ANSWER FOR 1B 2016

NAOKI YANO

(1)

$$\frac{x^3-3}{x^3-x^2-x+1} = \frac{x^3-x^2-x+1+(x^2+x-4)}{x^3-x^2-x+1}$$
$$= 1+\frac{x^2+x-4}{x^3-x^2-x+1}$$

ここで

$$x^3-x^2-x+1=(x-1)^2(x+1)$$

だから

$$\frac{x^2+x-4}{x^3-x^2-x+1} = \frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x+1}$$

とおけて,

$$x^2+x-4 = A(x+1)+B(x+1)(x-1)+C(x-1)^2$$
$$x^2+x-4 = (B+C)x^2+(A-2B)x+A-B+C$$

係数を比較して,

$$\begin{cases} B+C=1 \\ A-2B=1 \\ A-B-C=-4 \end{cases}$$

これを拡大係数行列にして連立方程式を解くと,

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & -1 & -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\therefore C=3, B=-2, A=-3$$

$$I = \int 1dx + \int \left(\frac{-3}{(x-1)^2}+\frac{-2}{x-1}+\frac{1}{x+1}\right)dx$$
$$= x+\frac{3}{x-1}+\log\left|\frac{x+1}{(x-1)^2}\right| + \text{const.}$$

(2)

$$\mathcal{D} := \{(x,y) \mid 0 \le y \le 1, y^3 \le x \le 2-y^2\}$$

これをグラフに書くと

$$\mathcal{D}' = \left\{(x,y) \middle| 0 \le x \le 2, \begin{cases} 0 \le y \le \sqrt[3]{x} & (0 \le x \le 1) \\ 0 \le y \le \sqrt{2-x} & (1 \le x \le 2) \end{cases}\right\}$$

従って定積分は

$$I = \int_0^1 \left(\int_0^{\sqrt[3]{x}} \frac{y^2}{x\sqrt{2-x}}dy\right)dx + \int_1^2 \left(\int_0^{\sqrt{2-x}} \frac{y^2}{x\sqrt{2-x}}dy\right)dx$$

$$= \int_0^1 \frac{1}{3\sqrt{2-x}}dx + \int_1^2 \frac{2-x}{3x}dx = \left[-\frac{2}{3}\sqrt{2-x}\right]_0^1 + \left[\frac{2}{3}\log x\right]_1^2 - \left[\frac{1}{3}x\right]_1^2$$

Date: January 21, 2018.

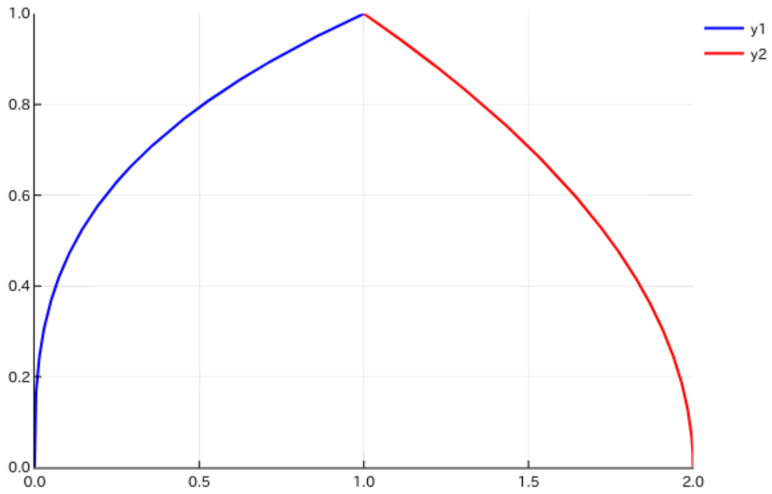

FIGURE 1. $y_1 : y = \sqrt[3]{x}, y_2 : y = \sqrt{2-x}$ のグラフ.

$$= \frac{2}{3}(\sqrt{2} + \log 2) - 1$$

(3)

$$\mathcal{D} := \{(x, y) \mid x^2 + y^2 \le 4\}$$

$$f_x = \frac{x}{2\sqrt{x^2+y^2}} \exp \frac{\sqrt{x^2+y^2}}{2} - \frac{x}{2\sqrt{x^2+y^2}} \exp - \frac{\sqrt{x^2+y^2}}{2}$$

$$f_y = \frac{y}{2\sqrt{x^2+y^2}} \exp \frac{\sqrt{x^2+y^2}}{2} - \frac{y}{2\sqrt{x^2+y^2}} \exp - \frac{\sqrt{x^2+y^2}}{2}.$$

曲面積は

$$S = \iint_{\mathcal{D}} \sqrt{1 + f_x^2 + f_y^2} dxdy = \iint_{\mathcal{D}} \sqrt{1 + \frac{1}{4}\left(\exp \frac{\sqrt{x^2+y^2}}{2} - \exp - \frac{\sqrt{x^2+y^2}}{2}\right)^2} dxdy.$$

