Introduction to probability for CS - Assignment 4

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1 Exercise 6

1.1 Q 3 b:

3. Find $E(X)$ and $V(X)$ if X is distributed according to the following probability functions:

(a)

$$p(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})}, & x = 1, 2, \dots, \ (\lambda > 0), \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$p(x) = \begin{cases} \frac{(1-p)^x}{-x \ln p}, & x = 1, 2, \dots, \ (0 < p < 1), \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

(b) (i)

$$E(X) = \sum_{x=0}^{\infty} p(x) \cdot x = \sum_{x=0}^{\infty} \frac{(1-p)^x}{-x \ln p} \cdot x = \sum_{x=1}^{\infty} \frac{(1-p)^x}{-x \ln p} \cdot x = -\frac{1}{\ln p} \sum_{x=1}^{\infty} (1-p)^x = \boxed{\frac{p-1}{p \ln p}}$$

(ii) (1)

$$E(X^2) = \sum_{x=0}^{\infty} p(x) \cdot x^2 = \sum_{x=0}^{\infty} \frac{(1-p)^x}{-x \ln p} \cdot x^2 = \sum_{x=1}^{\infty} \frac{(1-p)^x}{-x \ln p} \cdot x^2 =$$

$$= -\frac{1}{\ln p} \sum_{x=1}^{\infty} x(1-p)^x = -\frac{1}{\ln p} \cdot \frac{1-p}{p^2} = \frac{p-1}{p^2 \ln p}$$

(2)

$$E(X)^2 = \left(\frac{p-1}{p \ln p}\right)^2 = \frac{p^2 - 2p + 1}{p^2 \ln^2 p}$$

In total:

$$V(X) = E(X^2) - E(X)^2 = \frac{p-1}{p^2 \ln p} - \frac{(p-1)^2}{p^2 \ln^2 p} = (p-1)\left(\frac{1}{p^2 \ln p} - \frac{p-1}{p^2 \ln^2 p}\right) = \boxed{(p-1)\frac{\ln p - p + 1}{p^2 \ln^2 p}}$$

1.2 Q 8:

8. Find the variance of the number of letters sent to the right destination in the absent-minded secretary problem.

Solution:
Let us define an indicator random veriable:

$$X_i = \begin{cases} 1, & \text{if the } i\text{'th letter reached its destination} \\ 0, & \text{otherwise} \end{cases}$$

Where $X = \sum_{i=1}^{n} X_i$, therefore:

$$V(X) = cov(X,X) = cov\Big(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\Big) = n \cdot V(X_1) + n(n-1) \cdot cov(X_1, X_2) =$$

$$= n\Big(E(X_1^2) - E(X_1)^2\Big) + n(n-1)\Big(E(X_1X_2) - E(X_1)E(X_2)\Big) =$$

$$= n\Big(\frac{1}{n} - \frac{1}{n^2}\Big) + n(n-1)\Big(P(X_1 = 1, X_2 = 1) - \frac{1}{n^2}\Big) = 1 - \frac{1}{n} + n(n-1)\Big(\frac{1}{n(n-1)} - \frac{1}{n^2}\Big) =$$

$$= 1 - \frac{1}{n} + 1 - 1 + \frac{1}{n} = \boxed{1}$$

2 Exercise 7

2.1 Q 1:

1. The probability function of (X,Y) is given by the following table:

$y \backslash x$	1	2	3
1	1/12	1/6	1/12
2	1/6	1/4	1/12
3	1/12	1/12	0

Find:

(a) the probability for X to be odd.
(b) the probability for XY to be odd.
(c) $P(X + Y \geq 4)$.
(d) $E(X + Y)$.
(e) $V(X + Y)$.

Solution:

(a)

$$P(X \text{ is odd}) = P(x=1) + P(x=3) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \boxed{\frac{1}{2}}$$

(b)

$$P(XY \text{ is odd}) = P(1,1) + P(1,3) + P(3,1) + P(3,3) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + 0 = \boxed{\frac{1}{4}}$$

(c)

$$P(X+Y \geq 4) = P(1,3) + P(2,2) + P(2,3) + P(3,1) + P(3,2) + P(3,3) =$$

$$= \frac{1}{12} + \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + 0 = \boxed{\frac{7}{12}}$$

(d)

$$E(X+Y) = E(X) + E(Y) = 2\Big(1 \cdot (\frac{1}{12} + \frac{1}{6} + \frac{1}{12}) + 2 \cdot (\frac{1}{6} + \frac{1}{4} + \frac{1}{12}) + 3 \cdot (\frac{1}{12} + \frac{1}{12})\Big) = \boxed{\frac{11}{3}}$$

(e)

$$V(X+Y) = E((X+Y)^2) - E(X+Y)^2 = 4 \cdot \frac{1}{12} + 9 \cdot \frac{1}{3} + 16 \cdot \frac{5}{12} + 25 \cdot \frac{1}{6} - \left(\frac{11}{3}\right)^2 = \boxed{\frac{13}{18}}$$

2.2 Q 5:

5. A die is rolled n times. Let X_1 and X_2 be the number of 1-s and of 2-s, respectively. Find:
   (a) the probability function of (X_1, X_2).
   (b) $\mathrm{Cov(X_1, X_2)}$.

