Probability Statistics And Advanced Graph theory

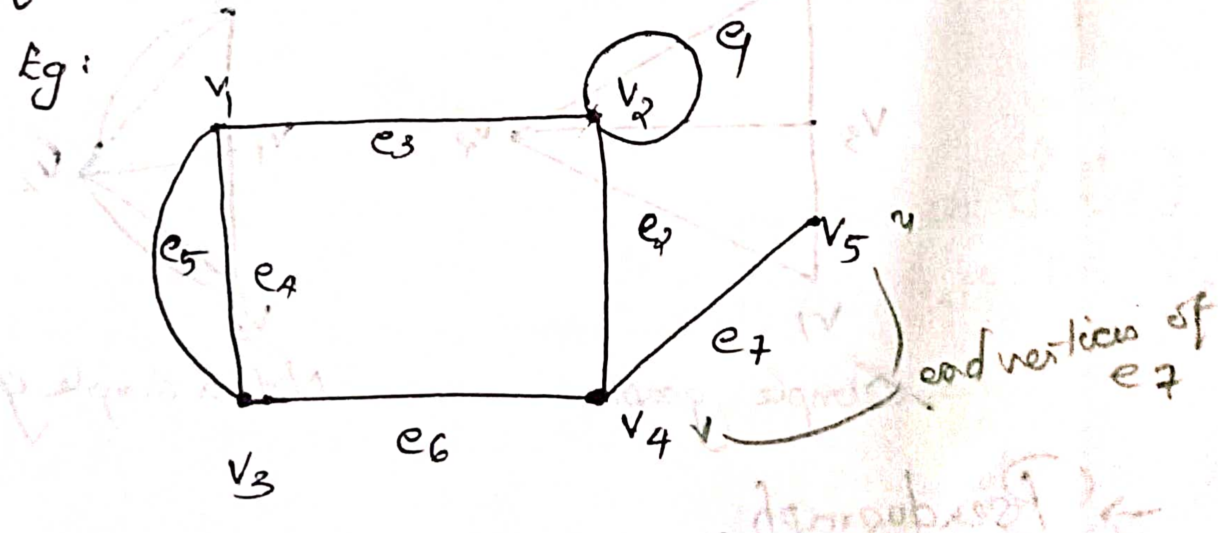
MAT 208

Module - 4

Graphs

A graph $G = (V, E)$ Consists of a set of objects, $V = \{v_1, v_2, v_3 \ldots\}$ Called vertices and another set $E = \{e_1, e_2, e_3 \ldots\}$ whose elements are Called edges. Such that each edge e_k is identified with an unordered pair (v_i, v_j).
(v_i, v_j) associated with e_k are called end vertices of e_k.

Eg:



$V = \{v_1, v_2, v_3, v_4, v_5\}$.

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

→ Loop

An edge of a graph G is called loop (self loop) if both the end vertices of the edge are same

e_1 is a loop in the above graph.

→ Parallel edges

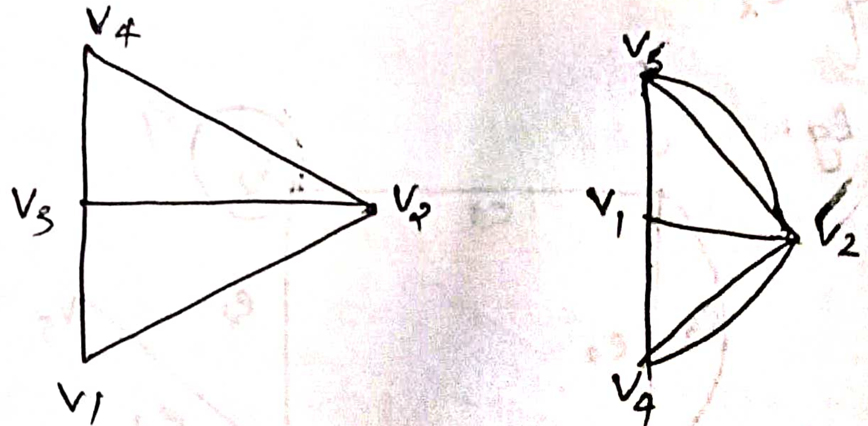
Two edges are called parallel edges if they have the same end vertices.

e_4 and e_5 are parallel edges in the above graph.

→ Simple Graph.

A graph with no loops and parallel edges

eg:



Simple graph.   Not a simple graph.

→ Pseudograph

Graphs with both loops and multiple edges.

→ Multigraph

Graphs where multiple edges are allowed.

→ Adjacent vertices

Vertices (u, v) in a graph G are called adjacent in G if u and v are end vertices of an edge e. Such an edge is called incident with u and v. and e is said to connect u and v.

Eg: V_5, V_4 are adjacent. V_2, V_5 are not

→ Neighbourhood of Vertex V

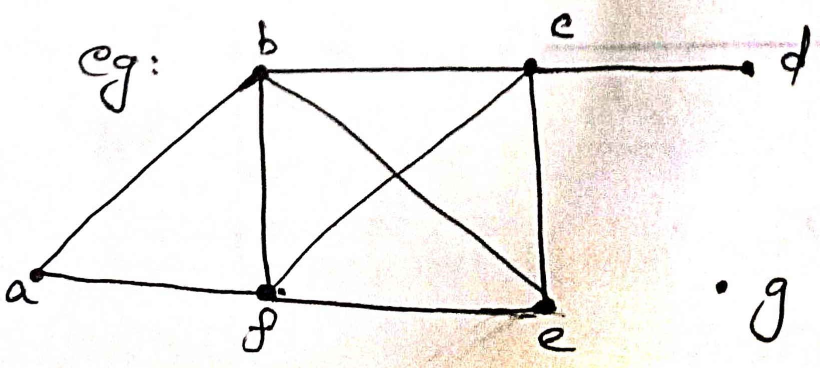
Set of all vertices adjacent to vertex V in G are called neighbourhood of V denoted as N(V).

Eg: $N(V_4) = \{V_3, V_2, V_5\}$

* → Degree of a Vertex

→ Degree of a Vertex, V denoted as deg(V) is the number of edges incident with V. where each loop is counted twice.

eg:



deg(a) = 2.
deg(b) = 4
deg(d) = 1
deg(g) = 0

→ Pendent Vertex

Vertex with degree equal to 1.
Eg: d.

→ Isolated Vertex

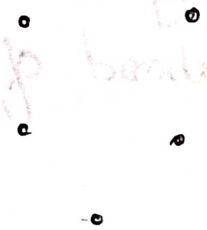
Vertex with degree zero.

Eg: g

→ Null graph

A graph with every vertex is an isolated vertex.

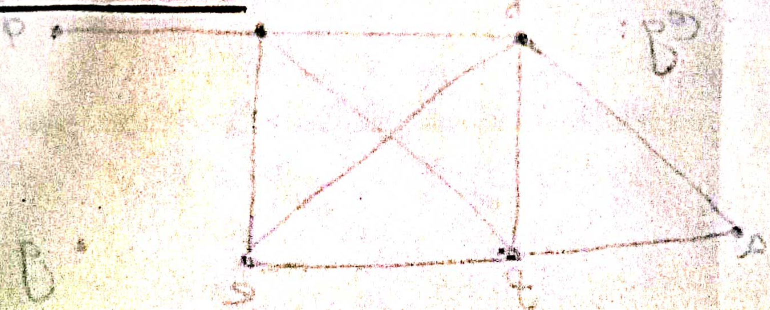
Eg:



→ Applications of Graph ⟶ Assignment - I (5)

* → THE HANDSHAKING THEOREM

Let $G = (V, E)$ be a graph with m edges, n vertices then $\sum_{i=1}^{n} d(v_i) = 2m$.



→ Proof

Each edge is counted twice while adding the degrees of the vertices.

i) How many edges are there in a graph with 10 vertices, each of degree 6.

Ans: given $n = 10$.

$\deg(v_i) = 6$

$m = ?$

add 6 → ten times.

$V = 10$

$60 = 2m$

$30 = m$

$$\sum_{i=1}^{n} \deg(v_i) = 2m.$$

$10 \times 6 = 2m.$

$60 = 2m$

$\Rightarrow \underline{30 = m}$

*→ Theorem (Odd vertex = vertex with odd degree)

even " = vertex with even degree.

A graph has even no: of odd vertices

Proof

Let $v_1, v_2, \ldots v_n$ be the vertices of G.

Let $V_1 = \{$set of ~~odd~~ even vertices$\}$.

$V_2 = \{$set of odd vertices$\}$

We know that by handshaking theorem

$$2m = \sum_{v \in V} \deg(v)$$

$$= \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

$$\Rightarrow \underbrace{2m - \sum_{v \in V_1} \deg(v)}_{} = \sum_{v \in V_2} \deg(v)$$

LHS = even

We have LHS is an even number. because both $2m$ and $\sum_{v \in V_1} \deg(v)$ are even.

