Question-3.4.2.3

EE24BTECH11048-NITHIN.K

Question:
The sum of the two digits of a two-digit number is 9. Also nine times this number is twice the number obtained by reversing the order of the digits.
Theoretical Solution:

$$x + y = 9 \tag{0.1}$$

$$9(10x + y) = 2(10y + x) \tag{0.2}$$

$$8x - y = 0 \tag{0.3}$$

The system of linear equations can be written as:

$$A\mathbf{x} = \mathbf{b} \tag{0.4}$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 8 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \tag{0.5}$$

Dolittle's Algorithm:
The LU decomposition splits A into a lower triangular matrix L and an upper triangular matrix U such that:

$$A = LU \tag{0.6}$$

Where:

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ L_{21} & 1 & 0 & \cdots & 0 \\ L_{31} & L_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ L_{n1} & L_{n2} & L_{n3} & \cdots & 1 \end{pmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & U_{n-1,n} \\ 0 & 0 & 0 & \cdots & U_{nn} \end{pmatrix}. \tag{0.7}$$

The Doolittle algorithm is computed as follows:
Elements of the U Matrix:
For each column j:

$$U_{ij} = A_{ij} \text{ if } i = 0, \tag{0.8}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \text{ if } i > 0. \tag{0.9}$$

Elements of the L Matrix:
For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \text{ if } j = 0, \tag{0.10}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \text{ if } j > 0. \tag{0.11}$$

Performing LU Decomposition using Dolittle's Algorithm, we get:

$$L = \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 0 & -9 \end{pmatrix} \tag{0.12}$$

Now we solve the system in two steps using forward substitution and backward substitution.
First Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$:

$$\begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \tag{0.13}$$

$$y_1 = 9 \tag{0.14}$$

$$8y_1 + y_2 = 0, \quad y_2 = -72 \tag{0.15}$$

$$\mathbf{y} = \begin{pmatrix} 9 \\ -72 \end{pmatrix} \tag{0.16}$$

Solve for $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$:

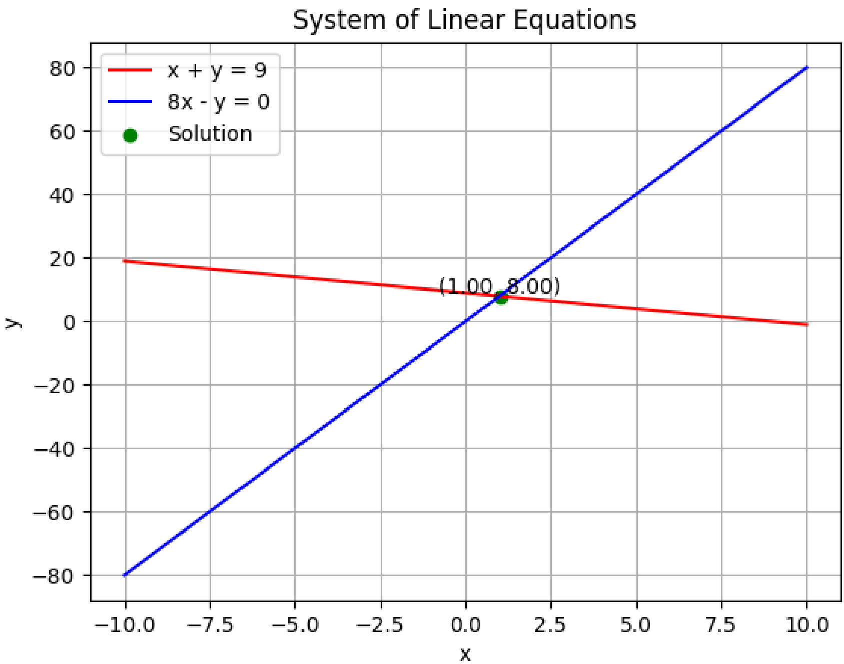
$$\begin{pmatrix} 1 & 1 \\ 0 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ -72 \end{pmatrix} \tag{0.17}$$

$$-9y = -72, \quad y = 8 \tag{0.18}$$

$$x + y = 9 \tag{0.19}$$

$$x = 1 \tag{0.20}$$

Thus the solution is $x = 1, y = 8$


System of Linear Equations
x + y = 9
8x - y = 0
Solution
(1.00, 8.00)
80
60
40
20
0
−20
−40
−60
−80
y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x