Question-9.3.11.3

EE24BTECH11048-NITHIN.K

Question:
Solve the differential equation $\frac{d^2y}{dx^2} + 1 = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 0$
Theoritical Solution:
Laplace Transform :

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{0.1}$$

Properties of Laplace tranform

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}(1) = \frac{1}{s} \tag{0.3}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = x^2 u(x) \tag{0.4}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{0.5}$$

Applying the properties to the given equation

$$y'' + 1 = 0 \tag{0.6}$$

$$\mathcal{L}(y'') + \mathcal{L}(1) = 0 \tag{0.7}$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \frac{1}{s} = 0 \tag{0.8}$$

Substituting the initial conditions gives

$$s^3\mathcal{L}(y) + 1 = 0 \tag{0.9}$$

$$\mathcal{L}(y) = \frac{-1}{s^3} \tag{0.10}$$

Bilinear Transform:
The bilinear transform maps the s-domain to the z-domain as:

$$s = \frac{2}{h}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{0.11}$$

Substitute s in $\mathcal{L}(y)$:

$$\mathcal{L}(y) = -\left(\frac{h}{2}\right)^3 \left(\frac{1+z^{-1}}{1-z^{-1}}\right)^3 \tag{0.12}$$

Let the z-transform of $y(n)$ be $Y(z)$. The z-transform relation becomes:

$$Y(z) = -\left(\frac{h}{2}\right)^3 \left(\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+3z^{-2}-z^{-3}}\right) \tag{0.13}$$

$$Y(z)\left(1-3z^{-1}+3z^{-2}-z^{-3}\right) = -\left(\frac{h}{2}\right)^3 \left(1+3z^{-1}+3z^{-2}+z^{-3}\right) \tag{0.14}$$

Multiply the above equation by z^3 and taking the inverse z-transform yields the difference equation:
ROC: $z \neq 0$

$$y(n+3) = -3y(n+1) + y(n) + 3y(n+2) + h^2\delta(n) - \left(\frac{h}{2}\right)^3 \delta(n) \tag{0.15}$$

Computational Solution:
Finite Difference Approximation
The second derivative $\frac{d^2y}{dx^2}$ is replaced by:

$$\frac{d^2y}{dx^2} \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} \tag{0.16}$$

Substitute this into the differential equation:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + 1 = 0 \tag{0.17}$$

$$y_{n+1} - 2y_n + y_{n-1} + h^2 = 0 \tag{0.18}$$

$$y_{n+1} = 2y_n - y_{n-1} - h^2 \tag{0.19}$$

If $y(0) = 0$, then $y_0 = 0$.
To compute y_1, use the derivative approximation:

$$y'(0) \approx \frac{y_1 - y_0}{h} \tag{0.20}$$

$$y_1 = 0 \tag{0.21}$$

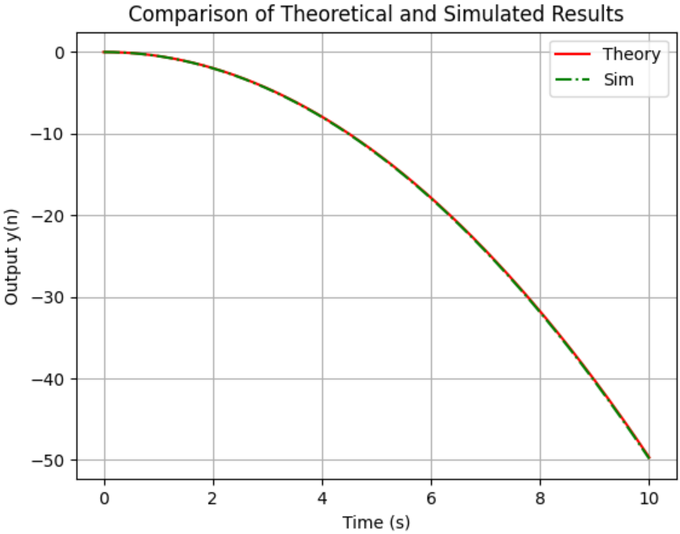
Comparison of Theoretical and Simulated Results
Theory
Sim
Output y(n)
Time (s)
0
−10
−20
−30
−40
−50
0
2
4
6
8
10

Fig. 0.1: Plot