

# Bilinear Transformation

EE24BTECH11048-Nithin.K

January 21, 2025

# Problem Statement

**Given Differential Equation:**

$$\frac{d^2y}{dx^2} + 1 = 0 \quad (1)$$

**Initial Conditions:**

$$y(0) = 0, \quad y'(0) = 0 \quad (2)$$

# Laplace Transform Method

## Laplace Transform:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (3)$$

## Properties:

$$\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - sy(0) - y'(0) \quad (4)$$

(5)

$$\mathcal{L}\{1\} = \frac{1}{s} \quad (6)$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = x^2 u(x) \quad (7)$$

## Substituting the initial conditions gives

$$y'' + 1 = 0 \quad (8)$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{1\} = 0 \quad (9)$$

# Inverse Laplace Transform

$$s^2 \mathcal{L}\{y\} - sy(0) - y'(0) + \frac{1}{s} = 0 \quad (10)$$

$$\mathcal{L}\{y\} = \frac{-1}{s^3} \quad (11)$$

$$y = \frac{-x^2}{2} u(x) \quad (12)$$

**Mapping to z-domain:**

$$s = \frac{2}{h} \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \quad (13)$$

**Substituting:**

$$Y(z) = - \left( \frac{h}{2} \right)^3 \left( \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - 3z^{-1} + 3z^{-2} - z^{-3}} \right) \quad (14)$$

$$Y(z) (1 - 3z^{-1} + 3z^{-2} - z^{-3}) = - \left( \frac{h}{2} \right)^3 (1 + 3z^{-1} + 3z^{-2} + z^{-3}) \quad (15)$$

Multiply the above equation by  $z^3$  and taking the inverse z-transform yields the difference equation:

ROC:  $z \neq 0$

$$Y(z)z^3 - zy[2] - z^2y[1] - z^3y[0] - 3Y(z)z^2 + 3zy[1] + z^2y[0] + 3Y(z)z - 3zy[0] - Y(z) = -\left(\frac{h}{2}\right)^3 [\delta(n+3) + 3\delta(n+2) + 3\delta(n+1) + \delta(n)] \quad (16)$$

**Difference Equation:**

$$y(n+3) = -3y(n+1) + y(n) + 3y(n+2) + h^2\delta(n) - \left(\frac{h}{2}\right)^3 \delta(n) \quad (17)$$

# Finite Difference Approximation

## Second Derivative Approximation:

$$\frac{d^2y}{dx^2} \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad (18)$$

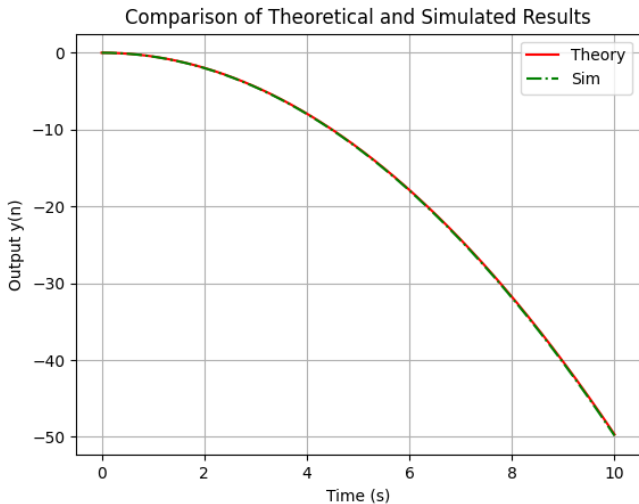
## Substituting:

$$y_{n+1} = 2y_n - y_{n-1} - h^2 \quad (19)$$

## Initial Conditions:

$$y_0 = 0, \quad y_1 = 0 \quad (20)$$

# Plot of the Solution





# Conclusion

- Laplace Transform provides an analytical solution.
- Bilinear Transform maps it to the z-domain for digital processing.
- Finite Difference Method allows computational approximation.
- Reference :  
<https://github.com/nithin769/EE1030/sciprogramming/question-9.3.11.3/codes>