Ma311matrixmethod

STABILITY ANALYSIS USING MATRIX METHOD

BTCS

Consider PDE $u_t = u_{xx}$ and BTCS discretization.

$$u_i^{n+1} - u_i^n = d\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right), \quad d = \frac{\Delta t}{\Delta x^2}$$

ie $$-d\,u_{i-1}^{n+1} + (1+2d)\,u_i^{n+1} - d\,u_{i-1}^{n+1} = u_i^n.$$

for $i = 1$, to $N-1$

$$-d\,u_0^{n+1} + (1+2d)\,u_1^{n+1} - d\,u_{i-1}^{n+1} = u_1^n \qquad i=1$$

$\vdots$

$$-d\,u_{N-2}^{n+1} + (1+2d)\,u_{N-1}^{n+1} - d\,u_N^{n+1} = u_{N-1}^n \qquad i=N-1$$

Using the boundary condition we get

$$A\,U^{n+1} = B\,U^n + b^{n+1}, \quad \text{where } b^{n+1} = \left(d u_0, \ldots d u_N^{n+1}\right)^T$$

$$A = \begin{pmatrix} 1+2d & -d & 0 & 0 & \cdots & 0 \\ -d & 1+2d & -d & \cdots & & 0 \\ 0 & -d & 1+2d & -d & & 0 \\ 0 & 0 & 0 & \ddots & \ddots & -d \\ 0 & 0 & 0 & 0 & -d & 1+2d \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} = I. \quad u^n = (U_1^n, \ldots U_{N-1}^n)^T$$

Now $U^{n+1} = A^{-1} B U^n + A^{-1} b^{n+1}$, we use $\tilde{A} = A^{-1} b^{n+1}$

Now $U^{n+1} = A^{-1} B U^n + \tilde{A}^{n+1}$

$$U^{n+1} = A^{-1}B\left(A^{-1}BU^{n-1} + \tilde{A}^n\right) + \tilde{A}^{n+1}$$

$$= (A^{-1}B)^2 U^{n-1} + (A^{-1}B)\tilde{A}^n + \tilde{A}^{n+1}, \quad A^{-1}B = Q$$

$$= Q^2 U^{n-1} + Q\tilde{A}^n + \tilde{A}^{n+1}$$

$$= Q^2\left(QU^{n-2} + \tilde{A}^{n-1}\right) + Q\tilde{A}^n + \tilde{A}^{n+1}.$$

$$= Q^3 U^{n-2} + Q^2\tilde{A}^{n-1} + Q\tilde{A}^n + \tilde{A}^{n+1}$$

$$= Q^{n+1} U^0 + Q^n \tilde{A}^0 + \tilde{Q}^{n-1} \tilde{A}^2 \cdots + \tilde{Q} \tilde{A}^n + \tilde{A}^{n+1}$$

$$\therefore \quad U^{n+1} = Q^{n+1} U^0 + Q^n \tilde{A} + \tilde{Q}^{n-1} \tilde{A}^2 + \cdots + \tilde{Q} \tilde{A}^n + \tilde{A}^{n+1} \quad (1)$$

$$\therefore \|U^{n+1}\| \le \|Q^{n+1} U^0\| + \|Q^n \tilde{A}\| + \|\tilde{Q}^{n-1} \tilde{A}^2\| + \cdots + \|\tilde{A}^{n+1}\|$$

ie $\|U^{n+1}\| \le \|Q^{n+1}\| \, \|U^0\| + \|Q^n\| \, \|\tilde{A}\| + \cdots + \|\tilde{A}^{n+1}\|.$

Therefore for the solution to remain bounded we need $\|Q\| \le 1$, and hence the stability condition reduces to $\|Q\| \le 1$; or we say the method is stable if $\|Q\| \le 1$.

$\therefore$ For BTCS scheme we have

$$Q = A^{-1}B = A^{-1} \quad \text{as } B = I, \text{ the identity}$$

matrix.

Note the fact that A^{-1} exist as A is strictly diagonally dominant matrix.

Now we consider the $\| \ \|_2$ norm stability

$\|Q\|_2 = \|A^{-1}\|$, it is easy to prove that the spectral radius $\rho(A^{-1}) = \dfrac{1}{\rho(A)}$

since $A = \text{diag}(-d, 1+2d, -d)$

eigenvalues of A, $\lambda_i = 1 + 2d - 2d\cos\theta_i$, $\theta_i = \dfrac{i\pi}{N}$

$$\therefore \rho(A) = \max_i |1 + 2d + 2d\cos\theta_i|$$

$$\|Q\|_2 = \rho(A^{-1}) = \frac{1}{\rho(A)} = \frac{1}{\max\limits_{1 \le i \le N-1} |1+2d-2d\cos\theta_i|} \le 1$$

for any $d > 0$

$\therefore$ $\boxed{\text{BTCS is unconditionally stable in } \| \ \|_2 \text{ norm.}}$

Round-off error stability.

Let us say the initial datum U^0 is perturbed as U^0_*, and the corresponding solution U^{n+1}_* is

$$U^{n+1}_* = Q^{n+1} U^0_* + Q^n \tilde{A}_* + \cdots + \tilde{Q} \tilde{A}^n_* + \tilde{A}^{n+1}_* \quad — (2)$$

For simplicity if we take the zero boundary condition, ① − ② gives

$$U^{n+1} - U^{n+1}_* = Q^{n+1} (U^0 - U^0_*)$$

$e^{n+1} = u_*^{n+1} - u^{n+1}$, is the corresponding round-off error

now $\|e^{n+1}\| \leq \|Q^{n+1}\| \, \|u_*^0 - u^0\|$

$\|e^{n+1}\|$ remain bounded when $\|Q^n\| \leq 1$

ie iff $\|Q\|^n \leq 1$, ie iff $\|Q\| \leq 1$.

