R, Q dynamics

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1 Preliminaries

We have episodes of length T where at each time step t we sample D-dimensional $x_t \sim N(0,1)$. We have teacher w^* and student w.

Definitions:

$$R = \frac{w^* w}{D} \tag{1}$$

$$Q = \frac{ww}{D} \tag{2}$$

$$y_t = wx_t \tag{3}$$

$$\hat{y}_t = w^* x_t \tag{4}$$

$$w^H = \frac{\frac{w^*}{\|w^*\|} + \frac{w}{\|w\|}}{\left\| \frac{w^*}{\|w^*\|} + \frac{w}{\|w\|} \right\|} \tag{5}$$

$$w_{inc} = \frac{\frac{w}{\|w\|} - \frac{w^*}{\|w^*\|}}{\left\| \frac{w}{\|w\|} - \frac{w^*}{\|w^*\|} \right\|} \tag{6}$$

$$p(correct) = (1 - \theta/\pi) = \left(1 - \arccos\left(\frac{w^* w}{\|w^*\|\|w\|}\right)/\pi\right) \tag{7}$$

$$\tag{8}$$

In the following, we find the Q and R dynamical equations by following the method prescribed in Engel's. The general update rule is always of the following form:

$$w^{\mu+1} = w^{\mu} + \frac{\eta}{\sqrt{D}} \left(\frac{1}{T} \sum_{t=1}^{T} x_t y_t \mathbb{1}(\text{some condition}) \right)^{\mu} \tag{9}$$

where $\mathbb{1}$ is the indicator function, and μ denotes the μth episode (instead of the μth example as used in Engel's). Multiplying both sides by the teacher, and iteratively summing over l episodes, we obtain:

$$\frac{D(R^{\mu+l} - R^{\mu})}{l} = \frac{\eta}{\sqrt{D}} \frac{1}{l} \sum_{i=0}^{l-1} \left(\frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \mathbb{1}(\text{some condition}) \right)^{\mu+i} \tag{10}$$

we take the thermodynamic limit $D \to \infty$, $l \to \infty$, and $l/D = d\alpha \to 0$, where α acts as a continuous time variable. This equates to taking the expectation, Eq 10 becomes:

$$\frac{dR}{d\alpha} = \frac{\eta}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \mathbb{1}(\text{some condition}) \right\rangle \tag{11}$$

Similarly, taking the square of Eq 9, and taking the thermodynamic limit, we equation for Q:

$$\frac{dQ}{d\alpha} = \frac{2\eta}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w x_t y_t \mathbb{1}(\text{some condition}) \right\rangle + \frac{\eta^2}{D} \left\langle \frac{1}{T^2} \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \mathbb{1}(\text{some condition}) \right\rangle \tag{12}$$

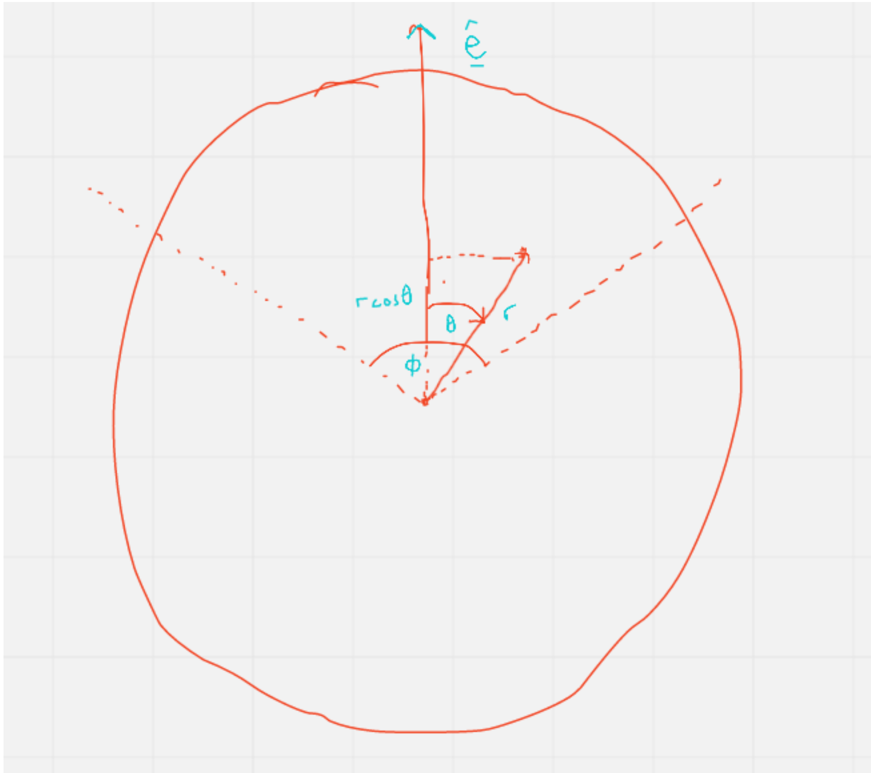

Figure 1: Caption

Some useful results:

The expected value of x given that it lies in a hypersector subtended by angle ϕ: it points in the direction $\hat{e}$, which denotes the unit vector pointing in the centre of the hypersector.

$$\langle x|\text{inside hypersector of } \phi\rangle = \frac{\int_0^\infty \int_{-\phi/2}^{\phi/2} rcos(\theta)\exp(-\frac{r^2}{2})rdrd\theta}{\int_0^\infty \int_{-\phi/2}^{\phi/2} \exp(-\frac{r^2}{2})rdrd\theta}\hat{e} \tag{13}$$

$$= \sqrt{\frac{\pi}{2}}\frac{\sin(\phi/2)}{\phi/2}\hat{e} \tag{14}$$

$$:= C_\phi \hat{e} \tag{15}$$

This allows us to calculate the following:

$$\langle xy|\text{correct}\rangle = C_{\pi-\theta} w^H \tag{16}$$

$$\langle xy|\text{incorrect}\rangle = C_\theta w_{inc} \tag{17}$$

$$\langle xy\rangle = C_\pi \frac{w}{\|w\|} = \sqrt{\frac{2}{\pi}}\frac{w}{\|w\|} \tag{18}$$

See figure 1 for a diagram. 4 further results we will use:

$$w^* w^H = \frac{\|w^*\|}{\sqrt{2}} \sqrt{1 + \frac{R}{\|w^*\|}\sqrt{\frac{D}{Q}}} \tag{19}$$

$$w w^H = \sqrt{\frac{DQ}{2}} \sqrt{1 + \frac{R}{\|w^*\|}\sqrt{\frac{D}{Q}}} \tag{20}$$

$$w^* w_{inc} = -\frac{\|w^*\|}{\sqrt{2}} \sqrt{1 - \frac{R}{\|w^*\|}\sqrt{\frac{D}{Q}}} \tag{21}$$

$$w w_{inc} = \sqrt{\frac{DQ}{2}} \sqrt{1 - \frac{R}{\|w^*\|}\sqrt{\frac{D}{Q}}} \tag{22}$$