$\therefore$ LHS must be an even number.

But each term in the $\sum$ summation is an odd number. $\therefore$ there must be even no: of terms in that sum.

$\Rightarrow$ Is even no: of odd degree vertices.

2) ~~Construct a simp~~

Note 1

Maximum degree of any vertex with in a simple graph with n vertices is atmost $n-1$.

Note 2

Maximum no: of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

using handshaking theorem we know that,

$$\sum d(v) = 2e$$

$$\Rightarrow n(n-1) = 2e$$

$$\Rightarrow e = \frac{n(n-1)}{2}$$

2) Construct a simple graph of 12 vertices with two of them having degree 1, 3 having degree 3 and the remaining 7 having degree 10.

Ans: Given :-

$n = 12$.

2 vertices = deg (1) } odd

3 " = deg (3) }

7 " = deg (10). } even.

There 5 vertices are of odd degree

ie ~~its~~ not possible since for any graph G, the no: of odd degree vertices is always an even no:.

3) What is the largest no: of vertices in a graph with 35 edges. if all vertices are of degree atleast 5.

Given :-

~~no. of~~ $e = 35$

for any vertex, $d(v) \geq 3$

$n = ?$

→ Use,

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

$$\Rightarrow \sum d(V) = 2 \times 35$$

$$\sum d(V) = 70$$

$$\Rightarrow 70 = \sum d(V)$$

$$70 = d(v_1) + d(v_2) \ldots + d(V_n)$$

$$= 23 + 3 + \ldots 3$$

$$70 \geq 3n$$

$$\frac{70}{3} \geq n$$

$$n \leq \underline{\underline{23.3}}$$

$\underline{\underline{n = 23}}$

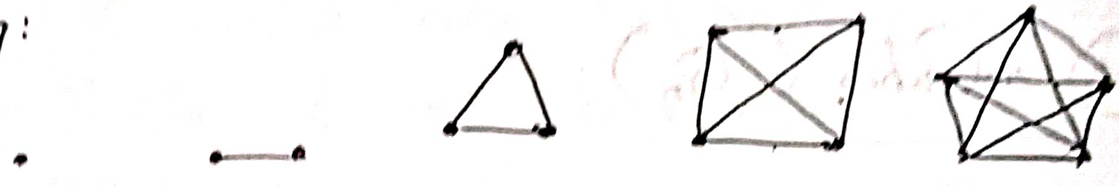
∴ largest possible value of n is 23.

Some Special Simple Graphs

① Complete graph on n vertices (K_n)

Graph which has exactly one edge between every pair of distinct vertices.

Eg:

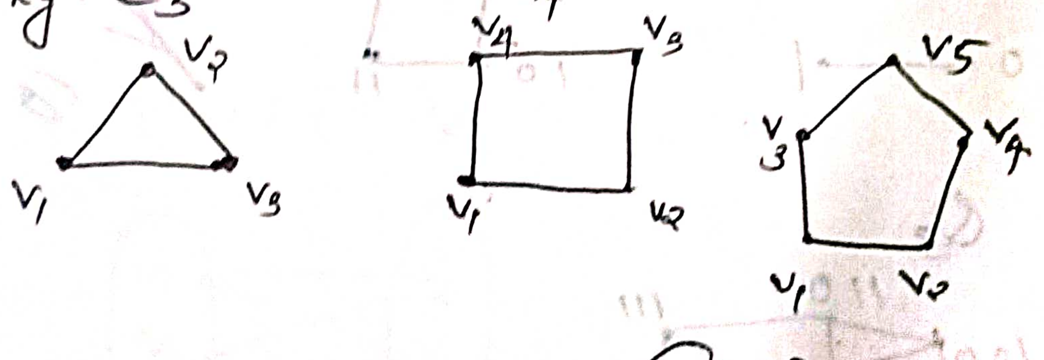


K_1 K_2 K_3 K_4 K_5

② Cycles on n vertices (C_n) ($n \geq 3$)

Consists of n vertices $v_1, v_2 \dots v_n$ and edges $(v_1, v_2), (v_2, v_3), \dots (v_{(n-1)}, v_n), (v_n, v_1)$.

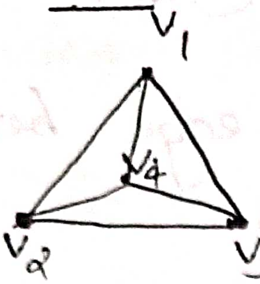
Eg: C_3 C_4 C_5



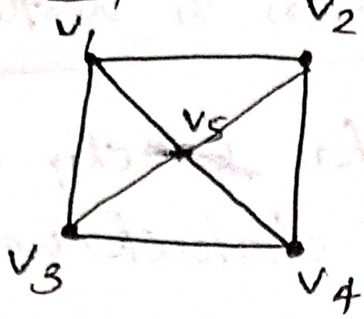
③ Wheels on n Vertices (W_n)

We obtain W_n by adding a new vertex to C_n ($n \geq 3$) and connecting this new vertex to every vertex of C_n by new edges.

W_3



W_4

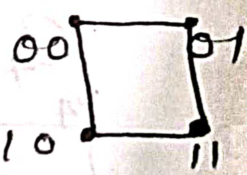


④ n-Cubes (Q_n)

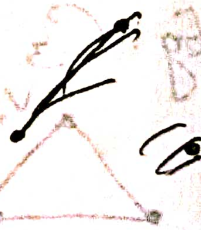
It has vertices representing 2^n with bit strings of length n. Two vertices are adjacent if and only if the bit string they represent differ only in one bit position.

Eg: Q_1

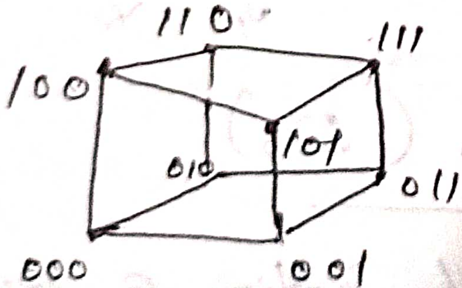
Q_2



Q_3



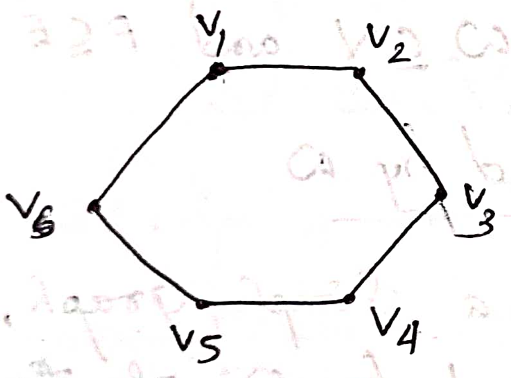
Q_3



Bipartite graph.

A simple graph G is said to be Bipartite if its vertex V can be partitioned into two disjoint sets. V_1 and V_2. Such that every edge in the graph connects a vertex in V_1 and a vertex V_2 (no two vertices in V_1 are adjacent also no two vertices in V_2 are adjacent).

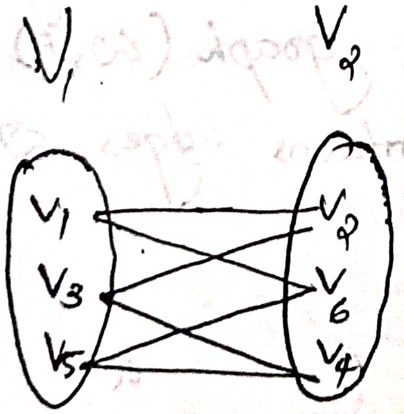
Eg: C_6 is bipartite.



$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$V_1 = \{v_1, v_5, v_3\}$

$V_2 = \{v_2, v_6, v_4\}$.

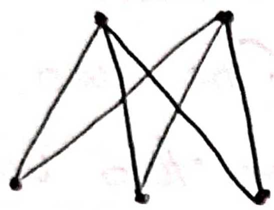


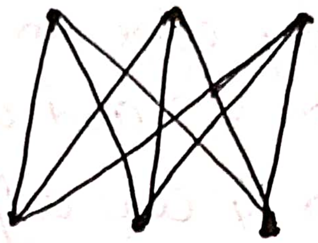
Complete Bipartite Graph ($K_{m,n}$)

A Bipartite graph whose vertex set V

is partitioned into V_1, V_2 one with m and other ~~m~~ with n, no: of vertices and every vertex in V_1 is joined to every ~~every~~ vertex in V_2.

Ex: $K_{2,3}$. $K_{3,3}$





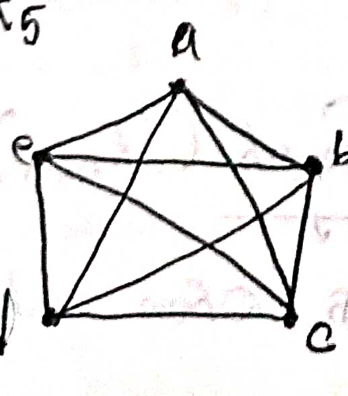
→ Subgraph of a Graph, G.

