ST-302
Design, Planning and Analysis of Experiments

Dr. (Mrs.) Kirtee Kiran Kamalja
Department of Statistics, School of Mathematical Sciences,
Kavayitri Bahinbai Chaudhari North Maharashtra University, Jalgaon

Topic 8: Analysis of Covariance

What is Analysis of Covariance?

Objective: To study the effect of some factors on the response variable as usual. But information about some correlated variables with response variable is additionally available. This additional information may be useful to improve the results of main objective.

- Use when you have some categorical factors and some quantitative predictors.
- ANCOVA is really "ANOVA with covariates" or, more simply, a combination of ANOVA and regression.

What is Analysis of Covariance?

- ANCOVA model is mixture/combination of ANOVA (non-full rank model) and Regression models (full rank model) .
- It will contain the qualitative variables as factors (as in ANOVA model) and quantitative variables as covariate (as in regression model)

What is Analysis of Covariance?

- Continuous variables are referred to as covariates or concomitant variables
- Concomitant variables are not necessarily of primary interest, but still their inclusion in the model will help explain more of the response, and hence reduce the error variance.
- In some situations, if important covariates are not included in model may yield misleading results.

Idea behind Analysis of Covariance?

- ANOVA models are based on dummy/categorical/qualitative variables while regression models are based on quantitative variables.
- Note that the indicator variables do not provide as much information as the quantitative variables.
- If the model includes some quantitative variables they will provide more information.
- Covariates (concomitant variables) can reduce unexplained variation (SSE) and thereby reduce the MSE.

Idea behind Analysis of Covariance?

- F -statistic depends on the SSE as $F - statistic = \frac{\left(\frac{SSH_0}{df_{H_0}}\right)}{\left(\frac{SSE}{df_E}\right)}$ and Smaller SSE ⇒large $F - Statistic$ ⇒ more chance of rejection
- Inclusion of covariate in model will result in increasing power for testing. Hence it is absolutely necessary in order to get accurate analysis.

Idea behind Analysis of Covariance?

- ANCOVA is useful for improving the precision of an experiment.
- Suppose response Y is linearly related to covariate X (or concomitant variable). Suppose experimenter cannot control X but can observe it.
- ANCOVA involves adjusting Y for the effect of X. If such an adjustment is not made, then the X can inflate the error mean square and makes the true differences is Y due to treatment harder to detect.

Assumptions

- Ideally the covariate will not be related to the treatment variables (factors) but correlated with the response variable.
- We assume that the covariate will be linearly related to the response and that the relationship will be the same for all levels of the factor (no interaction between covariate and factor)

Example 1

- Suppose our interest is to compare several different kinds of feed for their ability to put weight on for certain dairy animals.
- If we use ANOVA, then we our response variable will be the final weights at the end of experiment or increase in weights.
- However, final weights of the animals depend upon the initial weight of the animals at the beginning of the experiment as well as upon the difference in feeds.
- Thus we must consider here an initial weight as an covariate along with treatments.
- **Use of ANCOVA models enables us to adjust or correct these initial differences**

Example 1…

- In this example growth rate of the animals may also depend upon the **age of the animals**
- Thus we can consider here an age of animal one covariate along with treatments.
- **Use of ANCOVA models enables us to adjust or correct these initial differences**

Example: One way ANCOVA

- Factor A has k levels / k treatments are to be tested for significance

- Each treatment is replicated n times

- y_{ij} : the j^{th} observation on the **response variable** in the i^{th} group

- x_{ij} : the j^{th} observation on **the covariate** in the i^{th} group

One-way classification model with one concomitant variable

Model:

$$y_{ij} = \mu + \tau_i + \gamma x_{ij} + \varepsilon_{ij}, \quad i = 1,2,\dots,k; j = 1,2,\dots,n$$

y_{ij}: the j^{th} observation on the **response variable** in the i^{th} group

μ: Common effect

τ_i: Effect of i^{th} treatment

-γ: Regressor parameter associated with covariate

x_{ij}: the j^{th} observation on **the covariate** in the i^{th} group

ε_{ij}: Random error associated with y_{ij}

Assumptions

- Mean error is zero i.e $E(\varepsilon_{ij}) = 0$
- Variance of errors is constant i.e $var(\varepsilon_{ij}) = \sigma^2 \ for\ all\ i, j$
- Covariances between errors is zero $var(\underline{\varepsilon}) = \sigma^2 I_n$
- $\varepsilon_{ij} \sim N(0, \sigma^2)$

Implications

- $E(y_{ij}) = \mu + \tau_i + \gamma x_{ij}$
- $var(y_{ij}) = \sigma^2$
- y_{ij} are independently distributed but not identical
- $y_{ij} \sim N(\mu + \tau_i + \gamma x_{ij}, \sigma^2)$

