ST-302
Design, Planning and Analysis of Experiments

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Topic 8: Analysis of Covariance for one-way model

One-way ANCOVA model with one concomitant variable

- Consider Factor A with k levels (treatments) and are to be tested for significance
- Suppose each treatment is replicated n times
- y_{ij} : the j^{th} observation on the **response variable** in the i^{th} group
- x_{ij} : the j^{th} observation on **the covariate** in the i^{th} group

One-way classification model with one concomitant variable

Model:

$$y_{ij} = \mu + \tau_i + \gamma x_{ij} + \varepsilon_{ij}, \quad i = 1,2, \dots, k; j = 1,2, \dots, n$$

y_{ij}: the j^{th} observation on the **response variable** in the i^{th} group

μ: Common effect

τ_i: Effect of i^{th} treatment

γ: Regressor parameter associated with covariate

x_{ij}: the j^{th} observation on **the covariate** in the i^{th} group

ε_{ij}: Random error associated with y_{ij}

Assumptions

- Mean error is zero i.e $E(\varepsilon_{ij}) = 0$
- Variance of errors is constant i.e $var(\varepsilon_{ij}) = \sigma^2 \; for \; all \; i, j$
- Covariances between errors is zero $var(\underline{\varepsilon}) = \sigma^2 I_n$
- $\varepsilon_{ij} \sim N(0, \sigma^2)$

Implications

- $E(y_{ij}) = \mu + \tau_i + \gamma x_{ij}$
- $var(y_{ij}) = \sigma^2$
- y_{ij} are independently distributed but not identical
- $y_{ij} \sim N(\mu + \tau_i + \gamma x_{ij}, \sigma^2)$

Data for one-way ANCOVA model

Factor/ Group	Observations on the **response variable**				Observations on the **covariate**			
A_1	y_{11}	y_{12}	$\ldots$	y_{1n}	x_{11}	x_{12}	$\ldots$	x_{1n}
A_2	y_{21}	y_{22}	$\ldots$	y_{2n}	x_{21}	x_{22}	$\ldots$	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
A_k	y_{k1}	y_{k2}	$\ldots$	y_{kn}	x_{k1}	x_{k2}	$\ldots$	x_{kn}

Model details

- Number of observations=nk
- Number of parameters=$k+2$

  $\mu,$

  $\tau_1, \tau_2, \dots, \tau_k,$

  γ
- Here $nk > k+2$ ($n > p$ assumption in GLM)

Reparametrization of the Model

$$y_{ij} = \mu + \tau_i + \gamma x_{ij} + \varepsilon_{ij}, \quad i = 1,2,\dots,k; j = 1,2,\dots,n$$

$$= \mu + \tau_i + \gamma\left(x_{ij} - \bar{x}_{..}\right) + \gamma\bar{x}_{..} + \varepsilon_{ij}$$

$$= (\mu + \gamma\bar{x}_{..}) + \tau_i + \gamma\left(x_{ij} - \bar{x}_{..}\right) + \varepsilon_{ij}$$

$$= \mu^0 + \tau_i + \gamma\left(x_{ij} - \bar{x}_{..}\right) + \varepsilon_{ij},$$

Here $\mu^0 = \mu + \gamma\bar{x}_{..}$. The parameter transformation is as follows.

$$\begin{pmatrix} \mu \\ \underline{\tau} \\ \gamma \end{pmatrix} \rightarrow \begin{pmatrix} \mu^0 = \mu + \gamma\bar{x}_{..} \\ \underline{\tau} \\ \gamma \end{pmatrix}$$

Derivation of normal equations for ANCOVA Model

$$\phi = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \hat{\mu}^0 - \hat{\tau}_i - \hat{\gamma}\left(x_{ij} - \bar{x}_{..}\right)\right)^2$$

Differentiate with respect to unknown parameters.