We overload θ to represent the step function and the angle between teacher and student

2 Rule: Update if all correct

2.1 R equation

$$\begin{aligned}
\frac{dR}{d\alpha} &= \frac{\eta}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle & (23)\\
&= \frac{\eta}{\sqrt{D}} \langle w^* x_t y_t \theta(y_t \hat{y}_t) \rangle \left\langle \prod_{s \neq t}^{T} \theta(y_s \hat{y}_s) \right\rangle & (24)\\
&= \frac{\eta}{\sqrt{D}} w^* \langle x_t y_t | correct \rangle p(correct) p(correct)^{T-1} & (25)\\
&= \boxed{\frac{\eta}{\sqrt{D}} C_{\pi-\theta} w^* w^H p(correct)^T} & (26)
\end{aligned}$$

2.2 Q Equation

$$\frac{dQ}{d\alpha} = \frac{2\eta w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle + \frac{\eta^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle \tag{27}$$

We compute each term individually, first term:

$$\frac{2\eta w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle = \frac{2\eta}{\sqrt{D}} C_{\pi-\theta} w w^H p(correct)^T \tag{28}$$

second term:

$$\begin{aligned}
\frac{\eta^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle &= \frac{\eta^2}{T^2 D} \left\langle \left(\sum_{t=1}^{T} x_t x_t (y_t)^2 + 2 \sum_{t=2}^{T} \sum_{t'=1}^{t-1} x_t x_{t'} y_t y_{t'} \right) \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle & (29)\\
= \frac{\eta^2}{T^2 D} \left(T \langle x_t x_t | correct \rangle + T(T-1) \langle x_t y_t | correct \rangle \langle x_{t'} y_{t'} | correct \rangle \right) p(correct)^T & & (30)\\
= \frac{\eta^2}{T^2 D} (TD + T(T-1) C_{\pi-\theta}^2 w^H w^H) p(correct)^T & & (31)
\end{aligned}$$

combining gives:

$$\boxed{\frac{dQ}{d\alpha} = \eta \left(\frac{2}{\sqrt{D}} C_{\pi-\theta} w w^H + \frac{\eta}{T} + \frac{\eta}{TD} (T-1) C_{\pi-\theta}^2 \right) p(correct)^T} \tag{32}$$

3 Rule: Update if first n correct

3.1 R equation

$$\frac{dR}{d\alpha} = \frac{\eta}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \tag{33}$$

$$= \frac{\eta}{\sqrt{D}} \left\langle \frac{1}{T} \left(\sum_{t=1}^{n} w^* x_t y_t + \sum_{t=n+1}^{T} w^* x_t y_t \right) \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \tag{34}$$

$$= \frac{\eta}{T\sqrt{D}} \left(\left\langle \sum_{t=1}^{n} w^* x_t y_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle + \left\langle \sum_{t=n+1}^{T} w^* x_t y_t \right\rangle \left\langle \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \right) \tag{35}$$

$$= \frac{\eta}{T\sqrt{D}} \left(nC_{\pi-\theta} w^* w^H + (T-n) w^* \langle xy \rangle \right) p(correct)^n \tag{36}$$

$$= \boxed{\frac{\eta}{T\sqrt{D}} \left(nC_{\pi-\theta} w^* w^H + (T-n) C_\pi w^* \frac{w}{\|w\|} \right) p(correct)^n} \tag{37}$$

3.2 Q equation

$$\frac{dQ}{d\alpha} = \frac{2\eta w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle + \frac{\eta^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \tag{38}$$

observing the R-equation, we can write down the result of the first term:

$$\frac{2\eta w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle = \frac{2\eta}{T\sqrt{D}} \left(nC_{\pi-\theta} w w^H + (T-n) C_\pi \|w\| \right) p(correct)^n \tag{39}$$

The second term:

$$\begin{aligned}
&\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \\
&= \left\langle \sum_{t=1}^{n} x_t x_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle + \left\langle \sum_{t=n+1}^{T} x_t x_t \right\rangle \left\langle \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \\
&+ 2 \left\langle \sum_{t=2}^{n} \sum_{t'=1}^{t-1} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle + 2 \left\langle \sum_{t=n+2}^{T} \sum_{t'=n+1}^{t-1} x_t x_{t'} y_t y_{t'} \right\rangle \left\langle \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \\
&+ 2 \left\langle \sum_{t=1}^{n} x_t y_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle \left\langle \sum_{t'=n+1}^{T} x_{t'} y_{t'} \right\rangle \\
&= (n\langle xx|correct\rangle + (T-n)\langle xx\rangle + n(n-1)\langle xy|correct\rangle\langle xy|correct\rangle \\
&\quad + (T-n)(T-n-1)\langle xy\rangle\langle xy\rangle + 2n(T-n)\langle xy|correct\rangle\langle xy\rangle) p(correct)^n \\
&= \left(TD + n(n-1)C_{\pi-\theta}^2 + (T-n)(T-n-1)C_\pi^2 + 2n(T-n)C_\pi C_{\pi-\theta} \frac{ww^H}{\|w\|} \right) p(correct)^n
\end{aligned}$$

Combining terms 1 and 2:

$$\begin{aligned}
\frac{dQ}{d\alpha} &= \frac{2\eta}{T\sqrt{D}} \left(nC_{\pi-\theta} w w^H + (T-n) C_\pi \|w\| \right) p(correct)^n \\
&+ \frac{\eta^2}{T^2 D} \left(TD + n(n-1)C_{\pi-\theta}^2 + (T-n)(T-n-1)C_\pi^2 + 2n(T-n)C_\pi C_{\pi-\theta} \frac{ww^H}{\|w\|} \right) p(correct)^n
\end{aligned} \tag{40}$$

4 Rule: Update if any n in a row correct

This case is very tedious. When taking expectations we need to count the ways in which a row of n can occur. I couldn't find a neat way to count without double counting lots of situations, so I have resorted to calculating explicitly.

4.1 R equation

We take $T/2 < n \leq T-2$

$$\frac{dR}{d\alpha} = \frac{\eta}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \mathbb{1}(\text{any n in a row correct}) \right\rangle \tag{41}$$

We would like to count the ways in which episodes containing rows of size greater than n may occur, however, sets for different row lengths are not disjoint. We separate them by considering each row size individually, and requiring that a row of correct decisions must be preceded and followed by 1 incorrect decision (there are also the edge cases in which the correct row either begins or ends an episode, in these cases we just require 1 incorrect decision following or preceding the row respectively).