Solution:

(a)

$$P(X_1 = i, X_2 = j) = \binom{n}{i}\left(\frac{1}{6}\right)^i \cdot \binom{n-i}{j}\left(\frac{1}{6}\right)^j \cdot \left(\frac{4}{6}\right)^{(n-i-j)} =$$

$$= \boxed{\binom{n}{i, j, n-i-j}\left(\frac{1}{6}\right)^{(i+j)} \cdot \left(\frac{4}{6}\right)^{(n-i-j)}}$$

(b) (i)

$$X_1, X_2 \sim B(n, \frac{1}{6}) \implies E(X_1) = E(X_2) = n \cdot \frac{1}{6} = \frac{n}{6}$$

(ii) Let $X_{1,i}$ be an indicator random variable, where $X_{1,i} = 1$ if the outcome on the i'th position is 1 and 0 otherwise. Let $X_{2,i}$be an indicator random variable, where $X_{1,i} = 1$ if the outcome on the i'th position is 2 and 0 otherwise. We know that $X_1 = \sum_{i=1}^n X_{1,i}$ and $X_2 = \sum_{i=1}^n X_{2,i}$. Therefore:

$$E(X_1 X_2) = E(\sum_{i=1}^n X_{1,i} \cdot \sum_{j=1}^n X_{2,i}) = \sum_{i,j=1}^n E(X_{1,i} X_{2,j}) =$$

$$\sum_{i \neq j}^n E(X_{1,i})E(X_{2,j}) + \cancelto{0}{\sum_{i=1}^n E(X_{1,i}, X_{2,i})} = \frac{n}{6}\frac{n-1}{6} = \frac{n(n-1)}{36}$$

In total: $Cov(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2) = \frac{n(n-1)}{36} - \frac{n^2}{36} = \boxed{-\frac{n}{36}}$

3 Exercise 8:

3.1 Q 2 a:

2. Let X_i, $i = 1, 2, \dots, m$, be independent random variables distributed according to one of the following possibilities:

   (a) $X_i \sim P(\lambda_i)$.

   (b) $X_i \sim B(n_i, p)$.

   (c) $X_i \sim G(p)$.

   Let $Y = \sum_{i=1}^{m} X_i$. Prove that, correspondingly:

   (a) $Y \sim P\left(\sum_{i=1}^{m} \lambda_i\right)$.

   (b) $Y \sim B\left(\sum_{i=1}^{m} n_i, p\right)$.

   (c) $Y \sim \overline{B}(m, p)$.

Solution: Recall the multinomial theorem:

$$\boxed{(x_1 + x_2 + \dots + x_m)^n = \sum_{k_1+k_2+\dots+k_m=n;k_1,k_2,\dots,k_m\geq 0} \binom{n}{k_1, k_2, \dots, k_m} \prod_{i=1}^{m} x_i^{k_i}}$$

$$X_i \sim P(\lambda_i) \implies P(X_i = k) = \frac{\lambda_i^k e^{-\lambda_i}}{k!}$$

$$P(Y = k) = P(X_1 + X_2 + \dots + X_m = k)$$

Since X_i's are independent we get:

$$P(Y = k) = P(X_1 + X_2 + \dots + X_m = k) = \sum_{k_1+\dots+k_m=k} P(X_1 = k_1)P(X_2 = k_2) \cdot \dots \cdot P(X_m = k_m) =$$

$$= \sum_{k_1+\dots+k_m=k} \frac{\lambda_1^{k_1} e^{-\lambda_1}}{k_1!} \cdot \dots \cdot \frac{\lambda_m^{k_m} e^{-\lambda_m}}{k_m!} = e^{-(\lambda_1+\lambda_2+\dots+\lambda_m)} \cdot \Big(\sum_{k_1+\dots+k_m=k} \frac{\lambda_1^{k_1}}{k_1!} \cdot \dots \cdot \frac{\lambda_m^{k_m}}{k_m!} \Big) \cdot \frac{k!}{k!} =$$

$$= \frac{e^{-(\lambda_1+\lambda_2+\dots+\lambda_m)}}{k!} \cdot \Big(\sum_{k_1+\dots+k_m=k} \frac{k!}{k_1! \cdot k_2! \cdot \dots \cdot k_m!} \cdot (\lambda_1^{k_1} \cdot \lambda_2^{k_2} \cdot \dots \cdot \lambda_m^{k_m}) \Big) =$$

$$= \frac{e^{-(\lambda_1+\lambda_2+\dots+\lambda_m)}}{k!} \cdot \Big(\sum_{k_1+\dots+k_m=k} \binom{k}{k_1, k_2, \dots, k_m} \cdot \prod_{i=1}^{m} \lambda_i^{k_i} \Big) = \frac{e^{-(\lambda_1+\lambda_2+\dots+\lambda_m)}}{k!} \cdot (\lambda_1 + \lambda_2 + \dots + \lambda_m)^k =$$

$$= \frac{e^{-(\sum_{i=1}^{m} \lambda_i)} \cdot (\sum_{i=1}^{m} \lambda_i)^k}{k!} \implies Y \sim P\Big(\sum_{i=1}^{m} \lambda_i\Big) \blacksquare$$

