ST-302
Design, Planning and Analysis of Experiments

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Topic 3: Two-way Classification Model with r observations per cell

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$

- H_{01} can be rewritten in the form of estimable parametric functions.

$$\left.\begin{aligned} H_{01}: \alpha_1^* &= \alpha_2^* \\ \alpha_1^* &= \alpha_3^* \\ &\vdots \\ \alpha_1^* &= \alpha_p^* \end{aligned}\right\} \quad (p-1)\ l.i.e.p.f.$$

- Let $\alpha_1^* = \alpha_2^* = \cdots = \alpha_p^* = \alpha^*\ (say)$
- Here $\alpha_i^* = (\alpha_i + \bar{\gamma}_{i.})$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$

- **Original model**

Model : $E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij}$

Solution of

Normal equations: $\hat{\mu} = \bar{y}_{...}$,

$$\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...},$$

$$\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...},$$

$$\hat{\gamma}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

Fitted value : $\hat{y}_{ijk} = \bar{y}_{ij.}$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$

- **SSE for Original model**

$$SSE = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}\left(y_{ijk} - \hat{y}_{ijk}\right)^2$$

$$SSE = \left(\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r} y_{ijk}^2 - \frac{y_{\dots}^2}{pqr}\right) - \left(\sum_{i=1}^{p}\sum_{j=1}^{q}\frac{{y_{ij\cdot}}^2}{r} - \frac{y_{\dots}^2}{pqr}\right)$$

$$SSE = SST - SS_{cells}$$

DF for SSE : $df_{SSE} = pq(r-1)$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$

- **Reduced model**

$$E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

$$E(y_{ijk}) = \mu + (\alpha_i + \bar{\gamma}_{i.}) + \beta_j + (\gamma_{ij} - \bar{\gamma}_{i.})$$

$$E(y_{ijk}) = \mu + \alpha_i^* + \beta_j + \gamma_{ij}^0 \qquad \text{where } \gamma_{ij}^0 = \gamma_{ij} - \bar{\gamma}_{i.}$$

$$E(y_{ijk}) = \mu^0 + \beta_j + \gamma_{ij}^0 \qquad \text{where } \mu^0 = \mu + \alpha^*$$

Observe that the parameters γ_{ij}^0 are restricted. That is γ_{ij}^0 are subject to the following p conditions.

$$\sum_{j=1}^{q} \gamma_{ij}^0 = \sum_{j=1}^{q} (\gamma_{ij} - \bar{\gamma}_{i.})$$

$$= \gamma_{i.} - q\bar{\gamma}_{i.} = 0, \qquad i = 1,2,\ldots,p$$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$

- Thus under H_0 model reduces to

$$E(y_{ijk}) = \mu^0 + \beta_j + \gamma_{ij}^0$$

where $\sum_{j=1}^{q} \gamma_{ij}^0 = 0$, for $i = 1,2, \dots, p$

- To obtain normal equations and their solutions, minimize the function ϕ with respect to $\hat{\mu}^0, \hat{\beta}_j, \hat{\gamma}_{ij}^0$

$$\phi = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right)^2$$

subject to the p conditions $\sum_{j=1}^{q} \gamma_{ij}^0 = 0, i = 1,2, \dots, p$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$

- To obtain normal equations and their solutions, minimize the function ϕ with respect to $\hat{\mu}^0, \hat{\beta}_j, \hat{\gamma}_{ij}^0$

$$\phi = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right)^2$$

subject to the p conditions $\sum_{j=1}^{q} \gamma_{ij}^0 = 0, i = 1,2, \dots, p$

- The p conditions can be added to the ϕ function to be minimized by adding Langrangian multipliers λ_i for each of the p conditions $\sum_{j=1}^{q} \gamma_{ij}^0 = 0, i = 1,2, \dots, p$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$...

