Stability of finite difference schemes

von Neumann method

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In this note we study the stability criteria for the finite difference schemes applied to diffusion equations.

Fourier series

If $f : [0, 2\pi] \to \mathbb{R}$, then the Fourier series representation of f if it exist is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where $a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos ntdt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin ntdt$

Using the Euler formula $e^{i\theta} = \cos\theta + i \sin\theta$ we can express the Fourier series of the given f in the following form

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx},$$

where

$$\alpha_n = \frac{a_n - ib_n}{2}, \quad n = 1, 2, \cdots,.$$
$$\alpha_n = \frac{a_n + ib_n}{2}, \quad n = -1, -2, \cdots,.$$

and $\alpha_0 = \frac{a_0}{2}$.

If $f : [a, b] \to \mathbb{R}$, then the Fourier series representation of f if it exist is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos[2\pi n \frac{(x-a)}{(b-a)}] + b_n \sin[2\pi n \frac{(x-a)}{(b-a)}],$$

and the corresponding complex form is

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi n \frac{x-a}{b-a}},$$

If we take the domain $D = [a, b] \times [0, \infty)$ to solve the diffusion equation, we can represent the approximated solution at n th time step with $N + 1$ spacial grid points $x_0 = a, x_1, \cdots x_N = b$ as

$$u(x_i, t^n) \approx u_i^n = \sum_{m=-N/2}^{N/2} c_m^n e^{\frac{2\pi I m i}{N}},$$

For convenience we used the symbol $I = \sqrt{-1}$. Also we use the fact that the solution is periodic, i.e., $u_0^n = u_N^n$ for all n. It follows that

$$u_i^{n+1} = \sum_{m=-N/2}^{N/2} c_m^{n+1} e^{\frac{2\pi I m i}{N}}.$$

$$u_{i\pm 1}^{n+1} = \sum_{m=-N/2}^{N/2} c_m^{n+1} e^{\frac{2\pi I m(i\pm 1)}{N}}.$$

Note that the Fourier coefficients c_m^n depends on both m and n, and the exact solution of the finite difference is completely determined by these coefficients.

Remark: (Alternate formulation) The collection

$$\left\{(1, e^{\frac{2\pi I m 2}{N}}, \cdots, e^{\frac{2\pi I m N}{N}})\right\}_{m=-N/2}^{N/2} \tag{1}$$

forms a set of $N+1$ linearly independent vectors. Thus any solution vector $\boldsymbol{u}^n = (u_0^n, \cdots, u_N^n)$ can be uniquely expressed as a linear combination of elements of set (1), i.e. there exists constants c_m^n such that

$$\boldsymbol{u}^n = \sum_{m=-N/2}^{N/2} c_m^n \boldsymbol{v}_m; \qquad \boldsymbol{v}_m = (1, e^{\frac{2\pi I m 2}{N}}, \cdots, e^{\frac{2\pi I m N}{N}}) \tag{2}$$

so components u_i^n of $\boldsymbol{u}^n$ can be expressed as:

$$u_i^n = \sum_{m=-N/2}^{N/2} c_m^n e^{\frac{2\pi I m i}{N}}.$$

Now consider the FTCS scheme for the diffusion equation $u_t = u_{xx}$:

$$u_i^{n+1} = u_i^n + d(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad d = \frac{\Delta t}{\Delta x^2}.$$

Substituting the Fourier expressions we get

$$\sum_{m=-N/2}^{N/2} c_m^{n+1} e^{\frac{2\pi Imi}{N}} = \sum_{m=-N/2}^{N/2} \left(d(e^{\frac{2\pi Im}{N}} + e^{-\frac{2\pi Im}{N}} - 2) + 1\right) c_m^n e^{\frac{2\pi Imi}{N}},$$

$$\sum_{m=-N/2}^{N/2} \left(c_m^{n+1} - c_m^n (d(e^{\frac{2\pi Im}{N}} + e^{-\frac{2\pi Im}{N}} - 2) + 1)\right) e^{\frac{2\pi Imi}{N}} = 0,$$

Since the set $\{e^{\frac{2\pi Imi}{N}}\}_{m=-N/2}^{N/2}$ is linearly independent it follows that

$$c_m^{n+1} - c_m^n (d(e^{\frac{2\pi Im}{N}} + e^{-\frac{2\pi Im}{N}} - 2) + 1) = 0, \quad m = -N/2, \cdots, N/2.$$