$$\frac{d\phi}{d\hat{\mu}^0} = 0 \Rightarrow -2\sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \hat{\mu}^0 - \hat{\tau}_i - \hat{\gamma}\left(x_{ij} - \bar{x}_{..}\right)\right) = 0 \qquad (1)$$

$$\frac{d\phi}{d\hat{\tau}_i} = 0 \Rightarrow -2\sum_{j=1}^{n}\left(y_{ij} - \hat{\mu}^0 - \hat{\tau}_i - \hat{\gamma}\left(x_{ij} - \bar{x}_{..}\right)\right) = 0 \qquad i = 1,2,\ldots,k \quad (A)$$

$$\frac{d\phi}{d\hat{\gamma}} = 0 \Rightarrow -2\sum_{i=1}^{k}\sum_{j=1}^{n}\left\{\left(y_{ij} - \hat{\mu}^0 - \hat{\tau}_i - \hat{\gamma}\left(x_{ij} - \bar{x}_{..}\right)\right)\left(x_{ij} - \bar{x}_{..}\right)\right\} = 0 \qquad (2)$$

Derivation of normal equations for ANCOVA Model…

$(1) \Rightarrow \quad y_{..} = nk\hat{\mu}^0 + n\sum_{i=1}^{k}\hat{\tau}_i + \hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..}) \quad (3)$

$(A) \Rightarrow \quad y_{i.} = n\hat{\mu}^0 + n\hat{\tau}_i + \hat{\gamma}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..}) \quad (B)$

$(2) \Rightarrow \quad \sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}(x_{ij} - \bar{x}_{..}) = nk\hat{\mu}^0\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..})$

$$+\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..})\hat{\tau}_i + \hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..})^2$$

$\Rightarrow \quad \sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}(x_{ij} - \bar{x}_{..}) = \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..})\hat{\tau}_i + \hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{..})^2 \quad (4)$

Derivation of normal equations for ANCOVA Model…

$$y_{..} = nk\hat{\mu}^0 + n\sum_{i=1}^{k}\hat{\tau}_i + \hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{..}\right) \qquad (3)$$

$$y_{i.} = n\hat{\mu}^0 + n\hat{\tau}_i + \hat{\gamma}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{..}\right) \qquad (\mathrm{B})$$

$$\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}\left(x_{ij} - \bar{x}_{..}\right) = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{..}\right)\hat{\tau}_i$$

$$+\hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{..}\right)^2 \qquad (4)$$

- Observe that (3), (B) and (4) are $2 + k$ equations in $2 + k$ variables.
- These equations are called normal equations.
- But all are not linearly independent.
- Hence the solution to normal equations would not be unique.

Solutions of normal equations

- Only $(k+2)-1$ of these are linearly independent as
  - ❖ $\sum_{i=1}^{k}(B) = (3)$ i.e. addition of k equations in (A) gives (3)
- Hence we need one additional equation which is linearly independent with (3), (4) and (B).
- Let this equation be $\sum_{i=1}^{k} \hat{\tau}_i = 0$ (5)

Using (5) in (3):

$$y_{..} = nk\hat{\mu}^0 + n\sum_{i=1}^{k} \hat{\tau}_i + \hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{..}\right)$$

$$\Rightarrow \qquad \hat{\mu}^0 = \bar{y}_{..}$$

Solutions of normal equations…

$$y_{i.} = n\hat{\mu}^0 + n\hat{\tau}_i + \hat{\gamma}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{..}\right) \qquad \text{(B)}$$

$$\Rightarrow \quad y_{i.} = n\hat{\mu}^0 + n\hat{\tau}_i + \hat{\gamma}n(\bar{x}_{i.} - \bar{x}_{..})$$

$$\Rightarrow \quad \bar{y}_{i.} = \hat{\mu}^0 + \hat{\tau}_i + \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})$$

$$\Rightarrow \quad \bar{y}_{i.} - \bar{y}_{..} = \hat{\tau}_i + \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..}) \qquad \because \hat{\mu}^0 = \bar{y}_{..}$$

$$\Rightarrow \quad (\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..}) = \hat{\tau}_i$$

$$\Rightarrow \quad \hat{\tau}_i = (\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})$$

