Characterizations of general two-way Block Design

ST-302: Design, Planning and Analysis of Experiments

Outline of the Topic

1. Definition and example Connected GTWBD
2. Necessary and sufficient condition for estimability
3. Necessary and sufficient condition for connectedness
4. Concept of balanced design
5. Necessary and sufficient condition for balancedness
6. Concept of orthogonal block design
7. Necessary and sufficient condition for orthogonality
8. Example of orthogonal BD

Concept of Connected Block Design

Definition

A general two-way block design is said to be connected if and only if all elementary contrasts are estimable.

- A block design connected if all linear combinations of treatments $\underline{l}'\underline{\tau}$ are estimable whenever $\underline{l}'E_{v1} = 0$.
- If a block design not connected then it is said to be disconnected block design.
- If a block design can be divided in two or more subparts such that treatments appearing in one part do not appear in other part then the BD is disconnected design.

Concept of Connected Block Design

Physical meaning of connectedness of a design.

- A contrast in pair of treatment is estimable if they appear together in a block.
- Given any two treatment effects τ_{i1} and τ_{i2}, it is possible to have a chain of treatment effects $\tau_{i1}, \tau_{1j}, \tau_{2j}, \ldots, , \tau_{nj}, \tau_{i2}$ such that two adjoining treatments in this chain occur in the same block.
- To show the connectivity of treatments in a BD, we use connectivity graph in which pair of treatments are connected via lines if they appear together in a block.
- A design is connected if every treatment can be reached from every treatment via lines in the connectivity graph.

Example of disconnected BD

Consider a GTWBD with $v = 4$ treatments and $b = 5$ blocks as follows.

B_1	B_2	B_3	B_4	B_5
A	A	C	C	C
A	B	D	D	D
B	B			

Here the incidence matrix is as follows.

$$N = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Here observe that not every pair of treatment appears together in a block (for example A and C never appear together. Hence for this BD not all pairs of treatment contrasts are estimable and the design is disconnected

Example of disconnected BD

Consider a GTWBD with $v = 4$ treatments and $b = 5$ blocks as follows.

B_1	B_2	B_3	B_4	B_5
A	A	C	C	C
A	B	D	D	D
B	B			

- $1 \leftrightarrow 2$ (As 1 and 2 appear together in a block)
- $3 \leftrightarrow 4$ (As 3 and 4 appear together in a block)
- But $1 \not\leftrightarrow 3$ and $1 \not\leftrightarrow 4$ (As (1,3) and (1,4) do not appear together in a block)
- Also $2 \not\leftrightarrow 3and2 \not\leftrightarrow 4$ (As (2,3) and (2,4) do not appear together in a block)

Example of disconnected BD

Consider a GTWBD with $v = 4$ treatments and $b = 5$ blocks as follows.

B_1	B_2	B_3	B_4	B_5
A	A	C	C	C
A	B	D	D	D
B	B			

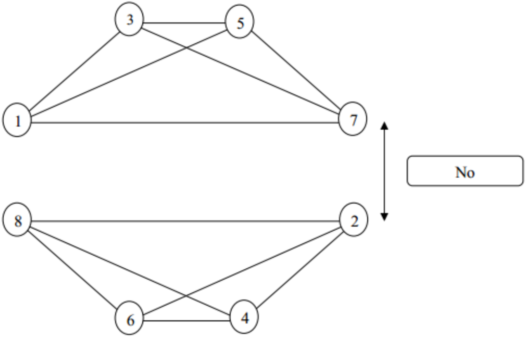
- $1 \leftrightarrow 2$ (As 1 and 2 appear together in a block)
- $3 \leftrightarrow 4$ (As 3 and 4 appear together in a block)
- But $1 \nleftrightarrow 3$ and $1 \nleftrightarrow 4$ (As (1,3) and (1,4) do not appear together in a block)
- Also $2 \nleftrightarrow 3 and 2 \nleftrightarrow 4$ (As (2,3) and (2,4) do not appear together in a block)

Example of disconnected BD

Consider a GTWBD with $v = 8$ and $b = 8$ as follows.

