

ANSWER FOR 1B 2016

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(1)

$$\begin{aligned}\frac{x^3 - 3}{x^3 - x^2 - x + 1} &= \frac{x^3 - x^2 - x + 1 + (x^2 + x - 4)}{x^3 - x^2 - x + 1} \\ &= 1 + \frac{x^2 + x - 4}{x^3 - x^2 - x + 1}\end{aligned}$$

ここで

$$x^3 - x^2 - x + 1 = (x - 1)^2(x + 1)$$

だから

$$\frac{x^2 + x - 4}{x^3 - x^2 - x + 1} = \frac{A}{(x - 1)^2} + \frac{B}{x - 1} + \frac{C}{x + 1}$$

とおけて、

$$x^2 + x - 4 = A(x + 1) + B(x + 1)(x - 1) + C(x - 1)^2$$

$$x^2 + x - 4 = (B + C)x^2 + (A - 2B)x + A - B + C$$

係数を比較して、

$$\begin{cases} B + C = 1 \\ A - 2B = 1 \\ A - B - C = -4 \end{cases}$$

これを拡大係数行列にして連立方程式を解くと、

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & -1 & -1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\therefore C = 3, B = -2, A = -3$$

$$\begin{aligned}I &= \int 1 dx + \int \left(\frac{-3}{(x - 1)^2} + \frac{-2}{x - 1} + \frac{1}{x + 1} \right) dx \\ &= x + \frac{3}{x - 1} + \log \left| \frac{x + 1}{(x - 1)^2} \right| + \text{const.}\end{aligned}$$

(2)

$$\mathcal{D} := \{(x, y) \mid 0 \leq y \leq 1, y^3 \leq x \leq 2 - y^2\}$$

これをグラフに書くと

$$\mathcal{D}' = \left\{ (x, y) \mid 0 \leq x \leq 2, \begin{cases} 0 \leq y \leq \sqrt[3]{x} & (0 \leq x \leq 1) \\ 0 \leq y \leq \sqrt{2 - x} & (1 \leq x \leq 2) \end{cases} \right\}$$

従って定積分は

$$\begin{aligned}I &= \int_0^1 \left(\int_0^{\sqrt[3]{x}} \frac{y^2}{x\sqrt{2-x}} dy \right) dx + \int_1^2 \left(\int_0^{\sqrt{2-x}} \frac{y^2}{x\sqrt{2-x}} dy \right) dx \\ &= \int_0^1 \frac{1}{3\sqrt{2-x}} dx + \int_1^2 \frac{2-x}{3x} dx = \left[-\frac{2}{3}\sqrt{2-x} \right]_0^1 + \left[\frac{2}{3}\log x \right]_1^2 - \left[\frac{1}{3}x \right]_1^2\end{aligned}$$

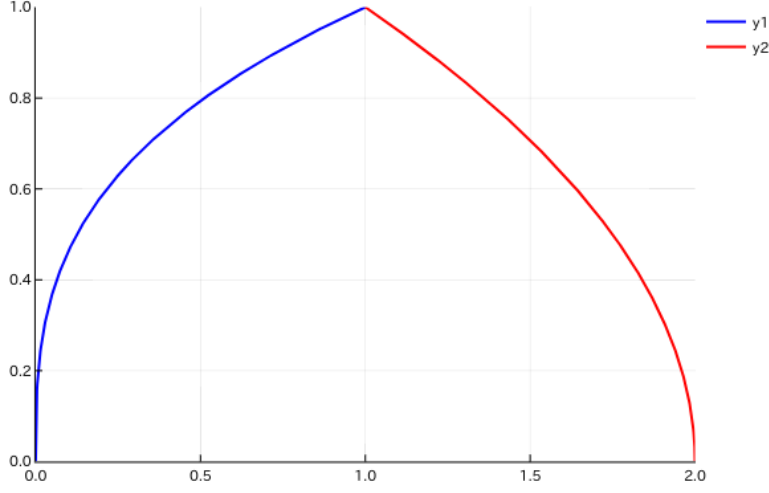


FIGURE 1. $y_1 : y = \sqrt[3]{x}, y_2 : y = \sqrt{2-x}$ のグラフ.