Data for one-way ANCOVA model

Factor/ Group	Observations on the **response variable**				Observations on the **covariate**			
A_1	y_{11}	y_{12}	$\dots$	y_{1n}	x_{11}	x_{12}	$\dots$	x_{1n}
A_2	y_{21}	y_{22}	$\dots$	y_{2n}	x_{21}	x_{22}	$\dots$	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
A_k	y_{k1}	y_{k2}	$\dots$	y_{kn}	x_{k1}	x_{k2}	$\dots$	x_{kn}

Model details

- Number of observations=nk
- Number of parameters=$k+2$

$$\mu,$$

$$\tau_1, \tau_2, \dots, \tau_k,$$

$$\gamma$$

- Here $nk > k + 2$ ($n > p$ assumption in GLM)

General Linear model for ANCOVA

Model: $\underline{Y} = X\underline{\beta} + Z\underline{\gamma} + \underline{\varepsilon}$

$\underline{Y}: n \times 1$ vector of known observations

$X: n \times p$ matrix whose columns are values of indicator variables

$Z: n \times m$ matrix whose columns are values of physical variables

$\underline{\beta}: p \times 1$ vector of unknown parameters

$\underline{\gamma}: m \times 1$ vector of covariate/Regressor parameters

$\underline{\varepsilon}: n \times 1$ vector of random error associated with y_{ij}

Assumptions in General Linear model for ANCOVA

Model: $\underline{Y} = X\underline{\beta} + Z\underline{\gamma} + \underline{\varepsilon}$

- All the m covariates or regressor variable are independent of each other.
- That is all m columns of Z matrix are linearly independent.
- $\rho(Z) = m$
- $\underline{\varepsilon} \sim N_n(\underline{0}, \sigma^2 I_n)$

Reparametrization of the Model

$$\underline{Y} = X\underline{\beta} + Z\underline{\gamma} + \underline{\varepsilon}$$

$$= X\underline{\beta} + Z\underline{\gamma} - XS^{-}X'Z\underline{\gamma} + XS^{-}X'Z\underline{\gamma} + \underline{\varepsilon}$$

$$= X\underline{\beta} - XS^{-}X'Z\underline{\gamma} + Z\underline{\gamma} + XS^{-}X'Z\underline{\gamma} + \underline{\varepsilon}$$

$$= X(\underline{\beta} - S^{-}X'Z\underline{\gamma}) + (I - XS^{-}X')Z\underline{\gamma} + \underline{\varepsilon}$$

$$= X\underline{\beta}^{*} + (I - XS^{-}X')Z\underline{\gamma} + \underline{\varepsilon}$$

Here $\underline{\beta}^{*} = \underline{\beta} - S^{-}X'Z\underline{\gamma}$

Characterization of $P = XS^{-}X'$ matrix

- $P = XS^{-}X'$ is $n \times n$ symmetric matrix, that is, $XS^{-}X' = XS^{-\prime}X'$
- P is idempotent with $\rho(P) = \rho(X'X)$
- $(I - P)$ is also symmetric matrix, i.e. $I - XS^{-}X' = I - XS^{-\prime}X'$
- $(I - P)$ is idempotent matrix, that is, $(I - XS^{-}X')^2 = (I - XS^{-}X')$
- P and $(I - P)$ are orthogonal to each other i.e. $P(I - P) = O$
- $(I - P)$ and X are orthogonal to each other. That is,

  $(I - XS^{-}X')X = O$ or $X'(I - XS^{-}X') = O$
- P is singular with $\rho(I - XS^{-}X') = n - \rho(X'X) = n - \rho(X)$
- Further $\rho(P) + \rho(I - P) = n$

Purpose of Reparametrization of the Model

- The parameters $\begin{pmatrix} \underline{\beta} \\ \hline \underline{\gamma} \end{pmatrix}$ are transformed to a new set of parameters $\begin{pmatrix} \underline{\beta}^* \\ \hline \underline{\gamma} \end{pmatrix}$ where $\underline{\beta}^* = \underline{\beta} - S^- X' Z \underline{\gamma}$.
- The coefficient matrices X and Z of parameters $\underline{\beta}$ and $\underline{\gamma}$ were not orthogonal for original model $\underline{Y} = X\underline{\beta} + Z\underline{\gamma} + \underline{\varepsilon}$.
- While the coefficient matrices X and $(I - XS^- X')Z$ of new transformed parameters $\underline{\beta}^*$ and $\underline{\gamma}$ are orthogonal for the reparametrized model $\underline{Y} = X\underline{\beta}^* + (I - XS^- X')Z\underline{\gamma} + \underline{\varepsilon}$

Derivation of normal equations for ANCOVA Model

$$\underline{Y} = X\underline{\beta}^* + (I - XS^-X')Z\underline{\gamma} + \underline{\varepsilon}$$

$$= \begin{pmatrix} X & (I - XS^-X')Z \end{pmatrix} \begin{pmatrix} \underline{\beta}^* \\ \underline{\gamma} \end{pmatrix} + \underline{\varepsilon}$$

