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Assignment - 4

1) 4.1)

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

The average no. of imperfections per 10 meters is given by, μ_x

$\Rightarrow \mu_x = E[x] = \sum_x x f_x(x)$

$= 0(0.41) + 1(0.37) + 2(0.16) + 3(0.05) + 4(0.01)$

$= 0 + 0.37 + 0.32 + 0.15 + 0.04$

$\therefore \boxed{E[x] = 0.88}$

2) 4.7) given, probability of profit = 0.3 (\$4000)

probability of loss = 0.7 (\$1000)

Expected gain of the person is the expected value of profit and loss

$\Rightarrow$ Expected gain = $E[x]$

$E[x] = 0.3(4000) + 0.7(-1000)$

$= 500$

The expected gain of the person is \$500

3) 4.9) given, insurance for airplane = \$200000

probability for total (100%) loss = 0.002 = $f(0)$

probability for 50% loss = 0.01 = $f(1)$

probability for 25% loss = 0.1 = $f(2)$

profit expected = \$500

The ~~per~~ premium should the insurance company ~~chagre~~ charge each year = ~~$E[ax] = \sum a f(x)$~~

$= \sum_x g(x) f_x(x)$

$\Rightarrow$ ~~$= 200000(0.002)$~~

~~$= a f_x(0) + a f_x(1) + a f_x(2)$~~

$= 200000(0.002)(1) + 200000(0.01)(0.5)$
$+ 200000(0.1)(0.25)$

$= 400 + 1000 + 5000 = 6400$

$\therefore$ The total amount to be charged,
$= \$6400 + \$500 = \$6900$

4) 4.13) X = number of hours a vacuum cleaner runs

given,
$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Expected value of # hours per year $= \mu_x = E[x]$

$\Rightarrow$ $E[x] = \left[\int_{-\infty}^{\infty} x\, f_x(x)\,dx\right] \times 100$

$$= \left[\int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx + 0\right] \times 100$$

$$= \left[\int_0^1 x^2\,dx + \int_1^2 2x - x^2\right] \times 100$$

$$= \left(\frac{x^3}{3}\Big|_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2\right) \times 100$$

$$= \left\{\left(\frac{1}{3} - 0\right) + \left[\left(\frac{\cancel{2}(2)^2}{\cancel{2}} - \frac{\cancel{2}(1)^2}{\cancel{2}}\right) - \left(\frac{2^3}{3} - \frac{1^3}{3}\right)\right]\right\} \times 100$$

$$= \left\{\frac{1}{3} + \left[(4-1) - \left(\frac{8-1}{3}\right)\right]\right\} \times 100$$

$$= \left\{\frac{1}{3} + \left[3 - \frac{7}{3}\right]\right\} \times 100$$

$$= \left(\frac{1}{3} + \frac{2}{3}\right) \times 100$$

$$= \left(\frac{\cancel{3}^1}{\cancel{3}}\right) \times 100$$

$$= \boxed{100 \text{ hours}}$$

5) 4.35)

x	2	3	4	5	6
$f(x)$	0.01	0.25	0.4	0.3	0.04

$\sigma_x^2 = E[x^2] - \mu_x^2$

$\Rightarrow \mu_x = \sum_x x f_x(x)$

$= (2)(0.01) + (3)(0.25) + 4(0.4) + 5(0.3) + 6(0.04)$

$= 0.02 + 0.75 + 1.6 + 1.5 + 0.24$

$= 4.11$

$E[x^2] = \cancel{g(0.01)} + \sum_x x^2 f_x(x)$

$= (2^2)(0.01) + (3^2)(0.25) + (4^2)(0.4) + (5^2)(0.3) + (6^2)(0.04)$

$= 0.04 + 2.25 + 6.4 + 7.5 + 1.44$

$= 17.63$

$\therefore \sigma_x^2 = 17.63 - (4.11)^2 = 17.63 - 16.8921$

Varience, $\boxed{\sigma_x^2 = 0.7379}$

6) 4.39)

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Variance $= \sigma_x^2 = E[x^2] - \mu_x^2$

$\Rightarrow \mu_x = \int_{-\infty}^{\infty} x f_x(x) dx$

$= \int_0^1 x \cdot x\, dx + \int_1^2 x \cdot (2-x)\, dx + 0$

$= \frac{x^3}{3}\Big|_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$

$= \frac{1}{3} + \left[\frac{\cancel{2}(2)^2}{\cancel{2}} - \frac{\cancel{2}(1)^2}{\cancel{2}} - \frac{8}{3} - \frac{1}{3}\right]$