Some notations for one-way ANCOVA model

$k = Number\ of\ treatments \quad n = Replication\ of\ each\ treatments$

$nk = Number\ of observations$

$T_{yy} = \frac{\sum_{i=1}^{k} y_{i.}^2}{n} - \frac{y_{..}^2}{nk},$ $T_{xx} = \frac{\sum_{i=1}^{k} x_{i.}^2}{n} - \frac{x_{..}^2}{nk},$ $T_{xy} = \frac{\sum_{i=1}^{k} x_{i.} y_{i.}}{n} - \frac{x_{..} y_{..}}{nk},$

$G_{yy} = \sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{nk},$ $G_{xx} = \sum_{i=1}^{k} \sum_{j=1}^{n} x_{ij}^2 - \frac{x_{..}^2}{nk},$ $G_{xy} = \sum_{i=1}^{k} \sum_{j=1}^{n} x_{ij} y_{ij} - \frac{x_{..} y_{..}}{nk}$

$E_{yy} = G_{yy} - T_{yy}$ $E_{xx} = G_{xx} - T_{xx}$ $E_{xy} = G_{xy} - T_{xy}$

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Solutions of normal equations…

Observe that,

$$G_{xx} = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{..}\right)^2 = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}^2 - \frac{x_{..}^2}{nk},$$

$$G_{xy} = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})\left(x_{ij} - \bar{x}_{..}\right)$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}\left(x_{ij} - \bar{x}_{..}\right)$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}y_{ij} - \frac{x_{..}y_{..}}{nk},$$

$$T_{xx} = \sum_{i=1}^{k}\sum_{j=1}^{n}(\bar{x}_{i.} - \bar{x}_{..})^2 = \sum_{i=1}^{k} n(\bar{x}_{i.} - \bar{x}_{..})^2 = \frac{\sum_{i=1}^{k} x_{i.}^2}{n} - \frac{x_{..}^2}{nk}$$

$$T_{xy} = \sum_{i=1}^{k}\sum_{j=1}^{n}(\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..})$$

$$= \sum_{i=1}^{k} n(\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..})$$

$$= \frac{\sum_{i=1}^{k} x_{i.}y_{i.}}{n} - \frac{x_{..}y_{..}}{nk}$$

Solutions of normal equations…

$$\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}\left(x_{ij}-\bar{x}_{..}\right)=\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)\hat{\tau}_i+\hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)^2 \qquad (4)$$

$\Rightarrow \quad G_{xy}=\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)\hat{\tau}_i+\hat{\gamma}G_{xx}$

$\Rightarrow \quad G_{xy}=\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)\{(\bar{y}_{i.}-\bar{y}_{..})-\hat{\gamma}(\bar{x}_{i.}-\bar{x}_{..})\}+\hat{\gamma}G_{xx} \qquad \because \hat{\tau}_i=(\bar{y}_{i.}-\bar{y}_{..})-\hat{\gamma}(\bar{x}_{i.}-\bar{x}_{..})$

$\Rightarrow \quad G_{xy}=\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)(\bar{y}_{i.}-\bar{y}_{..})-\hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{..}\right)(\bar{x}_{i.}-\bar{x}_{..})+\hat{\gamma}G_{xx}$

$\Rightarrow \quad G_{xy}=\sum_{i=1}^{k}n(\bar{x}_{i.}-\bar{x}_{..})(\bar{y}_{i.}-\bar{y}_{..})-\hat{\gamma}\sum_{i=1}^{k}n(\bar{x}_{i.}-\bar{x}_{..})(\bar{x}_{i.}-\bar{x}_{..})+\hat{\gamma}G_{xx}$

$\Rightarrow \quad G_{xy}=\sum_{i=1}^{k}n(\bar{x}_{i.}-\bar{x}_{..})(\bar{y}_{i.}-\bar{y}_{..})-\hat{\gamma}\sum_{i=1}^{k}n(\bar{x}_{i.}-\bar{x}_{..})^2+\hat{\gamma}G_{xx}$