Normal equations in GLM setup are : $X'\underline{Y} = X'X\,\underline{\hat{\beta}}$

$$\begin{pmatrix} X' \\ Z'(I - XS^-X') \end{pmatrix} \underline{Y} = \begin{pmatrix} X' \\ Z'(I - XS^-X') \end{pmatrix} \begin{pmatrix} X & (I - XS^-X')Z \end{pmatrix} \begin{pmatrix} \underline{\hat{\beta}}^* \\ \underline{\hat{\gamma}} \end{pmatrix}$$

Derivation of normal equations for ANCOVA Model…

$$\Rightarrow \quad \begin{pmatrix} X' \\ Z'(I - XS^{-}X') \end{pmatrix} \underline{Y} = \begin{pmatrix} X' \\ Z'(I - XS^{-}X') \end{pmatrix} \begin{pmatrix} X & (I - XS^{-}X')Z \end{pmatrix} \begin{pmatrix} \underline{\hat{\beta}^{*}} \\ \underline{\hat{\gamma}} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} X'\underline{Y} \\ Z'(I - XS^{-}X')\underline{Y} \end{pmatrix} = \begin{pmatrix} X'X & \boldsymbol{X}'(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{S}^{-}\boldsymbol{X}')Z \\ Z'(\boldsymbol{I} - \boldsymbol{X}\boldsymbol{S}^{-}\boldsymbol{X}')\boldsymbol{X} & Z'(I - XS^{-}X')Z \end{pmatrix} \begin{pmatrix} \underline{\hat{\beta}^{*}} \\ \underline{\hat{\gamma}} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} X'\underline{Y} \\ Z'(I - XS^{-}X')\underline{Y} \end{pmatrix} = \begin{pmatrix} X'X & O \\ O & Z'(I - XS^{-}X')Z \end{pmatrix} \begin{pmatrix} \underline{\hat{\beta}^{*}} \\ \underline{\hat{\gamma}} \end{pmatrix}$$

$$\Rightarrow \qquad X'\underline{Y} = X'X\underline{\hat{\beta}^{*}} \qquad \text{(A)}$$

$$Z'(I - XS^{-}X')\underline{Y} = Z'(I - XS^{-}X')Z\underline{\hat{\gamma}} \qquad \text{(B)}$$

Derivation of normal equations for ANCOVA Model…

$$\Rightarrow \qquad X'\underline{Y} = X'X\underline{\hat{\beta}}^{*} \qquad \text{(A)}$$

$$Z'(I - XS^{-}X')\underline{Y} = Z'(I - XS^{-}X')Z\underline{\hat{\gamma}} \qquad \text{(B)}$$

- (A) Contains p Equations only in terms of $\underline{\hat{\beta}}^{*}$.
- (B) Contains m Equations only in terms of $\underline{\hat{\gamma}}$.
- Due to orthogonality of X and $(I - P)$ the normal equations are not in terms of both parameters.

Derivation of normal equations for ANCOVA Model…

- Let $E_{yy} = \underline{Y}'(I - XS^{-}X')\underline{Y}$: SSE for model $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$
- Then

$$E_{zy} = Z'(I - XS^{-}X')\underline{Y} : m \times 1 \text{ vector}$$

$$E_{yz} = \underline{Y}'(I - XS^{-}X')Z : 1 \times m \text{ vector}$$

$$E_{zz} = Z'(I - XS^{-}X')Z : m \times m \text{ matrix}$$

- Note the following.
  - ➢ $E_{zy} = E'_{yz}$
  - ➢ E_{zz} is symmetric and non-singular with $\rho(E_{zz}) = m$
  - ➢ E_{zz}^{-1} exists

Solutions of normal equations for ANCOVA Model

$$X'\underline{Y} = X'X\underline{\hat{\beta}}^* \qquad (A)$$

$$Z'(I - XS^{-}X')\underline{Y} = Z'(I - XS^{-}X')Z\underline{\hat{\gamma}} \qquad (B)$$

- Hence the normal equations are:

$$X'\underline{Y} = X'X\underline{\hat{\beta}}^* \qquad (A)$$

$$E_{zy} = E_{zz}\underline{\hat{\gamma}} \qquad (B)$$

- Their Solutions are:

(A) $\Rightarrow$ $\quad \underline{\hat{\beta}}^* = (X'X)^{-}X'\underline{Y}$

(B) $\Rightarrow$ $\quad \underline{\hat{\gamma}} = E_{zz}^{-1}E_{zy}$

Solutions of normal equations for ANCOVA Model…

To obtain solution for the original parameter $\underline{\beta}$ use the following transformation done while reprametrization.

