# 3 Elliptic equation

Consider the Dirichlet problem

$$Pu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \text{ in } \Omega = (0,1)\times(0,1) \tag{3.1}$$

with boundary condition

$$u = g(x,y) \text{ on the boundary of } \Omega = \partial\Omega \tag{3.2}$$

Let us denote $u(x_i.y_j) = u_i^j, f(x_i,y_j) = f_i^j$and on boundary $u_i^j = v_i^j = g(x_i,y_j)$. Then the numerical scheme corresponding to (3.1) can be written as

$$P_{\Delta x,\Delta y}v = \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta x)^2} + \frac{(v_i^{j+1} - 2v_i^j + v_i^{j-1})}{(\Delta y)^2} = f_i^j \tag{3.3}$$

which is second order accurate. Because if $u$ is a solution of (3.1) then

$$P_{\Delta x,\Delta y} = O(\Delta x^2) + O(\Delta y^2)$$

Since the ratio of the mesh plays an insignificant role in the theory of elliptic problems. to study the above problem we take $\Delta x = \Delta y = h$ for simplicity. Then (3.3) becomes

$$v_{i+1}^j - 2v_i^j + v_{i-1}^j + (v_i^{j+1} - 2v_i^j + v_i^{j-1}) = h^2 f_i^j$$

i.e.,

$$4v_i^j = v_{i+1}^j + v_{i-1}^j + v_i^{j+1} + v_i^{j-1} - h^2 f_i^j, \quad 1 \le i,j \le M-1 \tag{3.4}$$

Let $x_0 = y_0 = 0$ $\qquad x_i = i\Delta x = ih, i = 0,\dots,M$
$x_M = y_M = 1$ $\qquad y_i = i\Delta y = ih, i = 0,\dots,M$
Let us consider the following simple cases to understand the scheme (3.4)

Let $M = 3$
Then by (3.4)

$$\begin{aligned}
4v_1^1 &- (v_2^1 + v_0^1 + v_1^2 + v_1^0) = -h^2 f_1^1\\
4v_2^1 &- (v_3^1 + v_1^1 + v_2^2 + v_2^0) = -h^2 f_2^1\\
4v_1^2 &- (v_2^2 + v_0^2 + v_1^3 + v_1^1) = -h^2 f_1^2\\
4v_2^2 &- (v_3^2 + v_1^2 + v_2^3 + v_2^1) = -h^2 f_2^2
\end{aligned}$$

As on boundary $v_i^j = g_i^j$,this can be written as a linear system

$$Av = b_{g,f}$$

i.e.

$$Av = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} g_0^1 + g_1^0 - h^2 f_1^1 \\ g_3^1 + g_2^0 - h^2 f_2^1 \\ g_0^2 + g_1^3 - h^2 f_1^2 \\ g_3^2 + g_2^3 - h^2 f_2^2 \end{bmatrix} = bg, f.$$

Now A can be written as

$$A = \begin{bmatrix} B & -I \\ -I & B \end{bmatrix}, \quad \text{where } B = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A$ is a strictly diagonally dominant matrix. Hence $A$ is invertible. Therefore above linear system can be solved uniquely.
Now let us consider the case $M = 4$

$$\begin{aligned}
4v_1^1 - v_2^1 - v_1^2 &= v_1^0 + v_0^1 - h^2 f_1^1 \\
4v_2^1 - v_3^1 - v_2^1 - v_1^1 &= v_2^0 - h^2 f_2^1 \\
4v_3^1 - v_3^2 - v_2^2 &= v_3^0 + v_4^1 - h^2 f_3^1 \\
4v_1^2 - v_2^2 - v_1^3 - v_1^1 &= v_0^2 - h^2 f_1^2 \\
4v_2^2 - v_3^2 - v_1^2 - v_2^3 - v_2^1 &= 0 - h^2 f_2^2 \\
4v_3^2 - v_2^2 - v_3^1 - v_3^3 &= v_4^2 - h^2 f_3^2 \\
4v_1^3 - v_2^3 - v_1^2 &= v_0^3 + v_1^4 - h^2 f_1^3 \\
4v_2^3 - v_3^3 - v_2^2 - v_1^3 &= v_2^4 - h^2 f_2^3 \\
4v_3^3 - v_2^3 - v_3^2 &= v_4^3 + v_3^4 - h^2 f_3^3
\end{aligned}$$