$\Rightarrow \quad G_{xy}=T_{xy}-\hat{\gamma}T_{xx}+\hat{\gamma}G_{xx}$

$\Rightarrow \quad G_{xy}-T_{xy}=\hat{\gamma}(G_{xx}-T_{xx})$

$\Rightarrow \quad E_{xy}=\hat{\gamma}E_{xx} \qquad \because G_{xy}-T_{xy}=E_{xy}$

Solutions of normal equations…

$$\Rightarrow \qquad G_{xy} - T_{xy} = \hat{\gamma}(G_{xx} - T_{xx})$$

$$\Rightarrow \qquad E_{xy} = \hat{\gamma} E_{xx}$$

$$\Rightarrow \qquad \hat{\gamma} = E_{xx}^{-1} E_{xy}$$

$$\Rightarrow \qquad \hat{\gamma} = E_{xx}^{-1} E_{xy}$$

$$\Rightarrow \qquad \hat{\gamma} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_{i.})(y_{ij}-\bar{y}_{i.})}{\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_{i.})^2}$$

$\because E_{xx} = \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})^2$ and

$E_{xy} = \sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})(y_{ij} - \bar{y}_{i.})$

Estimability of linear parametric function

- Model 1: Model without covariate : $y_{ij} = \mu^0 + \tau_i + \varepsilon_{ij},$
- Model 2: Model with covariate : $y_{ij} = \mu^0 + \tau_i + \gamma\left(x_{ij} - \bar{x}_{..}\right) + \varepsilon_{ij},$
- Contrasts in $\tau_i's$ are estimable in Model 1. Hence they will also be estimable in Model 2.
- BLUE's of contrasts
  - ❖ $\tau_i - \tau_j$ are estimable for all $i \neq j$

$$\widehat{\tau_i - \tau_j}$$

$$= \hat{\tau}_i - \hat{\tau}_j$$

$$= (\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..}) - \left\{\left(\bar{y}_{j.} - \bar{y}_{..}\right) - \hat{\gamma}\left(\bar{x}_{j.} - \bar{x}_{..}\right)\right\} \qquad \because \hat{\tau}_i = (\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})$$

$$= \left(\bar{y}_{i.} - \bar{y}_{j.}\right) - \hat{\gamma}\left(\bar{x}_{i.} - \bar{x}_{j.}\right)$$

Estimability of linear parametric function

- Variance of BLUE's of contrasts
  - $\tau_i - \tau_j$ are estimable for all $i \neq j$

$$\widehat{\tau_i - \tau_j} = (\bar{y}_{i.} - \bar{y}_{j.}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{j.})$$

$$var\left(\widehat{\tau_i - \tau_j}\right)$$

$$= var(\bar{y}_{i.} - \bar{y}_{j.}) + var(\hat{\gamma})(\bar{x}_{i.} - \bar{x}_{j.})^2 - 2cov\left(\hat{\gamma}, (\bar{y}_{i.} - \bar{y}_{j.})\right)$$

$$= \frac{2\sigma^2}{n} + \frac{\sigma^2(\bar{x}_{i.} - \bar{x}_{j.})^2}{\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})^2} \qquad \because var(\hat{\gamma}) = \frac{\sigma^2}{\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})^2}$$

Estimability of linear parametric function

- γ is estimable in model $y_{ij} = \mu^0 + \tau_i + \gamma\left(x_{ij} - \bar{x}_{..}\right) + \varepsilon_{ij}$,

- BLUE of $\gamma = \hat{\gamma} = E_{xx}^{-1} E_{xy} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{i.}\right)\left(y_{ij}-\bar{y}_{i.}\right)}{\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{i.}\right)^2}$

- $var(\hat{\gamma}) = E_{xx}^{-1}\sigma^2 = \frac{\sigma^2}{\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{i.}\right)^2}$