$$= \frac{2}{3}(\sqrt{2} + \log 2) - 1$$

(3)

$$\mathcal{D} := \{(x, y) \mid x^2 + y^2 \leq 4\}$$

$$f_x = \frac{x}{2\sqrt{x^2 + y^2}} \exp \frac{\sqrt{x^2 + y^2}}{2} - \frac{x}{2\sqrt{x^2 + y^2}} \exp -\frac{\sqrt{x^2 + y^2}}{2}$$

$$f_y = \frac{y}{2\sqrt{x^2 + y^2}} \exp \frac{\sqrt{x^2 + y^2}}{2} - \frac{y}{2\sqrt{x^2 + y^2}} \exp -\frac{\sqrt{x^2 + y^2}}{2}.$$

曲面積は

$$S = \iint_{\mathcal{D}} \sqrt{1 + f_x^2 + f_y^2} dx dy = \iint_{\mathcal{D}} \sqrt{1 + \frac{1}{4} \left(\exp \frac{\sqrt{x^2 + y^2}}{2} - \exp -\frac{\sqrt{x^2 + y^2}}{2} \right)^2} dx dy.$$

ここで

$$x = r \cos x, y = \sin x$$

と変数変換すると,

$$\begin{aligned} S &= \int_0^{2\pi} d\theta \int_0^2 r dr \sqrt{\frac{1}{2} + \frac{1}{4} (\exp r + \exp(-r))} \\ &= \int_0^{2\pi} d\theta \int_0^2 r dr \frac{1}{2} \sqrt{(\exp r + 2 + \exp(-r))} = \int_0^{2\pi} d\theta \int_0^2 r dr \frac{1}{2} \left(\exp \frac{r}{2} + \exp \frac{-r}{2} \right) \\ &= \int_0^{2\pi} d\theta \int_0^1 2u (\exp u + \exp(-u)) du = 4\pi [u \exp u - \exp u - u \exp -u - \exp -u]_0^1 \\ &= 2\pi(2 - 2e^{-1}) = 8\pi(1 - e^{-1}) \end{aligned}$$

(4) (a)

$$\varphi(x, y, z) = \exp(-x^2 - y^2) - z$$

とおくと, 問題の曲面は $\varphi = 0$ で表される曲面である. この曲面の法線ベクトルは

$$\nabla \varphi = \begin{bmatrix} -2x \exp(-x^2 - y^2) \\ -2y \exp(-x^2 - y^2) \\ -1 \end{bmatrix}$$

従って求める単位法線ベクトルは

$$\mathbf{n} = \frac{1}{\sqrt{4(x^2 + y^2) \exp(-2(x^2 + y^2)) + 1}} \begin{bmatrix} 2x \exp(-x^2 - y^2) \\ 2y \exp(-x^2 - y^2) \\ 1 \end{bmatrix}$$

(b)

$$\mathbf{f} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

だから

$$\mathbf{f} \cdot \mathbf{n} = \frac{\{2(x^2 + y^2) + 1\} \exp(-x^2 - y^2)}{\sqrt{4(x^2 + y^2) \exp(-2(x^2 + y^2)) + 1}}$$

また dS は

$$dS = \sqrt{1 + \varphi_x^2 + \varphi_y^2} dx dy = \sqrt{4(x^2 + y^2) \exp(-2(x^2 + y^2)) + 1}$$

従って求める面積分は

$$S = \iint_{\mathcal{D}} \mathbf{f} \cdot \mathbf{n} dS = \iint_{\mathcal{D}} \{2(x^2 + y^2) + 1\} \exp(-x^2 - y^2)$$

ここで

$$x = r \cos \theta, y = r \sin \theta$$

と変数変換すると,

$$\begin{aligned} S &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} r dr (2r^2 + 1) \exp(-r^2) = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} (2r^2 + 1) r \exp(-r^2) dr \\ &= \int_0^{\frac{\pi}{2}} d\theta \left\{ \left[-\frac{1}{2} (2r^2 + 1) \exp(-r^2) \right]_0^{\infty} + \int_0^{\infty} \frac{1}{2} (4r) \exp(-r^2) dr \right\} \\ &= \int_0^{\frac{\pi}{2}} d\theta \left(\frac{1}{2} + [-\exp(-r^2)]_0^{\infty} \right) = \frac{3}{2} \frac{\pi}{2} \\ &= \frac{3}{4} \pi \end{aligned}$$