- Minimize the function ϕ with respect to $\hat{\mu}^0, \hat{\beta}_j, \hat{\gamma}_{ij}^0, \lambda_i$

$$\phi = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right)^2 + 2 \sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{q} \hat{\gamma}_{ij}^0\right)$$

$$\frac{d\phi}{d\hat{\mu}^0} = 0$$

$$\Rightarrow \quad -2 \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right) = 0$$

$$\Rightarrow \quad y_{...} = pqr\hat{\mu}^0 + pr \sum_{j=1}^{q} \hat{\beta}_j + r \sum_{i=1}^{p} \sum_{j=1}^{q} \hat{\gamma}_{ij}^0 \qquad (1)$$

$$\Rightarrow \quad E(y_{...}) = pqr\mu^0 + pr \sum_{j=1}^{q} \beta_j + r \sum_{i=1}^{p} \sum_{j=1}^{q} \gamma_{ij}^0$$

$$\Rightarrow \quad E(\bar{y}_{...}) = \mu + \bar{\alpha} + \bar{\beta} + \bar{\gamma}_{..}$$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$

$$\phi = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right)^2 + 2 \sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{q} \hat{\gamma}_{ij}^0\right)$$

$$\frac{d\phi}{d\hat{\beta}_j} = 0$$

$$\Rightarrow \quad -2 \sum_{i=1}^{p} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right), \; j = 1 : q$$

$$\Rightarrow \quad y_{.j.} = pr\hat{\mu}^0 + pr\hat{\beta}_j + r \sum_{i=1}^{p} \hat{\gamma}_{ij}^0, \qquad j = 1 : q \qquad \text{(A)}$$

$$\Rightarrow \quad \mathrm{E}(y_{.j.}) = pr\mu^0 + pr\beta_j + r \sum_{i=1}^{p} \gamma_{ij}, \quad j = 1 : q$$

$$\Rightarrow \quad \mathrm{E}\left(\bar{y}_{.j.}\right) = \mu^0 + \beta_j + \bar{\gamma}_{.j}, \qquad j = 1 : q$$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$

$$\phi = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right)^2 + 2 \sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{q} \hat{\gamma}_{ij}^0\right)$$

$$\frac{d\phi}{d\hat{\gamma}_{ij}^0} = 0$$

$$\Rightarrow \quad -2 \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right) + 2\lambda_i = 0$$

$$\Rightarrow \quad y_{ij.} = r\hat{\mu}^0 + r\hat{\beta}_j + r\hat{\gamma}_{ij}^0 + \lambda_i, \qquad i = 1{:}\,p, j = 1{:}\,q \qquad \text{(B)}$$

$$\Rightarrow \quad E\left(y_{ij.}\right) = r\mu^0 + r\beta_j + r\gamma_{ij}^0 + \lambda_i, \qquad i = 1{:}\,p, j = 1{:}\,q$$

$$\Rightarrow \quad \mathrm{E}\left(\bar{y}_{ij.}\right) = \mu^0 + \beta_j + \gamma_{ij}^0 + \lambda_i/r,\,, \qquad i = 1{:}\,p, j = 1{:}\,q$$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$

$$\phi = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{\mu}^0 - \hat{\beta}_j - \hat{\gamma}_{ij}^0\right)^2 + 2 \sum_{i=1}^{p} \lambda_i \left(\sum_{j=1}^{q} \hat{\gamma}_{ij}^0\right)$$

$$\frac{d\phi}{d\lambda_i} = 0$$

$$\Rightarrow \quad 2 \sum_{j=1}^{q} \hat{\gamma}_{ij}^0 = 0$$

$$\Rightarrow \quad \sum_{j=1}^{q} \hat{\gamma}_{ij}^0 = 0, \qquad i = 1:p \qquad \text{(C)}$$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^* \ldots$

Normal equations for the reduced model:

$$y_{...} = pqr\hat{\mu}^0 + pr\sum_{j=1}^{q}\hat{\beta}_j + r\sum_{i=1}^{p}\sum_{j=1}^{q}\hat{\gamma}_{ij}^0 \qquad (1)$$

$$y_{.j.} = pr\hat{\mu}^0 + pr\hat{\beta}_j + r\sum_{i=1}^{p}\hat{\gamma}_{ij}^0, \qquad j = 1:q \qquad (A)$$

$$y_{ij.} = r\hat{\mu}^0 + r\hat{\beta}_j + r\hat{\gamma}_{ij}^0 + \lambda_i, \qquad i = 1:p, j = 1:q \qquad (B)$$