- $var\left(\hat{\gamma}, \left(\bar{y}_{i.} - \bar{y}_{j.}\right)\right) = 0$

Fitted value

$$
\begin{aligned}
\hat{y}_{ij} &= \hat{\mu}^0 + \hat{\tau}_i + \hat{\gamma}\left(x_{ij} - \bar{x}_{..}\right), \\
&= \bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..}) + \hat{\gamma}\left(x_{ij} - \bar{x}_{..}\right) \qquad \hat{\tau}_i = (\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..}) \\
&= \bar{y}_{i.} + \hat{\gamma}\left(x_{ij} - \bar{x}_{i.}\right)
\end{aligned}
$$

$$
\begin{aligned}
SSE &= \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \hat{y}_{ij}\right)^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\left(y_{ij} - \bar{y}_{i.}\right) - \hat{\gamma}\left(x_{ij} - \bar{x}_{i.}\right)\right)^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{i.}\right)^2 + \hat{\gamma}^2\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{i.}\right)^2 \\
&\qquad - 2\hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{i.}\right)\left(x_{ij} - \bar{x}_{i.}\right)
\end{aligned}
$$

SSE for one-way ANCOVA model with one concomitant variable

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \hat{y}_{ij}\right)^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\left(y_{ij} - \bar{y}_{i.}\right) - \hat{\gamma}\left(x_{ij} - \bar{x}_{i.}\right)\right)^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{i.}\right)^2 + \hat{\gamma}^2\sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij} - \bar{x}_{i.}\right)^2 - 2\hat{\gamma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{i.}\right)\left(x_{ij} - \bar{x}_{i.}\right)$$

$$= E_{yy} + \hat{\gamma}^2 E_{xx} - 2\hat{\gamma}E_{xy}$$

$$= E_{yy} + \left(E_{xx}^{-1}E_{xy}\right)^2 E_{xx} - 2E_{xx}^{-1}E_{xy}E_{xy}$$

$$= E_{yy} + E_{xx}^{-1}\left(E_{xy}\right)^2 - 2E_{xx}^{-1}E_{xy}^2$$

$$= E_{yy} - E_{xx}^{-1}E_{xy}^2$$

SSE for one-way ANCOVA model with one concomitant variable

$$SSE$$

$$= E_{yy} - E_{xx}^{-1} E_{xy}^{2}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n} \left(y_{ij} - \bar{y}_{i.}\right)^{2} - \frac{\left\{\sum_{i=1}^{k} \sum_{j=1}^{n} \left(y_{ij} - \bar{y}_{i.}\right)\left(x_{ij} - \bar{x}_{i.}\right)\right\}^{2}}{\sum_{i=1}^{k} \sum_{j=1}^{n} \left(x_{ij} - \bar{x}_{i.}\right)^{2}}$$

Rank of estimation and error space

Rank of estimation space

$= Number\ of\ linearly\ independent\ normal\ equations$

$= k + 1$

Rank of error space

$= nk -$ Rank of estimation space

$= nk - k - 1$

$= k(n - 1) - 1$

Testing of Hypothesis $H_0 : \gamma = 0$

The hypothesis $H_0 : \gamma = 0$ is testable as γ is estimable.

Original model : $y_{ij} = \mu^0 + \tau_i + \gamma\left(x_{ij} - \bar{x}_{..}\right) + \varepsilon_{ij}$,

Reduced model : $y_{ij} = \mu^0 + \tau_i + \varepsilon_{ij}$

SSE(original model)$= E_{xx} - E_{yx} E_{xx}^{-1} E_{xy}$

SSE(Reduced model)$= E_{yy}$

SSH_0

$= SSE(\text{Reduced model}) - SSE(\text{original model})$

$= E_{yx} E_{xx}^{-1} E_{xy}$

Testing of Hypothesis $H_0 : \gamma = 0$

Degrees of freedom for $SSH_0 = 1$

Degrees of freedom for $SSE = k(n-1)-1$

$$Test\ Statistic = \frac{SSH_0/1}{SSE/(k(n-1)-1)} = \frac{E_{xy}^2/E_{xx}}{\left(E_{yy}-\frac{E_{xy}^2}{E_{xx}}\right)/(k(n-1)-1)} \sim F_{1,(k(n-1)-1)}$$

where $E_{yy} = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij}-\bar{y}_{i.}\right)^2$

$$E_{xx} = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(x_{ij}-\bar{x}_{i.}\right)^2$$

$$E_{xy} = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(y_{ij}-\bar{y}_{i.}\right)\left(x_{ij}-\bar{x}_{i.}\right)$$