$$\underline{\beta}^* = \underline{\beta} - S^- X' Z \underline{\gamma}$$

$$\Rightarrow \quad \underline{\beta} = \underline{\beta}^* + S^- X' Z \underline{\gamma}$$

$$\Rightarrow \quad \underline{\hat{\beta}} = \underline{\hat{\beta}}^* + S^- X' Z \underline{\hat{\gamma}}$$

$$\Rightarrow \quad \underline{\hat{\beta}} = (X'X)^- X' \underline{Y} + S^- X' Z \underline{\hat{\gamma}} \qquad \text{use } \underline{\hat{\beta}}^* = (X'X)^- X' \underline{Y}$$

$$\Rightarrow \quad \underline{\widehat{\boldsymbol{\beta}}} = (\boldsymbol{X'X})^- \boldsymbol{X'}\underline{\boldsymbol{Y}} + \boldsymbol{S^- X' Z E_{zz}^{-1} E_{zy}} \quad \text{use } \underline{\hat{\gamma}} = E_{zz}^{-1} E_{zy}$$

where $E_{zy} = Z'(I - XS^- X')\underline{Y}$ and $E_{zz} = Z'(I - XS^- X')Z$

Estimability of linear parametric function

- ANCOVA model is mixture of two models: One model is FRM (with parameters $\underline{\gamma}$) and other is NFRM (with parameters $\underline{\beta}$)
- As coefficient matrix Z of $\underline{\gamma}$ is FCRM, all individual elements of $\underline{\gamma}$ are estimable and will have unique solution.
- Hence $\underline{\gamma}$ are estimable as these correspond to regression parameters and its BLUE is $\underline{\hat{\gamma}} = E_{zz}^{-1} E_{zy}$

Estimability of linear parametric function…

- For estimability of $\underline{\lambda}'\underline{\beta}$ consider the following two models.

Model 1: Model without covariate : $\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$

Model 2: Model with covariates : $\underline{Y} = X\underline{\beta} + Z\underline{\gamma} + \underline{\varepsilon}$

- In GLM setup, $\underline{\lambda}'\underline{\beta}$ is estimable in Model 1 if $\underline{\lambda}' = \underline{\lambda}'H$ and its BLUE is $\underline{\lambda}'\underline{\hat{\beta}}$ where $\underline{\hat{\beta}}$ is solution of normal equations $X'\underline{Y} = X'X\underline{\hat{\beta}}$
- Now $\underline{\lambda}'\underline{\beta}$ is estimable in Model 2 if it is estimable in Model 1 and its BLUE will be:

$$\underline{\lambda}'\underline{\hat{\beta}} = \underline{\lambda}'\left(\underline{\hat{\beta}}^* + S^- X'Z\underline{\hat{\gamma}}\right) \text{ where } \underline{\hat{\beta}}^* = (X'X)^- X'\underline{Y}$$

VCM of BLUE of estimable linear parametric functions

- $\underline{\gamma}$ is estimable and its BLUE is $\underline{\hat{\gamma}} = E_{zz}^{-1} E_{zy}$
- First obtain $cov\left(E_{zy}\right)$

$cov\left(E_{zy}\right)$

$= cov(Z'(I - XS^{-}X')\underline{Y})$

$= Z'(I - XS^{-}X')\, cov\left(\underline{Y}\right)\left(Z'(I - XS^{-}X')\right)'$

$= Z'(I - XS^{-}X')\left(Z'(I - XS^{-}X')\right)' \sigma^2 \quad \because\ cov\left(\underline{Y}\right) = \sigma^2 I_n$

$= Z'(I - XS^{-}X')(I - XS^{-}X')Z\sigma^2 \quad \because\ (I - P)$ is symmetric

$= Z'(I - XS^{-}X')Z\sigma^2 \quad \because\ (I - P)$ is idempotent

$= E_{zz}\sigma^2 \quad \because\ \mathrm{E}_{zz} = Z'(I - XS^{-}X')Z$

Variance of BLUE of estimable linear parametric functions …

$$cov\left(\underline{\hat{\gamma}}\right)$$

$= cov(E_{zz}^{-1} E_{zy})$ here E_{zy} is random

$= E_{zz}^{-1} cov(E_{zy}) E_{zz}^{-1}$ as E_{zz}^{-1} is symmetric

$= E_{zz}^{-1} E_{zz} \sigma^2 E_{zz}^{-1}$ as $cov\left(E_{zy}\right) = E_{zz}\sigma^2$

$= E_{zz}^{-1}\sigma^2$

VCM of BLUE of estimable linear parametric functions …

$$cov\left(\underline{\hat{\beta}}^*\right)$$

$$= cov(S^- X' \underline{Y}) \qquad\qquad S^- = (X'X)^-$$

$$= S^- S S^{-\prime} \sigma^2$$

$$= H S^{-\prime} \sigma^2$$

VCM of BLUE of estimable linear parametric functions …

$$cov\left(\underline{\hat{\gamma}}, \underline{\hat{\beta}}^*\right)$$

$$= cov(E_{zz}^{-1}E_{zy}, S^{-}X'\underline{Y})$$

$$= cov(E_{zz}^{-1}Z'(I-P)\underline{Y}, S^{-}X'\underline{Y})$$

$$= E_{zz}^{-1}Z'(I-P)cov\left(\underline{Y}\right)(S^{-}X')'$$

$$= E_{zz}^{-1}Z'(I-P)(S^{-}X')'\,\sigma^2 \qquad cov\left(\underline{Y}\right) = \sigma^2 I_n$$

$$= E_{zz}^{-1}Z'(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{X}(S^{-})'\,\sigma^2 \qquad (I-P)X = O$$

$$= O$$

Meaning: Each $\hat{\beta}_i$ is independently distributed of $\hat{\gamma}_j$ i.e. $cov\left(\hat{\beta}_i, \hat{\gamma}_j\right) = 0, i = 1,2, \ldots, p; j = 1,2, \ldots, m$