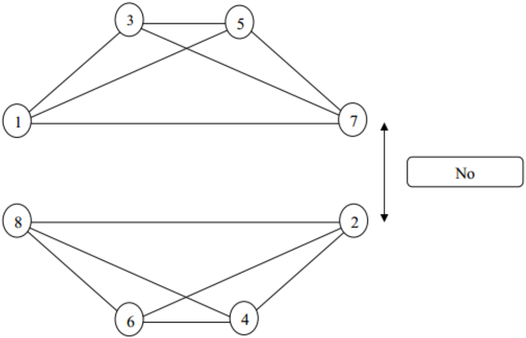
B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
1	2	3	4	5	6	7	8
3	4	5	6	7	8	1	2
5	6	7	8	1	2	3	4

Connectivity graph for the design



Example of disconnected BD

The blocks of the design represented graphically:



1. Note that it is not possible to reach the treatment, 7 from 2, 3 from 4, 6 from 1 etc. So the design is not connected.
2. A design is connected if every treatment can be reached from every treatment via lines in the connectivity graph.

Example of connected BD

Consider a GTWBD with $v = 5$ treatments and $b = 5$ blocks as follows.

B_1	B_2	B_3	B_4	B_5
A	B	C	D	E
B	C	D	E	A
C	D	E	A	B
D	E	A	B	C

Here every pair of treatment appear together in a block. Hence the BD all pairs of treatment contrasts are estimable and the design is connected

Necessary and sufficient condition for estimability

Theorem

For two-way BD a linear parametric function of treatment effects $\underline{l}'\underline{\tau}$ is estimable if and only if the vector of coefficients in a linear parametric function $\underline{l}'$ is linear combination of rows of C-matrix in the reduced normal equations $\underline{Q} = C\hat{\underline{\tau}}$.

That is,

$\underline{l}'\underline{\tau}$ *is estimable iff* $\underline{l}' = \underline{m}'C$ *for some* $\underline{m} \in \Re^v$

Necessary and sufficient condition for estimability

Proof

Reduced normal equations in terms of treatment effects are
$\underline{Q} = C\underline{\hat{\tau}}$

$$
\begin{aligned}
\Rightarrow E\left(\underline{Q}\right) &= C\underline{\tau} \\
&= \begin{pmatrix} C'_{(1)}\underline{\tau} \\ C'_{(2)}\underline{\tau} \\ \vdots \\ C'_{(v)}\underline{\tau} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_v \end{pmatrix} \\
&= \underline{\xi}
\end{aligned}
$$

That is $\underline{\xi}$ is collection of v estimable parametric function in τ's and $\underline{Q}$ is BLUE of $\underline{\xi} = C\underline{\tau}$

Necessary and sufficient condition for estimability

$\underline{l}'\underline{\tau}$ is estimable if and only if it is linear combination of ξ_i's.

$$\begin{aligned}
\Leftrightarrow \underline{l}'\underline{\tau} &= \sum_{i=1}^{v} m_i \xi_i \\
&= \underline{m}'\underline{\xi} \\
&= \underline{m}' C \underline{\tau} \qquad \because \underline{\xi} = C\underline{\tau} \\
\Leftrightarrow \underline{l}' &= \underline{m}' C
\end{aligned}$$

$\Leftrightarrow \underline{l}'$ is linear combination of rows of C-matrix.

N and S condition for connectedness of BD

Theorem

A block design (BD) is connected if and only if $\rho(C) = v - 1$

Claim: A BD is connected $\Leftrightarrow \rho(C) = v - 1$

If part: If BD is connected $\Rightarrow \rho(C) = v - 1$

Let the block design be connected.

$\Rightarrow$ All contrasts in τ's are estimable.

Let $\underline{\lambda}'_{(i)}\underline{\tau}, i = 1, 2, ..., v-1$ be l.i. estimable parametric functions and

$$\Lambda = \begin{pmatrix} \underline{\lambda}'_{(1)} \\ \underline{\lambda}'_{(2)} \\ \vdots \\ \underline{\lambda}'_{(v-1)} \end{pmatrix}$$ be $(v-1) \times v$ matrix with all $(v-1)$ rows l.i.