$$0 = \sum_{j=1}^{q}\hat{\gamma}_{ij}^0, \qquad i = 1:p \qquad (C)$$

Number of normal equations $=1 + p + q + pq$

Number of parameters $= 1 + p + q + pq$

Reduced Model details

- Number of observations=$n = pqr$
- Number of parameters=$1 + p + q + pq$

$$\mu,$$

$$\lambda_1, \lambda_2, \ldots, \lambda_p,$$

$$\beta_1, \beta_2, \ldots, \beta_q,$$

$$\gamma_{11}^0, \gamma_{12}^0, \ldots, \gamma_{1q}^0$$

$$\gamma_{21}^0, \gamma_{22}^0, \ldots, \gamma_{2q}^0$$

$$\vdots$$

$$\gamma_{p1}^0, \gamma_{p2}^0, \ldots, \gamma_{pq}^0$$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$...

Number of equations which are linearly independent :

- (1) is linearly dependent as $\sum_{i=1}^{p}(A) = (1)$
- (C) is linearly dependent as $\sum_{i=1}^{p}(C) = 0$ i.e. $\sum_{i=1}^{p}\sum_{j=1}^{q}\hat{\gamma}_{ij}^{0} = 0$
- Consider an additional equation $\sum_{j=1}^{q}\hat{\beta}_j = 0$

$$y_{\ldots} = pqr\hat{\mu}^0 + pr\sum_{j=1}^{q}\hat{\beta}_j + r\sum_{i=1}^{p}\sum_{j=1}^{q}\hat{\gamma}_{ij}^{0} \qquad (1)$$

$\Rightarrow \qquad \hat{\mu} = \bar{y}_{\ldots}$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$...

$$y_{ij.} = r\hat{\mu}^0 + r\hat{\beta}_j + r\hat{\gamma}_{ij}^0 + \lambda_i, \qquad i = 1:p, j = 1:q \qquad \text{(B)}$$

- Summing equations in (B) over j and then using (C)

$\sum_{j=1}^{q}(A)$ along with (C)

$$\Rightarrow \quad y_{i..} = qr\hat{\mu}^0 + r\sum_{j=1}^{q}\hat{\beta}_j + r\sum_{j=1}^{q}\hat{\gamma}_{ij}^0 + q\lambda_i$$

$$\Rightarrow \quad q\lambda_i = y_{i..} - qr\hat{\mu}^0$$

$$\Rightarrow \quad \frac{\lambda_i}{r} = \bar{y}_{i..} - \hat{\mu}^0 \qquad \Rightarrow \qquad \lambda_i = r(\bar{y}_{i..} - \bar{y}_{...})$$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$…

$$\lambda_i = r(\bar{y}_{i..} - \bar{y}_{...}), \qquad i = 1:p$$

Substituting in (B)

$$y_{ij.} = r\hat{\mu}^0 + r\hat{\beta}_j + r\hat{\gamma}_{ij}^0 + \lambda_i \qquad \text{(B)}$$

$$\Rightarrow \qquad y_{ij.} = r\hat{\mu}^0 + r\hat{\beta}_j + r\hat{\gamma}_{ij}^0 + r(\bar{y}_{i..} - \bar{y}_{...}) \quad i = 1:p, j = 1:q$$

$$\Rightarrow \qquad r\hat{\gamma}_{ij}^0 = y_{ij.} - r\hat{\mu}^0 - r\hat{\beta}_j - r(\bar{y}_{i..} - \bar{y}_{...})$$

$$\Rightarrow \qquad \hat{\gamma}_{ij}^0 = \bar{y}_{ij.} - \hat{\mu}^0 - \hat{\beta}_j - (\bar{y}_{i..} - \bar{y}_{...})$$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$...

$$\Rightarrow \quad \hat{\gamma}_{ij}^0 = (\bar{y}_{ij.} - \bar{y}_{...}) - (\bar{y}_{.j.} - \bar{y}_{...}) - (\bar{y}_{i..} - \bar{y}_{...})$$

$$\Rightarrow \quad \hat{\gamma}_{ij}^0 = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

Solution of normal equations for the reduced model:

$E(y_{ijk}) = \mu^0 + \beta_j + \gamma_{ij}^0$ subject to $\sum_{j=1}^{q} \gamma_{ij}^0 = 0, \quad i = 1,2,\ldots,p$