SSE for ANCOVA model $\underline{\boldsymbol{Y}} = \boldsymbol{X}\underline{\boldsymbol{\beta}}^* + (\boldsymbol{I} - \boldsymbol{X}\boldsymbol{S}^-\boldsymbol{X}')\boldsymbol{Z}\underline{\boldsymbol{\gamma}} + \underline{\boldsymbol{\varepsilon}}$

$$SSE = \underline{Y}'\underline{Y} - \begin{pmatrix} \underline{\hat{\beta}^*} \\ \underline{\hat{\gamma}} \end{pmatrix}' \underline{q}$$

$\underline{q} = LHS\ of\ normal\ equations$

$$= \underline{Y}'\underline{Y} - (\underline{\hat{\beta}}^{*\prime} \quad \underline{\hat{\gamma}}') \begin{pmatrix} X'\underline{Y} \\ E_{zy} \end{pmatrix}$$

$$\underline{q} = \begin{pmatrix} X'\underline{Y} \\ Z'(I - XS^-X')\underline{Y} \end{pmatrix}$$

$$= \underline{Y}'\underline{Y} - ((S^-X'\underline{Y})' \quad (E_{zz}^{-1}E_{zy})') \begin{pmatrix} X'\underline{Y} \\ E_{zy} \end{pmatrix}$$

$$= \underline{Y}'\underline{Y} - (\underline{Y}'XS^{-\prime} \quad E_{yz}E_{zz}^{-1}) \begin{pmatrix} X'\underline{Y} \\ E_{zy} \end{pmatrix}$$

SSE for ANCOVA model $\underline{\boldsymbol{Y}} = \boldsymbol{X}\underline{\boldsymbol{\beta}}^{*} + (\boldsymbol{I} - \boldsymbol{X}\boldsymbol{S}^{-}\boldsymbol{X}')\boldsymbol{Z}\underline{\boldsymbol{\gamma}} + \underline{\boldsymbol{\varepsilon}}$

$$SSE = \underline{Y}'\underline{Y} - \left(\underline{Y}'XS^{-\prime} \quad E_{yz}E_{zz}^{-1}\right)\begin{pmatrix} X'\underline{Y} \\ E_{zy} \end{pmatrix}$$

$$= \underline{Y}'\underline{Y} - \left(\underline{Y}'XS^{-\prime}X'\underline{Y} + E_{yz}E_{zz}^{-1}E_{zy}\right)$$

$$= \left(\underline{Y}'\underline{Y} - \underline{Y}'XS^{-\prime}X'\underline{Y}\right) - E_{yz}E_{zz}^{-1}E_{zy}$$

$$= \underline{Y}'(I - XS^{-\prime}X')\underline{Y} - E_{yz}E_{zz}^{-1}E_{zy}$$

$$= \underline{Y}'(I - XS^{-}X')\underline{Y} - E_{yz}E_{zz}^{-1}E_{zy} \qquad \because XS^{-\prime}X' \textit{ is } \text{symmetric}$$

$$= E_{yy} - E_{yz}E_{zz}^{-1}E_{zy} \qquad \because E_{yy} = \underline{Y}'(I - XS^{-}X')\underline{Y}$$

SSE for ANCOVA model $\underline{\boldsymbol{Y}} = \boldsymbol{X}\underline{\boldsymbol{\beta}}^{*} + (\boldsymbol{I} - \boldsymbol{X}\boldsymbol{S}^{-}\boldsymbol{X}')\boldsymbol{Z}\underline{\boldsymbol{\gamma}} + \underline{\boldsymbol{\varepsilon}}$

SSE for ANCOVA model $= E_{yy} - E_{yz}E_{zz}^{-1}E_{zy}$

Note the following for SSE

- $E_{yy} = \underline{Y}'(I - XS^{-}X')\underline{Y}$ is SSE for model without covariate i.e. $\underline{Y} = X\underline{\beta}^{*} + \underline{\varepsilon}$
- $E_{yz}E_{zz}^{-1}E_{zy}$ is a positive definite Quadratic form in $\underline{Y}$ and hence $E_{yz}E_{zz}^{-1}E_{zy} \geq 0$ for all $\underline{Y}$.
- Hence SSE for ANCOVA model will always be less than SSE for model without covariate as

$$\mathrm{E}_{yy} > E_{yy} - E_{yz}E_{zz}^{-1}E_{zy}$$

Home Work

- Show that $E_{yz}E_{zz}^{-1}E_{zy}$ is a positive definite Quadratic form in $\underline{Y}$