N and S condition for connectedness of BD

Then, $\rho(\Lambda) = v - 1$
Now $\underline{\lambda}'_{(i)}$ must be linear combination of rows of C matrix.
That is,

$$\underline{\lambda}'_{(i)} = \underline{m}'_{(i)} C, i = 1, 2, ..., v - 1$$

Hence,

$$\Lambda = \begin{pmatrix} \underline{\lambda}'_{(1)} \\ \underline{\lambda}'_{(2)} \\ \vdots \\ \underline{\lambda}'_{(v-1)} \end{pmatrix} = \begin{pmatrix} \underline{m}'_{(1)} \\ \underline{m}'_{(2)} \\ \vdots \\ \underline{m}'_{(v-1)} \end{pmatrix} C = M_{(v-1)\times v} * C_{v\times v}$$

N and S condition for connectedness of BD

$$
\begin{aligned}
\Lambda &= M * C \\
\Rightarrow (v-1) &= \rho(\Lambda) \\
&= \rho(M * C) \\
&\leq \rho(C) \\
\Rightarrow (v-1) &\leq \rho(C) \\
\rho(C) &\leq v-1 \qquad \because \text{ C matrix is singular} \\
\text{Hence,} v-1 \leq \rho(C) &\leq v-1
\end{aligned}
$$

$$\Rightarrow \rho(C) = v-1$$

Thus if BD is connected $\Rightarrow \rho(C) = v-1$

N and S condition for connectedness of BD

Only if part: If $\rho(C) = v - 1 \Rightarrow$ A BD is connected
Let $\rho(C) = v - 1$ and $C'_{(i)}$ be the i^{th} row of C-matrix
Now $\quad E(\underline{Q}) = C\underline{\tau} \Rightarrow E(Q_i) = C'_{(i)}\underline{\tau}, i = 1, 2, ..., v$
Further $CE_{v1} = \underline{0} \Rightarrow C'_{(i)}CE_{v1} = 0, i = 1, 2, ..., v$

- All rows of C matrix add to zero.
- Hence $C'_{(i)}\underline{\tau}$ represent contrast in $\underline{\tau}'s$
- $C'_{(i)}\underline{\tau}$ is estimable as $E(Q_i) = C'_{(i)}\underline{\tau}, i = 1, 2, ..., v$
- $C'_{(i)}\underline{\tau}, i = 1, 2, ..., v$ are $v - 1$ linearly independent estimable contrast in $\underline{\tau}'s$
- As all contrasts are estimable $\Rightarrow$ all elementary contrasts in $\underline{\tau}'s$ are estimable.
- A block design is connected.

Thus if $\rho(C) = v - 1 \Rightarrow$ the BD is connected.

Balanced design

Definition

A connected block design is said to be balanced design if all elementary contrasts are estimated with equal variance or all normalized contrasts are estimated with equal variance.

Example: Consider a two-way block design with Factor-A at p levels and Factor-B at q levels. Such a design is balanced due to the following.

- $v(\alpha_i \hat{-} \alpha_u) = var(\overline{y}_{i.} - \overline{y}_{u.}) = 2\sigma^2/q \quad \forall i \neq u$.
- $v(\beta_j \hat{-} \beta_k) = var(\overline{y}_{.j} - \overline{y}_{.k}) = 2\sigma^2/p \quad \forall j \neq k$
- All elementary contrasts in treatment effects are estimated with equal variance.
- All elementary contrasts in block effects are estimated with equal variance.

Necessary and sufficient condition for balancedness

Theorem

A connected BD is variance balanced if and only if all non-zero eigen values of C-matrix are equal.

Proof:

Consider a connected Block design.

$\Leftrightarrow \rho(C) = v - 1$

$\Rightarrow v - 1$ eigen values of C-matrix would be non-zero and one eigen value is zero.

Let $\theta_1, \theta_2, ..., \theta_{v-1}$ be non-zero eigen values of C-matrix and $\underline{\xi}_1, \underline{\xi}_2, ..., \underline{\xi}_{v-1}$ be the corresponding orthonormal eigen vectors.

Let $\Lambda = diag(\theta_1, \theta_2, ..., \theta_{v-1})$ be a $(v-1) \times (v-1)$ matrix.

