Computation of Harmonics

Let the Fourier series for the function $y = f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx$; $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos nx \cdot dx$; $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin nx \cdot dx$ (2)

Let the function is given by a graph or a table; then the mean value of the function $y = f(x)$ over the range (a, b) is $\frac{1}{b-a}\int_a^b f(x)\cdot dx$.

$\therefore$ The equation (2) gives $a_0 = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot dx = 2$[mean value of $f(x)$ in $(0, 2\pi)$];

$$a_n = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot \cos nx \cdot dx = 2[\text{mean value of } f(x)\cdot\cos nx \text{ in } (0, 2\pi)]; \qquad (3)$$

$$b_n = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot \sin nx \cdot dx = 2[\text{mean value of } f(x)\cdot\sin nx \text{ in } (0, 2\pi)];$$

In (1), the term $(a_1 \cos x + b_1 \sin x)$ is called the fundamental or **first harmonic**, the term $(a_2 \cos 2x + b_2 \sin 2x)$ is called the **second harmonic**, and so on.

Example 1. The displacement y of a part of a mechanism is tabulated with corresponding angular movement x° of the crank. Express y as a Fourier series neglecting the harmonic above the third:

x°	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	0.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

Solution: Let the Fourier series up to the third harmonic representing y in $(0, 2\pi)$ be

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + ... \qquad (1)$$

To evaluate the coefficients, we form the following table.

x°	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$\cos 3x$	$\sin 3x$	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$	$y\cos 3x$	$y\sin 3x$
0	1.00	0.00	1.00	0.00	1	0	1.80	1.80	0.00	1.80	0.00	1.80	0.00
30	0.87	0.50	0.50	0.87	0	1	1.10	0.96	0.55	0.55	0.96	0.00	1.10
60	0.50	0.87	-0.50	0.87	-1	0	0.30	0.15	0.26	-0.15	0.26	-0.30	0.00
90	0.00	1.00	-1.00	0.00	0	-1	0.16	0.00	0.16	-0.16	0.00	0.00	-0.16
120	-0.50	0.87	-0.50	-0.87	1	0	0.50	-0.25	0.43	-0.25	-0.43	0.50	0.00
150	-0.87	0.50	-0.50	-0.87	0	1	1.30	-1.13	0.65	0.65	-1.13	0.00	1.30
180	-1.00	0.00	1.00	0.00	-1	0	2.16	-2.16	0.00	2.16	-0.00	-2.16	0.00
210	-0.87	-0.50	0.50	0.87	0	-1	1.25	-1.09	-0.63	0.63	1.09	0.00	-1.25
240	-0.50	-0.87	-0.50	0.87	1	0	1.30	-0.65	-1.13	-0.65	1.13	1.30	0.00
270	0.00	-1.00	-1.00	0.00	0	1	1.52	0.00	-1.52	-1.52	0.00	0.00	1.52
300	0.50	-0.87	-0.50	-0.87	-1	0	1.76	0.88	-1.53	-0.88	-1.53	-1.76	0.00
330	0.87	-0.50	0.50	-0.87	0	-1	2.00	1.74	-1.00	1.00	-1.74	0.00	-2.00
						$\sum =$	15.15	-3.76	0.25	-1.39	3.18	0.51	-0.62

$$a_0 = 2\times\frac{\sum y}{12} = \frac{15.15}{6} = 2.53; \qquad a_1 = 2\times\frac{\sum y\cos x}{12} = \frac{0.25}{6} = 0.04;$$

$$a_2 = 2\times\frac{\sum y\cos 2x}{12} = \frac{3.18}{6} = 0.53; \qquad a_3 = 2\times\frac{\sum y\cos 3x}{12} = \frac{-0.62}{6} = -0.10;$$

$$b_1 = 2\times\frac{\sum y\sin x}{12} = \frac{-3.76}{6} = -0.63; \qquad b_2 = 2\times\frac{\sum y\sin 2x}{12} = \frac{-1.39}{6} = -0.23;$$

$$b_3 = 2\times\frac{\sum y\sin 3x}{12} = \frac{0.51}{6} = 0.085.$$

Substituting the values of $a_0,\, a_1,\, a_2,\, a_3,\, b_1,\, b_2$ & b_3 (1), we get

$$y = 1.26 + 0.04\cos x + 0.53\cos 2x - 0.10\cos 3x - 0.63\sin x - 0.23\sin 2x + 0.85\sin 3x + ...$$

Example 2. The following table gives the variations of periodic current over a period.

t Sec	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A Amp.	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a direct current part of 0.75 Amp. In the variable current and obtain the amplitude of the first harmonic.