Rank of estimation space

Normal equations are:

$$\begin{pmatrix} X'\underline{Y} \\ Z'(I - XS^{-}X')\underline{Y} \end{pmatrix} = \begin{pmatrix} X'X & O \\ O & Z'(I - XS^{-}X')Z \end{pmatrix} \begin{pmatrix} \underline{\hat{\beta}^*} \\ \underline{\hat{\gamma}} \end{pmatrix}$$

$$\begin{pmatrix} X'\underline{Y} \\ E_{zy} \end{pmatrix} = \begin{pmatrix} X'X & O \\ O & E_{zz} \end{pmatrix} \begin{pmatrix} \underline{\hat{\beta}^*} \\ \underline{\hat{\gamma}} \end{pmatrix}$$

Rank of estimation space $= \rho \begin{pmatrix} X'X & O \\ O & E_{zz} \end{pmatrix}$

$$= \rho(X'X) + \rho(E_{zz})$$

$$= r + m \qquad \because E_{zz} \text{ is } m \times m \text{ nonsingular}$$

Rank of Error Space

Rank of estimation space

$=n$-Rank of estimation space

$$= n - r - m$$

Testing of Hypothesis $H_0 : \underline{\gamma} = \underline{0}$

The hypothesis $H_0 : \underline{\gamma} = \underline{0}$ is testable as $\underline{\gamma}$ is estimable.

Original model : $\underline{Y} = X\underline{\beta}^* + (I - XS^{-}X')Z\underline{\gamma} + \underline{\varepsilon}$

Reduced model : $\underline{Y} = X\underline{\beta}^* + \underline{\varepsilon}$

SSE(original model)$= E_{yy} - E_{yz}E_{zz}^{-1}E_{zy}$

SSE(Reduced model)$= E_{yy}$

SSH_0

$= SSE$(Reduced model) $- SSE$(original model)

$= E_{yz}\ E_{zz}^{-1}E_{zy}$

Testing of Hypothesis $H_0: \underline{\gamma} = \underline{0}$

Degrees of freedom for $SSH_0 = m$

Degrees of freedom for $SSE = n - r - m$

$$Test\ Statistic = \frac{SSH_0/m}{SSE/df_E}$$

$$= \frac{E_{yz}E_{zz}^{-1}E_{zy}/m}{(E_{yy} - E_{yz}E_{zz}^{-1}E_{zy})\ /(n-r-m)} \sim F_{m,(n-r-m)}$$

Testing of Hypothesis $\boldsymbol{H_0: \Lambda\underline{\beta} = \underline{d}}$

Let Λ be $k \times p$ full row rank matrix with $\rho(\Lambda) = k$ such that $\Lambda\underline{\beta}$ are k linearly independent estimable parametric functions. Hence the hypothesis $H_0: \Lambda\underline{\beta} = \underline{d}$ is testable.

Consider the following models:

M_1 (Original Model) : $\underline{Y} = X\underline{\beta}^* + (I - XS^{-}X')Z\underline{\gamma} + \underline{\varepsilon}$

M_1^c (OM subject to H_0) : M_1 subject to H_0

M_2 (Model without covariate) : $\underline{Y} = X\underline{\beta}^* + \underline{\varepsilon}$

M_2^c (Model without covariate subject to H_0) : M_2 subject to H_0

Testing of Hypothesis $H_0 : \Lambda\underline{\beta} = \underline{d}$…

Note that:

- $SSE(M_1) \quad = E_{yy} - E_{yz}E_{zz}^{-1}E_{zy}$

$$= SSE(M_2) - E_{yz}E_{zz}^{-1}E_{zy}$$

- Thus to calculate $SSE(M_1)$ we need to calculate first SSE of the model without covariate i.e. $SSE(M_2)$
- Similarly to calculate $SSE(M_1^c)$, we will need to first calculate SSE of the model without covariate subject to H_0 i.e. $SSE(M_2^c)$. Let it be E_{yy}^c.
- $SSE(M_1^c) = E_{yy}^c - E_{yz}^c {E_{zz}^c}^{-1} E_{zy}^c$