(5) Green の定理から

$$\begin{aligned} I &= \int_{\Gamma} (\sin x + e^x) \sin y dx + (\cos x + e^x) \cos y dy = \iint_S \left\{ -\frac{\partial}{\partial y} (\sin x + e^x) \sin y + \frac{\partial}{\partial x} (\cos x + e^x) \cos y \right\} dx dy \\ &= \iint_S \{ -(\sin x + e^x) \cos y + (-\sin x + e^x) \cos y \} dx dy = \iint_S (-2 \sin x \cos y) dx dy. \end{aligned}$$

ここで

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(u - v) \\ \frac{1}{\sqrt{2}}(u + v) \end{bmatrix}$$

と変数変換すると, 積分範囲は

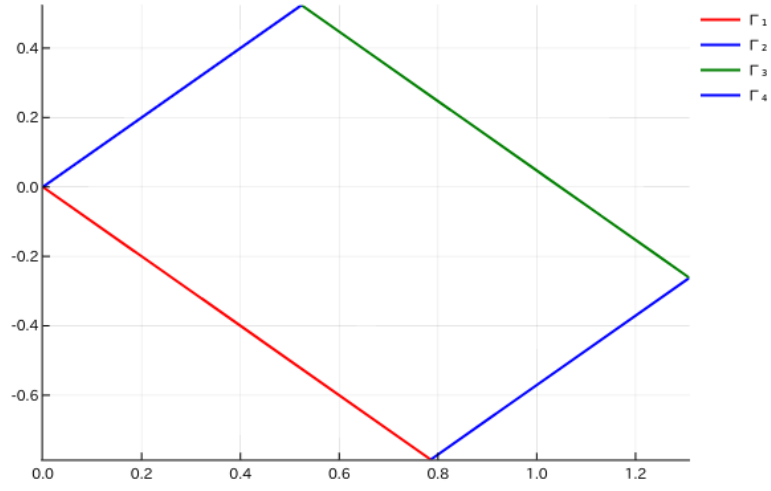
$$S = \left\{ (u, v) \mid 0 \leq u \leq \frac{\sqrt{2}\pi}{4}, 0 \leq v \leq \frac{\sqrt{2}\pi}{6} \right\}$$

ヤコビ行列式は

$$J(u, v) = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$$

したがって求める積分は

$$\therefore I = \int_0^{\frac{\sqrt{2}\pi}{4}} du \int_0^{\frac{\sqrt{2}\pi}{6}} dv \left(-2 \sin \frac{1}{\sqrt{2}}(u - v) \cos \frac{1}{\sqrt{2}}(u + v) \right)$$

FIGURE 2. 経路 Γ .

積和の公式から

$$\sin \frac{1}{\sqrt{2}}(u-v) \cos \frac{1}{\sqrt{2}}(u+v) = \frac{1}{2}(\sin \sqrt{2}u - \sin \sqrt{2}v)$$

だから,

$$\begin{aligned} I &= \int_0^{\frac{\sqrt{2}\pi}{4}} du \int_0^{\frac{\sqrt{2}\pi}{6}} dv (-\sin \sqrt{2}u + \sin \sqrt{2}v) \\ &= \int_0^{\frac{\sqrt{2}\pi}{4}} \left[-v \sin \sqrt{2}u + \frac{1}{\sqrt{2}} \cos \sqrt{2}v \right]_0^{\frac{\sqrt{2}\pi}{6}} du = \int_0^{\frac{\sqrt{2}\pi}{4}} \left[-\frac{\sqrt{2}\pi}{6} \sin \sqrt{2}u + \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}} \right] du \\ &= \left[\frac{\pi}{6} \cos \sqrt{2}u - \frac{u}{\sqrt{2}} \right]_0^{\frac{\sqrt{2}\pi}{4}} = 0 - \frac{\pi}{4} - \frac{\pi}{6} \\ &= -\frac{\pi}{12}. \end{aligned}$$