Convolution of $f(t) = \log t$ with a Rectangular Kernel

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1 Problem Statement

Compute the convolution of the signal $f(t) = \log t$ with the rectangular kernel:

$$h(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

The convolution is defined as:

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)\, d\tau$$

We also analyze the system behavior for various values of T, and consider modified and shifted kernels.

Derivation of $y(t)$

Since $h(t - \tau)$ is nonzero only when $-T \leq t - \tau \leq T$, it follows that:

$$t - T \leq \tau \leq t + T$$

Thus, the convolution becomes:

$$y(t) = \int_{t-T}^{t+T} \log \tau \, d\tau$$

Using the integral:

$$\int \log \tau \, d\tau = \tau \log \tau - \tau + C$$

we obtain:

$$y(t) = [\tau \log \tau - \tau]_{t-T}^{t+T}$$

Expanding:

$$y(t) = (t + T)\log(t + T) - (t + T) - (t - T)\log(t - T) + (t - T)$$

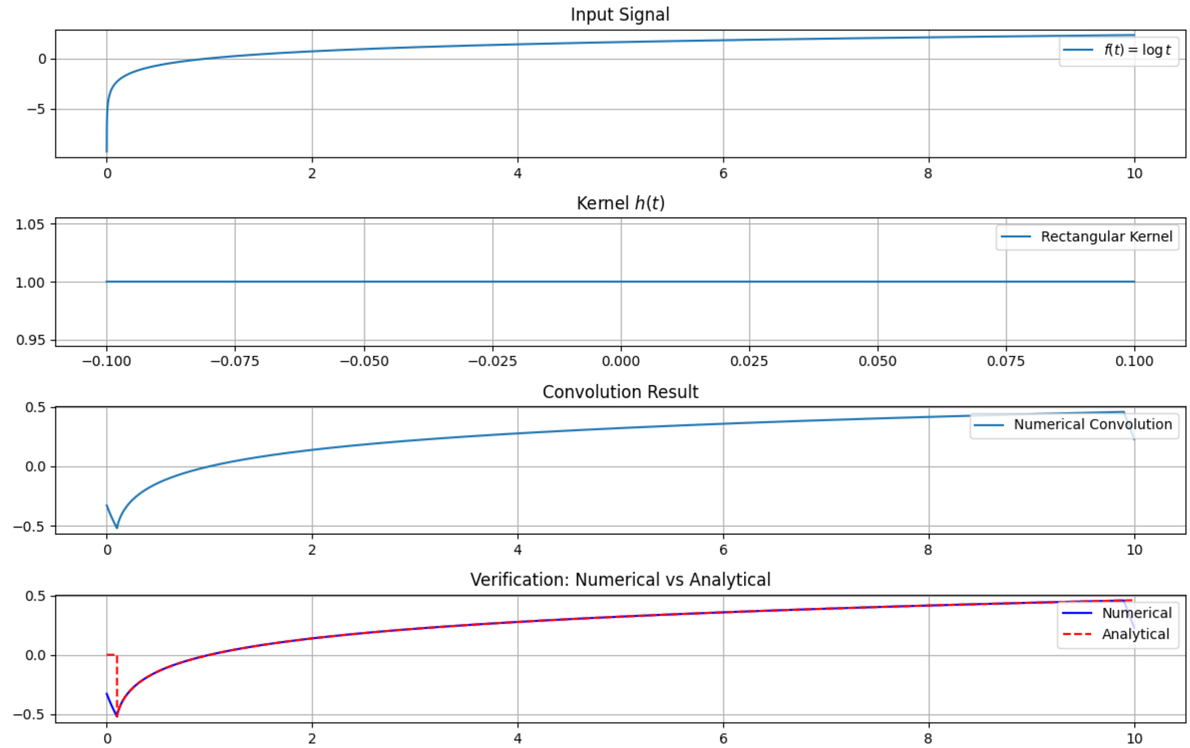
Simplifying:

