ANSWER FOR 1B 2016

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(1)

$$\frac{x^3-3}{x^3-x^2-x+1} = \frac{x^3-x^2-x+1+(x^2+x-4)}{x^3-x^2-x+1}$$
$$= 1 + \frac{x^2+x-4}{x^3-x^2-x+1}$$

ここで

$$x^3-x^2-x+1 = (x-1)^2(x+1)$$

だから

$$\frac{x^2+x-4}{x^3-x^2-x+1} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}$$

とおけて，

$$x^2+x-4 = A(x+1) + B(x+1)(x-1) + C(x-1)^2$$
$$x^2+x-4 = (B+C)x^2 + (A-2B)x + A - B + C$$

係数を比較して，

$$\begin{cases} B+C=1 \\ A-2B=1 \\ A-B-C=-4 \end{cases}$$

これを拡大係数行列にして連立方程式を解くと，

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & -1 & -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\therefore C=3, B=-2, A=-3$$

$$I = \int 1dx + \int \left(\frac{-3}{(x-1)^2} + \frac{-2}{x-1} + \frac{1}{x+1} \right) dx$$
$$= x + \frac{3}{x-1} + \log\left| \frac{x+1}{(x-1)^2} \right| + \text{const.}$$

(2)

$$\mathcal{D} := \{(x,y) \mid 0 \le y \le 1, y^3 \le x \le 2-y^2\}$$

これをグラフに書くと

$$\mathcal{D}' = \left\{ (x,y) \middle| 0 \le x \le 2, \begin{cases} 0 \le y \le \sqrt[3]{x} & (0 \le x \le 1) \\ 0 \le y \le \sqrt{2-x} & (1 \le x \le 2) \end{cases} \right\}$$

従って定積分は

$$I = \int_0^1 \left(\int_0^{\sqrt[3]{x}} \frac{y^2}{x\sqrt{2-x}} dy \right) dx + \int_1^2 \left(\int_0^{\sqrt{2-x}} \frac{y^2}{x\sqrt{2-x}} dy \right) dx$$

$$= \int_0^1 \frac{1}{3\sqrt{2-x}} dx + \int_1^2 \frac{2-x}{3x} dx = \left[-\frac{2}{3}\sqrt{2-x} \right]_0^1 + \left[\frac{2}{3}\log x \right]_1^2 - \left[\frac{1}{3}x \right]_1^2$$

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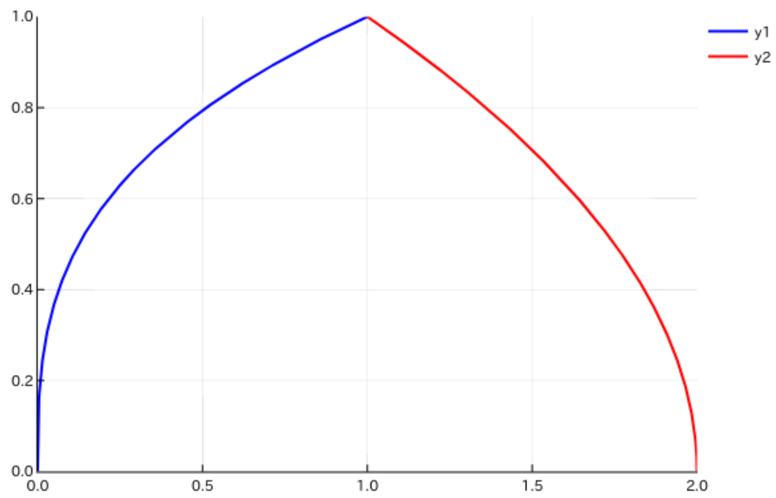

FIGURE 1. $y_1 : y = \sqrt[3]{x}, y_2 : y = \sqrt{2-x}$ のグラフ.

$$= \frac{2}{3}(\sqrt{2} + \log 2) - 1$$

(3)

$$\mathcal{D} := \{(x, y) \mid x^2 + y^2 \leq 4\}$$

$$f_x = \frac{x}{2\sqrt{x^2+y^2}} \exp \frac{\sqrt{x^2+y^2}}{2} - \frac{x}{2\sqrt{x^2+y^2}} \exp - \frac{\sqrt{x^2+y^2}}{2}$$

$$f_y = \frac{y}{2\sqrt{x^2+y^2}} \exp \frac{\sqrt{x^2+y^2}}{2} - \frac{y}{2\sqrt{x^2+y^2}} \exp - \frac{\sqrt{x^2+y^2}}{2}.$$

曲面積は

$$S = \iint_{\mathcal{D}} \sqrt{1 + f_x^2 + f_y^2} dxdy = \iint_{\mathcal{D}} \sqrt{1 + \frac{1}{4}\left(\exp \frac{\sqrt{x^2+y^2}}{2} - \exp - \frac{\sqrt{x^2+y^2}}{2}\right)^2} dxdy.$$

