Bilinear Transformation

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January 20, 2025

Problem Statement

Given Differential Equation:

$$\frac{d^2y}{dx^2} + 1 = 0 \tag{1}$$

Initial Conditions:

$$y(0) = 0, \quad y'(0) = 0 \tag{2}$$

Laplace Transform:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \tag{3}$$

Properties:

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) \tag{4}$$

$$\tag{5}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \tag{6}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = x^2 u(x) \tag{7}$$

Substituting the initial conditions gives

$$y'' + 1 = 0 \tag{8}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{1\} = 0 \tag{9}$$

Inverse Laplace Transform

$$s^2\mathcal{L}\{y\} - sy\,(0) - y'\,(0) + \frac{1}{s} = 0 \tag{10}$$

$$\mathcal{L}\{y\} = \frac{-1}{s^3} \tag{11}$$

$$y = \frac{-x^2}{2}u\,(x) \tag{12}$$

Mapping to z-domain:

$$s = \frac{2}{h}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{13}$$

Substituting:

$$Y(z) = -\left(\frac{h}{2}\right)^3 \left(\frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 3z^{-1} + 3z^{-2} - z^{-3}}\right) \tag{14}$$

$$Y(z)\left(1 - 3z^{-1} + 3z^{-2} - z^{-3}\right) = -\left(\frac{h}{2}\right)^3 \left(1 + 3z^{-1} + 3z^{-2} + z^{-3}\right) \tag{15}$$

Multiply the above equation by z^3 and taking the inverse z-transform yields the difference equation:
ROC: $z \neq 0$

$$Y(z)z^3 - zy[2] - z^2y[1] - z^3y[0] - 3Y(z)z^2 + 3zy[1] + z^2y[0] + 3Y(z)z$$
$$-3zy[0] - Y(z) = -\left(\frac{h}{2}\right)^3 [\delta(n+3) + 3\delta(n+2) + 3\delta(n+1) + \delta(n)] \tag{16}$$

Difference Equation:

$$y(n+3) = -3y(n+1) + y(n) + 3y(n+2) + h^2\delta(n) - \left(\frac{h}{2}\right)^3 \delta(n) \tag{17}$$

Finite Difference Approximation

Second Derivative Approximation:

$$\frac{d^2y}{dx^2} \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \tag{18}$$

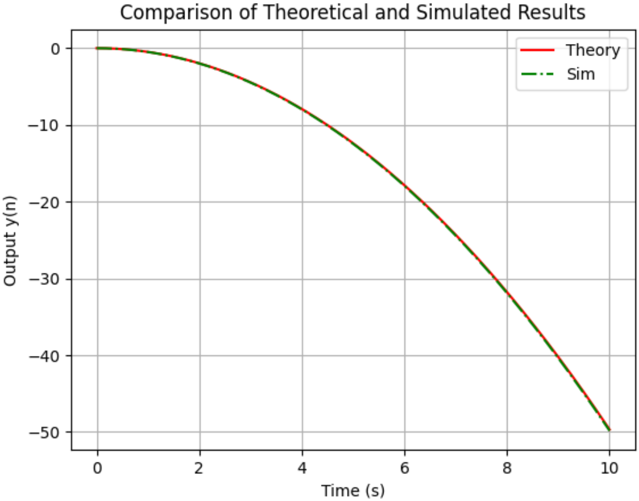
Substituting:

$$y_{n+1} = 2y_n - y_{n-1} - h^2 \tag{19}$$

Initial Conditions:

$$y_0 = 0, \quad y_1 = 0 \tag{20}$$

Plot of the Solution



Conclusion

- Laplace Transform provides an analytical solution.
- Bilinear Transform maps it to the z-domain for digital processing.
- Finite Difference Method allows computational approximation.
- Reference : https://github.com/nithin769/EE1030/sciprogramming/question-9.3.11.3/codes