$= \frac{1}{3} + \left[3 - \frac{7}{3}\right] = \frac{1}{3} + \frac{2}{3} = 1$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2 \cdot (2-x)\,dx + 0$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3)\,dx$$

$$= \frac{x^4}{4}\Big|_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} - 0 + \left[\left(\frac{2(2)^3}{3} - \frac{2(1)^3}{3}\right) - \left(\frac{(2)^4}{4} - \frac{(1)^4}{4}\right)\right]$$

$$= \frac{1}{4} + \left[\left(\frac{16}{3} - \frac{2}{3}\right) - \left(\frac{16}{4} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{14}{3} - \frac{15}{4} = \cancel{3+24-45}$$

$$= \frac{3+56-45}{12} = \frac{14}{12} = \frac{7}{6}$$

$$\sigma_x^2 = \frac{7}{6} - (1)^2 = \frac{7-6}{6} = \frac{1}{6}$$

$$\boxed{\sigma_x^2 = \frac{1}{6}}$$

7) 4.45)

$f(x,y)$	x = 1	2	3	$f_Y(y)$
y = 1	0.05	0.05	0.10	0.2
3	0.05	0.10	0.35	0.5
5	0.00	0.20	0.10	0.3
$f_X(x)$	0.1	0.35	0.55	1.0

Covariance $= \sigma_{XY} = cov(X,Y) = E[XY] - \mu_X \mu_Y$

$$E[XY] = (1)(1)(0.05) + (2)(1)(0.05) + (3)(1)(0.10) +$$
$$(1)(3)(0.05) + (2)(3)(0.10) + (3)(3)(0.35) +$$
$$(1)(5)(0.00) + (2)(5)(0.20) + (3)(5)(0.10)$$

$$= 0.05 + 0.1 + 0.3 + 0.15 + 0.6 + 3.15 + 2 + 1.5$$

$$= 7.85$$

$\mu_x = E[x] = \sum_x x \cdot f(x) = (1)(0.1) + (2)(0.35) + (3)(0.55)$
$= 0.1 + 0.7 + 1.65 = 2.45$

$\mu_y = E[Y] = \sum_y y \cdot f(y) = (1)(0.2) + (3)(0.5) + (5)(0.3)$
$= 0.2 + 1.5 + 1.5 = 3.2$

Covariance $= 7.85 - [(2.45)(3.2)]$
$= 7.85 - 7.84 = \underline{0.01}$

8) 4.53)

x	2	3	4	5	6
$f(x)$	0.01	0.25	0.4	0.3	0.04

given, $Z = 3x - 2$

Mean of $Z = E[3x-2]$
$= 3E[x] - 2$

$E[x] = \sum_x x f(x)$
$= (2)(0.01) + (3)(0.25) + (4)(0.4) + (5)(0.3) + (6)(0.04)$
$= 0.02 + 0.75 + 1.6 + 1.5 + 0.24$
$= 4.11$

Mean of $Z = 3E[x] - 2$
$= 3(4.11) - 2 = \underline{10.33}$

Variance of $Z = V(ax+b)$ ~~$= V[3x-2]$~~
$= a^2 V(x)$

$\Rightarrow V(3x-2) = 3^2 V(x)$

$V(x) = E[x^2] - \mu_x^2$
$E[x^2] = (2^2)(0.01) + (3^2)(0.25) + (4^2)(0.4) + (5^2)(0.3)$
$+ (6^2)(0.04) = 0.04 + 2.25 + 6.4 + 7.5 + 1.44$
$= 17.63$

$V(x) = 17.63 - (4.11)^2 = 17.63 - 16.8921$
$V(x) = 0.7379$

Variance of $Z = 3^2 V(x) = 9(0.7379) = \underline{6.6411}$

9) 4.57)

x	-3	6	9
$f(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

$$E(X) = \sum_x x \cdot f(x)$$
$$= (-3)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{2}\right) + (9)\left(\frac{1}{3}\right)$$
$$= -\frac{1}{2} + 3 + 3 = \frac{11}{2}$$

$$E(X^2) = \sum_x x^2 \cdot f(x)$$
$$= (-3)^2\left(\frac{1}{6}\right) + (6)^2\left(\frac{1}{2}\right) + (9)^2\left(\frac{1}{3}\right)$$
$$= \frac{9}{6} + \frac{36}{2} + \frac{81}{3} = \frac{3}{2} + 18 + 27 = \frac{93}{2}$$

$$E[(2X+1)^2] = E[4X^2 + 2(2X)(1) + 1]$$
$$= E[4X^2 + 4X + 1]$$
$$= 4E[X^2] + 4E[X] + 1$$
$$= 4\left(\frac{93}{2}\right) + 4\left(\frac{11}{2}\right) + 1$$
$$= 186 + 22 + 1 = \underline{209}$$