ここで

$$x = r\cos x, y = \sin x$$

と変数変換すると,

$$S = \int_0^{2\pi} d\theta \int_0^2 rdr \sqrt{\frac{1}{2} + \frac{1}{4}(\exp r + \exp(-r))}$$

$$= \int_0^{2\pi} d\theta \int_0^2 rdr \frac{1}{2}\sqrt{(\exp r + 2 + \exp(-r))} = \int_0^{2\pi} d\theta \int_0^2 rdr \frac{1}{2}\left(\exp \frac{r}{2} + \exp \frac{-r}{2}\right)$$

$$= \int_0^{2\pi} d\theta \int_0^1 2u(\exp u + \exp(-u))\, du = 4\pi[u\exp u - \exp u - u\exp -u - \exp -u]_0^1$$

$$= 2\pi(2 - 2e^{-1}) = 8\pi(1 - e^{-1})$$

(4) (a)

$$\varphi(x, y, z) = \exp(-x^2 - y^2) - z$$

とおくと, 問題の曲面は $\varphi = 0$ で表される曲面である. この曲面の法線ベクトルは

$$\nabla\varphi = \begin{bmatrix} -2x\exp(-x^2 - y^2) \\ -2y\exp(-x^2 - y^2) \\ -1 \end{bmatrix}$$

従って求める単位法線ベクトルは

$$\boldsymbol{n} = \frac{1}{\sqrt{4(x^2+y^2)\exp(-2(x^2+y^2))+1}}\begin{bmatrix}2x\exp(-x^2-y^2)\\2y\exp(-x^2-y^2)\\1\end{bmatrix}$$

(b)

$$\boldsymbol{f} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

だから

$$\boldsymbol{f}\cdot\boldsymbol{n} = \frac{\{2(x^2+y^2)+1\}\exp(-(x^2+y^2))}{\sqrt{4(x^2+y^2)\exp(-2(x^2+y^2))+1}}$$

また dS は

$$dS = \sqrt{1+\varphi_x^2+\varphi_y^2}dxdy = \sqrt{4(x^2+y^2)\exp -2(x^2-y^2)+1}$$

従って求める面積分は

$$S = \iint_{\mathcal{D}} \boldsymbol{f}\cdot\boldsymbol{n}dS = \iint_{\mathcal{D}}\{2(x^2+y^2)+1\}\exp(-x^2-y^2)$$

ここで

$$x = r\cos x,\ y = \sin x$$

と変数変換すると,

$$S = \int_0^{\frac{\pi}{2}} d\theta\int_0^\infty rdr(2r^2+1)\exp(-r^2) = \int_0^{\frac{\pi}{2}} d\theta\int_0^\infty(2r^2+1)r\exp(-r^2)dr$$

$$= \int_0^{\frac{\pi}{2}} d\theta\left\{\left[-\frac{1}{2}(2r^2+1)\exp(-r^2)\right]_0^\infty + \int_0^\infty\frac{1}{2}(4r)\exp(-r^2)dr\right\}$$

$$= \int_0^{\frac{\pi}{2}} d\theta\left(\frac{1}{2}+[-\exp(-r^2)]_0^\infty\right) = \frac{3}{2}\frac{\pi}{2}$$

$$= \frac{3}{4}\pi$$

(5) Green の定理から

$$I = \int_\Gamma(\sin x+e^x)\sin ydx+(\cos x+e^x)\cos ydy = \iint_S\left\{-\frac{\partial}{\partial y}(\sin x+e^x)\sin y+\frac{\partial}{\partial x}(\cos x+e^x)\cos y\right\}dxdy$$

$$= \iint_S\{-(\sin x+e^x)\cos y+(-\sin x+e^x)\cos y\}dxdy = \iint_S(-2\sin x\cos y)dxdy.$$

ここで

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}(u-v)\\ \frac{1}{\sqrt{2}}(u;v)\end{bmatrix}$$

と変数変換すると,

$$S = \left\{(u,v)\middle|0\le u\le\frac{\sqrt{2}\pi}{4}, 0\le v\le\frac{\sqrt{2}\pi}{6}\right\}$$

$$J(u,v) = \begin{vmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{vmatrix} = 1$$

$$\therefore I = \int_0^{\frac{\sqrt{2}\pi}{4}} du\int_0^{\frac{\sqrt{2}\pi}{6}} dv -2\sin\frac{1}{\sqrt{2}}(u-v)\cos\frac{1}{\sqrt{2}}(u+v)$$

積和の公式から

$$\sin\frac{1}{\sqrt{2}}(u-v)\cos\frac{1}{\sqrt{2}}(u+v)=\frac{1}{2}(\sin\sqrt{2}u-\sin\sqrt{2}v)$$

だから,

$$I=\int_0^{\frac{\sqrt{2}\pi}{4}}du\int_0^{\frac{\sqrt{2}\pi}{6}}dv(-\sin\sqrt{2}u+\sin\sqrt{2}v)$$

$$=\int_0^{\frac{\sqrt{2}\pi}{4}}\left[-v\sin\sqrt{2}u+\frac{1}{\sqrt{2}}\cos\sqrt{2}v\right]_0^{\frac{\sqrt{2}\pi}{6}}du=\int_0^{\frac{\sqrt{2}\pi}{4}}\left[-\frac{\sqrt{2}\pi}{6}\sin\sqrt{2}u+\frac{1}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\right]du$$

$$=\left[\frac{\pi}{6}\cos\sqrt{2}u-\frac{u}{\sqrt{2}}\right]_0^{\frac{\sqrt{2}\pi}{4}}=0-\frac{\pi}{4}-\frac{\pi}{6}$$

$$=-\frac{\pi}{12}.$$