General Two-way Block Design: GTWBD

ST-302: Design, Planning and Analysis of Experiments

Outline of the Topic

Normal equations of GTWBD ...

Normal equations are,

$$\begin{pmatrix} G \\ \underline{T} \\ \underline{B} \end{pmatrix} = \begin{pmatrix} n & \underline{r}' & \underline{k}' \\ \underline{r} & R & N \\ \underline{k} & N' & K \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \underline{\hat{\tau}} \\ \underline{\hat{\beta}} \end{pmatrix}$$

These can be written as

$$G = n\hat{\mu} + \underline{r}'\underline{\hat{\tau}} + \underline{k}'\underline{\hat{\beta}}$$

$$\underline{T} = \underline{r}\hat{\mu} + R\underline{\hat{\tau}} + N\underline{\hat{\beta}}$$

$$\underline{B} = \underline{k}\hat{\mu} + N'\underline{\hat{\tau}} + K\underline{\hat{\beta}}$$

Reduced normal equations...

1. We know that all $v+b+1$ equations in the above set of equations are not linearly independent.
2. We will obtain the other set of reduced normal equations in terms of $\underline{\hat{\beta}}$ eliminating $\underline{\hat{\tau}}$
3. To eliminate $\underline{\hat{\tau}}$ from normal equations consider the second set of equations

$$\underline{\hat{\tau}} = R^{-1}\left(\underline{T} - \underline{r}\hat{\mu} - N\underline{\hat{\beta}}\right)$$

Reduced normal equations...

Substituting in the third set of equations,

$$\begin{aligned}
\underline{B} &= \underline{k}\hat{\mu} + N'\underline{\hat{\tau}} + K\underline{\hat{\beta}} \\
&= \underline{k}\hat{\mu} + N'R^{-1}\left(\underline{T} - \underline{r}\hat{\mu} - N\underline{\hat{\beta}}\right) + K\underline{\hat{\beta}} \\
&= \underline{k}\hat{\mu} + N'R^{-1}\underline{T} - N'R^{-1}\underline{r}\hat{\mu} - N'R^{-1}N\underline{\hat{\beta}} + K\underline{\hat{\beta}} \\
&= \underline{k}\hat{\mu} + N'R^{-1}\underline{T} - \underline{k}\hat{\mu} - N'R^{-1}N\underline{\hat{\beta}} + K\underline{\hat{\beta}} \\
\underline{B} - N'R^{-1}\underline{T} &= K\underline{\hat{\beta}} - N'R^{-1}N\underline{\hat{\beta}} \\
\underline{B} - N'R^{-1}\underline{T} &= \left(K - N'R^{-1}N\right)\underline{\hat{\beta}} \\
\underline{P} &= D\underline{\hat{\beta}}
\end{aligned}$$

Here $\underline{P} = \underline{B} - N'R^{-1}\underline{T}$ and $D = \left(K - N'R^{-1}N\right)$

Reduced Normal Equations...

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The reduced normal equations for blocks are:

$$\underline{P} = D\underline{\hat{\beta}}$$

where $\underline{P} = \underline{B} - N'R^{-1}\underline{T}$ and $D = \left(K - N'R^{-1}N\right)$

- $\underline{P}$ is $b \times 1$ vector and is called vector of block totals adjusted for treatments.
- D is $b \times b$ symmetric matrix
- $\underline{P}$ is also called LHS of Reduced Normal Equations
- The difference between the two block totals:
  $\underline{B}$ is vector of treatment totals (unadjusted)
  $\underline{P}$ is vector of adjusted (for treatment) blocks totals

Characterizations of Reduced Normal Equations...

1. D is a symmetric matrix.

$$\begin{aligned} D' &= \left(K - N'R^{-1}N\right)' \\ &= \left(K - N'R^{-1}N\right) \\ &= D \end{aligned}$$

Characterizations of Reduced Normal Equations...

2. D is a singular matrix.

$$\begin{aligned}
DE_{b1} &= \left(K - N'R^{-1}N\right)E_{b1} \\
&= KE_{b1} - N'R^{-1}NE_{b1} \\
&= \underline{k} - N'R^{-1}\underline{r} && \because NE_{b1} = \underline{r} \\
&= \underline{k} - N'E_{v1} && \because R^{-1}\underline{r} = E_{v1} \\
&= \underline{k} - \underline{k} && \because N'E_{v1} = \underline{k} \\
&= \underline{0}
\end{aligned}$$

Thus $DE_{b1} = \underline{0}$ interprets that all row sums of D-matrix are zero.
Or sum of all columns of DC-matrix is $\underline{0}$ and all columns are not linearly independent.
Hence $rank\,(D) < b$
That is D is symmetric singular matrix.

