Question-4.1.1

EE24BTECH11048-NITHIN.K

Question:
Find the roots of the quadratic equation $(x+1)^2 = 2(x-3)$
Theoretical Solution:
Simplifying the terms,

$$x^2 + 1 + 2x = 2x - 6 \tag{0.1}$$

$$x^2 + 7 = 0 \tag{0.2}$$

The roots are,

$$x = \pm\sqrt{7}j \tag{0.3}$$

Computational Solution:
Eigenvalues of Companion Matrix:
The roots of a polynomial equation $x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0$ is given by finding eigenvalues of the companion matrix (C).

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix} \tag{0.4}$$

Here $b_0 = 7$, $b_1 = 0$

$$C = \begin{pmatrix} 0 & 1 \\ -7 & 0 \end{pmatrix} \tag{0.5}$$

We find the eigenvalues using the QR algorithm. The basic principle behind this algorithm is a similarity transform,

$$A' = X^{-1}AX \tag{0.6}$$

which does not alter the eigenvalues of the matrix A.
We use this to get the Schur Decomposition,

$$A = Q^{-1}UQ = Q^*UQ \tag{0.7}$$

where Q is a unitary matrix $\left(Q^{-1} = Q^*\right)$ and U is an upper triangular matrix whose diagonal entries are the eigenvalues of A.
To efficiently get the Schur Decomposition, we first householder reflections to reduce it

to an upper hessenberg form.
A householder reflector matrix is of the form,

$$P = I - 2\mathbf{u}\mathbf{u}^* \tag{0.8}$$

Householder reflectors transforms any vector $\mathbf{x}$ to a multiple of $\mathbf{e_1}$,

$$P\mathbf{x} = \mathbf{x} - 2\mathbf{u}\left(\mathbf{u}^*\mathbf{x}\right) = \alpha\mathbf{e_1} \tag{0.9}$$

P is unitary, which implies that,

$$\|P\mathbf{x}\| = \|\mathbf{x}\| \tag{0.10}$$

$$\implies \alpha = \rho\,\|\mathbf{x}\| \tag{0.11}$$

$$\tag{0.12}$$

As $\mathbf{u}$ is unit norm,

$$\mathbf{u} = \frac{\mathbf{x} - \rho\,\|\mathbf{x}\|\,\mathbf{e_1}}{\|\mathbf{x} - \rho\,\|\mathbf{x}\|\,\mathbf{e_1}\|} = \frac{1}{\|\mathbf{x} - \rho\,\|\mathbf{x}\|\,\mathbf{e_1}\|}\begin{pmatrix} x_1 - \rho\,\|\mathbf{x}\| \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{0.13}$$

Selection of ρ is flexible as long as $|\rho| = 1$. To ease out the process, we take $\rho = \frac{x_1}{|x_1|}$, $x_1 \neq 0$. If $x_1 = 0$, we take $\rho = 1$.
Householder reflector matrix (P_i) is given by,

$$P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^* \\ \mathbf{0} & I_{n-i} - 2\mathbf{u_i}\mathbf{u_i}^* \end{bmatrix} \tag{0.14}$$

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \tag{0.15}$$

Next step is to do Given's rotation to get the QR Decomposition.
The Givens rotation matrix $G(i, j, c, s)$ is defined by

$$G(i, j, c, s) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\bar{s} & \cdots & \bar{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{0.16}$$

where $|c|^2 + |s|^2 = 1$, and G is a unitary matrix.
Say we take a vector $\mathbf{x}$, and $\mathbf{y} = G(i, j, c, s)\,\mathbf{x}$, then

$$y_k = \begin{cases} cx_i + sx_j, & k = i \\ -\overline{s}x_i + \overline{c}x_j, & k = j \\ x_k, & k \neq i, j \end{cases} \tag{0.17}$$

For y_j to be zero, we set

$$c = \frac{\overline{x_i}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = c_{ij} \tag{0.18}$$

$$s = \frac{\overline{x_j}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = s_{ij} \tag{0.19}$$

Using this Givens rotation matrix, we zero out elements of subdiagonal in the hessenberg matrix H.

$$\begin{aligned} H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} &\xrightarrow{G(1,2,c_{12},s_{12})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \\ &\xrightarrow{G(2,3,c_{23},s_{23})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,4,c_{34},s_{34})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \\ &\xrightarrow{G(4,5,c_{45},s_{45})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} = R \end{aligned} \tag{0.20}$$

where R is upper triangular. For the given companion matrix,

Let $G_k = G(k, k+1, c_{k,k+1}, s_{k,k+1})$, then we deduce that

$$G_4G_3G_2G_1H = R \tag{0.21}$$

$$H = G_1^*G_2^*G_3^*G_4^*R \tag{0.22}$$

$$H = QR, \text{ where } Q = G_1^*G_2^*G_3^*G_4^* \tag{0.23}$$

Using this QR algorithm, we get the following update equation,

$$A_k = Q_kR_k \tag{0.24}$$

$$A_{k+1} = R_kQ_k \tag{0.25}$$

$$= (G_n \dots G_2G_1)\,A_k\,(G_1^*G_2^* \dots G_n^*) \tag{0.26}$$

Running the eigenvalue code we get

$$x_1 = 0.0 + 2.6457513110645907j \tag{0.27}$$

$$x_2 = 0.0 - 2.6457513110645907j \tag{0.28}$$

Newton-Raphson iterative method:

$$f(x) = x^2 + 7 \tag{0.29}$$

$$f'(x) = 2x \tag{0.30}$$

Difference equation,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.31}$$

$$x_{n+1} = x_n - \frac{x_n^2 + 7}{2x_n} \tag{0.32}$$

$$x_{n+1} = \frac{x_n}{2} - \frac{7}{2x_n} \tag{0.33}$$

Picking two initial guesses,

$$x_0 = 0 + i \text{ converges to } 0.0 + 2.6457513110645907j \tag{0.34}$$

$$x_0 = 0 - i \text{ converges to } 0.0 - 2.6457513110645907j \tag{0.35}$$

Acknowledgments:
This project uses code from (GitHub: Dwarak A), available at [https://github.com/Dwarak-A/sprog/tree/c6f873e29ab9910f22e49f7421dfc8e5fd5c60fc/ncert/10/4/1/1/2/codes].