3.2 Q 4:

4. A die is rolled twice. Let X_i denote the outcome of the ith roll, and put $S = X_1 + X_2$ and $D = |X_1 - X_2|$.

   (a) Show that $E(SD) = E(S)E(D)$.

   (b) Are S and D independent?

Solution:

(a)

$$E(S) = E(X_1 + X_2) = E(X_1) + E(X_2) = 3.5 + 3.5 = 7$$

$$E(D) = E(|X_1 - X_2|) = \sum_{i=0}^{5} P(|X_1 - X_2| = i) \cdot i = 0 \cdot \frac{6}{36} + 1 \cdot \frac{10}{36} + 2 \cdot \frac{8}{36} + 3 \cdot \frac{6}{36} + 4 \cdot \frac{4}{36} + 5 \cdot \frac{2}{36} = \frac{35}{18}$$

$$E(SD) = E((X_1 + X_2)|X_1 - X_2|) = E(X_1 \cdot |X_1 - X_2|) + E(X_2 \cdot |X_1 - X_2|) = 2 \cdot E(X_1 \cdot |X_1 - X_2|)$$

The probability function $P(X_1, |X_1 - X_2|)$ is given by the following table:

X_1 \ $\|X_1 - X_2\|$	0	1	2	3	4	5
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/18	1/36	1/36	1/36	0
3	1/36	1/18	1/18	1/36	0	0
4	1/36	1/18	1/18	1/36	0	0
5	1/36	1/18	1/36	1/36	1/36	0
6	1/36	1/36	1/36	1/36	1/36	1/36

Therefore:

$$E(SD) = 2 \cdot \sum_{i=1}^{6} \sum_{j=0}^{5} i \cdot j \cdot P(X_1 = i, |X_1 - X_2| = j) = \frac{245}{18} = 7 \cdot \frac{35}{18} = E(S) \cdot E(D) \blacksquare$$

(b)

$$P(X_1 + X_2 = 7, |X_1 - X_2| = 0) = 0 \neq \frac{6}{36} \cdot \frac{6}{36} = P(X_1 + X_2 = 7) \cdot P(|X_1 - X_2| = 0)$$

Therefore S and D are not independent.

3.3 Q 7:

7. A tetrahedron, with the numbers 1, 2, 3, and 4 marked on its faces, is tossed twice. Let X be the smaller of the two outcomes and Y the larger. Find $\rho(X, Y)$.

Solution:

1. $P(X = k) = \frac{9-2k}{16}$
2. $E(X) = \sum_{k=1}^{4} k \cdot \frac{9-2k}{16} = \frac{15}{8}$
3. $E(X^2) = \sum_{k=1}^{4} k^2 \cdot \frac{9-2k}{16} = \frac{35}{8}$
4. $V(X) = \frac{35}{8} - \left(\frac{15}{8}\right)^2 = \frac{55}{64}$
5. $P(Y = k) = \frac{2k-1}{16}$
6. $E(Y) = \sum_{k=1}^{4} k \cdot \frac{2k-1}{16} = \frac{25}{8}$
7. $E(Y^2) = \sum_{k=1}^{4} k^2 \cdot \frac{2k-1}{16} = \frac{85}{8}$
8. $V(Y) = \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64}$

The probability function $P(X, Y)$ is given by the following table:

X \ Y	1	2	3	4
1	1/16	1/8	1/8	1/8
2	0	1/16	1/8	1/8
3	0	0	1/16	1/8
4	0	0	0	1/16

$$E(XY) = P(1,1) + 2P(1,2) + 3P(1,3) + 4\Big(P(1,4) + P(2,2)\Big) +$$
$$+6P(2,3) + 8P(2,4) + 9P(3,3) + 12P(3,4) + 16P(4,4) =$$
$$= \frac{1}{16} + \frac{2}{8} + \frac{3}{8} + \frac{4}{8} + \frac{4}{16} + \frac{6}{8} + \frac{8}{8} + \frac{9}{16} + \frac{12}{8} + \frac{16}{16} = \frac{25}{4}$$

In total we get:

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{(V(X)V(Y))}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{(V(X)V(Y))}} = \frac{\frac{25}{4} - \frac{15}{8} \cdot \frac{25}{8}}{\sqrt{\frac{55}{64} \cdot \frac{55}{64}}} = \frac{\frac{25}{64}}{\frac{55}{64}} = \boxed{\frac{5}{11}}$$