Rank of error space using normal equations of GTWBD

3. ρ(Estimation space)$=\rho\left(D\right)+v$

$$
\begin{aligned}
&\rho\left(Estimationspace\right)\\
&= \rho\begin{pmatrix} n & \underline{r}' & \underline{k}' \\ \underline{r} & R & N \\ \underline{k} & N' & K \end{pmatrix}\\
&= \rho\begin{pmatrix} R & N \\ N' & K \end{pmatrix} \quad \text{performing} \quad \begin{pmatrix} N \\ K \end{pmatrix} - \begin{pmatrix} R \\ N' \end{pmatrix} R^{-1}N\\
&= \rho\begin{pmatrix} R & N - RR^{-1}N \\ N' & K - N'R^{-1}N \end{pmatrix}\\
&= \rho\begin{pmatrix} R & O \\ N' & D \end{pmatrix}\\
&= \rho\left(D\right)+\rho\left(R\right)\\
&= \rho\left(D\right)+v
\end{aligned}
$$

Rank of error space using normal equations of GTWBD

4. ρ(Estimation space)$=\rho\left(C\right)+b$

$$\rho\left(Estimationspace\right)$$

$$=\ \rho\begin{pmatrix} n & \underline{r}' & \underline{k}' \\ \underline{r} & R & N \\ \underline{k} & N' & K \end{pmatrix}$$

$$=\ \rho\begin{pmatrix} R & N \\ N' & K \end{pmatrix} \quad \text{performing} \quad \begin{pmatrix} R \\ N' \end{pmatrix} - \begin{pmatrix} N \\ K \end{pmatrix} K^{-1}N'$$

$$=\ \rho\begin{pmatrix} R-NK^{-1}N' & N \\ N'-KK^{-1}N' & K \end{pmatrix}$$

$$=\ \rho\begin{pmatrix} C & N \\ O & K \end{pmatrix}$$

$$=\rho\left(C\right)+\rho\left(K\right)$$

$$=\rho\left(C\right)+b$$

Characterizations of Reduced Normal Equations...

5. $E\left(\underline{P}\right) = D\underline{\beta}$

$$\begin{aligned}
E\left(\underline{P}\right) &= E\left(\underline{B} - N'R^{-1}\underline{T}\right) \\
&= E\left(D'\underline{Y} - N'R^{-1}\Delta'\underline{Y}\right) \\
&= \left(D' - N'R^{-1}\Delta'\right) E\left(\underline{Y}\right) \\
&= \left(D' - N'R^{-1}\Delta'\right) E\left(\mu E_{n1} + \Delta\underline{\tau} + D\underline{\beta} + \underline{\varepsilon}\right) \\
&= \left(D' - N'R^{-1}\Delta'\right)\left(\mu E_{n1} + \Delta\underline{\tau} + D\underline{\beta}\right) \\
&= D'\left(\mu E_{n1} + \Delta\underline{\tau} + D\underline{\beta}\right) - N'R^{-1}\Delta'\left(\mu E_{n1} + \Delta\underline{\tau} + D\underline{\beta}\right) \\
&= \mu D'E_{n1} + D'\Delta\underline{\tau} + D'D\underline{\beta} \\
&- N'R^{-1}\Delta'\mu E_{n1} - N'R^{-1}\Delta'\Delta\underline{\tau} - N'R^{-1}\Delta'D\underline{\beta} \\
&= \mu\underline{k} + N\underline{\tau} + K\underline{\beta} - \underline{k}\mu - N'R^{-1}N\underline{\beta} - N\underline{\beta} \\
&= \left(K - N'R^{-1}N\right)\underline{\beta} \\
&= D\underline{\beta}
\end{aligned}$$

Characterizations of Reduced Normal Equations...

- $D\underline{\beta}$ contains b linear parametric functions which are estimable.
- If $D'_{(j)}$ is the j^{th} row of D then $D'_{(j)}\underline{\beta}, j = 1, 2, ..., b$ are estimable parametric functions.
- $D'_{(j)}\underline{\beta}, j = 1, 2, ..., b$ are not linearly independent estimable parametric functions as $rank\,(D) \leq b - 1$
- $D'_{(j)}\underline{\beta}, j = 1, 2, ..., b$ are contrast (linear parametric function whose coefficients add to zero) as each row sum of D-matrix is zero.
- $D\underline{\beta}$ contains b linear estimable parametric functions which are contrast in β_j's.
- If P_j is the j^{th} element of $\underline{P}$ then P_j is BLUE of $D'_{(j)}\underline{\beta}$ or $D'_{(j)}\hat{\underline{\beta}} = P_j$, for $j = 1, 2, ..., b$

6. $rank\,(D) \leq b - 1$
   As D is $b \times b$ singular matrix, $rank\,(D) < b \leq b - 1$

Characterizations of Reduced Normal Equations...