- Consider the case of any i in a row being correct, with $n \leq i \leq T-2$. There are $(T-i-1)$ ways of positioning the row in the episode where it can be preceded and followed by an incorrect decision, and there are 2 ways of placing the row where it either can be preceded or followed by an incorrect decision (the beginning and the end).
- For the case of $T-1$ in a row being correct, there are only 2 possible situations when this can occur (either the first or last decision in the episode is incorrect)
- For the case of T in a row being correct, this reduces to the original rule, and can only occur in 1 way

Putting these cases together, we can write:

$$\begin{aligned}
&\frac{dR}{d\alpha} \\
&= \frac{\eta w^*}{T\sqrt{D}} \sum_{i=n}^{T-2} \Bigg[(T-i-1) \left\langle \sum_{t=1}^{T} x_t y_t \left(\prod_{s=1}^{i} \theta(y_s \hat{y}_s) \right) \theta(-y_a \hat{y}_a) \theta(-y_b \hat{y}_b) \right\rangle \\
&+ 2 \left\langle \sum_{t=1}^{T} x_t y_t \left(\prod_{s=1}^{i} \theta(y_s \hat{y}_s) \right) \theta(-y_a \hat{y}_a) \right\rangle \Bigg] \\
&+ \frac{\eta w^*}{T\sqrt{D}} \left(2 \left\langle \sum_{t=1}^{T} x_t y_t \left(\prod_{s=1}^{T-1} \theta(y_s \hat{y}_s) \right) \theta(-y_a \hat{y}_a) \right\rangle \right. \\
&\left. + \left\langle \sum_{t=1}^{T} x_t y_t \left(\prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right) \right\rangle \right)
\end{aligned} \tag{42}$$

(where the subscripts a and b refer to the timesteps requiring an incorrect decision). By expanding the summation terms to $\sum_{t=1}^{T} x_t y_t = \sum_{t=1}^{i} x_t y_t + x_a y_a + x_b y_b + \sum_{t=i+3}^{T} y_t y_t$. From now on we also switch to the notation $p(correct) = (1-\theta/\pi)$ and $p(incorrect) = \theta/\pi$. The first term of Eq 42 can be written

$$\begin{aligned}
\text{1st} &= \frac{\eta w^*}{T\sqrt{D}} \sum_{i=n}^{T-2} [\left\{ (T-i-1) \left(i \langle xy|correct \rangle + 2\langle xy|incorrect \rangle + (T-i-2)\langle xy \rangle \right) \left(1-\frac{\theta}{\pi}\right)^i \left(\frac{\theta}{\pi}\right)^2 \right\} \\
&+ 2 \left\{ \left(i\langle xy|correct \rangle + \langle xy|incorrect \rangle + (T-i-1)\langle xy \rangle \right) \left(1-\frac{\theta}{\pi}\right)^i \left(\frac{\theta}{\pi}\right) \right\}] \\
&= \frac{\eta w^*}{T\sqrt{D}} \sum_{i=n}^{T-2} [(T-i-1) \left(iC_{\pi-\theta} w^H + 2C_\theta w_{inc} + (T-i-2)C_\pi \frac{w}{\|w\|} \right) \left(1-\frac{\theta}{\pi}\right)^i \left(\frac{\theta}{\pi}\right)^2 \\
&+ 2 \left(iC_{\pi-\theta} w^H + C_\theta w_{inc} + (T-i-1)C_\pi \frac{w}{\|w\|} \right) \left(1-\frac{\theta}{\pi}\right)^i \left(\frac{\theta}{\pi}\right)]
\end{aligned}$$

Overall, this gives us 3 terms:

$$= \frac{\eta}{T\sqrt{D}} \left[\sum_{i=n}^{T-2} i \left((T-i-1)\left(\frac{\theta}{\pi}\right) + 2 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) C_{\pi-\theta} w^* w^H$$
$$+ \frac{\eta}{T\sqrt{D}} \left[\sum_{i=n}^{T-2} \left((T-i-1)\left(\frac{\theta}{\pi}\right) + 1 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) 2C_{\theta} w^* w_{inc}$$
$$+ \frac{\eta}{T\sqrt{D}} \left[\sum_{i=n}^{T-2} (T-i-1) \left((T-i-2)\left(\frac{\theta}{\pi}\right) + 2 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) C_{\pi} w^* \frac{w}{\|w\|}$$

We now compute the second term of Eq 42

$$= \frac{\eta w^*}{T\sqrt{D}} \left(2 \left\langle \sum_{t=1}^{T} x_t y_t \left(\prod_{s=1}^{T-1} \theta(y_s \hat{y}_s) \right) \theta(-y_a \hat{y}_a) \right\rangle + \left\langle \sum_{t=1}^{T} x_t y_t \left(\prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right) \right\rangle \right)$$
$$= \frac{\eta w^*}{T\sqrt{D}} 2\{(T-1)\langle xy|correct\rangle + \langle xy|incorrect\rangle\} \left(1-\frac{\theta}{\pi}\right)^{T-1} \left(\frac{\theta}{\pi}\right)$$
$$+ \frac{\eta w^*}{T\sqrt{D}} T\langle xy|correct\rangle \left(1-\frac{\theta}{\pi}\right)^{T}$$
$$= \frac{\eta}{T\sqrt{D}} \left[\left(2(T-1)\left(\frac{\theta}{\pi}\right) + T\left(1-\frac{\theta}{\pi}\right) \right) \left(1-\frac{\theta}{\pi}\right)^{T-1} C_{\theta-\pi} w^* w^H + 2\left(1-\frac{\theta}{\pi}\right)^{T-1} \left(\frac{\theta}{\pi}\right) C_{\theta} w^* w_{inc} \right]$$
$$= \frac{\eta}{T\sqrt{D}} \left[\left((T-2)\left(\frac{\theta}{\pi}\right) + T \right) \left(1-\frac{\theta}{\pi}\right)^{T-1} C_{\pi-\theta} w^* w^H + 2\left(1-\frac{\theta}{\pi}\right)^{T-1} \left(\frac{\theta}{\pi}\right) C_{\theta} w^* w_{inc} \right]$$

Combining with the previous term, we obtain:

$$\frac{dR}{d\alpha} = \frac{\eta}{T\sqrt{D}} \left\{ \left[\sum_{i=n}^{T-2} i \left((T-i-1)\left(\frac{\theta}{\pi}\right) + 2 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) + \left((T-2)\left(\frac{\theta}{\pi}\right) + T \right) \left(1-\frac{\theta}{\pi}\right)^{T-1} \right\} C_{\pi-\theta} w^* w^H$$
$$+ \frac{\eta}{T\sqrt{D}} \left[\sum_{i=n}^{T-1} \left((T-i-1)\left(\frac{\theta}{\pi}\right) + 1 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) 2C_{\theta} w^* w_{inc}$$
$$+ \frac{\eta}{T\sqrt{D}} \left[\sum_{i=n}^{T-2} (T-i-1) \left((T-i-2)\left(\frac{\theta}{\pi}\right) + 2 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) C_{\pi} w^* \frac{w}{\|w\|}$$