$$\boxed{y(t) = (t + T)\log(t + T) - (t - T)\log(t - T) - 2T}$$

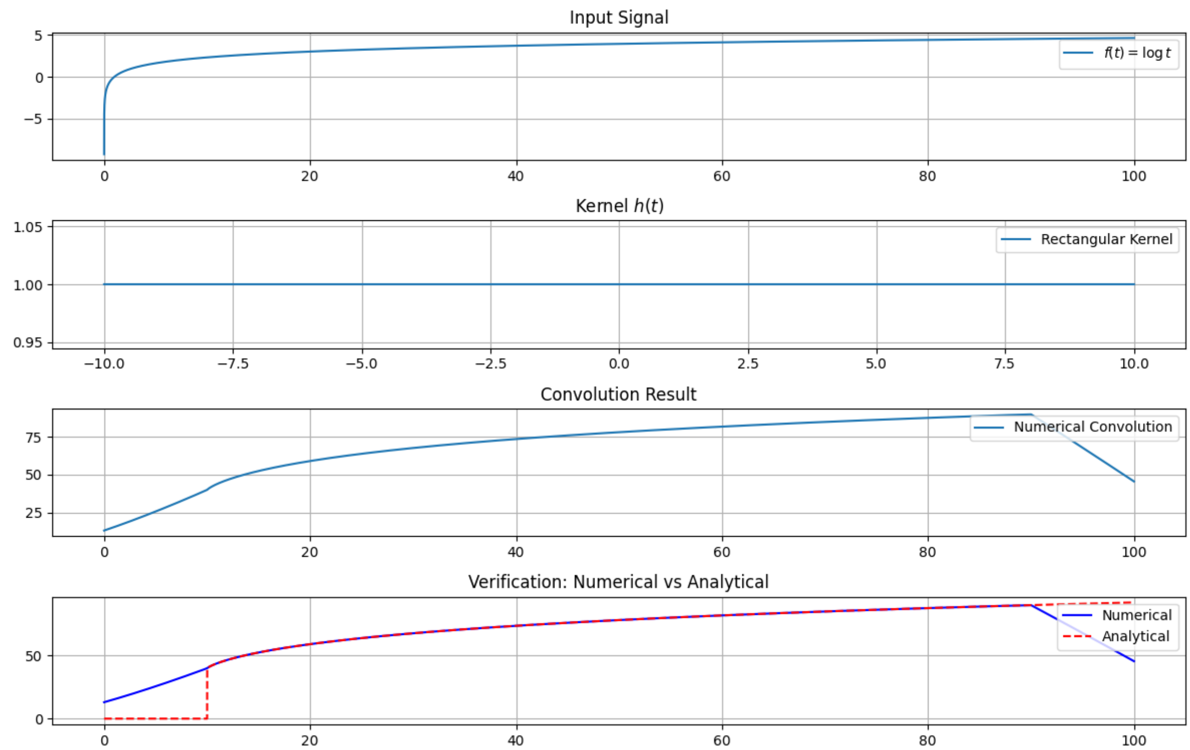
Behavior Analysis for Different T

Near $t = 0$: $\log 0$ is undefined; restrict domain to $t > T$.

- **Small** T: Localized smoothing around t. Mean squared error between analytical and simulated plots: 0.14497976045750576



- **Large** T: Broad averaging, heavy smoothing. Mean squared error between analytical and simulated plots: 0.00206248319793766



Modifications

(a) Kernel Considering Only $t > 0$

Modified kernel:

$$h(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Nonzero range:

$$t - T \leq \tau \leq t$$

Thus:

$$y(t) = \int_{t-T}^{t} \log \tau \, d\tau$$

Evaluating:

$$y(t) = [\tau \log \tau - \tau]_{t-T}^{t}$$

$$y(t) = t \log t - t - (t - T) \log(t - T) + (t - T)$$

Simplifying:

$$\boxed{y(t) = t \log t - (t - T) \log(t - T) - T}$$

Impact: The system becomes *causal*, depending only on past values.

(b) Shifted Kernel by τ_0

Shifted kernel $h(t - \tau_0)$ is nonzero when:

$$\tau_0 - T \leq t \leq \tau_0 + T$$

In convolution:

$$y(t) = \int_{t-\tau_0-T}^{t-\tau_0+T} \log \tau \, d\tau$$

Evaluating:

$$y(t) = [\tau \log \tau - \tau]_{t-\tau_0-T}^{t-\tau_0+T}$$

$$y(t) = (t - \tau_0 + T) \log(t - \tau_0 + T) - (t - \tau_0 + T) - (t - \tau_0 - T) \log(t - \tau_0 - T) + (t - \tau_0 - T)$$

Simplifying:

$$\boxed{y(t) = (t - \tau_0 + T) \log(t - \tau_0 + T) - (t - \tau_0 - T) \log(t - \tau_0 - T) - 2T}$$

Impact: Introducing a shift τ_0 causes a *time delay* in the output.

Summary

- Original convolution averages symmetrically.
- Modified kernel ($t > 0$) yields a causal system.
- Shifting the kernel results in delayed outputs, significant in time-delayed systems.
- Care must be taken near $t = 0$ due to $\log 0$ being undefined.