Some notations for ANCOVA for one-way model

$k = Number\ of\ treatments$ $\qquad n = Replication\ of\ each\ treatments$

$nk = Number\ of observations$

$$T_{yy} = \frac{\sum_{i=1}^{k} y_{i.}^2}{n} - \frac{y_{..}^2}{nk}, \qquad T_{xx} = \frac{\sum_{i=1}^{k} x_{i.}^2}{n} - \frac{x_{..}^2}{nk}, \qquad T_{xy} = \frac{\sum_{i=1}^{k} x_{i.}y_{i.}}{n} - \frac{x_{..}y_{..}}{nk},$$

$$G_{yy} = \sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{nk}, \qquad G_{xx} = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}^2 - \frac{x_{..}^2}{nk}, \qquad G_{xy} = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}y_{ij} - \frac{x_{..}y_{..}}{nk}$$

$$E_{yy} = G_{yy} - T_{yy} \qquad E_{xx} = G_{xx} - T_{xx} \qquad E_{xy} = G_{xy} - T_{xy}$$

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Testing the hypothesis of no covariate

Steps to test the hypothesis $\mathbf{H_{01}}: \boldsymbol{\gamma} = \mathbf{0}$

- *Calculate* E_{yy}, E_{yx}, E_{xx}
- *Calculate* $SSE = E_{yy} - E_{yx}E_{xx}^{-1}E_{xy} = T$
- *Calculate* $MSE = T/(k(n-1)-1)$
- *Calculate* $SSH_0 = SSE_c - SSE = E_{yy} - T$
- *Calculate* $F_C = \frac{SSH_0}{MSE} = \frac{MSH_0}{MSE} \sim F_{1,\,(k(n-1)-1)}$

Testing the hypothesis of equality of treatment effects

Steps to test the hypothesis H_{02}: $\boldsymbol{\tau_1 = \tau_2 = \cdots = \tau_k}$

- $Calculate\ E_{yy}, E_{yx}, E_{xx}$
- $Calculate\ SSE = E_{yy} - E_{yx}E_{xx}^{-1}E_{xy} = T$
- $Calculate\ MSE = T/(k(n-1)-1)$
- $Calculate\ E_{yy}^{c} = T_{yy} + E_{yy} = G_{yy}$
- $Calculate\ SSE_c = E_{yy}^{c} - E_{yx}^{c}{E_{xx}^{c}}^{-1}E_{xy}^{c}$

  $$= G_{yy} - G_{yx}G_{xx}^{-1}G_{xy} = T_1$$
- $Calculate\ SSH_0 = SSE_c - SSE = T_1 - T$
- $Calculate\ F_C = \frac{SSH_0/(k-1)}{MSE} = \frac{MSH_0}{MSE} \sim F_{k-1,\,(k(n-1)-1)}$

ANCOVA for one-way model with one concomitant variable

S.V.	df	Sum of squares			df_{adj}	SS_{adj}	MS_{adj}	F-Ratio	Hypothesis
Treat	(k – 1)	T_{yy}	T_{xx}	T_{xy}					
Error	k(n – 1)	E_{yy}	E_{xx}	E_{xy}		$T = E_{yy} - E_{yx}E_{xx}^{-1}E_{xy}$			
Total	(nk – 1)	G_{yy}	G_{xx}	G_{xy}		$T_1 = G_{yy} - G_{yx}G_{xx}^{-1}G_{xy}$			
Treat (Adj.)					$k - 1$	$T_1 - T$	$\frac{T_1 - T}{k - 1}$	$\frac{MS_{trea(adj)}}{MSE_{adj}}$	$H_{01}: \tau_1 = \cdots = \tau_k$
Error (Adj.)					k(n – 1) – 1	T	$\frac{T}{k(n-1)-1}$		
Regr.					1	$E_{yy} - T$	$E_{yy} - T$	$\frac{MS_{Reg(adj)}}{MSE_{adj}}$	$H_{02}: \gamma = 0$