This can be written as a linear system

$$Av = b_{g,f}$$

i,e.,

$$Av = \begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \\ v_1^2 \\ v_2^2 \\ v_3^2 \\ v_1^3 \\ v_2^3 \\ v_3^3 \end{bmatrix} = b_{g,f}$$

Now A can be written as

$$A = \begin{bmatrix} B & -I & O \\ -I & B & -I \\ O & -I & B \end{bmatrix}, \quad \text{where } B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

$A$ is $9 \times 9$ matrix, $B$, $O$(zero matrix) and $I$ are $3 \times 3$ matrices.
In general (3.4) can be written as

$$Av = b_{g,f}$$

where

$$A = \begin{bmatrix} B & -I & & & O \\ -I & B & -I & & \\ & -I & B & -I & \\ & & . & . & . \\ O & & & -I & B \end{bmatrix}$$

where

$$B = \begin{bmatrix} 4 & -1 & & O \\ -1 & 4 & -1 & \\ & . & . & . \\ O & & -1 & 4 \end{bmatrix}$$

with $A$ is a matrix of order $(M-1)^2$. $B$ , $O$ and $I$ are matrices of order $M-1$. $v$ is a vector given by

$$v = (v_1^1, v_2^1, \ldots, v_{M-1}^1, v_1^2, v_2^2, \ldots, v_{M-1}^2, \ldots, v_1^{M-1}, \ldots, v_{M-1}^{M-1})^T$$

and $b_g$ is a vector depends on the boundary values.

A is symmetric and positive definite. As A is tridioganal block matix there are several methods like direct methods or iterative methods to solve the above system.

**Convergence :** Let $u = u(x,y)$ be the actual solution of the problem and let $\epsilon_i^j = u_i^j - v_i^j$ where $u_i^j = u(x_i, y_j) = u(ih, jh)$ and $v_i^j$ is obtained from (3.4). Since $\Delta u = f$ we have

$$4u_i^j - (u_{i+1}^j + u_{i-1}^j + u_i^{j+1} + u_i^{j-1} - h^2 f_i^j) + O(h^4) = 0$$ (is obtained by Teylar method)

Therefore we have, by subtracting 3.4 from above eqn.

$$L\epsilon = \epsilon_i^j - \frac{1}{4}(\epsilon_{i+1}^j + \epsilon_{i-1}^j + \epsilon_i^{j+1} + \epsilon_i^{j-1}) = O(h^4) \leq Mh^4$$

for some $M > 0$. On the boundary $\epsilon_i^j = u_i^j - v_i^j \equiv 0$. Let $w(x,y) = x^2 + y^2$ and $w_i^j = w(ih, jh)$ Then

$$Lw = w_i^j - \frac{1}{4}(w_{i+1}^j + w_{i-1}^j + w_i^{j+1} + w_i^{j-1}) = -h^2$$

Define

$$\tilde{\epsilon}^j{}_i = \epsilon_i^j + Mh^2 w_i^j. \text{ Then}$$

$$L\tilde{\epsilon} = L\epsilon + Mh^2 Lw \leq Mh^4 - Mh^4 = 0$$

$\Rightarrow$

$$\tilde{\epsilon}_i^j \leq \frac{1}{4}(\tilde{\epsilon}_{i+1}^j + \tilde{\epsilon}_{i-1}^j + \tilde{\epsilon}_i^{j+1} + \epsilon_i^{j-1})$$

$\Rightarrow$ $\tilde{\epsilon}_i^j$ attains maxima on the boundary

Let $r$ denotes the radius of a circle about the origin enclosing the region $\Omega = (0,1) \times (0,1)$. Then

$\tilde{\epsilon}_i^j \leq$ maximum of $\epsilon_i^j$ on the boundary $+Mh^2r^2$

$= 0 + Mh^2 r = Mh^2 r^2$( because $\epsilon_i^j \equiv 0$ on boundary)

Now define $\underline{\epsilon}_i^j = \epsilon_i^j - Mh^2 w_i^j$. By similar arguments one can show that

$$\underline{\epsilon}_i^j \geq -Mr^2h^2$$

$$-Mr^2h^2 \leq \underline{\epsilon}_i^j \leq \epsilon_i^j \leq \tilde{\epsilon}_i^j \leq Mh^2r^2$$
$$\Rightarrow |\epsilon_i^j| \leq Mh^2r^2$$

Hence as mesh size goes to zero numerical solution goes to actual solution.