10) 4.65)

X = red die

Y = green die

$X = Y$ (since both dice have same sample space)

y: x	1	2	3	4	5	6
$f(y)$: $f(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E[X] = E[Y] = (1)\left(\frac{1}{6}\right) + (2)\left(\frac{1}{6}\right) + (3)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$

$$= \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$$

a) $E(x+y) = E[x] + E[y]$

$= \frac{7}{2} + \frac{7}{2} = 7$

b) $E(x-y) = E[x] - E[y]$

$= \frac{7}{2} - \frac{7}{2} = 0$

c) $E[xy]$

Since throwing one die is independent of throwing other die, $E[xy] = E[x] \cdot E[y]$

$\therefore E[xy] = \left(\frac{7}{2}\right)\left(\frac{7}{2}\right) = \frac{49}{4}$

11) 4.77) given, x = random variable

$\mu = 10, \sigma^2 = 4 \Rightarrow \sigma = 2$

a) $P(|x-10| \geq 3)$

$= 1 - P(|x-10| < 3) = \cancel{1 - P(-6 < x-8 < 6)}$

$= 1 - P(-3 < x - 10 < 3)$

$= 1 - P(10-3 < x < 10+3)$

$\cancel{= 1 - P[10}$

$k\sigma = 3$

$k(2) = 3 \Rightarrow k = \frac{3}{2}$

$= 1 - P\left(10 - \frac{3}{2}(2) < x < 10 + \frac{3}{2}(2)\right)$

$\geq 1 - \left(1 - \frac{1}{k^2}\right) = \frac{1}{k^2} = \frac{1}{\left(\frac{3}{2}\right)^2} = \boxed{\frac{4}{9}}$

b) $P(|x-10| < 3)$

$= P(-3 < x - 10 < 3)$

$= P(10-3 < x < 10+3)$

from above $k = \frac{3}{2}$

$$= P\left[10 - \frac{3}{2}(2) < X < 10 + \frac{3}{2}(2)\right] \geq 1 - \frac{1}{k^2}$$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{\left(\frac{3}{2}\right)^2} = 1 - \frac{1}{\frac{9}{4}} = 1 - \frac{4}{9}$$

$$= \frac{9-4}{9} = \boxed{\frac{5}{9}}$$

c) $P(5 < X < 15)$

given mean $= 10$, $\sigma^2 = 4 \Rightarrow \sigma = 2$

$\Rightarrow P[10 - k(2) < X\ 10 + k(2)] \geq 1 - \frac{1}{k^2}$

$\Rightarrow 10 - k(2) = 5$

$\Rightarrow -k(2) = 5 - 10$

$\cancel{-}k(2) = \cancel{-}5$

$\Rightarrow k = \frac{5}{2}$

$$P\left(10 - \frac{5}{2}(2) < X < 10 + \frac{5}{2}(2)\right) \geq 1 - \frac{1}{k^2}$$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{\left(\frac{5}{2}\right)^2} = 1 - \frac{1}{\frac{25}{4}} = 1 - \frac{4}{25} = \frac{21}{25}$$

$$\Rightarrow P(5 < X < 15) = \boxed{\frac{21}{25}}$$

d) find c in $P(|X - 10| \geq c) \leq 0.04$

$P(|X - 10| \geq c) = 1 - P(|X - 10| < c)$

$= 1 - P(-c < X - 10 < c) = 1 - P(10 - c < X < 10 + c)$

$\Rightarrow c = k\sigma$

from question $y - \left(x - \frac{1}{k^2}\right) = 0.04$

$\Rightarrow \frac{1}{k^2} = 0.04$

$\Rightarrow k^2 = \frac{1}{0.04} = 25 \Rightarrow k = \sqrt{25}$

$\boxed{k = 5}$

$\Rightarrow \boxed{c = 5 \times 2 = 10}$