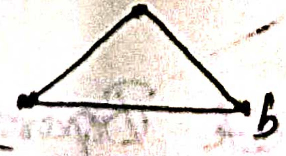
Let $G = (V, E)$ then $H = (\omega, F)$ is a Subgraph of G if $\omega \subseteq V$ and $F \subseteq E$.

• Eg: Subgraph induced by ω

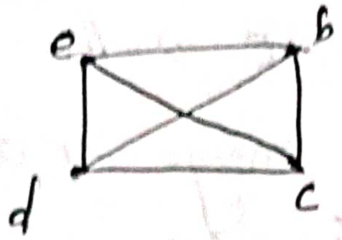
Let $G = (V, E)$ be a simple graph, the subgraph induced by ω of ~~the~~ a ~~vertex~~ sub set V is the graph (ω, F) where the edge set F contains edges whose endvertices both are in ω.

Eg: K_5



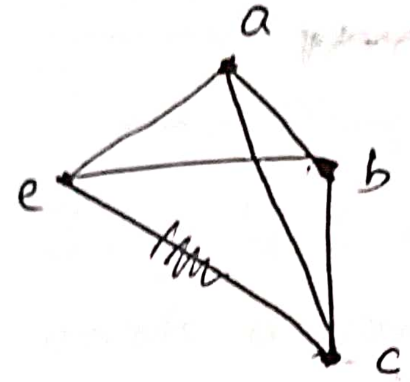


Subgraph of G



Subgraph of G.

→ Let $\omega = \{a, b, c, e\}$



Subgraph induced by ω.

→ <u>Addition of edges or removal of edges.</u>

Let $G = (V, E)$

* add edge e → $G + e = \{V, E \cup e\}$
* removal edge e → $G - e = \{V, E - \{e\}\}$

→ <u>Removal of Vertices</u>

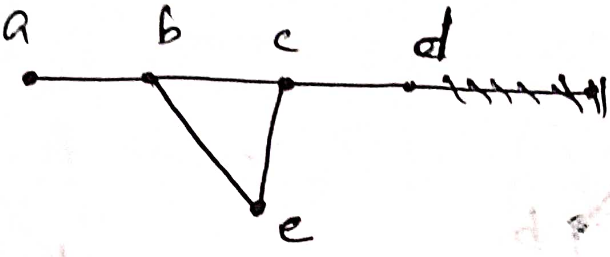
Let $G = (V, E)$

- $G - V = \{V - \{v\}, E'\}$
- E' is the edge set in which no edge incident with V_k is present.

→ <u>Edge Contraction</u>

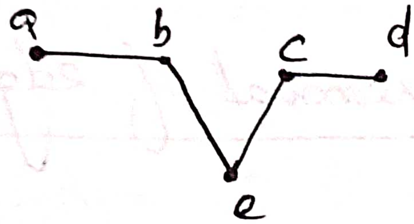
Which remove an edge E with endpoints u and v and merge u and v into a new single vertex, w.

Eg:

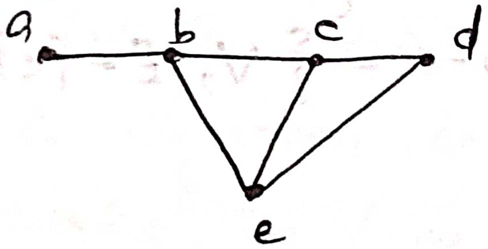


G

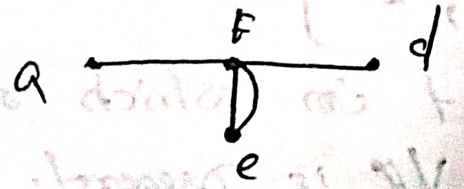
→ G - bc



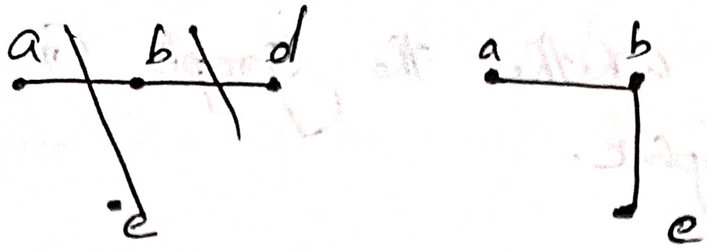
→ G + ed



→ Contracting be and replace with F



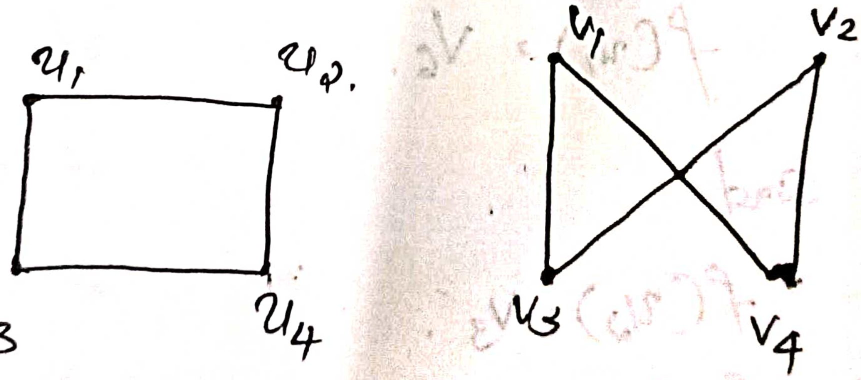
→ G - c.



*→ **Isomorphism of two graphs.**

Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-one f^n from V_1 to V_2 with ~~corresponds to~~ the property that ~~if a in G_1, and b in G_2~~ if a and b are adjacent in G_1, then $f(a)$ and $f(b)$ must be adjacent in G_2, $\forall$ vertices a, b in V_1.

Eg:



G_1 G_2

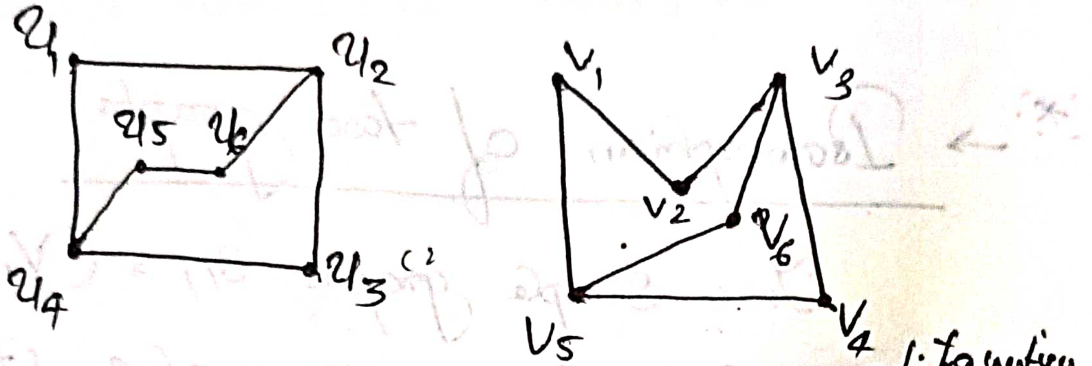
define $f(u_1) = v_1$

$f(u_2) = v_3$

$f(u_3) = v_4$

$f(u_4) = v_2$

(i) determine whether the graph G and H are isomorphic.



adj to vertices with edges

Ans: $f(u_1) =$

Choose u_1 → V_1 (2, 3), V_4 → V_3 : 3, V_5 3., V_6 → V_5, V_3 (3,3), V_2 → V_1, V_3

But u_1 is adjacent to two 3 degree vertices. So u_1 can be mapped to V_4 and V_6. 2 3

Let,

u_1 → V_4, V_6

$f(u_1) = V_6$.

And

$f(u_2) = V_3$.

$f(u_4) = V_5$.

Now choose u_3, (deg 2).

