期末試験解答例　　100点満点　解答内容に応じて部分点があります

1.
15点

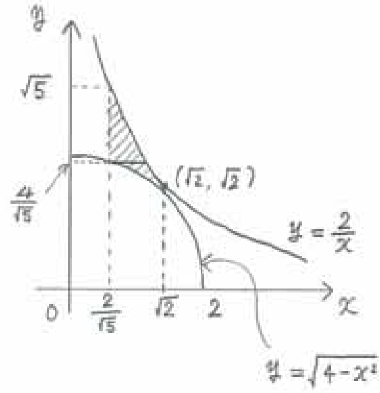
$$\frac{1}{x^3+x^2-2} = \frac{1}{5}\left(\frac{1}{x-1} - \frac{x+3}{x^2+2x+2}\right)$$

$$\therefore \int \frac{dx}{x^3+x^2-2} = \frac{1}{5}\left(\int \frac{dx}{x-1} - \int \frac{\frac{1}{2}(2x+2)+2}{x^2+2x+2}\,dx\right)$$

$$= \frac{1}{5}\left(\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{(x^2+2x+2)'}{x^2+2x+2}\,dx - 2\int \frac{dx}{(x+1)^2+1}\right)$$

$$= \frac{1}{5}\left(\log|x-1| - \frac{1}{2}\log|x^2+2x+2| - 2\,\mathrm{Tan}^{-1}(x+1)\right) + C$$

2.
15点



$$I = \int_{\sqrt{2}}^{4/\sqrt{5}} \left[\int_{\sqrt{4-y^2}}^{2/y} f(x,y)\,dx\right] dy + \int_{4/\sqrt{5}}^{\sqrt{5}} \left[\int_{2/\sqrt{5}}^{2/y} f(x,y)\,dx\right] dy$$

3.
20点

$\begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases}$ とおくと, $\begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le a \end{cases}$, $|J(r,\varphi,\theta)| = r^2\sin\theta$

$$\therefore I = \int_0^a \left[\int_0^{\pi/2} \left\{\int_0^{\pi/2} r\sin\varphi\sin\theta \cdot r\cos\theta \cdot r^2\sin\theta\,d\theta\right\} d\varphi\right] dr$$

$$= \int_0^a r^4\,dr \cdot \int_0^{\pi/2} \sin\varphi\,d\varphi \int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta$$

$$= \frac{a^5}{15}$$

計 20点

4. (1) 10点 $\begin{cases} x = st \\ y = (1-s)t \end{cases}$ より, $J(s,t) = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} t & s \\ -t & 1-s \end{pmatrix}$

(2) $0 \leq s \leq 1$, $1 \leq t \leq 2$, $|J(s,t)| = t$ より,

10点 $I = \int_0^1 \left[\int_1^2 st \; e^{s^2} t \, dt \right] ds = \int_0^1 s e^{s^2} ds \int_1^2 t^2 dt = \frac{7}{6}(e-1)$

5. $z_x = -\sqrt{a}\, y$, $z_y = -\sqrt{a}\, x$ より, $D:\ x, y \geq 0,\ x^2+y^2 \leq 1$ とおくと,

10点

$$S = \iint_D \sqrt{1 + (-\sqrt{a}\,y)^2 + (-\sqrt{a}\,x)^2}\, dx dy$$

$$= \iint_D \sqrt{1 + a(x^2+y^2)}\, dx dy$$

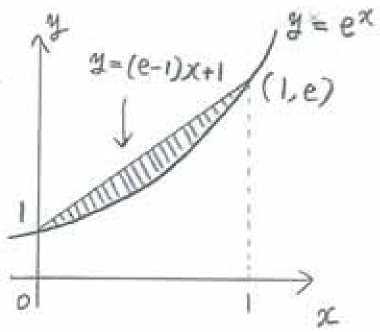
$$= \int_0^1 \left[\int_0^{\pi/2} \sqrt{1+r^2} \cdot r \, d\theta \right] dr \quad \left(極座標変換 \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases},\ 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2} \right)$$

$$= \int_0^1 r\sqrt{1+r^2}\, dr \cdot \int_0^{\pi/2} d\theta$$

$$= \frac{\pi}{6a} \left\{ (1+a)^{3/2} - 1 \right\}$$

6. $f_1(x,y) = xy^2$, $f_2(x,y) = x + x^2 y$ とおき,

20点 Γで囲まれる領域をDとすると, グリーンの定理より,



$$I = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy$$

$$= \int_0^1 \left[\int_{e^x}^{(e-1)x+1} \{(1+2xy) - 2xy\} dy \right] dx$$

$$= \int_0^1 \left[\int_{e^x}^{(e-1)x+1} dy \right] dx$$

$$= \int_0^1 \{(e-1)x + 1 - e^x\} dx$$

$$= \frac{1}{2}(3-e)$$

[別解]

$$\Gamma_1 : \begin{cases} x = t \\ y = e^t \end{cases}, \quad 0 \leq t \leq 1, \quad \frac{dx}{dt} = 1, \quad \frac{dy}{dt} = e^t$$

$$\therefore \ I_1 = \int_{\Gamma_1} xy^2 dx + (x + x^2 y) dy$$

$$= \int_0^1 \{ t e^{2t} \cdot 1 + (t + t^2 e^t) \cdot e^t \} dt$$

$$= \frac{e^2}{2} + 1$$

$$\Gamma_2 : \begin{cases} x = 1 - t \\ y = (e-1)(1-t) + 1 \end{cases}, \quad 0 \leq t \leq 1, \quad \frac{dx}{dt} = -1, \quad \frac{dy}{dt} = 1 - e$$

$$\therefore \ I_2 = \int_{\Gamma_2} xy\, dx + (x + x^2 y) dy$$

$$= \int_0^1 \Big[(1-t) \{ (e-1)(1-t) + 1 \}^2 \cdot (-1)$$

$$+ \{ (1-t) + (1-t)^2 ((e-1)(1-t) + 1) \} (1-e) \Big] dt$$

$$= \frac{1}{2} (-e^2 - e + 1)$$

$$\therefore \ I = I_1 + I_2 = \frac{1}{2}(3 - e)$$