or Cab with probabilities 3/8 , 1/4 , 3/8. Probabilities of being late are 1/5 , 1/3 , 1/6 respectively. If the engineer is late, what is the probability he used the bus?

Solution :

Let

- C: Car, B: Bus, T: Cab
- L: Event that the engineer is **late**

Given:

$$P(C) = \frac{3}{8}, \quad P(B) = \frac{1}{4}, \quad P(T) = \frac{3}{8}$$
$$P(L|C) = \frac{1}{5}, \quad P(L|B) = \frac{1}{3}, \quad P(L|T) = \frac{1}{6}$$

Step 1: Find the total probability that the engineer is late

Using Total Probability Theorem:

$$P(L) = P(C)P(L|C) + P(B)P(L|B) + P(T)P(L|T)$$
$$P(L) = \frac{3}{8} \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{1}{3} + \frac{3}{8} \cdot \frac{1}{6}$$
$$P(L) = \frac{3}{40} + \frac{1}{12} + \frac{3}{48}$$

LCM = 240:

$$P(L) = \frac{18}{240} + \frac{20}{240} + \frac{15}{240} = \frac{53}{240}$$

Step 2: Apply Bayes' Theorem

Required probability:

$$P(B|L) = \frac{P(B)P(L|B)}{P(L)}$$
$$P(B|L) = \frac{\frac{1}{4} \cdot \frac{1}{3}}{\frac{53}{240}} = \frac{\frac{1}{12}}{\frac{53}{240}} = \frac{1}{12} \cdot \frac{240}{53} = \frac{20}{53}$$

Assignment

Question 1 :
A man is known to speak the truth 2 out of 3 times. He throws a die and reports that the number obtained is a four. Find the probability that the number obtained is actually a four.

Question 2 :
A bag I contains 4 white and 6 black balls while another Bag II contains 4 white and 3 black balls. One ball is drawn at random from one of the bags, and it is found to be black. Find the probability that it was drawn from Bag I.

Question 3 :
Consider there are three machines. All the machines can produce 1000 pins at a time. The rate of producing a faulty pin from Machine 1 is 10%, from Machine 2 is 20%, and from Machine 3 is 5%. What is the probability that a produced pin will be faulty and it will be from the first machine?

Question 4 :
You have been given a dice and a pack of 52 cards. You have to throw a dice, and then you have to pick up a card. What is the probability that you picked up a red card and threw 6 on the dice?

Question 5 :
You have been given a dice. You tell a lie 4 out of 5 times. It is reported that an odd number appears on the dice. Find out the probability that an odd number actually appears.

Answer Sheet

Q1: 2/3
Q2: 7/12
Q3: 1/30
Q4: 1/12
Q5: 13/25