Note that $\cos\theta = (e^{I\theta} + e^{-I\theta})/2$, together with this and using the notation $\phi_m = \frac{2\pi m}{N}$ the last expression becomes

$$c_m^{n+1} - c_m^n(2d(\cos\phi_m - 1) + 1) = 0,$$

$$c_m^{n+1} = G_m c_m^n, \quad G_m = 2d(\cos\phi_m - 1) + 1, \tag{3}$$

G_m is the **amplification factor**, which is independent of the factor n. For the solution to remain bounded we need $|G_m| \leq 1$, i.e

$$-1 \leq 2d(\cos\phi_m - 1) + 1 \leq 1 \quad \Longrightarrow \quad d \leq \frac{1}{1 - \cos\phi_m},$$

$$d \leq \min_{\phi_m} \frac{1}{1 - \cos\phi_m} = \frac{1}{2}.$$

There for the condition for stability reduces to $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$.

Thus the FTCS is conditionally stable.

It is enough to consider a single Fourier term;

$$u_i^n = c_m^n e^{I\phi_m i},$$

eventually we drop the subscript m and we write

$$u_i^n = c^n e^{I\phi i},$$

From (3) we assert that $c_m^n = G_m.G_m.G_m....c_m^0$. Using a convenient choice of $c_m^0 = 1$, (any other choice except 0 would work with out effecting the result) we write $c_m^n = G_m^n$, again dropping the subscript m, we write

$$c_m^n = G^n,$$

note that here the superscript n is the power of G.
Now our aim is to find the condition for which $|G| \leq 1$.

Again we consider the FTCS scheme for the diffusion equation $u_t = u_{xx}$:

$$u_i^{n+1} = u_i^n + d(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad d = \frac{\Delta t}{\Delta x^2}.$$

Substituting $u_i^n = G^n e^{I\phi i}$ in the above expression we get

$$G^{n+1} e^{I\phi i} = G^n e^{I\phi i} + d(G^n e^{I\phi(i+1)} - 2G^n e^{I\phi i} + G^n e^{I\phi(i-1)}),$$

Dividing throughout by $G^n e^{I\phi i}$ we get

$$\begin{aligned} G &= 1 + d(e^{I\phi} + e^{-I\phi} - 2), \\ &= 1 + 2d(\cos\phi - 1). \end{aligned}$$

Finally $|G| \leq 1$ if $d \leq \frac{1}{2}$.

Exercise: Compute the amplification factor G for the BTCS and Crank-Nicolson schemes applied to the diffusion equation $u_t = \alpha^2 u_{xx}$ and find the corresponding stability conditions.

Hint: The amplification factor for BTCS scheme is

$$G = \frac{1}{1 + 2d(1 - \cos\phi)}, \quad d = \frac{\alpha^2 \Delta t}{\Delta x^2}.$$

The amplification factor for Crank-Nicolson scheme is

$$G = \frac{1 - d(1 - \cos\phi)}{1 + d(1 - \cos\phi)}.$$

Alternate way:

We are now considering only one term of the Fourier expansion of the component u_i^n,

$$u_i^n = c_m^n e^{I\phi_m i}$$

We see that $u_{i\pm1}^n = e^{\pm I\phi_m} u_i^n$. In order to see the growth of the computed solution u_i^{n+1} we express this as

$$u_i^{n+1} = G u_i^n$$

and we find the expression for G, the stability condition now becomes $|G| \leq 1$. Finally we drop the index m as usual.

FTCS for diffusion equation $u_t = u_{xx}$

$$u_i^{n+1} = u_i^n + d(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad d = \frac{\Delta t}{\Delta x^2}.$$

Substituting $u_{i\pm 1}^n = e^{\pm I\phi_m} u_i^n$ in the above scheme we get

$$\begin{aligned} u_i^{n+1} &= u_i^n + d(e^{I\phi} u_i^n - 2u_i^n + e^{-I\phi} u_i^n) \\ &= \Big(1 + d(e^{I\phi} + e^{-I\phi} - 2)\Big) u_i^n. \\ &= G u_i^n, \end{aligned}$$

and thus the amplification factor is obtained as
$G = \Big(1 + d(e^{I\phi} + e^{-I\phi} - 2)\Big)$,

$$G = 1 + 2d(\cos\phi - 1)$$