- $\hat{\mu}^0 = \bar{y}_{...}$,
- $\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...}$
- $\hat{\gamma}_{ij}^0 = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$…

Fitted value of the reduced model:

$$\hat{y}_{ijk} = \hat{\mu}^0 + \hat{\beta}_j + \hat{\gamma}_{ij}^0$$

$$= \bar{y}_{\dots} + \left(\bar{y}_{.j.} - \bar{y}_{\dots}\right) + \left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{\dots}\right)$$

$$= \bar{y}_{\dots} + \left(\bar{y}_{ij.} - \bar{y}_{\dots}\right) - (\bar{y}_{i..} - \bar{y}_{\dots})$$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*$...

$=SSE_c$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \hat{y}_{ijk}\right)^2$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \left(\bar{y}_{...} + \left(\bar{y}_{ij.} - \bar{y}_{...}\right) - \left(\bar{y}_{i..} - \bar{y}_{...}\right)\right)\right)^2$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(\left(y_{ijk} - \bar{y}_{...}\right) - \left(\bar{y}_{ij.} - \bar{y}_{...}\right) + \left(\bar{y}_{i..} - \bar{y}_{...}\right)\right)^2$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \bar{y}_{...}\right)^2 - \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(\bar{y}_{ij.} - \bar{y}_{...}\right)^2$$

$$+ \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} (\bar{y}_{i..} - \bar{y}_{...})^2$$

$$= SST - SS_{cells} + SSA$$

Testing of hypothesis $\boldsymbol{H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^*}$...

- $\boldsymbol{SSH_0}$ **for the hypothesis**

$$\mathrm{SS}H_0$$

$$= \mathrm{SSE_c} - \mathrm{SSE}$$

$$= (SST - SS_{cells} + SSA) - (SST - SS_{cells})$$

$$= SSA$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} (\bar{y}_{i..} - \bar{y}_{...})^2$$

$$= \frac{\sum_{i=1}^{p} y_{i..}^2}{qr} - \frac{y_{...}^2}{pqr}$$

Testing of hypothesis $H_{01}: \alpha_1^* = \alpha_2^* = \cdots = \alpha_p^* \ldots$

- **Test Statistic**

$$SSH_0 = \sum_{i=1}^{p} qr(\bar{y}_{i..} - \bar{y}_{...})^2 \sim \sigma^2 \chi^2_{(p-1)}$$

$$SSE = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \bar{y}_{ij.}\right)^2$$

$$SSE \sim \sigma^2 \chi^2_{pq(r-1)}$$

$$MSH_0 = \sum_{i=1}^{p} qr(\bar{y}_{i..} - \bar{y}_{...})^2 / (p-1)$$

$$MSE = SSE/(pq(r-1))$$

$$Test - Statistic = \frac{MSH_0}{MSE} \sim F_{(p-1),pq(r-1)}$$

Testing of hypothesis $\boldsymbol{H_{02}}: \boldsymbol{\beta_1^* = \beta_2^* = \cdots = \beta_q^*}$

- **Test Statistic**

$$\mathrm{SS}H_0 = \sum_{j=1}^{q} pr\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2 \sim \sigma^2 \chi^2_{(q-1)}$$

$$SSE = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} \left(y_{ijk} - \bar{y}_{ij.}\right)^2$$

$$SSE \sim \sigma^2 \chi^2_{pq(r-1)}$$

$$\mathrm{MS}H_0 = \sum_{j=1}^{q} pr\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2 / (q-1)$$

$$MSE = SSE/(pq(r-1))$$

$$Test - Statistic = \frac{MSH_0}{MSE} \sim F_{(q-1),pq(r-1)}$$

Estimation of error variance

$$SSE = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}\left(y_{ijk} - \bar{y}_{ij.}\right)^2$$

$$SSE \sim \sigma^2 \chi^2_{(pq(r-1))}$$

$$E(SSE) = \sigma^2(pq(r-1))$$

$$E\left(\frac{SSE}{pq(r-1)}\right) = \sigma^2$$

$$E(MSE) = \sigma^2 \qquad \because \text{MSE} = \text{SSE}/\, pq(r-1)$$

$$\hat{\sigma}^2 = MSE = \frac{\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}\left(y_{ijk}-\bar{y}_{ij.}\right)^2}{pq(r-1)}$$