Then $\underline{\xi}_i' \underline{\xi}_j = 0$ for all $i \neq j$ and $\underline{\xi}_i' \underline{\xi}_i = 1$ for $i = 1, 2, ..., v - 1$

Let $\theta_v = 0$ and $\underline{\xi}_v = E_{v1}$ be the corresponding eigen vector.

Necessary and sufficient condition for balancedness

Proof...

Further $\underline{\xi}_i'\underline{\xi}_v = \underline{\xi}_i' E_{v1} = 0$ for $i = 1, 2, ..., v-1$

$\Rightarrow \underline{\xi}_i'\underline{\tau}$ is normalized contrast.

Thus $\underline{\xi}_i'\underline{\tau}, i = 1, 2, ..., v-1$ are linearly independent normalized contrast which are estimable. Then

$$H\underline{\tau} = \begin{pmatrix} \underline{\xi}_{(1)}' \\ \underline{\xi}_{(2)}' \\ \vdots \\ \underline{\xi}_{(v-1)}' \end{pmatrix} \underline{\tau} \text{ is estimable and its BLUE is } H\hat{\underline{\tau}}$$

As $\underline{\xi}_i'\underline{\tau}$ is estimable $\Rightarrow$ BLUE of $\underline{\xi}_i'\underline{\tau} = \underline{\xi}_i'\hat{\underline{\tau}}$

Hence $cov(H\hat{\underline{\tau}}) = Hginv(C)H'\sigma^2 = \Lambda^{-1}\sigma^2$

$cov(H\hat{\underline{\tau}}) = diag(\theta_1^{-1}, \theta_2^{-1}, ..., \theta_{v-1}^{-1})\sigma^2$

Necessary and sufficient condition for balancedness

Proof...

$\Rightarrow var(\underline{\xi}_i'\underline{\hat{\tau}}) = \theta_i^{-1}\sigma^2, i = 1, 2, ..., v-1$

Now by definition, a BD is variance balanced if and only if all normalized contrasts are estimable with equal variance.

Let BD be balanced.

$\Leftrightarrow var(\underline{\xi}_1'\underline{\hat{\tau}}) = var(\underline{\xi}_2'\underline{\hat{\tau}}) = ... = var(\underline{\xi}_{v-1}'\underline{\hat{\tau}})$

$\Leftrightarrow \theta_1^{-1} = \theta_2^{-1} = ... = \theta_{v-1}^{-1}$

$\Leftrightarrow \theta_1 = \theta_2 = ... = \theta_{v-1}$

$\Leftrightarrow$ All non-zero eigen values of C-matrix are equal.

Concept of Orthogonal Block Design

Definition

A general two-way block design is said to be orthogonal if every adjusted treatment total is orthogonal to every adjusted block total.

That is,

A BD is orthogonal if $Cov(Q_i, P_j) = 0 \quad \forall \quad i, j$ or

A BD is orthogonal if $Cov(\underline{Q}, \underline{P}) = O_{v \times b}$

- Designs which do not satisfy this condition are called non-orthogonal.
- A block design with at least one zero-entry in its incidence matrix is called an incomplete block design.
- It is clear from this result that if at least one entry of N is zero, the design cannot be orthogonal.

Necessary and sufficient condition for orthogonality

Theorem

A two-way BD is orthogonal if $n_{ij} = r_i \times k_j/n \quad \forall \quad i, j$
that is, $N = \underline{r} \times \underline{k}'/n$

Proof:

Let a connected BD be orthogonal.