4.2 Q equation

The Q equation is even more tedious!

$$\frac{dQ}{d\alpha} = \frac{2\eta}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w x_t y_t \mathbb{1}(\text{any n in row}) \right\rangle + \frac{\eta^2}{D} \left\langle \frac{1}{T^2} \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \mathbb{1}(\text{any n in row}) \right\rangle \tag{43}$$

We can write down the first term, having already done the calculation for R:

$$1\text{st} = \frac{2\eta}{T\sqrt{D}} \left\{ \left[\sum_{i=n}^{T-2} i \left((T-i-1)\left(\frac{\theta}{\pi}\right) + 2 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) + \left((T-2)\left(\frac{\theta}{\pi}\right) + T \right) \left(1-\frac{\theta}{\pi}\right)^{T-1} \right\} C_{\pi-\theta} w w^H$$
$$+ \frac{2\eta}{T\sqrt{D}} \left[\sum_{i=n}^{T-1} \left((T-i-1)\left(\frac{\theta}{\pi}\right) + 1 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) 2C_{\theta} w w_{inc}$$
$$+ \frac{2\eta}{T\sqrt{D}} \left[\sum_{i=n}^{T-2} (T-i-1) \left((T-i-2)\left(\frac{\theta}{\pi}\right) + 2 \right) \left(1-\frac{\theta}{\pi}\right)^i \right] \left(\frac{\theta}{\pi}\right) C_{\pi} \|w\|$$

For the second term, we separate into disjoint cases as we did in Eq 42

$$\frac{\eta^2}{T^2 D}\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \mathbb{1}(\text{any n in row})\right\rangle$$

$$\begin{aligned}
&= \frac{\eta^2}{T^2 D}\sum_{i=n}^{T-2}\Bigg[(T-i-1)\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'}\left(\prod_{s=1}^{i}\theta(y_s\hat{y}_s)\right)\theta(-y_a\hat{y}_a)\theta(-y_b\hat{y}_b)\right\rangle \\
&+ 2\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'}\left(\prod_{s=1}^{i}\theta(y_s\hat{y}_s)\right)\theta(-y_a\hat{y}_a)\right\rangle\Bigg] \qquad (44)\\
&+ \frac{\eta^2}{T^2 D}\Bigg(2\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'}\left(\prod_{s=1}^{T-1}\theta(y_s\hat{y}_s)\right)\theta(-y_a\hat{y}_a)\right\rangle \\
&+ \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'}\left(\prod_{s=1}^{T}\theta(y_s\hat{y}_s)\right)\right\rangle\Bigg)
\end{aligned}$$

While evaluating this expression, it would help to keep in mind the diagrams in figure 2. There are 4 different expectation terms above, we will consider them individually. For the first expectation, the diagram on the left of figure 2 is relevant.

1.

$$\begin{aligned}
&\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'}\left(\prod_{s=1}^{i}\theta(y_s\hat{y}_s)\right)\theta(-y_a\hat{y}_a)\theta(-y_b\hat{y}_b)\right\rangle \\
&= \langle [\sum_{t=1}^{i} x_t x_t + \sum_{t\in\{a,b\}} x_t x_t + \sum_{t=i+3}^{T} x_t x_t \\
&+ 2\sum_{t=2}^{i}\sum_{t'=1}^{t-1} x_t x_{t'} y_t y_{t'} + 2x_a y_a x_b y_b + 2\sum_{t=T-i-1}^{T}\sum_{t'=T-i-2}^{t-1} x_t x_{t'} y_t y_{t'} \\
&+ 2\sum_{t\in\{a,b\}}\sum_{t'=1}^{i} x_t x_{t'} y_t y_{t'} + 2\sum_{t=1}^{i}\sum_{t'=T-i-2}^{T} x_t x_{t'} y_t y_{t'} + 2\sum_{t\in\{a,b\}}\sum_{t'=T-i-2}^{T} x_t x_{t'} y_t y_{t'}] \\
&\times \left(\prod_{s=1}^{i}\theta(y_s\hat{y}_s)\right)\theta(-y_a\hat{y}_a)\theta(-y_b\hat{y}_b)\rangle \\
&= (i\langle xx|correct\rangle + 2\langle xx|incorrect\rangle + (T-i-2)\langle xx\rangle \\
&+ i(i-1)\langle xy|correct\rangle\langle xy|correct\rangle + 2\langle xy|incorrect\rangle\langle xy|incorrect\rangle + (T-i-2)(T-i-3)\langle xy\rangle\langle xy\rangle \\
&+ 2(2i)\langle xy|incorrect\rangle\langle xy|correct\rangle + 2(i(T-2-i))\langle xy|correct\rangle\langle xy\rangle + 2(2(T-i-2))\langle xy|incorrect\rangle\langle xy\rangle) \\
&\times \left(1-\frac{\theta}{\pi}\right)^i\left(\frac{\theta}{\pi}\right)^2 \\
&= (TD + i(i-1)C_{\pi-\theta}^2 + 2C_\theta^2 + (T-i-2)(T-i-3)C_\pi^2 + 4iC_{\pi-\theta}C_\theta \cancelto{0}{w^H w_{inc}} + 2i(T-2-i)C_{\pi-\theta}C_\pi\frac{ww^H}{\|w\|} \\
&+ 4(T-i-2)C_\theta C_\pi\frac{ww_{inc}}{\|w\|}) \times \left(1-\frac{\theta}{\pi}\right)^i\left(\frac{\theta}{\pi}\right)^2
\end{aligned}$$

For the second expectation the diagram on the right of figure 2 is relevant. We can easily use the result of expectation 1 to write down expectation 2.