ここで

$$x = r \cos x, y = \sin x$$

と変数変換すると,

$$S = \int_0^{2\pi} d\theta \int_0^2 r dr \sqrt{\frac{1}{2} + \frac{1}{4}(\exp r + \exp(-r))}$$

$$= \int_0^{2\pi} d\theta \int_0^2 r dr \frac{1}{2}\sqrt{(\exp r + 2 + \exp(-r))} = \int_0^{2\pi} d\theta \int_0^2 r dr \frac{1}{2}\left(\exp \frac{r}{2} + \exp \frac{-r}{2}\right)$$

$$= \int_0^{2\pi} d\theta \int_0^1 2u(\exp u + \exp(-u)) du = 4\pi[u \exp u - \exp u - u \exp -u - \exp -u]_0^1$$

$$= 2\pi(2 - 2e^{-1}) = 8\pi(1 - e^{-1})$$

(4) (a)

$$\varphi(x, y, z) = \exp(-x^2 - y^2) - z$$

とおくと, 問題の曲面は $\varphi = 0$ で表される曲面である. この曲面の法線ベクトルは

$$\nabla\varphi = \begin{bmatrix} -2x \exp(-x^2 - y^2) \\ -2y \exp(-x^2 - y^2) \\ -1 \end{bmatrix}$$

従って求める単位法線ベクトルは

$$\boldsymbol{n} = \frac{1}{\sqrt{4(x^2+y^2)\exp(-2(x^2+y^2))+1}}\begin{bmatrix} 2x\exp(-x^2-y^2) \\ 2y\exp(-x^2-y^2) \\ 1 \end{bmatrix}$$

(b)

$$\boldsymbol{f} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

だから

$$\boldsymbol{f}\cdot\boldsymbol{n} = \frac{\{2(x^2+y^2)+1\}\exp(-(x^2+y^2))}{\sqrt{4(x^2+y^2)\exp(-2(x^2+y^2))+1}}$$

また dS は

$$dS = \sqrt{1+\varphi_x^2+\varphi_y^2}dxdy = \sqrt{4(x^2+y^2)\exp-2(x^2-y^2)+1}$$

従って求める面積分は

$$S = \iint_{\mathcal{D}} \boldsymbol{f}\cdot\boldsymbol{n}dS = \iint_{\mathcal{D}} \{2(x^2+y^2)+1\}\exp(-x^2-y^2)$$

ここで

$$x = r\cos x, y = \sin x$$

と変数変換すると,

$$\begin{aligned} S &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} rdr(2r^2+1)\exp(-r^2) = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} (2r^2+1)r\exp(-r^2)dr \\ &= \int_0^{\frac{\pi}{2}} d\theta \left\{ \left[-\frac{1}{2}(2r^2+1)\exp(-r^2)\right]_0^{\infty} + \int_0^{\infty} \frac{1}{2}(4r)\exp(-r^2)dr \right\} \\ &= \int_0^{\frac{\pi}{2}} d\theta \left(\frac{1}{2} + [-\exp(-r^2)]_0^{\infty}\right) = \frac{3}{2}\frac{\pi}{2} \\ &= \frac{3}{4}\pi \end{aligned}$$

(5) Green の定理から

$$I = \int_{\Gamma} (\sin x+e^x)\sin ydx+(\cos x+e^x)\cos ydy = \iint_S \left\{-\frac{\partial}{\partial y}(\sin x+e^x)\sin y + \frac{\partial}{\partial x}(\cos x+e^x)\cos y\right\} dxdy$$

$$= \iint_S \{-(\sin x+e^x)\cos y + (-\sin x+e^x)\cos y\}dxdy = \iint_S (-2\sin x\cos y)dxdy.$$

ここで

$$u = \frac{1}{\sqrt{2}}(x+y), v = \frac{1}{\sqrt{2}}(-x+y)$$

と変数変換すると,

$$S = \left\{(u,v)\middle| 0 \le u \le \frac{\sqrt{2}\pi}{4}, 0 \le v \le \frac{\sqrt{2}\pi}{6}\right\}$$

$$x = \frac{1}{\sqrt{2}}(u-v), y = \frac{1}{\sqrt{2}}(u+v).$$

$$J(u,v) = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$$

$$\therefore I = \int_0^{\frac{\sqrt{2}\pi}{4}} du \int_0^{\frac{\sqrt{2}\pi}{6}} dv - 2\sin\frac{1}{\sqrt{2}}(u-v)\cos\frac{1}{\sqrt{2}}(u+v)$$

積和の公式から

$$\sin\frac{1}{\sqrt{2}}(u-v)\cos\frac{1}{\sqrt{2}}(u+v)=\frac{1}{2}(\sin\sqrt{2}u-\sin\sqrt{2}v)$$

だから,

$$I=\int_0^{\frac{\sqrt{2}\pi}{4}}du\int_0^{\frac{\sqrt{2}\pi}{6}}dv(-\sin\sqrt{2}u+\sin\sqrt{2}v)$$

$$=\int_0^{\frac{\sqrt{2}\pi}{4}}\left[-v\sin\sqrt{2}u+\frac{1}{\sqrt{2}}\cos\sqrt{2}v\right]_0^{\frac{\sqrt{2}\pi}{6}}du=\int_0^{\frac{\sqrt{2}\pi}{4}}\left[-\frac{\sqrt{2}\pi}{6}\sin\sqrt{2}u+\frac{1}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\right]du$$

$$=\left[\frac{\pi}{6}\cos\sqrt{2}u-\frac{u}{\sqrt{2}}\right]_0^{\frac{\sqrt{2}\pi}{4}}=0-\frac{\pi}{4}-\frac{\pi}{6}$$

$$=-\frac{\pi}{12}.$$