Thus $\hat{\sigma}^2$ is an unbiased estimator of σ^2 (i.e. Error variance)

ANOVA

Source of vari.	Degrees of Freedom	Sum of squares	Mean SS	F-ratio	Hypothesis
F-A	$(p-1)$	$SS_A = \frac{\sum_{i=1}^{p} y_{i..}^2}{qr} - \frac{y_{...}^2}{pqr}$	$MSA = \frac{SSA}{p-1}$	$\frac{MS_A}{MSE}$	$H_{01}: \alpha_1^* = \cdots = \alpha_p^*$
F-B	$(q-1)$	$SS_B = \frac{\sum_{i=1}^{p} y_{.j.}^2}{pr} - \frac{y_{...}^2}{pqr}$	$MSB = \frac{SSB}{q-1}$	$\frac{MS_B}{MSE}$	$H_{02}: \beta_1^* = \cdots = \beta_q^*$
Int-AB	$(p-1) \times (q-1)$	$SS_{AB} = SS_{cells} - SS_A - SS_B$	$MS_{AB} = \frac{SS_{AB}}{(p-1)(q-1)}$	$\frac{MS_{AB}}{MSE}$	$H_{03}: \gamma_{ij}^* = 0 \ \forall \ i, j$
Error	$pq(r-1)$	$SSE = SST - SS_{cells}$	$MSE = \frac{SSE}{pq(r-1)}$		
Total	$(pqr-1)$	$SST = \sum_{i,j,k} y_{ijk}^2 - \frac{y_{...}^2}{pqr}$			

ANOVA

- $SS_{cells} = \frac{\sum_{i,j} y_{ij.}^2}{r} - \frac{y_{...}^2}{pqr}$

- $\frac{MSA}{MSE} \sim F_{(p-1),pq(r-1)}$

- $\frac{MS\mathrm{B}}{MSE} \sim F_{(q-1),pq(r-1)}$

- $\frac{MS\mathrm{AB}}{MSE} \sim F_{(q-1)(q-1),pq(r-1)}$

- Further $E(MSE) = \sigma^2,$ that is MSE is unbiased estimator of σ^2

Decision about TOH

- p-value for $H_{01} = 1 - P\left(F_{(p-1),pq(r-1)} \leq \frac{MS_A}{MSE}\right)$

- p-value for $H_{02} = 1 - P\left(F_{(q-1),pq(r-1)} \leq \frac{MS_{\mathrm{B}}}{MSE}\right)$

- p-value for $H_{03} = 1 - P\left(F_{(p-1)(q-1),pq(r-1)} \leq \frac{MS_{\mathrm{AB}}}{MSE}\right)$

- If p-value for $H_{01} < \alpha$ then reject H_{01}
- If p-value for $H_{02} < \alpha$ then reject H_{02}
- If p-value for $H_{03} < \alpha$ then reject H_{03}

Testing the hypothesis with individual epf.

- $H_0: \alpha_1^* = \alpha_2^*$
- Rewrite H_0 as $\alpha_1^* - \alpha_2^* = 0$
- $\widehat{\alpha_1^* - \alpha_2^*} = \bar{y}_{1..} - \bar{y}_{2..} \sim N\left(\alpha_1^* - \alpha_2^*, \mathrm{var}(\widehat{\alpha_1 - \alpha_2})\right)$
- $$\bar{y}_{1..} - \bar{y}_{2..} \sim N\left(\alpha_1^* - \alpha_2^*, \frac{2\sigma^2}{qr}\right)$$
- $\frac{(\bar{y}_{1..} - \bar{y}_{2..}) - (\alpha_1^* - \alpha_2^*)}{\sqrt{\frac{2\sigma^2}{qr}}} \sim N(0,1)$
- $\left\{\frac{(\bar{y}_{1..} - \bar{y}_{2..}) - (\alpha_1^* - \alpha_2^*)}{\sqrt{\frac{2\sigma^2}{qr}}}\right\}^2 \sim \chi^2_{(1)}$

Testing the hypothesis with individual epf.

