Optimization

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***Abstract*—This document contains the solution of the question from NCERT 11th standard chapter 10 exercise 10.4 problem 4**

1 Exercise 10.4

1) What are the points on y axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.
The given line is,

$$\begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{1.0.1}$$

Let the required point on y axis be $\begin{pmatrix}0\\y\end{pmatrix}$, then the distance of this point from the given line is,

$$d = \frac{\left|\begin{pmatrix}4 & 3\end{pmatrix}\begin{pmatrix}0\\y\end{pmatrix} - 12\right|}{\sqrt{3^2+4^2}} \tag{1.0.2}$$

$$d = \frac{|3y-12|}{5} \tag{1.0.3}$$

$$d = 4 \implies \frac{|3y-12|}{5} = 4 \tag{1.0.4}$$

$$|3y-12| = 20 \tag{1.0.5}$$

$$y = 4+\frac{20}{3} = \frac{32}{3} \text{or} y = 4-\frac{20}{3} = -\frac{8}{3} \tag{1.0.6}$$

$$\tag{1.0.7}$$

The foot of perpendicular to the line from the point $(0, \frac{32}{3})$ is,

$$\mathbf{x}_0 = \min_{\mathbf{x}} \left\|\mathbf{x} - \begin{pmatrix}0\\ \frac{32}{3}\end{pmatrix}\right\|^2 \tag{1.0.8}$$

$$\text{s.t} \quad \begin{pmatrix}4 & 3\end{pmatrix}\mathbf{x} = 12 \tag{1.0.9}$$

This can be coverted into an unconstrained optimization problem,

$$\mathbf{x}_0 = \min_{\lambda} \left\|\lambda\begin{pmatrix}3\\-4\end{pmatrix} + \begin{pmatrix}0\\4\end{pmatrix} - \begin{pmatrix}0\\ \frac{32}{3}\end{pmatrix}\right\|^2 \tag{1.0.10}$$

$$\tag{1.0.11}$$

$$\left\|\lambda\begin{pmatrix}3\\-4\end{pmatrix} + \begin{pmatrix}0\\4\end{pmatrix} - \begin{pmatrix}0\\ \frac{32}{3}\end{pmatrix}\right\|^2 = \left\|\lambda\begin{pmatrix}3\\-4\end{pmatrix} + \begin{pmatrix}0\\ -\frac{20}{3}\end{pmatrix}\right\|^2 \tag{1.0.12}$$

$$= 25\lambda^2 + 2\lambda\left(\frac{80}{3}\right) + \left(\frac{20}{3}\right)^2 \tag{1.0.13}$$

A numerical solution for this can be obtained with,

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \tag{1.0.14}$$

Here λ_0 is the initial guess of lambda and α is the step size in gradient descent.

$$f'(\lambda_n) = 50\lambda_n + \frac{160}{3} \tag{1.0.15}$$

$$\tag{1.0.16}$$

So we get,

$$\lambda_{n+1} = \lambda_n - \alpha\left(50\lambda + \frac{160}{3}\right) \tag{1.0.17}$$

We get the optimal λ to be,

$$\lambda^* = -1.0667 \tag{1.0.18}$$

$$\mathbf{x}_0 = \begin{pmatrix}3\lambda\\ 4(1-\lambda)\end{pmatrix} \tag{1.0.19}$$

$$= \begin{pmatrix}-3.2\\ 8.2667\end{pmatrix} \tag{1.0.20}$$

The foot of perpendicular to the line from the point $(0, -\frac{8}{3})$ is,

Parameter	Value	Description
λ_0	1	Initial guess
α	0.01	step size
N	10000	Number of iterations
ϵ	10^{-7}	Tolerance in λ

TABLE 1

$$\mathbf{x}_0 = \min_{\mathbf{x}} \left\| \mathbf{x} - \begin{pmatrix} 0 \\ -\frac{8}{3} \end{pmatrix} \right\|^2 \tag{1.0.21}$$

$$\text{s.t} \quad \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \tag{1.0.22}$$

This can be coverted into an unconstrained optimization problem,

$$\mathbf{x}_0 = \min_{\lambda} \left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{8}{3} \end{pmatrix} \right\|^2 \tag{1.0.23}$$

$$\tag{1.0.24}$$

$$\left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{8}{3} \end{pmatrix} \right\|^2 = \left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{20}{3} \end{pmatrix} \right\|^2 \tag{1.0.25}$$

$$= 25\lambda^2 - 2\lambda \left(\frac{80}{3} \right) + \left(\frac{20}{3} \right)^2 \tag{1.0.26}$$

A numerical solution for this can be obtained with,

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \tag{1.0.27}$$

Here λ_0 is the initial guess of lambda and α is the step size in gradient descent.

$$f'(\lambda_n) = 50\lambda_n - \frac{160}{3} \tag{1.0.28}$$

$$\tag{1.0.29}$$

So we get,

$$\lambda_{n+1} = \lambda_n - \alpha \left(50\lambda - \frac{160}{3} \right) \tag{1.0.30}$$

We get the optimal λ to be,

$$\lambda^* = 1.0667 \tag{1.0.31}$$

$$\mathbf{x}_1 = \begin{pmatrix} 3.2 \\ -0.2667 \end{pmatrix} \tag{1.0.32}$$

Parameter	Value	Description
λ_0	2	Initial guess
α	0.01	step size
N	10000	Number of iterations
ϵ	10^{-7}	Tolerance in λ

TABLE 1