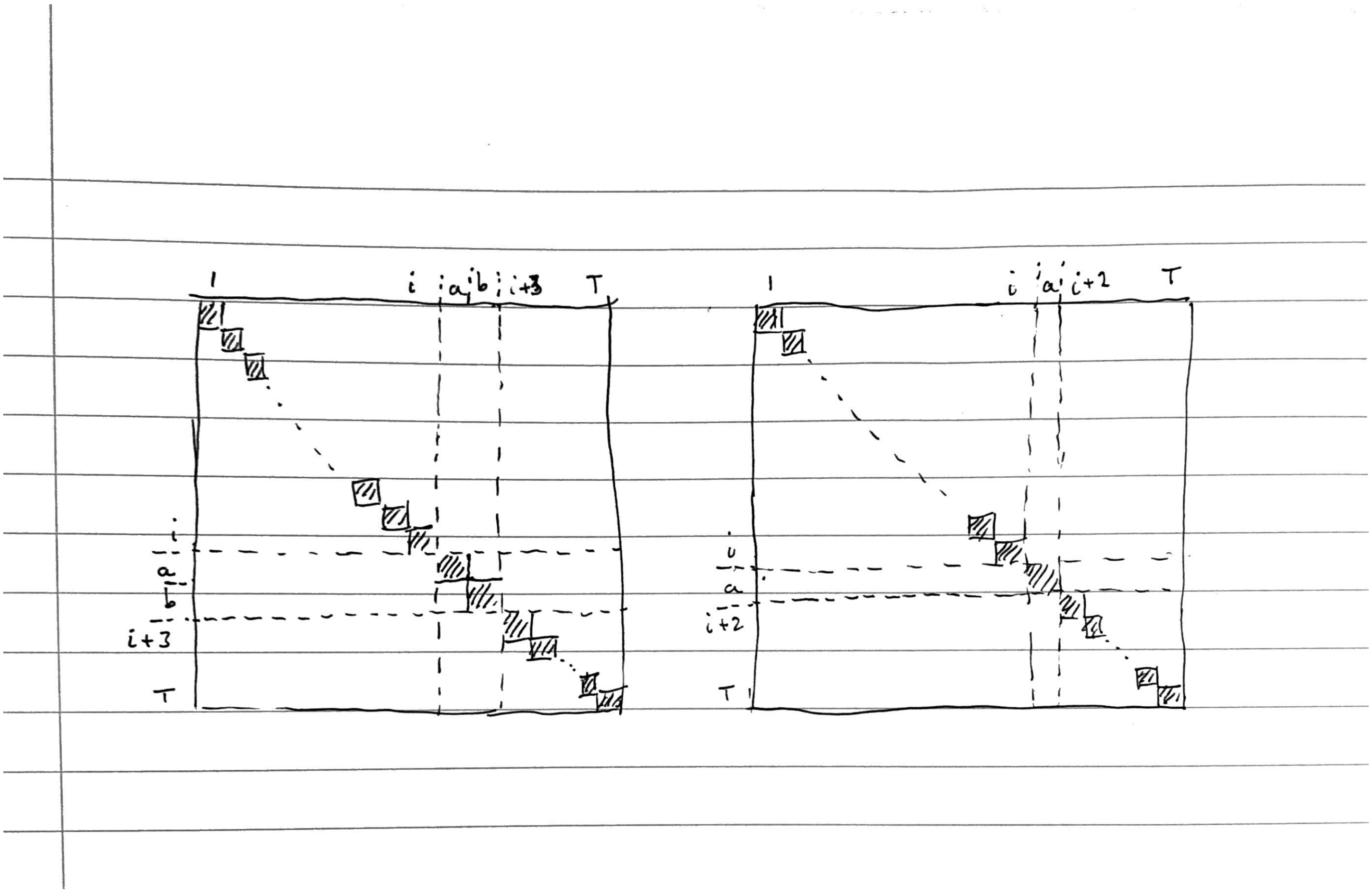

Figure 2: Caption

$$
\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \left(\prod_{s=1}^{i} \theta(y_s \hat{y}_s) \right) \theta(-y_a \hat{y}_a) \right\rangle
$$

$$
= [(TD + i(i-1)C_{\pi-\theta}^2 + (T-i-1)(T-i-2)C_\pi^2 + 2i(T-i-1)C_{\pi-\theta}C_\pi \frac{ww^H}{\|w\|} + 2(T-i-1)C_\theta C_\pi \frac{ww_{inc}}{\|w\|}]
$$

$$
\times \left(1 - \frac{\theta}{\pi}\right)^i \left(\frac{\theta}{\pi}\right)
$$

Expectation 3 is a case of expectation 2 with $i = T-1$. Expectation 4 is something we have calculated in the very first rule.

$$
(\mathbf{3}:) \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \left(\prod_{s=1}^{T-1} \theta(y_s \hat{y}_s) \right) \theta(-y_a \hat{y}_a) \right\rangle = (TD + (T-1)(T-2)C_{\pi-\theta}^2) \left(1 - \frac{\theta}{\pi}\right)^{T-1} \left(\frac{\theta}{\pi}\right)
$$

$$
(\mathbf{4}:) \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \left(\prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right) \right\rangle = (TD + T(T-1)C_{\pi-\theta}^2) \left(1 - \frac{\theta}{\pi}\right)^{T}
$$

Combining by collecting like terms, we have terms proportional to: $TD, C_{\pi-\theta}^2, C_\pi^2, C_\theta^2, C_{\pi-\theta}C_\pi, C_\theta C\pi$
TD term:

$$
= \frac{\eta^2}{T^2 D} \left(\left(1 - \frac{\theta}{\pi}\right)^{T-1} \left(1 + \frac{\theta}{\pi}\right) + \sum_{i=n}^{T-2} \left(1 - \frac{\theta}{\pi}\right)^i \left[(T-i-1)\left(\frac{\theta}{\pi}\right) + 2 \right] \left(\frac{\theta}{\pi}\right) \right) \tag{45}
$$

$C_{\pi-\theta}^2$ term:

$$= \frac{\eta^2}{T^2 D}\left((T-1)\left(1-\frac{\theta}{\pi}\right)^{T-1}\left(T+(T-4)\frac{\theta}{\pi}\right)+\sum_{i=n}^{T-2}\left(1-\frac{\theta}{\pi}\right)^i i(i-1)\left[(T-i-1)\left(\frac{\theta}{\pi}\right)+2\right]\left(\frac{\theta}{\pi}\right)\right) \tag{46}$$

C_θ^2 term:

$$= \frac{\eta^2}{T^2 D}\left(2\sum_{i=n}^{T-2}\left[(T-i-1)\left(1-\frac{\theta}{\pi}\right)^i\right]\left(\frac{\theta}{\pi}\right)^2\right) \tag{47}$$

C_π^2 term:

$$= \frac{\eta^2}{T^2 D}\left(\sum_{i=n}^{T-2}\left[(T-i-1)(T-i-2)\left(1-\frac{\theta}{\pi}\right)^i (2+(T-i-3)\frac{\theta}{\pi})\right]\left(\frac{\theta}{\pi}\right)\right) \tag{48}$$

$C_{\pi-\theta}C_\pi$ term:

$$= \frac{\eta^2}{T^2 D}\left(2\sum_{i=n}^{T-2}\left[i(T-i-1)\left(1-\frac{\theta}{\pi}\right)^i (2+(T-i-2)\frac{\theta}{\pi})\right]\left(\frac{\theta}{\pi}\right)\right)\frac{ww^H}{\|w\|} \tag{49}$$

$C_\theta C_\pi$ term:

$$= \frac{\eta^2}{T^2 D}\left(4\sum_{i=n}^{T-2}\left[(T-i-1)\left(1-\frac{\theta}{\pi}\right)^i (1+(T-i-2)\frac{\theta}{\pi})\right]\left(\frac{\theta}{\pi}\right)\right)\frac{ww_{inc}}{\|w\|} \tag{50}$$

