

Question-9.3.11.3

EE24BTECH11048-NITHIN.K

Question:

Solve the differential equation $\frac{d^2y}{dx^2} + 1 = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 0$

Theoretical Solution:

Laplace Transform :

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt \quad (0.1)$$

Properties of Laplace tranform

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - sy(0) - y'(0) \quad (0.2)$$

$$\mathcal{L}(1) = \frac{1}{s} \quad (0.3)$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = x^2 u(x) \quad (0.4)$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \quad (0.5)$$

Applying the properties to the given equation

$$y'' + 1 = 0 \quad (0.6)$$

$$\mathcal{L}(y'') + \mathcal{L}(1) = 0 \quad (0.7)$$

$$s^2 \mathcal{L}(y) - sy(0) - y'(0) + \frac{1}{s} = 0 \quad (0.8)$$

Substituting the initial conditions gives

$$s^3 \mathcal{L}(y) + 1 = 0 \quad (0.9)$$

$$\mathcal{L}(y) = \frac{-1}{s^3} \quad (0.10)$$

Bilinear Transform:

The bilinear transform maps the s-domain to the z-domain as:

$$s = \frac{2}{h} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (0.11)$$

Substitute s in $\mathcal{L}(y)$:

$$\mathcal{L}(y) = -\left(\frac{h}{2}\right)^3 \left(\frac{1+z^{-1}}{1-z^{-1}}\right)^3 \quad (0.12)$$

Let the z-transform of $y(n)$ be $Y(z)$. The z-transform relation becomes:

$$Y(z) = -\left(\frac{h}{2}\right)^3 \left(\frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-3z^{-1}+3z^{-2}-z^{-3}}\right) \quad (0.13)$$

$$Y(z)(1-3z^{-1}+3z^{-2}-z^{-3}) = -\left(\frac{h}{2}\right)^3 (1+3z^{-1}+3z^{-2}+z^{-3}) \quad (0.14)$$

Multiply the above equation by z^3 and taking the inverse z-transform yields the difference equation:

ROC: $z \neq 0$

$$y(n+3) = -3y(n+1) + y(n) + 3y(n+2) + h^2\delta(n) - \left(\frac{h}{2}\right)^3 \delta(n) \quad (0.15)$$

Computational Solution:

Finite Difference Approximation

The second derivative $\frac{d^2y}{dx^2}$ is replaced by:

$$\frac{d^2y}{dx^2} \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta x^2} \quad (0.16)$$

Substitute this into the differential equation:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + 1 = 0 \quad (0.17)$$

$$y_{n+1} - 2y_n + y_{n-1} + h^2 = 0 \quad (0.18)$$

$$y_{n+1} = 2y_n - y_{n-1} - h^2 \quad (0.19)$$

If $y(0) = 0$, then $y_0 = 0$.

To compute y_1 , use the derivative approximation:

$$y'(0) \approx \frac{y_1 - y_0}{h} \quad (0.20)$$

$$y_1 = 0 \quad (0.21)$$

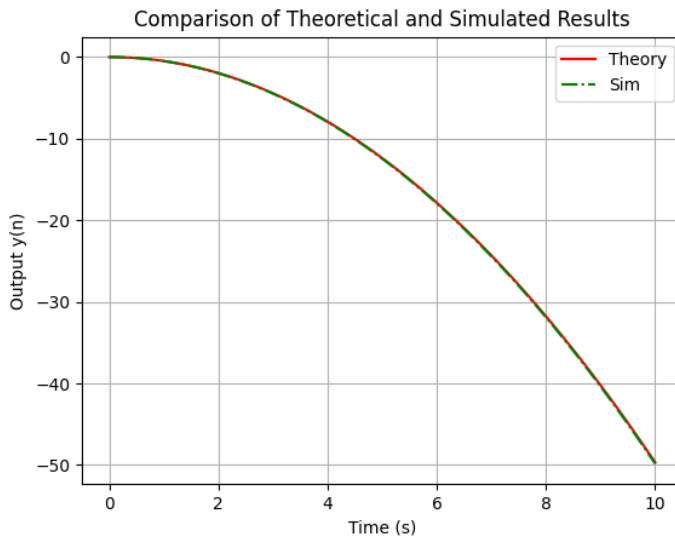


Fig. 0.1: Plot