A BD is connected

$\Rightarrow \rho(C) = v - 1$

$\Rightarrow \nu(C) = Nullity(C) = 1$

$\Rightarrow$ There is only one l.i. vector orthogonal to rows of C

But $E_{v1} \quad \perp \quad C_{i*} \quad \forall \quad i = 1, 2, ..., v \qquad \because C * E_{v1} = \underline{0}$

$\Rightarrow$ Any other vector orthogonal to rows of C would be l.d. with E_{v1}

Necessary and sufficient condition for orthogonality

Now a block design is orthogonal

$\Rightarrow Cov(\underline{Q}, \underline{P}) = O_{v \times b}$

$\Rightarrow -CR^{-1}N = O$

$\Rightarrow C(R^{-1}N) = O$

$\Rightarrow$ Columns of $R^{-1}N$ are orthogonal to rows of C-matrix

$\Rightarrow (R^{-1}N)_{*j} \quad \perp \quad C_{i*} \quad \forall \quad i = 1, 2, ..., v; j = 1, 2, ..., b$

$\Rightarrow (R^{-1}N)_{*j} \propto \quad E_{v1} \quad \forall \quad j = 1, 2, ..., b$

Necessary and sufficient condition for orthogonality

Now the j^{th} column of $R^{-1}N = (R^{-1}N)_{*j}$ and has elements as follows.

$$(R^{-1}N)_{*j} = \begin{pmatrix} n_{1j}/r_1 \\ n_{2j}/r_2 \\ \vdots \\ n_{vj}/r_v \end{pmatrix} \propto E_{v1}, j = 1, 2, ..., b$$

$$\begin{pmatrix} n_{1j}/r_1 \\ n_{2j}/r_2 \\ \vdots \\ n_{vj}/r_v \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{pmatrix}, j = 1, 2, ..., b \quad \text{for some real constant } \alpha$$

Necessary and sufficient condition for orthogonality

$$\Rightarrow n_{ij}/r_i = \alpha, i = 1, 2, ..., v$$
$$\Rightarrow n_{ij} = r_i\alpha, i = 1, 2, ..., v$$
$$\Rightarrow \sum_{i=1}^{v} n_{ij} = \alpha \sum_{i=1}^{v} r_i$$
$$\Rightarrow n_{.j} = \alpha n$$
$$\Rightarrow \alpha = k_j/n \quad \because n_{.j} = k_j$$

$$\begin{pmatrix} n_{1j}/r_1 \\ n_{2j}/r_2 \\ \vdots \\ n_{vj}/r_v \end{pmatrix} = \begin{pmatrix} k_j/n \\ k_j/n \\ \vdots \\ k_j/n \end{pmatrix}, j = 1, 2, ..., b$$

Necessary and sufficient condition for orthogonality

$$\Rightarrow n_{ij}/r_i = k_j/n, i = 1, 2, ..., v, , j = 1, 2, ..., b$$
$$\Rightarrow n_{ij} = r_i \times k_j/n, i = 1, 2, ..., v, , j = 1, 2, ..., b$$
$$\Rightarrow N = \underline{rk}'/n$$

Thus if a connected BD is orthogonal
$\Rightarrow n_{ij} = r_i \times k_j/n, i = 1, 2, ..., v, , j = 1, 2, ..., b$

Example of orthogonal BD

Consider RBD with v treatments and b blocks. Then for this BD

- Number of observations, $n = bv$
- Number of times i^{th} treatment appears in j^{th} block $= n_{ij} = 1$
- Replication of i^{th} treatment$=r_i = b, i = 1, 2, ..., v$
- Size of j^{th} block $= k_j = v, j = 1, 2, ..., b$
- $\underline{r} = bE_{v1}$
- $\underline{k} = vE_{b1}$
- $N = E_{vb}$

The RHS of condition of orthogonality $N = \underline{rk}'/n$ for this design is

$$\begin{aligned} RHS &= \underline{rk}'/n \\ &= (bE_{v1})(vE_{1b})/n \\ &= bvE_{v1}E_{1b}/n = bvE_{vb}/bv \\ &= Evb = N = LHS \end{aligned}$$

Example of non-orthogonal BD

Consider RBD with v treatments and b blocks but one treatment missing in one block. Then for this BD

- Number of times i^{th} treatment appears in j^{th} block $= n_{ij} = 0$ for that particular treatment and block
- The condition of orthogonality $N = \underline{rk}'/n$ will not be satisfied.

Hence RBD with one treatment missing in one block is not an orthogonal BD.

Example of non-orthogonal BD

- Any incomplete block design is always non-orthogonal as $= n_{ij} = 0$ for some pair (i, j)
- BIBD is also a non-orthogonal block design
- PBIBD is also a non-orthogonal block design