- Under the null hypothesis $\alpha_1^* - \alpha_2^* = 0$.
- Hence $\text{SSH}_0 = \frac{(\bar{y}_{1..} - \bar{y}_{2..})^2}{\left(\frac{2}{qr}\right)} \sim \sigma^2 \chi^2_{(1)}$
- $SSE = \sum_{i,j,k} \left(y_{ijk} - \bar{y}_{ij.}\right)^2 \sim \sigma^2 \chi^2_{(pq(r-1))}$
- Further $\text{SSH}_0$is independently distributed of SSE.
- Hence test statistic is:

$$\frac{\frac{(\bar{y}_{1..} - \bar{y}_{2..})^2}{\left(\frac{2}{qr}\right)}}{MSE} \sim F_{1,(pq(r-1))}$$

Testing the hypothesis with individual epf.

- There is no interaction between first level of factor A and second level of factor B
- $H_0: \gamma_{12}^* = 0$
- $\hat{\gamma}_{12}^* = \bar{y}_{12..} - \bar{y}_{1..} - \bar{y}_{2..} + \bar{y}_{...} \sim N\left(\gamma_{12}^*, \mathrm{var}(\hat{\gamma}_{12}^*)\right)$

$$(\bar{y}_{12..} - \bar{y}_{1..} - \bar{y}_{2..} + \bar{y}_{...}) \sim N\left(\gamma_{12}^*, \frac{\sigma^2(p-1)(q-1)}{pqr}\right)$$

- $\left\{\frac{(\bar{y}_{12..} - \bar{y}_{1..} - \bar{y}_{2..} + \bar{y}_{...}) - \gamma_{12}^*}{\sqrt{\frac{\sigma^2(p-1)(q-1)}{pqr}}}\right\}^2 \sim \chi_{(1)}^2$

Testing the hypothesis with individual epf.

- Under the null hypothesis $\frac{(\bar{y}_{12..}-\bar{y}_{1..}-\bar{y}_{2..}+\bar{y}_{...})^2}{\frac{(p-1)(q-1)}{pqr}} \sim \sigma^2\chi^2_{(1)}$

- $SSE = \sum_{i,j,k}\left(y_{ijk} - \bar{y}_{ij.}\right)^2 \sim \sigma^2\chi^2_{(pq(r-1))}$
- Further SSH_0 is independently distributed of SSE.
- Hence test statistic is:

$$\frac{\frac{(\bar{y}_{12..}-\bar{y}_{1..}-\bar{y}_{2..}+\bar{y}_{...})^2}{\frac{(p-1)(q-1)}{pqr}}}{MSE} \sim F_{1,(pq(r-1))}$$

General test statistic for Testing the individual epf.

- H_0: $\underline{\lambda}'\underline{\beta} = d$
- As $\underline{\lambda}'\underline{\hat{\beta}} \sim N\left(\underline{\lambda}'\underline{\beta},\ var\left(\underline{\lambda}'\underline{\hat{\beta}}\right)\right) \Rightarrow\ SSH_0 = \frac{\left(\underline{\lambda}'\underline{\widehat{\beta}} - d\right)^2}{var\left(\underline{\lambda}'\underline{\widehat{\beta}}\right)} \sim \chi^2_{(1)}$
- $\frac{SSE}{\sigma^2} \sim \chi^2_{pq(r-1)}$
- SSH_0 is independently distributed of SSE
- Hence $test - statistic \sim F_{1,pq(r-1)}$ distribution and is:

$$\frac{\left(BLUE\left(\underline{\lambda}'\underline{\beta}\right) - hypothetical\ value\ of\ \underline{\lambda}'\underline{\beta}\right)^2 / \left(var\left(\underline{\lambda}'\underline{\widehat{\beta}}\right) without\sigma^2\right)}{MSE}$$

What we have studied

- GLM for two-way classification model with interaction and r observations per cell
- Normal equations and their solutions
- Estimability conditions for linear parametric functions
- Estimable parametric functions, their BLUEs and variances of BLUEs.
- Rank of error space and estimation space
- A set of linearly independent estimable parametric functions
- Fitted value of y_{ij}, SSE and an unbiased estimator of error variance
- Testing of hypothesis: No interaction effect, equality of all row/column effects
- Testing of hypothesis: single epf
- ANOVA table specifying the hypothesis to be tested against each row of it.