(3,3) u_3 → V_1 (2, 3), V_2 (2, 3), V_4 (3, 3)

Since u_3's adjacent vertices are of $\overset{two}{3}$ degree we can map it into to V_4

$$f(u_3) = V_4.$$

→ Choose $f(u_5)$ (deg = 2),

$$(2,3)\,u_5 < \begin{matrix} v_1 \ (2,3) \\ v_2 \ (2,3) \end{matrix}$$

$$f(u_5) = V_1.$$

→ Choose $f(u_6)$,

$$f(u_6) = V_2.$$

→ $f(u_1) = V_6.$

→ $f(u_2) = V_3$

$f(u_3) = V_4$

$f(u_4) = V_5$

$f(u_5) = V_1$

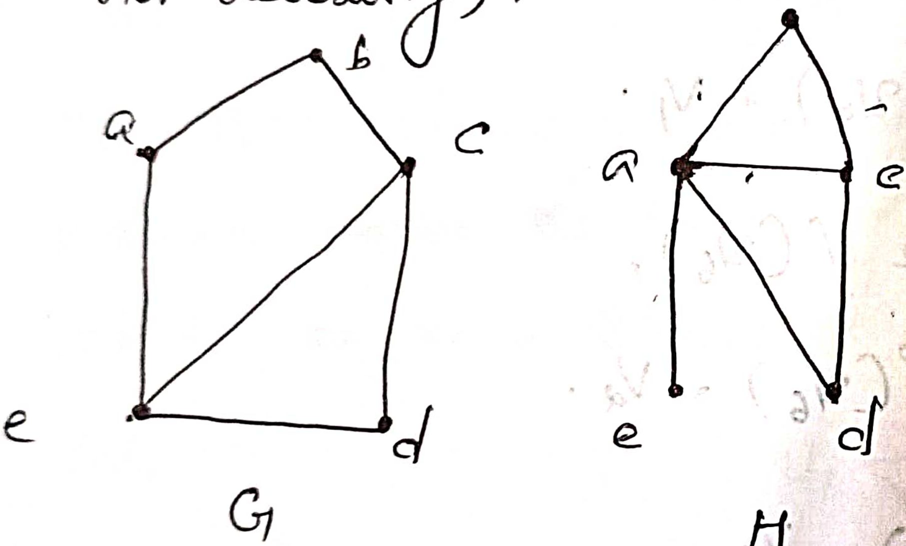
$f(u_6) = V_2$

Note: If two graphs are isomorphic they have the same no: of vertices edges and same no: of vertices of a particular degree.

If any of these is not satisfied two graphs cannot be isomorphic.

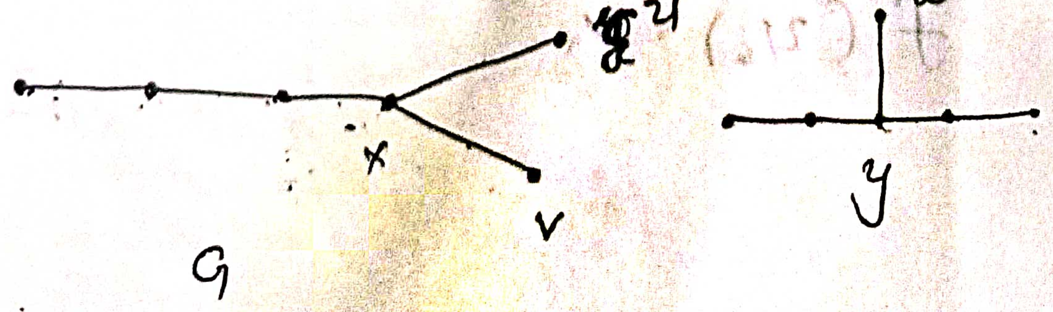
(above 3 conditions are sufficient conditions, not necessary).

Ex:



not isomorphic.

H has a vertex of degree 4 but G has no vertex of degree 4. Therefore G and H are not isomorphic.



not isomorphic

Here x is a 3 degree vertex in G, so it must be mapped to the only 3 degree vertex y in H. But x has two pendant vertices u and v and y has only one pendant vertex w. ∴ the mapping is not possible.

So G and H are not isomorphic.

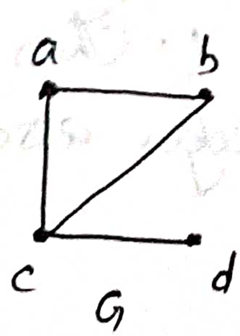
→ Representing graphs using matrix

- Adjacency matrix

Let $G = (V, E)$ be a simple graph with $|V| = n$. Suppose vertices are listed $v_1, v_2, \cdots v_n$. Then adjacency matrix A or A_G of G w.r.t this listing is an $n \times n$ zero-one matrix ie $A = [a_{ij}]$

$$= \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Ex:



Write the adjacency matrix of G w.r.t listing a, b, c, d.

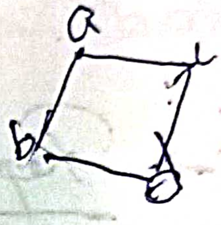
$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

4×4

→ <u>Note</u>: Adjacency matrix of a simple graph is symmetric ie, $a[ij] = a[ji]$ and since it has no loop $a_{ii} = 0$ ie, diagonal entries are always zero.

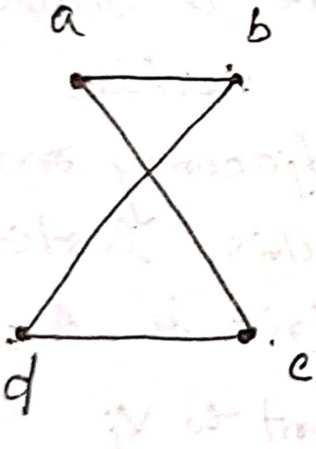
→ draw a graph with adjacency matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



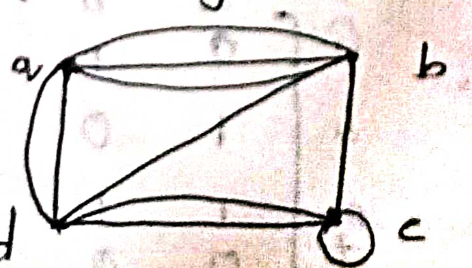
w.r.t ordering of vertices a, b, c, d.

Ans:



→ For a graph is is not simple, the entries need not be only 0 and 1. Then $(ij)^{th}$ entry will be no: of edges associated with v_i and v_j.

Ex:



with respect to listing of vertices a, b, c, d. the matrix of the given graph.

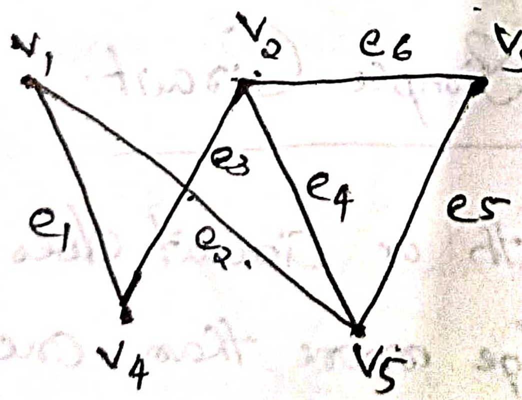
$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 3 & 0 & 2 \\
b & 3 & 0 & 1 & 1 \\
c & 0 & 1 & 1 & 2 \\
d & 2 & 1 & 2 & 0
\end{array}
$$

→ Incidence Matrix

Let $G = (V, E)$ be an undirected graph and let $v_1, v_2 \dots v_n$ are the vertices and $e_1, e_2 \dots e_m$ are the edges. Then incidence matrix $M = [m_{ij}]$ is an $n \times m$ matrix where

$$
[m_{ij}] = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with } v_i \\ 0, & \text{otherwise} \end{cases}
$$

Ex:



$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 1 & 1 & 0 & 1 \\
v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\
v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\
v_5 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

HW

→ ## Path in a Graph

Let n be a non negative integer and G be an undirected graph, a path of length n from a vertex u to v in G is a sequence of n edges. $e_1, e_2 \ldots e_n$ for which there exists a sequence of vertices $u = x_0, x_1 \ldots x_{n-1}$, $x_n = v$. Such that e_i has n points x_{i-1}, x_i. If a graph is simple we just represent vertices for a path.

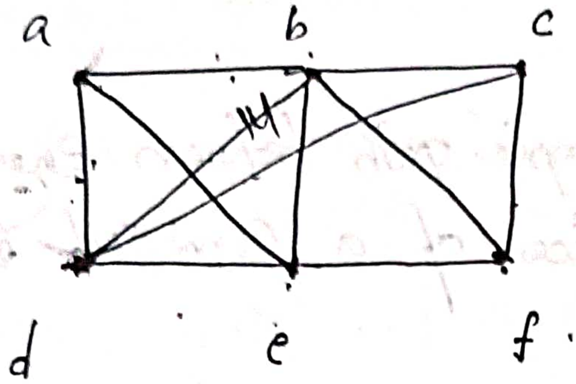
→ ## Circuit

A path is a circuit if it begins and ends at the same vertex ie, $u = v$.

→ ## Simple path or Simple Circuit.

If a path or circuit does not contain an edge more than once they are called simple path or simple circuit.