$$= SSE(M_2^c) - E_{yz}^c {E_{zz}^c}^{-1} E_{zy}^c$$

Testing of Hypothesis $H_0: \Lambda\underline{\beta} = \underline{d}$…

Model	M_1	M_2
GLM	$\underline{Y} = X\underline{\beta}^* + (I-P)Z\underline{\gamma} + \underline{\varepsilon}$	$\underline{Y} = X\underline{\beta}^* + \underline{\varepsilon}$
SSE	$SSE(M_1) = \boldsymbol{E_{yy}} - E_{yz}E_{zz}^{-1}E_{zy}$	$SSE(M_2) = \boldsymbol{E_{yy}}$
df_{SSE}	$df_{SSE(M_1)} = n - r - m$	$df_{SSE(M_2)} = n - r$
$RM: OM \mid H_0$	M_1^c: M_1 subject to $H_0: \Lambda\underline{\beta} = \underline{d}$	M_2^c: M_2 subject to H_0
SSE RM	$SSE(M_1^c) = \boldsymbol{E_{yy}^c} - E_{yz}^c {E_{zz}^c}^{-1} E_{zy}^c$	$SSE(M_2^c) = \boldsymbol{E_{yy}^c}$
SSH_0	$SSH_0 = \boldsymbol{SSE(M_1^c)} - SSE(M_1)$ $= \left(\boldsymbol{E_{yy}^c} - \boldsymbol{E_{yz}^c}\boldsymbol{E_{zz}^c}^{-1}\boldsymbol{E_{zy}^c}\right)$ $-(E_{yy} - E_{yz}E_{zz}^{-1}E_{zy})$	$SSH_0 = \boldsymbol{SSE(M_2^c)} - SSE(M_2)$ $H_{yy} = \boldsymbol{E_{yy}^c} - \boldsymbol{E_{yy}}$ (H_{yy} is known) $\Rightarrow \boldsymbol{E_{yy}^c} = \boldsymbol{H_{yy}} + \boldsymbol{E_{yy}}$
df_{SSH_0}	$df_{SSH_0} = k$	$df_{SSH_0} = k$

Testing of Hypothesis $H_0: \Lambda\underline{\beta} = \underline{d}$...

$$SSH_0 = SSE(M_1^c) - SSE(M_1)$$

$$= \left(E_{yy}^c - E_{yz}^c {E_{zz}^c}^{-1} E_{zy}^c\right) - (E_{yy} - E_{yz}E_{zz}^{-1}E_{zy})$$

$$= \left\{(H_{yy} + E_{yy}) - \left(H_{yz} + E_{yz}\right)(H_{zz} + E_{zz})^{-1}(H_{zy} + E_{zy})\right\}$$

$$-(E_{yy} - E_{yz}E_{zz}^{-1}E_{zy})$$

Here

$H_{yy} = SSH_0$ for model without covariate

$E_{yy} = SSE$ for model without covariate

Thus knowing H_{yy} and E_{yy} for model without covariate we can calculate SSE and SSH_0 for model with covariate.

Testing of Hypothesis $H_0 : \Lambda\underline{\beta} = \underline{d}$ …

$$Test\ Statistic = \frac{\left(\frac{SSH_0}{k}\right)}{\left(\frac{SSE}{n-r-m}\right)} \sim F_{k,n-r-m}$$

where

$$\mathrm{SSH_0} = \left\{(H_{yy} + E_{yy}) - \left(H_{yz} + E_{yz}\right)(H_{zz} + E_{zz})^{-1}(H_{zy} + E_{zy})\right\}$$

$$\mathrm{SSE} = (E_{yy} - E_{yz}E_{zz}^{-1}E_{zy})$$

$H_{yy} = SSH_0$ for model without covariate

$E_{yy} = SSE$ for model without covariate

Example: ANCOVA for one-way model

$k = Number\ of\ treatments \quad n = Replication\ of\ each\ treatments$

$nk = Number\ of\ observations$

$T_{yy} = \frac{\sum_{i=1}^{k} y_{i.}^2}{n} - \frac{y_{..}^2}{nk},$ $T_{xx} = \frac{\sum_{i=1}^{k} x_{i.}^2}{n} - \frac{x_{..}^2}{nk},$ $T_{xy} = \frac{\sum_{i=1}^{k} x_{i.} y_{i.}}{n} - \frac{x_{..} y_{..}}{nk},$

$G_{yy} = \sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{nk},$ $G_{xx} = \sum_{i=1}^{k} \sum_{j=1}^{n} x_{ij}^2 - \frac{x_{..}^2}{nk},$ $G_{xy} = \sum_{i=1}^{k} \sum_{j=1}^{n} x_{ij} y_{ij} - \frac{x_{..} y_{..}}{nk}$

$E_{yy} = G_{yy} - T_{yy}$ $E_{xx} = G_{xx} - T_{xx}$ $E_{xy} = G_{xy} - T_{xy}$

,

ANCOVA for one-way model with one concomitant variable

S.V.	df	Sum of squares			df_{adj}	SS_{adj}	MS_{adj}	F-Ratio	Hypothesis
Treat	(k − 1)	T_{yy}	T_{xx}	T_{xy}					
Error	k(n − 1)	E_{yy}	E_{xx}	E_{xy}		$T = E_{yy} - E_{yx}E_{xx}^{-1}E_{xy}$			
Total	(nk − 1)	G_{yy}	G_{xx}	G_{xy}		$T_1 = G_{yy} - G_{yx}G_{xx}^{-1}G_{xy}$			
Treat (Adj.)					$k - 1$	$T_1 - T$	$\frac{T_1 - T}{k - 1}$	$\frac{MS_{trea(adj)}}{MSE_{adj}}$	$H_{01}: \alpha_1 = \cdots = \alpha_p$
Error (Adj.)					$k(\mathrm{n} - 1) - 1$	T	$\frac{T}{k(n-1) - 1}$		
Regr.					1	$E_{yy} - T$	$E_{yy} - T$	$\frac{MS_{Reg(adj)}}{MSE_{adj}}$	$H_{02}: \gamma = 0$