5 Rule: Update if n or more correct

5.1 R equation

$$\begin{aligned}
\frac{dR}{d\alpha} &= \frac{\eta}{T\sqrt{D}}\left\langle\sum_{t=1}^{T} w^* x_t y_t \mathbb{1}(n \text{ or more correct})\right\rangle & (51)\\
&= \frac{\eta}{T\sqrt{D}}\sum_{i=n}^{T}\binom{T}{i}\left\langle\sum_{t=1}^{T} w^* x_t y_t \prod_{s=1}^{i}\theta(y_s\hat{y}_s)\prod_{s'=i+1}^{T}\theta(-y_{s'}\hat{y}_{s'})\right\rangle & (52)\\
&= \frac{\eta w^*}{T\sqrt{D}}\sum_{i=n}^{T}\binom{T}{i}\left[\left\langle\sum_{t=1}^{i} x_t y_t \prod_{s=1}^{i}\theta(y_s\hat{y}_s)\right\rangle\left(\frac{\theta}{\pi}\right)^{T-i}+\left\langle\sum_{t=i+1}^{T} x_t y_t \prod_{s=i+1}^{T}\theta(-y_s\hat{y}_s)\right\rangle\left(1-\frac{\theta}{\pi}\right)^{i}\right] & (53)\\
&= \boxed{\frac{\eta}{T\sqrt{D}}\sum_{i=n}^{T}\binom{T}{i}\left[iC_{\pi-\theta}w^*w^H+(T-i)C_\theta w^* w_{inc}\right]\left(1-\frac{\theta}{\pi}\right)^i\left(\frac{\theta}{\pi}\right)^{T-i}} & (54)
\end{aligned}$$

5.2 Q equation

$$\frac{dQ}{d\alpha} = \frac{2\eta}{\sqrt{D}}\left\langle\frac{1}{T}\sum_{t=1}^{T} w x_t y_t \mathbb{1}(\text{n or more correct})\right\rangle+\frac{\eta^2}{D}\left\langle\frac{1}{T^2}\sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'}\mathbb{1}(\text{n or more correct})\right\rangle \tag{55}$$

first term:

$$= \frac{2\eta}{T\sqrt{D}}\sum_{i=n}^{T}\binom{T}{i}\left[iC_{\pi-\theta}ww^H+(T-i)C_\theta w w_{inc}\right]\left(1-\frac{\theta}{\pi}\right)^i\left(\frac{\theta}{\pi}\right)^{T-i} \tag{56}$$

second term:

$$= \frac{\eta^2}{T^2 D} \sum_{i=n}^{T} \binom{T}{i} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{i} \theta(y_s \hat{y}_s) \prod_{s'=i+1}^{T} \theta(-y_{s'} \hat{y}_{s'}) \right\rangle \tag{57}$$

$$= \frac{\eta^2}{T^2 D} \sum_{i=n}^{T} \binom{T}{i} \left\langle \left(\sum_{t=1}^{T} x_t x_t + \sum_{t \neq t'}^{\text{block diag}} x_t x_{t'} y_t y_{t'} + 2 \sum_{t=2}^{T} \sum_{t'=1}^{t-1} x_t x_{t'} y_t y_{t'} \right) \prod_{s=1}^{i} \theta(y_s \hat{y}_s) \prod_{s'=i+1}^{T} \theta(-y_{s'} \hat{y}_{s'}) \right\rangle \tag{58}$$

$$= \frac{\eta^2}{T^2 D} \sum_{i=n}^{T} \binom{T}{i} \left(TD + i(i-1) C_{\pi-\theta}^2 + (T-i)(T-i-1) C_\theta^2 + 2 C_{\pi-\theta} \cancelto{0}{C_\theta w^H w_{inc}} \right) \left(1 - \frac{\theta}{\pi}\right)^i \left(\frac{\theta}{\pi}\right)^{T-i} \tag{59}$$

$$= \frac{\eta^2}{T^2 D} \sum_{i=n}^{T} \binom{T}{i} \left(TD + i(i-1) C_{\pi-\theta}^2 + (T-i)(T-i-1) C_\theta^2 \right) \left(1 - \frac{\theta}{\pi}\right)^i \left(\frac{\theta}{\pi}\right)^{T-i} \tag{60}$$

6 Positive and negative updates

We now consider a positive reward if some condition is fulfilled, and a negative reward otherwise (with a different learning rate in each case). The form of the update on the student is now:

$$w^{\mu+1} = w^{\mu} + \frac{\eta_1}{\sqrt{D}} \left(\frac{1}{T} \sum_{t=1}^{T} x_t y_t \mathbb{1}(\text{some condition}) \right) - \frac{\eta_2}{\sqrt{D}} \left(\frac{1}{T} \sum_{t=1}^{T} x_t y_t \left(1 - \mathbb{1}(\text{some condition})\right) \right) \tag{61}$$

The R update is now:

$$\frac{dR}{d\alpha} = \frac{\eta_1 + \eta_2}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \mathbb{1}(\text{some condition}) \right\rangle - \frac{\eta_2}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \right\rangle \tag{62}$$

The Q update is now:

$$\begin{aligned} \frac{dQ}{d\alpha} &= \frac{2(\eta_1 + \eta_2) w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \mathbb{1}(\text{condition}) \right\rangle - \frac{2\eta_2 w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \right\rangle \\ &\quad + \frac{\eta_1^2 - \eta_2^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \mathbb{1}(\text{condition}) \right\rangle + \frac{\eta_2^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \right\rangle \end{aligned}$$

7 Rule: Positive update if all correct, negative update otherwise

7.1 R equation

$$\frac{dR}{d\alpha} = \frac{\eta_1}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle - \frac{\eta_2}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \left(1 - \prod_{s=1}^{T} \theta(y_s \hat{y}_s)\right) \right\rangle \tag{63}$$

$$= \frac{\eta_1 + \eta_2}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} w^* x_t y_t \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle - \frac{\eta_2}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \right\rangle \tag{64}$$

$$= \boxed{\frac{\eta_1 + \eta_2}{\sqrt{D}} C_{\pi-\theta} w^* w^H p(correct)^T - \frac{\eta_2}{\sqrt{D}} C_\pi \frac{w^* w}{\|w\|}} \tag{65}$$

7.2 Q equation

$$\frac{dQ}{d\alpha} = \frac{2}{T\sqrt{D}} \left\langle \left((\eta_1 + \eta_2) \prod_{s=1}^{T} \theta(y_s \hat{y}_s) - \eta_2 \right) \left(\sum_{t=1}^{T} w x_t y_t \right) \right\rangle + \frac{1}{T^2 D} \left\langle \left((\eta_1 + \eta_2) \prod_{s=1}^{T} \theta(y_s \hat{y}_s) - \eta_2 \right)^2 \left(\sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \right) \right\rangle \tag{66}$$

$$\tag{67}$$

First term:

$$= \frac{2(\eta_1+\eta_2)}{\sqrt{D}} C_{\pi-\theta} w w^H p(correct)^T - \frac{2\eta_2}{\sqrt{D}} C_\pi \|w\|$$