* We consider the stability with respect to the natural norms $\| \ \|_2$ & $\| \ \|_\infty$.

FTCS

we derive the stability condition for the FTCS method applied to $u_t = u_{xx}$. with initial & boundary conditions. The FTCS scheme is

$$U_i^{n+1} = u_i^n + d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \quad d = \frac{\Delta t}{\Delta x^2}$$

$$u_i^{n+1} = d\,u_{i-1}^n + (1-2d)\,u_i^n + d\,u_{i+1}^n \qquad i = 1 \text{ to } N-1$$

$$u^{n+1} = A\,U^n + b^n \qquad b^n = (d\,u_0^n, \ldots d\,u_N^n)^T$$

$$U^{n+1} = (U_1^{n+1}, \ldots, U_{N-1}^{n+1})^T$$

$$A = \begin{pmatrix} 1-2d & d & 0 & \cdots & & \\ d & 1-2d & d & 0 & \cdots & \\ 0 & d & 1-2d & d & \cdots & \\ & & & & & d \\ & & & & d & 1-2d \end{pmatrix}$$

ie $A = \operatorname{diag}(d, 1-2d, d)$.

we derive the stability condition for the FTCS method applied to $u_t = u_{xx}$ with initial & boundary conditions. The FTCS scheme is

$$U_i^{n+1} = u_i^n + d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right), \quad d = \frac{\Delta t}{\Delta x^2}$$

$$u_i^{n+1} = d\,u_{i-1}^n + (1-2d)u_i^n + d\,u_{i+1}^n \qquad i = 1 \text{ to } N-1$$

$$u^{n+1} = A U^n + b^n \qquad b^n = (d u_0^n, \ldots d u_N^n)^T$$

$$U^{n+1} = (U_0^{n+1}, \ldots, U_{N-1}^{n+1})^T$$

$$A = \begin{pmatrix} 1-2d & d & 0 & \cdots & & \\ d & 1-2d & d & 0 & \cdots & \\ 0 & d & 1-2d & d & \cdots & \\ & & & & & d \\ & & & & d & 1-2d \end{pmatrix}$$

ie $A = \operatorname{diag}(d, 1-2d, d)$.

$$A = I_{N-1 \times N-1} + d T_{N-1}$$

$$T_{N-1} = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & & \end{pmatrix}_{N-1 \times N-1}$$

As we see A is a Hermition matrix, $\bar{A}^T = A$

$$\therefore \quad \|A\|_2 = \rho(A) = \max |\lambda_i|.$$

now, eigenvalues of T_{N-1} are $\alpha_i = -2 + 2\cos\frac{i\pi}{N}$
$= -2\left(1 - \cos\frac{i\pi}{N}\right)$

and therefore eigenvalues of A are

$\lambda_i = 1 + d\alpha_i \quad \therefore \quad \lambda_i = 1 - 2d\left(1 - \cos\frac{i\pi}{N}\right)$

$$\therefore \quad \max_i |\lambda_i| = \max_i \left|1 - 2d\left(1 - \cos\frac{i\pi}{N}\right)\right|$$

the scheme is stable if $\|A\|_2 \leq 1$

ie if

$$-1 \le 1 - 2d\left(1 - \cos\frac{i\pi}{N}\right) \le 1 \qquad \forall i = 1 \text{ to } N-1$$

ie $\quad -2 \le -2d\left(1-\cos\frac{i\pi}{N}\right) \le 0$

upper bound is always ~~satisfied~~ as $d > 0$

Now,

$\|A\|_2 \le 1$ if $d\left(1-\cos\frac{i\pi}{N}\right) \le 1$

$\therefore$ If $d \le 1/2$ $\qquad d\left(1-\cos\frac{i\pi}{N}\right) \le \frac{\left(1-\cos\frac{i\pi}{N}\right)}{2} \le \frac{2}{2} = 1$

$\therefore$ the stability condition is $\frac{\Delta t}{\Delta x^2} \le 1/2$.

<u>Alternate approach,</u>

$\therefore$ when $0 < d \le 1/2$

$d < 2d \le 1$

$1 - 2d \ge 0 \qquad \therefore \quad \|A\|_\infty = 1 - 2d + 2d = 1.$

when $\quad d > 1/2 \qquad 2d - 1 > 0$

$\therefore \quad \|A\|_\infty = 2d - 1 + 2d = 4d - 1 > 1$

$\therefore$ stable iff $0 < d \le 1/2$ ie if $\frac{\Delta t}{\Delta x^2} \le 1/2$.

Exercise: Using matrix method show that the Crank-Nicolson scheme is unconditionally stable.

Hint: Here $Q = A^{-1}B$,

$$A = \frac{1}{2}\,\mathrm{diag}(-\lambda, 2+2\lambda, -\lambda)$$

$$B = \frac{1}{2}\,\mathrm{diag}(\lambda, 2-2\lambda, \lambda)$$

$$\|Q\|_2 = \sqrt{\rho(Q^T Q)}$$