Ex :



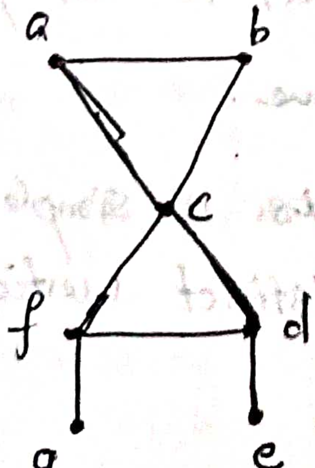
① a, d, c, f, e is a path from a to e of length 4 and is a simple path.

② a, b, e, d, a, b is ~~a path from a to~~ not a simple path.

→ Connected graph.

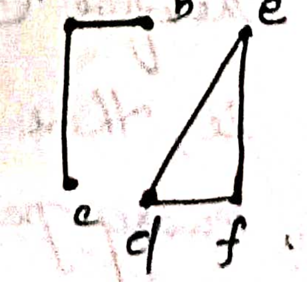
A graph is connected if there is a path between every pair of distinct vertices of the graph. A graph not connected is called a disconnected graph and it may have two or more connected components.

Ex :



Connected graph

disconnected graph



with 2 connected components.

Theorem

There is a simple path between every pair of distinct vertices of a connected undirected graph.

Let G be connected by definition there exists a path ~~from~~ between every pair of distinct vertices. Let u, v be two vertices. Let $u = x_0, x_1, x_2 \ldots x_{n-1}, x_n = v$ be a path of least length. Then this path must be simple. ⟶ (1)

Otherwise,

There exists two vertices in the sequence x_i, x_j $\therefore$ $x_i = x_j$, $0 \leq i \leq j$.

This implies, ∃ a path,

$$x_0, x_1, x_2 \ldots x_{i-1}, x_j, \ldots x_n$$

of shorter length that is not possible.

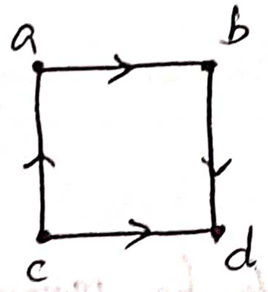
Therefore (1) is true.

ie, there exists ~~in~~ a simple path b/w every pair of distinct vertices of a connected graph.

Directed graphs.

A directed graph (di-graph) Consists of set of vertices $V = \{v_1, v_2 \dots etc\}$ set of edges $E = \{e_1, e_2, \dots\}$ and a mapping that maps every edge onto some ordered pair of vertices (v_i, v_j).

Ex :



→ Consider edge e_k in a directed graph $e_k \to (v_i, v_j)$.

- initial vertex of e_k :- The vertex v_i which edge e_k is incident out of : $v_i \xrightarrow{e_k} v_j$
- Terminal vertex of e_k : The vertex v_j which edge e_k is incident onto.
- ~~Set~~ Outdegree of vertex v_i $(d^+(v_i))$: Number of edges incident out of v_i
- indegree of vertex v_i $(d^-(v_i))$: Number of edges incident onto v_i

→ **Note**

indegree

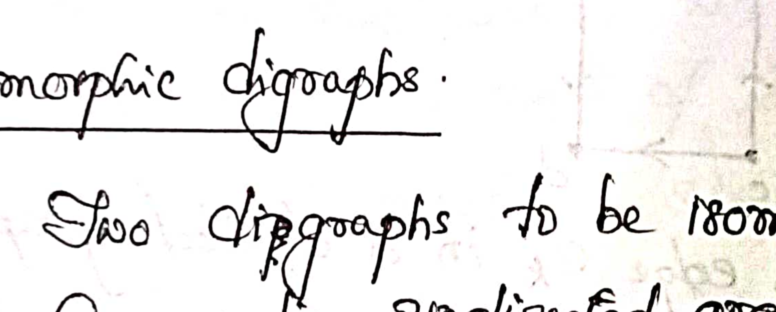
$$\sum d^+(v_i) = \sum d^-(v_i) = \text{no: of edges.}$$

→ **Parallel edges** ~~two~~

Two ~~of~~ edges are parallel in a directed graph if they are parallel by considering the direction also.

→ **Isomorphic digraphs.**

for Two digraphs to be isomorphic there corresponding undirected graphs must be isomorphic. But the directions of the corresponding edges must also agree.



Non isomorphic digraphs.

→ Paths in a directed graph.

We use the same definition of the for an undirected graph by considering the given directions.

→ Connectedness in directed graph.

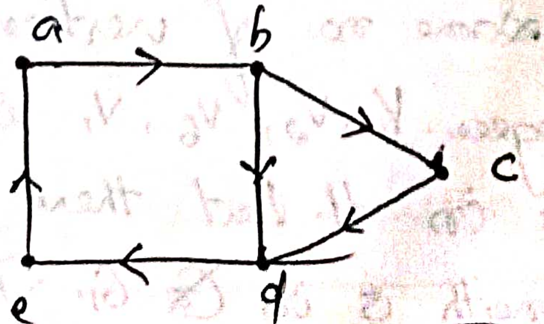
- **Strongly Connected :-**

A directed graph is strongly connected if there exists a path from a to b and from b to a. When a and b are vertices in the graph.

- **Weakly Connected:**

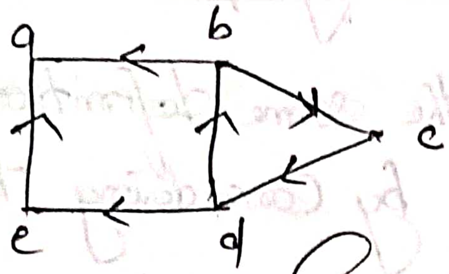
A directed graph is weakly connected if there exists a path between every pair of vertices in the underlying undirected graph.

Ex:



Strongly connected & weakly connected

Ex :



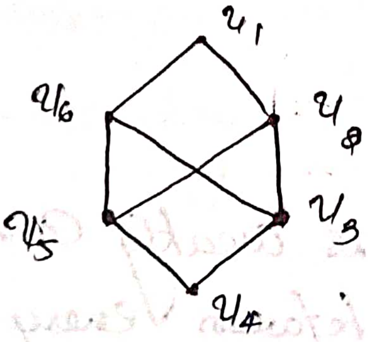
not strongly Connected

Since ∃ no path from a to b.

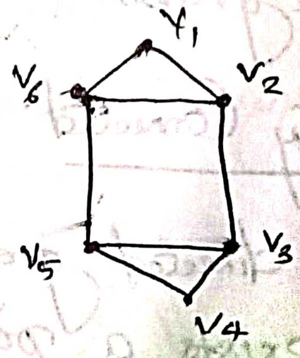
→ paths and isomorphism

① ~~Existe~~ Existence of a simple Circuit of particular length is invariant under isomorphism.

Ex :



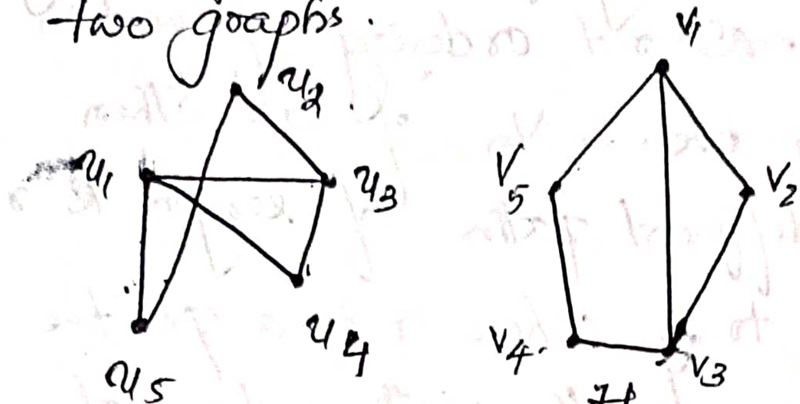
G



H

G and H have the same number of vertices, edges and same no. of vertices of a particular degree. v_1, v_2, v_6, v_1 is a cycle of length 3 in H. But there exists no cycle of length 3 in ~~G~~ G. Therefore G and H are not isomorphic.

② Use paths for finding isomorphism between two graphs.



G H

G and H have same no: of vertices, edges, vertices of particular degree.

Consider a path in G, $\overset{3}{u_1}, \overset{2}{u_4}, \overset{3}{u_3}, \overset{3}{u_2}, \overset{2}{u_5}$.

and a path in H, $\overset{3}{v_3}, \overset{2}{v_2}, \overset{3}{v_1}, \overset{3}{v_5}, \overset{2}{v_4}$

→ Both start at vertex of degree 3, go through vertices of degree 2, 3, 2 respectively and end at a vertex of degree 2.

We can define isomorphism so $f(u_1) = v_3$.

$f(u_4) = v_2$.

$f(u_3) = v_1$

$f(u_2) = v_5$

$f(u_5) = v_4$

→ Counting paths between vertices:

Theorem

Let G be a graph with adjacency matrix A. w.r.t ordering of vertices v_1, v_2, v_3 etc... v_n of G. Then no: of different paths of length 'r' from v_i to v_j where r is a positive integer equals the $(i,j)^{th}$ entry of A^r.