Second Term:

$$= \frac{1}{T^2 D} \left\langle \left(\eta_2^2 + (\eta_1^2 - \eta_2^2) \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right) \left(\sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \right) \right\rangle \tag{68}$$

$$= \frac{\eta_1^2 - \eta_2^2}{T^2 D} (TD + T(T-1) C_{\pi-\theta}^2 w^H w^H) p(correct)^T + \frac{\eta_2^2}{T^2 D} \left\langle \left(\sum_{t=1}^{T} x_t x_t + 2 \sum_{t=2}^{T} \sum_{t'=1}^{t-1} x_t x_{t'} y_t y_{t'} \right) \right\rangle \tag{69}$$

$$= \frac{\eta_1^2 - \eta_2^2}{TD} (D + (T-1) C_{\pi-\theta}^2) p(correct)^T + \frac{\eta_2^2}{TD} \left(D + (T-1) C_\pi^2 \right) \tag{70}$$

combining:

$$\boxed{\frac{dQ}{d\alpha} = \frac{\eta_2^2}{TD} \left(D + (T-1) C_\pi^2 \right) - \frac{2\eta_2}{\sqrt{D}} C_\pi \|w\| + \left(\frac{\eta_1^2 - \eta_2^2}{TD} (D + (T-1) C_{\pi-\theta}^2) + \frac{2(\eta_1+\eta_2) C_{\pi-\theta}}{\sqrt{D}} w w^H \right) p(correct)^T}$$

8 Rule: Positive update if first n correct, negative update otherwise

8.1 R equation

$$\frac{dR}{d\alpha} = \frac{\eta_1+\eta_2}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle - \frac{\eta_2}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} w^* x_t y_t \right\rangle \tag{71}$$

$$= \boxed{\frac{\eta_1+\eta_2}{T\sqrt{D}} \left(n C_{\pi-\theta} w^* w^H + (T-n) C_\pi w^* \frac{w}{\|w\|} \right) p(correct)^n - \frac{\eta_2}{\sqrt{D}} C_\pi w^* \frac{w}{\|w\|}} \tag{72}$$

where we used the result from section 3.1.

8.2 Q Equation

$$\frac{dQ}{d\alpha} = \frac{2(\eta_1+\eta_2) w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle - \frac{2\eta_2 w}{T\sqrt{D}} \left\langle \sum_{t=1}^{T} x_t y_t \right\rangle \tag{73}$$

$$+ \frac{\eta_1^2 - \eta_2^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{n} \theta(y_s \hat{y}_s) \right\rangle + \frac{\eta_2^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \right\rangle \tag{74}$$

first and second term:

$$= \frac{2(\eta_1+\eta_2)}{T\sqrt{D}} \left(n C_{\pi-\theta} w w^H + (T-n) C_\pi \|w\| \right) p(correct)^n - \frac{2\eta_2}{\sqrt{D}} C_\pi \|w\| \tag{75}$$

fourth term:

$$\frac{\eta_2^2}{TD} (D + (T-1) C_\pi^2) \tag{76}$$

third term:

$$\frac{\eta_1^2 - \eta_2^2}{T^2 D} \left(TD + n(n-1) C_{\pi-\theta}^2 + (T-n)(T-n-1) C_\pi^2 + 2n(T-n) C_\pi C_{\pi-\theta} \frac{w w^H}{\|w\|} \right) p(correct)^n \tag{77}$$

9 Rule: positive update if n or more correct, neg otherwise

9.1 R equation

$$\frac{dR}{d\alpha} = \frac{\eta_1+\eta_2}{\sqrt{D}}\left\langle \frac{1}{T}\sum_{t=1}^{T} w^* x_t y_t \mathbb{1}(\text{n or more correct})\right\rangle - \frac{\eta_2}{\sqrt{D}}\left\langle \frac{1}{T}\sum_{t=1}^{T} w^* x_t y_t\right\rangle \tag{78}$$

$$= \frac{\eta_1+\eta_2}{T\sqrt{D}}\sum_{i=n}^{T}\binom{T}{i}\left[iC_{\pi-\theta}w^* w^H + (T-i)C_\theta w^* w_{inc}\right]\left(1-\frac{\theta}{\pi}\right)^i\left(\frac{\theta}{\pi}\right)^{T-i} - \frac{\eta_2}{\sqrt{D}}C_\pi w^* \frac{w}{\|w\|} \tag{79}$$

9.2 Q equation

$$\frac{dQ}{d\alpha} = \frac{2(\eta_1+\eta_2)w}{T\sqrt{D}}\left\langle \sum_{t=1}^{T} x_t y_t \mathbb{1}(\text{n or more correct})\right\rangle - \frac{2\eta_2 w}{T\sqrt{D}}\left\langle \sum_{t=1}^{T} x_t y_t\right\rangle$$
$$+ \frac{\eta_1^2-\eta_2^2}{T^2 D}\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \mathbb{1}(\text{n or more correct})\right\rangle + \frac{\eta_2^2}{T^2 D}\left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'}\right\rangle$$

first and second terms:

$$= \frac{2(\eta_1+\eta_2)}{T\sqrt{D}}\sum_{i=n}^{T}\binom{T}{i}\left[iC_{\pi-\theta}ww^H + (T-i)C_\theta w w_{inc}\right]\left(1-\frac{\theta}{\pi}\right)^i\left(\frac{\theta}{\pi}\right)^{T-i} - \frac{2\eta_2}{\sqrt{D}}C_\pi\|w\| \tag{80}$$

3rd term:

$$= \frac{\eta_1^2-\eta_2^2}{T^2 D}\sum_{i=n}^{T}\binom{T}{i}\left(TD + i(i-1)C_{\pi-\theta}^2 + (T-i)(T-i-1)C_\theta^2\right)\left(1-\frac{\theta}{\pi}\right)^i\left(\frac{\theta}{\pi}\right)^{T-i} \tag{81}$$

4th term:

$$= \frac{\eta_2^2}{TD}(D + (T-1)C_\pi^2) \tag{82}$$

10 Bread crumb trails

We now consider the case where we receive a reward for completing a task (e.g make a correct decision at every timestep), and a different reward for every individual correct decision made. The update on the student is now:

$$w^{\mu+1} = w^\mu + \frac{\eta_1}{\sqrt{D}}\left(\frac{1}{T}\sum_{t=1}^{T} x_t y_t \prod_{s=1}^{T}\theta(y_s\hat{y}_s)\right) + \frac{\eta_2}{\sqrt{D}}\left(\frac{1}{T}\sum_{t=1}^{T} x_t y_t\left(\sum_{s=1}^{T}\theta(y_s\hat{y}_s)\right)\right) \tag{83}$$