Solution: The length of the interval is T. $\therefore C = \frac{T}{2}$

Then the Fourier Series is

$$A = \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T}\right) + a_2 \cos\left(\frac{4\pi t}{T}\right) + a_3 \cos\left(\frac{6\pi t}{T}\right) + b_1 \sin\left(\frac{2\pi t}{T}\right) + b_2 \sin\left(\frac{4\pi t}{T}\right) + b_3 \sin\left(\frac{6\pi t}{T}\right) + ... \tag{1}$$

To evaluate the coefficients, we form the following table.

t	$\frac{2\pi t}{T}$	$\cos\left(\frac{2\pi t}{T}\right)$	$\sin\left(\frac{2\pi t}{T}\right)$	A	$A\cos\left(\frac{2\pi t}{T}\right)$	$A\sin\left(\frac{2\pi t}{T}\right)$
0	0	1.0	0.000	1.98	1.980	0.000
$T/6$	$\frac{\pi}{3}$	0.5	0.866	1.30	0.650	1.126
$T/3$	$\frac{2\pi}{3}$	-0.5	0.866	1.05	-0.525	0.909
$T/2$	π	-1.0	0.000	1.30	-1.300	0.000
$2T/3$	$\frac{4\pi}{3}$	-0.5	-0.866	-0.88	0.440	0.762
$5T/6$	$\frac{5\pi}{3}$	0.5	-0.866	-0.25	-0.125	0.217
			$\sum =$	4.5	1.12	3.014

$a_0 = 2 \times \frac{\sum A}{6} = \frac{4.5}{3} = 1.5$; $\qquad a_1 = 2 \times \frac{\sum A\cos\left(\frac{2\pi t}{T}\right)}{6} = \frac{1.12}{3} = 0.373$;

$b_1 = 2 \times \frac{\sum A\sin\left(\frac{2\pi t}{T}\right)}{6} = \frac{3.014}{3} = 1.005$.

The direct current part in the variable current is $\frac{a_0}{2} = 0.75$ and amplitude of the first harmonic is

$\sqrt{\left(a_1^2 + b_1^2\right)} = \sqrt{(0.373)^2 + (1.005)^2} = 1.072$.

Example 3. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Solution: Taking the interval as 60°, we have

θ°	0	60	120	180	240	300
x	0	1	2	3	4	5
y	4	8	15	7	6	2

Fourier cosine series in the interval $(0,\ 2\pi)$ is

$$y=\frac{a_0}{2}+a_1\cos\theta+a_2\cos 2\theta+a_3\cos 3\theta+.... \qquad (1)$$

θ°	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	1.0	1.0	1	4	4	4	4
60	0.5	-0.5	-1	8	4	-4	-8
120	-0.5	-0.5	1	15	-7.5	-7.5	15
180	-1.0	1.0	-1	7	-7	7	-7
240	-0.5	-0.5	1	6	-3	-3	6
300	0.5	-0.5	-1	2	1	-1	-2
			$\sum=$	42	-8.5	-4.5	8

$$a_0=2\times\frac{\sum y}{6}=\frac{42}{3}=14; \qquad a_1=2\times\frac{\sum y\cos\theta}{6}=\frac{-8.5}{3}=-2.8;$$

$$a_2=2\times\frac{\sum y\cos 2\theta}{6}=\frac{-4.5}{3}=-1.5; \qquad a_3=2\times\frac{\sum y\cos 3\theta}{6}=\frac{8}{3}=2.7.$$

Substituting the values of $a_0,\ a_1,\ a_2$ & a_3 (1), we get

$$y=7-2.8\cos\theta-1.5\cos 2\theta+2.7\cos 3\theta+....$$

Example 4. The tuning moment T is given for a series of values of the crank angle $\theta=75^\circ$.

θ°	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T for $\theta=75^\circ$.

Solution: Let the Fourier sine series to represent T in $(0, \pi)$ is

$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + ...$ (1)

θ°	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$	T	$T\sin\theta$	$T\sin 2\theta$	$T\sin 3\theta$	$T\sin 4\theta$
0	0	0	0	0	0	0	0	0	0
30	0.500	0.866	1	0.866	5224	2627	4524	5224	4524
60	0.866	0.866	0	-0.866	8097	7012	7012	0	-7012
90	1.000	0	-1	0	7850	7850	0	-7850	0
120	0.866	-0.866	0	0.866	5499	4762	-4762	0	4762
150	0.500	-0.866	1	-0.866	2626	1313	-2274	2626	-2274
					$\sum =$	23550	4500	-8	0

$b_1 = 2\times\frac{\sum y\sin\theta}{6} = \frac{23550}{3} = 7850$; $b_2 = 2\times\frac{\sum y\sin 2\theta}{6} = \frac{4500}{3} = 1500$;

$b_3 = 2\times\frac{\sum y\sin 3\theta}{6} = \frac{-8}{3} = -2.7$; $b_4 = 2\times\frac{\sum y\sin 4\theta}{6} = \frac{0}{3} = 0$.

Substituting the values of b_1, b_2, b_3 & b_4 (1), we get

$y = 7850\sin\theta + 1500\sin 2\theta - 2.7\sin 3\theta + 0\sin 4\theta + ...$

At $\theta = 75^\circ$: $y = 7850\sin(75) + 1500\sin 2(75) - 2.7\sin 3(75) + 0\sin 4(75) + ...$

$\approx 7850(-0.39) + 1500(-0.71) - 2.7(-0.93)$

Therefore $y(75) = -4124$.

Exercise 1. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in terms of a Fourier series.

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$
y	0	92	14.4	17.8	17.3	11.7

Exercise 2. Obtain the constant term and coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table:

x	0	1	2	3	4	5
y	9	18	24	28	26	20