→ We use induction on r,

The result is true for $r=1$. because the no: of paths of length 1 from $v_i \to v_j$ ie no: of edges from $v_i \to v_j$ is same as the $(i,j)^{th}$ entry of $A^1 = A$. ~~By~~ by definition of adjacency matrix.

Let $A^r = [b_{ik}]$ and $A = [a_{ij}]$.

$\therefore (i,j)^{th}$ entry of $A^{r+1} = A^r \cdot A$

$$= b_{i1} a_{1j} + b_{i2} a_{2j} + \ldots b_{in} a_{nj} \longrightarrow (1)$$

→ Assume that no: of paths of length r from $v_i \to v_j$ is the $(i,j)^{th}$ entry of A^r.

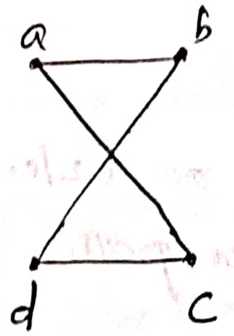
Let b_{ik} be the no: of paths of length r from $v_i \to v_k$.

and a_{kj} be the no: of edges joining $v_k \to v_j$.

Adding a vertex $v_j \to v_k$ and joining edge $v_k - v_j$ and taking all combinations of paths of length $r+1$ from $v_i \to v_j$ we get the total no: of paths as $b_{i1} a_{1j} + b_{i2} a_{2j} + \dots b_{in} a_{nj}$. which is same as eqn(1).

$\therefore (i,j)^{th}$ entry of A^{r+1} = no: of paths of length $r+1$ from $v_i \to v_j$.

Q: How many paths of length 4 from $A \to D$ in the given graph



$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 1 & 1 & 0 \end{array}$$

Ans.
path $a \to d = \underline{\underline{8}}$.

$$A^4 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 8 & 0 & 0 & 8 \\ b & 0 & 8 & 8 & 0 \\ c & 0 & 8 & 8 & 0 \\ d & 8 & 0 & 0 & 8 \end{array}$$

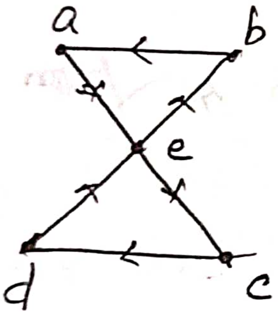
→ Euler path and Circuit

An Euler path in G is a simple path containing every edge of G (ie path containing every edge exactly ones)

Euler Circuit

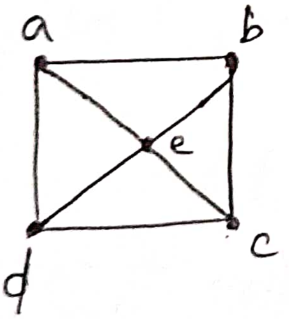
It is a circuit containing every edge exactly ones.

Ex :



a, e, c, d, e, b, a. is an Euler circuit

(2)



It has no Euler circuit and no Euler path.

✓*: Theorem

A connected multigraph with atleast two vertices has an Euler circuit if and only if each of vertices have even degree.

- Suppose that G contains an Euler circuit
- To prove every vertex has an even degree.

G has an Euler circuit that implies a circuit starting at a vertex say 'a' and ending at 'a', that passes through every edge of G exactly once.

While tracing this circuit we enter a vertex v through an edge and exit by another edge. Therefore degree of any internal vertex as well as vertex 'a' is a multiple of 2.

Conversely let degree of 6 = multiple of 2 for every vertex in G. To prove that G has an Euler circuit,

- Start from any vertex say a.
- & Trace through edges of G.
- Since degree of any vertex in G is even, we can enter through an edge and exit by another always.
- Trace as far as possible. If at first attempt we can trace through all edges and come back to 'a'.
- The closed path thus obtained is an Euler circuit.
- If not form a subgraph of G by deleting

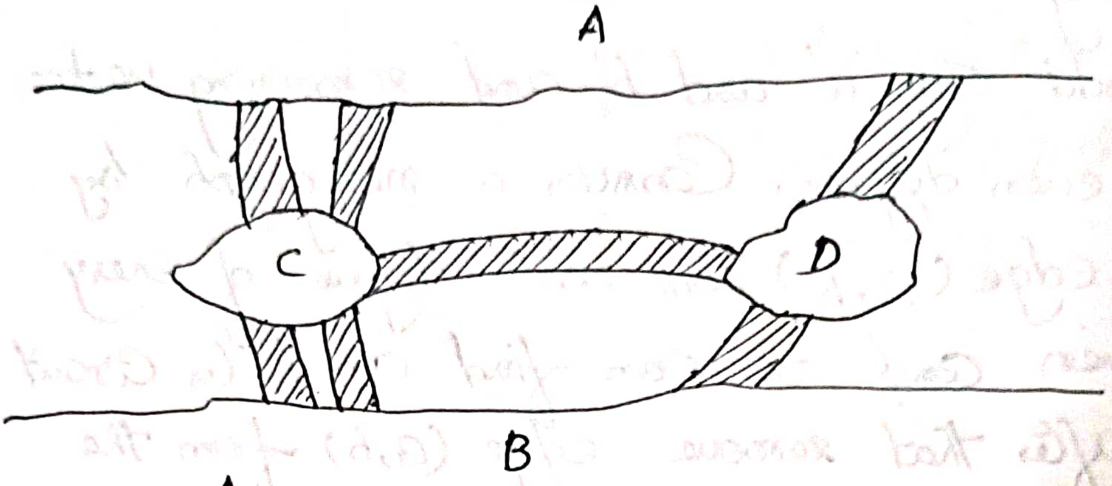
all the edges tracing the above process.

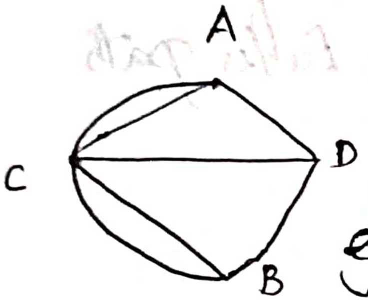
- Let H be the subgraph.
- It has a vertex in common with the circuit in the previous step. Since G is connected. Let w be that vertex. now start from w and find a circuit ending at w.
- Join the new circuit to the previous one. If all the edges are covered we obtain an Euler circuit. Otherwise repeat the process until all the edges are covered.

APPLICATION

→ **Königsberg Bridge Problem**

City of Königsberg has two islands C and D formed by Pregal River. Where there are two islands and two banks A and B are connected with seven bridges. Problem was to start from any of the land area A, B, C or D. Walk over each of the seven bridges exactly ones and return to the starting point.





Graphical Representation.

So problem is to find an Euler circuit that is not possible because by using previous theorem. Every vertex is not of even degree. So we cannot find Euler Circuit

Theorem

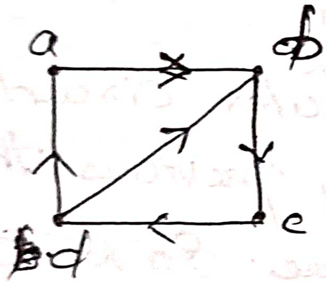
A connected multigraph has an Euler path but not an Euler Circuit if and only if it has exactly two vertices of ~~so~~ odd degree.

Proof

Let G has an Euler path starting at 'a' and ending at 'b'. Then every internal vertex will have degree multiple of 2 but 'a' and 'b' will have degree odd.

Conversely assume that let two vertices has degree odd say 'a' and 'b'. and remaining vertices are of even degree. Consider a new graph by joining edge (a, b) will make degree of every vertex even and we can find an Euler Circuit in it. after that remove edge (a, b) from the Euler Circuit. That will produce an Euler path in the original graph.

Ex:



deg(b) = deg(d) = 3
deg(a) = deg(c) = 2.

Euler path - d, a, b, c, d, b.

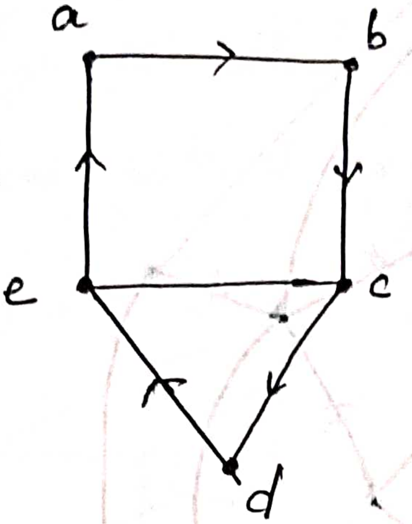
~~Euler~~.

→ Hamiltonian Path/circuit

Hamiltonian path: A simple path in a graph G that passes through every vertex exactly once.