Example: ANOVA for two-way model

Source of variation	Degrees of Freedom	Sum of squares	Mean SS	F-ratio	Hypothesis
Factor A	$(p-1)$	$SSA = \frac{\sum_{i=1}^{p} y_{i.}^2}{q} - \frac{y_{..}^2}{pq}$	$MSA = \frac{SSA}{p-1}$	$\frac{MSA}{MSE}$	$H_{01}: \alpha_1 = \cdots = \alpha_p$
Factor B	$(q-1)$	$SSB = \frac{\sum_{j=1}^{q} y_{.j}^2}{p} - \frac{y_{..}^2}{pq}$	$MSB = \frac{SSB}{q-1}$	$\frac{MSB}{MSE}$	$H_{02}: \beta_1 = \cdots = \beta_q$
Error	$(p-1)(q-1)$	$SSE = \mathrm{SST} - \mathrm{SSA} - \mathrm{SSB}$	MSE		
Total	$(pq-1)$	$\mathrm{SST} = \sum_{i=1}^{p} \sum_{j=1}^{q} y_{ij}^2 - \frac{y_{..}^2}{pq}$			

ANCOVA for two-way model

$$T_{yy} = \frac{\sum_{i=1}^{p} y_{i.}^2}{q} - \frac{y_{..}^2}{pq}, \qquad T_{xx} = \frac{\sum_{i=1}^{v} x_{i.}^2}{q} - \frac{x_{..}^2}{pq}, \qquad T_{xy} = \frac{\sum_{i=1}^{p} x_{i.} y_{i.}}{q} - \frac{x_{..} y_{..}}{pq},$$

$$B_{yy} = \frac{\sum_{j=1}^{q} y_{.j}^2}{p} - \frac{y_{..}^2}{pq}, \qquad B_{xx} = \frac{\sum_{j=1}^{q} x_{.j}^2}{p} - \frac{x_{..}^2}{pq}, \qquad B_{xy} = \frac{\sum_{j=1}^{q} x_{.j} y_{.j}}{p} - \frac{x_{..} y_{..}}{pq}$$

$$G_{yy} = \sum_{i=1}^{p} \sum_{j=1}^{q} y_{ij}^2 - \frac{y_{..}^2}{pq}, \qquad G_{xx} = \sum_{i=1}^{p} \sum_{j=1}^{q} x_{ij}^2 - \frac{x_{..}^2}{pq}, \qquad G_{xy} = \sum_{i=1}^{p} \sum_{j=1}^{q} x_{ij} y_{ij} - \frac{x_{..} y_{..}}{pq},$$

$$E_{yy} = G_{yy} - T_{yy} - B_{yy} \qquad E_{xx} = G_{xx} - T_{xx} - B_{xx} \qquad E_{xy} = G_{xy} - T_{xy} - B_{xy}$$

ANCOVA for two-way model with one concomitant variable

S.V.	df	SS			df_{adj}	SS_{adj}	MS_{adj}	F-Ratio
Treat	$(p-1)$	T_{yy}	T_{xx}	T_{xy}				
Block	$(q-1)$	B_{yy}	B_{xx}	B_{xy}				
Error	$(p-1)\times$ $(q-1)$	E_{yy}	E_{xx}	E_{xy}	$(p-1)\times$ $(q-1)-1$	$T=E_{yy}-E_{yx}E_{xx}^{-1}E_{xy}$		
Total	$(pq-1)$	G_{yy}	G_{xx}	G_{xy}				
Treat+Error		$T_{yy}+E_{yy}$	$T_{xx}+E_{xx}$	$T_{xy}+E_{xy}$		T_1		
Block+Err		$B_{yy}+E_{yy}$	$B_{xx}+E_{xx}$	$B_{xy}+E_{xy}$		T_2		
Treat (Adj.)					$p-1$	T_1-T	$\dfrac{T_1-T}{p-1}$	$\dfrac{MS_{trea(adj)}}{MSE_{adj}}$
Treat (Adj.)					$q-1$	T_2-T	$\dfrac{T_2-T}{q-1}$	$\dfrac{MS_{block(adj)}}{MSE_{adj}}$
Error (Adj.)					$(p-1)$ $(q-1)-1$	T	$\dfrac{T}{df_{error}}$	
Regr.					1	$E_{yy}-T$	$E_{yy}-T$	$\dfrac{MS_{Reg}}{MSE_{adj}}$