10.1 R equation

$$\frac{dR}{d\alpha} = \frac{\eta_1}{\sqrt{D}}\left\langle \frac{1}{T}\sum_{t=1}^{T} w^* x_t y_t \prod_{s=1}^{T}\theta(y_s\hat{y}_s)\right\rangle + \frac{\eta_2}{\sqrt{D}}\left\langle \frac{1}{T}\sum_{t=1}^{T} w^* x_t y_t\left(\sum_{s=1}^{T}\theta(y_s\hat{y}_s)\right)\right\rangle \tag{84}$$

first term:

$$= \frac{\eta_1}{\sqrt{D}}C_{\pi-\theta}w^* w^H p(correct)^T \tag{85}$$

second term:

$$= \frac{\eta_2}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t,t'=1}^{T} w^* x_t y_t \theta(y_{t'} \hat{y}_{t'}) \right\rangle \tag{86}$$

$$= \frac{\eta_2}{T\sqrt{D}} \left(\left\langle \sum_{t=1}^{T} w^* x_t y_t \theta(y_t \hat{y}_t) \right\rangle + T(T-1) \langle xy \rangle \langle \theta(y\hat{y}) \rangle \right) \tag{87}$$

$$= \frac{\eta_2}{\sqrt{D}} \left(C_{\pi-\theta} w^* w^H + (T-1) C_\pi \frac{w^* w}{\|w\|} \right) p(correct) \tag{88}$$

$$\frac{dR}{d\alpha} = \frac{\eta_1}{\sqrt{D}} C_{\pi-\theta} w^* w^H p(correct)^T + \frac{\eta_2}{\sqrt{D}} \left(C_{\pi-\theta} w^* w^H + (T-1) C_\pi \frac{w^* w}{\|w\|} \right) p(correct) \tag{89}$$

10.2 Q equation

$$\begin{aligned}
\frac{dQ}{d\alpha} &= \frac{2\eta_1 w}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} x_t y_t \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle + \frac{2\eta_2 w}{\sqrt{D}} \left\langle \frac{1}{T} \sum_{t=1}^{T} x_t y_t \left(\sum_{s=1}^{T} \theta(y_s \hat{y}_s) \right) \right\rangle \\
&+ \frac{\eta_1^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle + \frac{\eta_2^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \left(\sum_{s,s'=1}^{T} \theta(y_s \hat{y}_s) \theta(y_{s'} \hat{y}_{s'}) \right) \right\rangle \\
&+ \frac{2\eta_1 \eta_2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \left(\sum_{s=1}^{T} \theta(y_s \hat{y}_s) \prod_{s'=1}^{T} \theta(y_{s'} \hat{y}_{s'}) \right) \right\rangle
\end{aligned}$$

first and second terms

$$= \frac{2\eta_1}{\sqrt{D}} C_{\pi-\theta} w w^H p(correct)^T + \frac{2\eta_2}{\sqrt{D}} \left(C_{\pi-\theta} w w^H + (T-1) C_\pi \|w\| \right) p(correct) \tag{90}$$

3rd term:

$$\frac{\eta_1^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \prod_{s=1}^{T} \theta(y_s \hat{y}_s) \right\rangle = \frac{\eta_1^2}{TD} (D + (T-1) C_{\pi-\theta}^2 w^H w^H) p(correct)^T \tag{91}$$

4th term:

$$= \frac{\eta_2^2}{T^2 D} \left\langle \sum_{t,t'=1}^{T} x_t x_{t'} y_t y_{t'} \left(\sum_{s,s'=1}^{T} \theta(y_s \hat{y}_s) \theta(y_{s'} \hat{y}_{s'}) \right) \right\rangle \tag{92}$$

$$= \frac{\eta_2^2}{T^2 D} \left\langle \sum_{t=1}^{T} x_t x_t \left(\sum_{s=1}^{T} \theta(y_s \hat{y}_s) + \sum_{s \neq s'}^{T} \theta(y_s \hat{y}_s) \theta(y_{s'} \hat{y}_{s'}) \right) + \sum_{t \neq t'}^{T} x_t x_{t'} y_t y_{t'} \left(\sum_{s=1}^{T} \theta(y_s \hat{y}_s) + \sum_{s \neq s'}^{T} \theta(y_s \hat{y}_s) \theta(y_{s'} \hat{y}_{s'}) \right) \right\rangle \tag{93}$$

$$= \frac{\eta_2^2}{T^2 D} \left[\left(1 + (T-1) p(correct)) T^2 D p(correct) \right) + \left\langle \sum_{t \neq t'}^{T} x_t x_{t'} y_t y_{t'} \left(\sum_{s=1}^{T} \theta(y_s \hat{y}_s) + \sum_{s \neq s'}^{T} \theta(y_s \hat{y}_s) \theta(y_{s'} \hat{y}_{s'}) \right) \right\rangle \right] \tag{94}$$

$$= \frac{\eta_2^2}{T^2 D} \left[\text{—”—} + (T(T-1) \left[2 \langle xy\theta \rangle \langle xy \rangle + \langle xy \rangle \langle xy \rangle (T-2) \left\langle \theta(y\hat{y}) \right\rangle \right]) + \left\langle \sum_{t \neq t'}^{T} x_t x_{t'} y_t y_{t'} \left(\sum_{s \neq s'}^{T} \theta(y_s \hat{y}_s) \theta(y_{s'} \hat{y}_{s'}) \right) \right\rangle \right] \tag{95}$$

$$= \frac{\eta_2^2}{T^2 D} \left[\text{—”—} + T(T-1) \left[2 C_{\pi-\theta} C_\pi \frac{w w^H}{\|w\|} + (T-2) C_\pi^2 p(correct) \right] + \left\langle \sum_{t \neq t'}^{T} x_t x_{t'} y_t y_{t'} \left(\sum_{s \neq s'}^{T} \theta(y_s \hat{y}_s) \theta(y_{s'} \hat{y}_{s'}) \right) \right\rangle \right] \tag{96}$$

$$\frac{dQ}{d\alpha} = \frac{2\eta_1}{\sqrt{D}} C_{\pi-\theta} w w^H p(correct)^T + \frac{2\eta_2}{\sqrt{D}} \left(C_{\pi-\theta} w^* w^H + (T-1) C_\pi \|w\| \right) p(correct) \tag{97}$$

$$+ \frac{\eta_1^2}{TD} (D + (T-1) C_{\pi-\theta}^2 w^H w^H) p(correct)^T \tag{98}$$

$$+ \frac{\eta_2^2}{T^2 D} \left((1 + (T-1) p(correct)) T^2 D + T(T-1) \left[2 C_{\pi-\theta} C_\pi \frac{w w^H}{\|w\|} + (T-2) C_\pi^2 \right] \right) p(correct) \tag{99}$$

$$+ \frac{2\eta_1^2 \eta_2^2}{D} (D + (T-1) C_{\pi-\theta}^2 w^H w^H) p(correct)^T \tag{100}$$