Hamiltonian Circuit: A simple circuit that passes through every vertex exactly once.

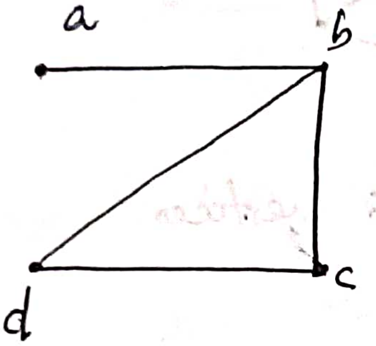
Ex:



a, b, c, d, e, a - circuit

a, b, c, d, e - path.

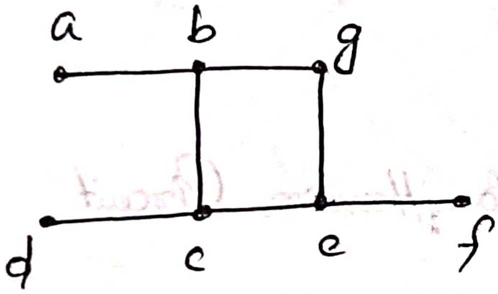
Ex :



a, b, c, d - path.

d, c, b, a - Hamiltonian path.

Ex :



No Hamiltonian path and circuit.

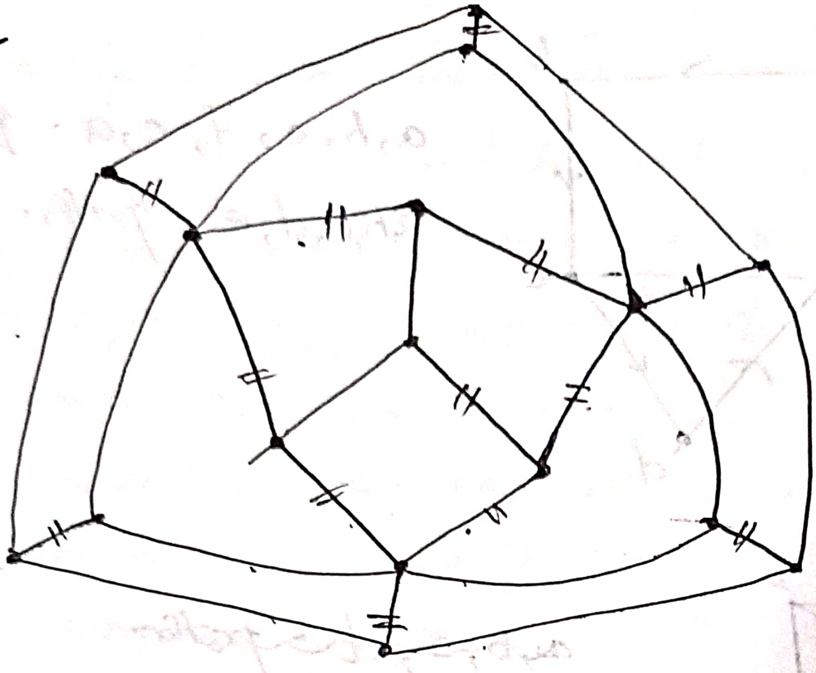
Note

~~A vertex of degree of odd one~~

- A graph of vertex one degree cannot have a Hamiltonian circuit, since once we enter we cannot exit.

- If a vertex has degree 2 both these edges must be part of the circuit.

Ex: HW



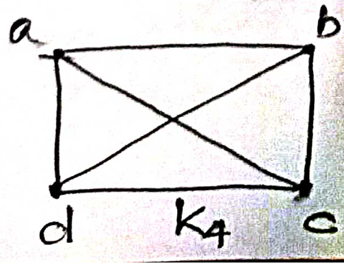
exclude = 13   we have 16 vertices.
include = 14.

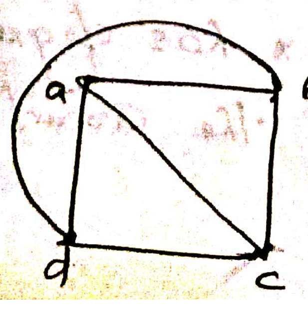
Note

Q. Show that K_n has a Hamiltonian Circuit when $n \geq 3$?

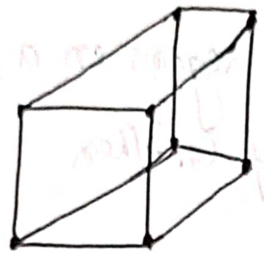
In K_n there are edges between any pair of vertices, so starting from any vertex we can trace through every vertex exactly once and come back to the starting vertex.

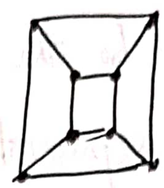
→ PLANAR GRAPHS





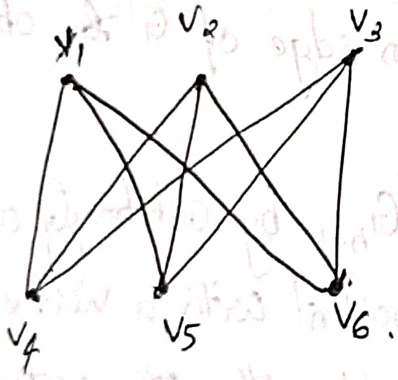
A graph is called planar if it can be drawn in the plane without any edges crossing.

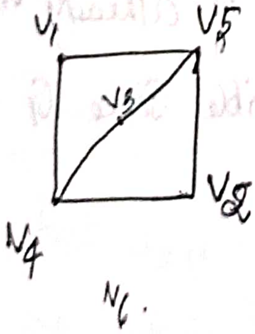




Q_3.

→ K_{33} is non planar.





Euler's Formula. (To prove graph is not planar)

Let G be a connected, planar, simple graph with e edges and v vertices. Let r be the no: of regions in a planar representation of G. Then

$r = e - v + 2$.

Proof:

* Arbitrarily pick an edge of G to obtain G_1.
* Obtain G_n from G_{n-1} by arbitrarily adding an edge that is incident with a vertex already in G_{n-1}, adding the other vertex incident with this edge if it is not already in G_{n-1}. This Construction is possible since G is connected.
* We use induction on 'n'.
* Let r_n, e_n, v_n represent the no: of regions, no: of edges, no: of vertices in planar representation of G_n.
* Now we prove the result is true for $n=1$. ie G_1.

$r_1 = 1$

$v_1 = 2$

$e_1 = 1$

$r_1 = e_1 - v_1 + 2 = 1 - 2 + 2 = \underline{\underline{1}} = r_1$

→ Result is true for $n = 1$.

* Now assume that the result is true for G_K.

$\Rightarrow r_K = e_K - v_K + 2 \longrightarrow ①$.

* Now to prove that the result is true for $n = K+1$.

→ ie to prove $r_{K+1} = e_{K+1} - v_{K+1} + 2$.

Cases Let (a_{K+1}, b_{K+1}) be the edge added to G_K to obtain G_{K+1}:

Case. 1

Both a_{K+1}, b_{K+1} are already in G_K.
Then these must be on the boundary of a common region otherwise we cannot add edge (a_{K+1}, b_{K+1}) into G_K without crossing the edges.

$r_{K+1} = r_K + 1$

$e_{K+1} = e_K + 1$

$v_{K+1} = v_K$

$e_{k+1} - v_{k+1} + 2 = e_k + 1 - v_k + 2$

$= \underbrace{e_k - v_k + 2}_{} + 1 \longrightarrow (1)$

$= r_k + 1$

$= r_{k+1}$

Case - 2

One of the two vertices are already in G_k. Let a_{k+1} is in G_k and we have to add b_{k+1} to add the edge (a_{k+1}, b_{k+1}).

$r_{k+1} = r_k \longrightarrow (a)$

$v_{k+1} = v_k + 1 \longrightarrow (b)$

$e_{k+1} = e_k + 1 \longrightarrow (c)$

RHS : $e_{k+1} - v_{k+1} + 2 = e_k + 1 - (v_k + 1) + 2$

$= e_k - v_k + 2 - 1 + 1$

$= r_k = r_{k+1}$

$r_k = r_{k+1}$.

$\therefore$ RHS = LHS

So the result is true for $n = k+1$

So we have $r_n = e_n - v_n + 2$.

Original graph is obtained after all the edges are added.

$\therefore$ We have $r = e - v + 2$.

(1) A connected planar graph has 20 vertices each of degree 3. Into how many regions does a representation of this planar graph split the plane?

given

$V = 20$ $\quad$ $r = ?$

$d(v) = \deg(3)$.

$r = e - v + 2$.

now use $\sum d(v) = 2 \times e$.

$3 \times \cancel{20}^{10} = \cancel{2} \times e$

$e = 30$.

By Euler formula;

$V = 20$

$e = 30$

$r = 30 - 20 + 2$.