7. $E_{1b}\underline{P} = 0$

$$\begin{aligned} E_{1b}\underline{P} &= E_{1b}\left(\underline{B} - N'R^{-1}\underline{T}\right) \\ &= E_{1b}\underline{B} - E_{1b}N'R^{-1}\underline{T} \\ &= G - \underline{r}'R^{-1}\underline{T} \\ &= G - E_{1v}\underline{T} \\ &= G - G \\ &= 0 \end{aligned}$$

It is same as $\underline{P}E_{b1} = 0$
It means sum of elements of vector of adjusted block total is 0.

Characterizations of Reduced Normal Equations...

8. $Cov\left(\underline{P}\right) = D\sigma^2$

$$
\begin{aligned}
& Cov\left(\underline{P}\right) \\
&= Cov\left(\underline{B} - N'R^{-1}\underline{T}\right) \\
&= Cov\left(D'\underline{Y} - N'R^{-1}\Delta'\underline{Y}\right) \\
&= \left(D' - N'R^{-1}\Delta'\right) Cov\left(\underline{Y}\right)\left(D' - N'R^{-1}\Delta'\right)' \\
&= \left(D' - N'R^{-1}\Delta'\right)\sigma^2 I_n\left(D - \Delta R^{-1}N\right) \\
&= \left(D'\left(D - \Delta R^{-1}N\right) - N'R^{-1}\Delta'\left(D - \Delta R^{-1}N\right)\right)\sigma^2 \\
&= \left(K - N'R^{-1}N - N'R^{-1}N - N'R^{-1}RR^{-1}N\right)\sigma^2 \\
&= \left(K - N'R^{-1}N\right)\sigma^2 \\
&= D\sigma^2
\end{aligned}
$$

Characterizations of Reduced Normal Equations...

9. $\underline{P} \sim N_b\left(D\underline{\beta}, D\sigma^2\right)$

$$\begin{aligned}\underline{P} &= \left(\underline{B} - N'R^{-1}\underline{T}\right) \\ &= \left(D'\underline{Y} - N'R^{-1}\Delta'\underline{Y}\right) \\ &= \left(D' - N'R^{-1}\Delta'\right)\underline{Y}\end{aligned}$$

Observe the characterizations of $\underline{P}$

- Each element of $\underline{P}$ is linear combination of $\underline{Y}$ i.e. $\underline{P}$ is collection of b linear combinations of $\underline{Y}$
- $\underline{Y} \sim N_n\left(\underline{0}, \sigma^2 I_n\right)$ distribution
- $E\left(\underline{P}\right) = D\underline{\beta}$ and $Cov\left(\underline{P}\right) = D\sigma^2$
- Hence $\underline{P} \sim N_b\left(D\underline{\beta}, D\sigma^2\right)$ distribution.

Characterizations of Reduced Normal Equations...

10. $\underline{P} = D\underline{\beta}$ is a consistent system of equations
    As the reduced normal equations are obtained by eliminating one set of parameters and original system of equations is consistent hence the set of reduced normal equations is consistent.

Dependance between adjusted treatment and block totals

11. $Cov\left(\underline{Q},\underline{P}\right)=-NK^{-1}D\sigma^2$

$$
\begin{aligned}
&Cov\left(\underline{Q},\underline{P}\right)\\
&=Cov\left(\underline{T}-NK^{-1}\underline{B},\underline{B}-N'R^{-1}\underline{T}\right)\\
&=Cov\left(\Delta'\underline{Y}-NK^{-1}D'\underline{Y},D'\underline{Y}-N'R^{-1}\Delta'\underline{Y}\right)\\
&=\left(\Delta'-NK^{-1}D'\right)Cov\left(\underline{Y}\right)\left(D'-N'R^{-1}\Delta'\right)'\\
&=\left(\Delta'-NK^{-1}D'\right)\sigma^2 I_n\left(D'-N'R^{-1}\Delta'\right)'\\
&=\left(\Delta'-NK^{-1}D'\right)\left(D-\Delta R^{-1}N\right)\sigma^2\\
&=\left(\Delta'\left(D-\Delta R^{-1}N\right)-NK^{-1}D'\left(D-\Delta R^{-1}N\right)\right)\sigma^2\\
&=\left(N-RR^{-1}N-NK^{-1}K-NK^{-1}N'R^{-1}N\right)\sigma^2\\
&=-NK^{-1}\left(K-N'R^{-1}N\right)\sigma^2\\
&=-NK^{-1}D\sigma^2
\end{aligned}
$$