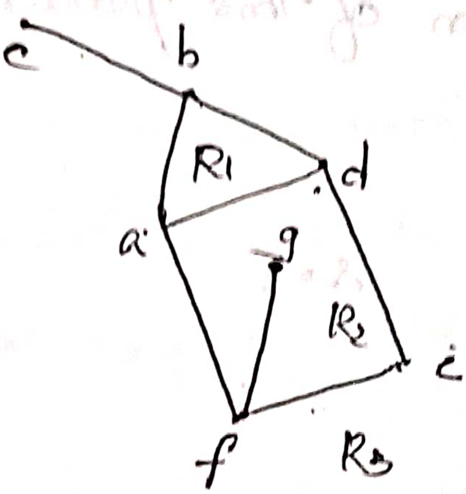
$= 10 + 2$

$= 12$.

Degree of a region

It is the ~~sum~~ no. of edges in the boundary of the region, an edge is counted twice if an edge occurs twice on the boundary.

Ex:



deg $(R_1) = 3$.

deg $(R_2) = 4 + 2 = 6$.

deg $(R_3) = 7$.

→ Corollary.

→ Corollary-I

If G is a connected planar simple graph with e edges and v vertices where ~~where~~ e v ≥ 3. then $e \leq 3v - 6$.

Proof

A connected planar graph drawn in a plane divides plane into r regions. Here degree of every region is atleast 3. (Since there are no loops or parallel edges) in the given graph) Also degree of unbounded region is atleast 3 because there are atleast 3 vertices in the graph which is connected.

Also we have $2e$ = sum of of the degrees.

$$\Rightarrow 2e \geq 3+3+\dots+3 \text{ (r times)}.$$

$$2e \geq 3r.$$

$$\frac{2}{3}e \geq r.$$

By Euler's theorem;

$$\Rightarrow \frac{2}{3}e \geq e-v+2.$$

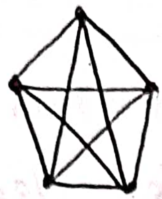
$$\Rightarrow 2e \geq 3e-3v+6.$$

$$\Rightarrow 3v-6 \geq 3e-2e$$

$$\Rightarrow 3v-6 \geq e.$$

$$\Rightarrow e \leq 3v-6$$

→ Show that K_5 is non-planar.



K_5

✓ 5 vertices
10 edges.

$r = 7$

$r = e - v + 2$
$= 10 - 5 + 2$
$= \underline{\underline{3}}$.

✓ $v \geq 3$
$e \leq 3v - 6$
$e \leq 3 \times 5 - 6$
$e \leq 15 - 6$
$e \leq 9$
$r = e - v + 2$
$r =$

Hence K_5 is non planar.

(II) Here $e > 3v - 6$.

→ $\therefore$ By Corollary I K_5 is non-planar.

→ (1) **Note**

K_n is non planar for $n \geq 5$.

→ **Corollary II**

If G is connected planar simple graph then G has a vertex of degree not exceeding 5.

- If G has one or two vertices the result is true.
- If G has atleast 3 vertices by Corollary I

$e \leq 3v - 6$ mul both sides by 2.

$\Rightarrow 2e \leq 6v - 12. \longrightarrow (1)$

$\rightarrow$ $2e \geq (6+6 \ldots\ldots 6)V$

$\rightarrow$ $2e \geq 6V$.

We have to prove that there exists atleast one vertex of degree not exceeding 5. Let us
$\rightarrow$ assume the contrary.
ie, degree of every vertex is atleast 6. or $deg(V) \geq 6$.
Now by using,

$$2e = \sum d(v)$$
$$\geq (6+6+\cdots+6)(V \text{ times})$$
$$2e \geq 6V$$

ie $2e \geq 6V \longrightarrow (2)$

Which is a contradiction to eqn (1).

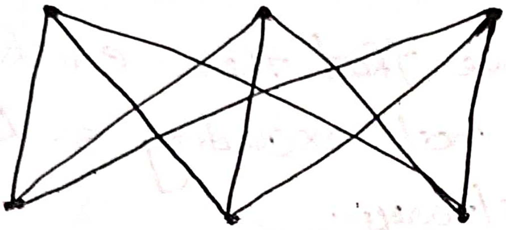
So our assumption was wrong.
$\therefore$ There exists a vertex of degree not exceeding 5.

$\rightarrow$ <u>Corollary III</u>

If a connected planar simple graph has atleast three edges and v vertices with $v \geq 3$ and no circuit of length 3 then <u>$e \leq 2V-4$</u>.

→ Show that $K_{3,3}$ is non planar.

$e(6-1)$



$e \leq 2v-4$

$2v-4 = 2(6)-4$

$= 8$ $\qquad$ $e = 9$

$e \nleq 2v-4$

∴ By Corollary III, $K_{3,3}$ is nonplanar.

→ Theorem

→ Kuratowski's Theorem

A graph is non planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

→ Homeomorphic graphs.

Graphs $G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$ are called Homeomorphic

if they can be obtained from the same graph by a sequence of elementary subdivisions.

- Removal of an edge $\{u, v\}$.
- Adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$.

* → Cut vertex, Cut edge (Bridge).

(1) Cut vertex

If the removal of a vertex from a graph makes it disconnected that vertex is called Cut-vertex.

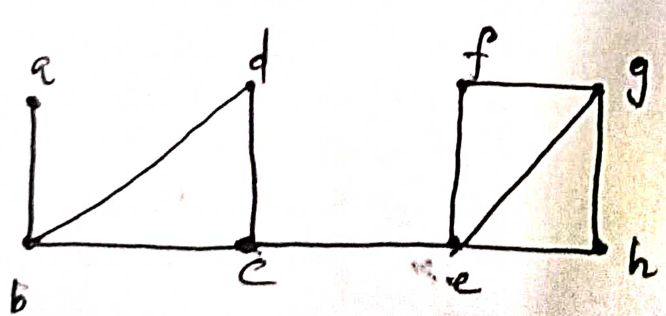
(2) Cut edge

If removal of an edge makes a graph disconnected the edge is known as Cut-edge or bridge.

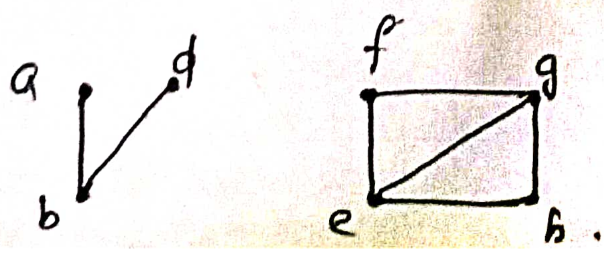
→ Non Seperable Graph

Connected graph without Cut vertices.

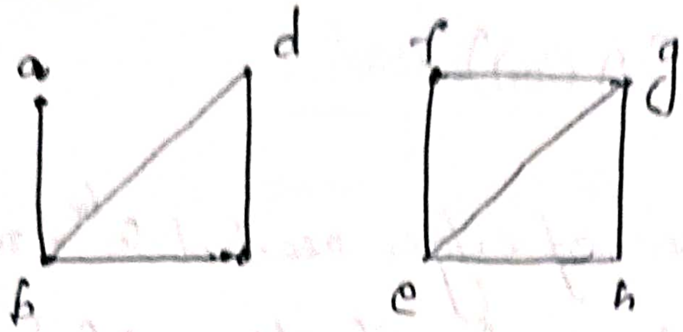
Ex: K_n.



G − C



cut vertices.



remove c

Cut edge.

→ Vertex - Cut

Subset V' of vertex set V such that $G - V'$ is disconnected.

→ Vertex Connectivity $K(G)$

It is the minimum no. of vertices to be removed to make a graph disconnected.

Note:

1. $K(K_n) = n-1$
2. $0 \le K(G) \le n-1$.
3. $K(G) = 0 \iff G$ is disconnected or $G = K_1$.
4. $K(G) = n-1 \iff G$ is Complete.

→ K-connected graph

A graph G is k-connected if and only if $K(G) \ge k$.

→ <u>Edge Connectivity ($\lambda(G)$)</u>.

Minimum no: of edges needed to be removed from a graph to make it disconnected.

- $\lambda(G) = 0$ if G is disconnected.
- $\lambda(G) = 0$ if G has single vertex.
- $\lambda(G) = n-1 \iff$ if and only $G = K_n$.
- $0 \leq \lambda(G) \leq n$.
- when G is not a complete graph $\lambda(G) \leq n-2$.

<u>Note</u>

1) $K(G) \leq \min_v \deg(v)$

   $\lambda(G) \leq \min_v \deg(v)$.

2) $K(K_n) = \lambda(K_n) = n-1$.

3) $K(G) \leq \lambda(G)$

4) $K(G) = \lambda(G) = 0$ when G is disconnected.

5) $K(G) \leq \lambda(G) \leq \min_v \deg(v)$.