Quiz 12

S Nithish

***Abstract*—This document contains the solution of the question from NCERT 9th standard chapter 10 exercise 10.6 problem 4**

1 Exercise 10.6

1) Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect the chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and DE at the centre.

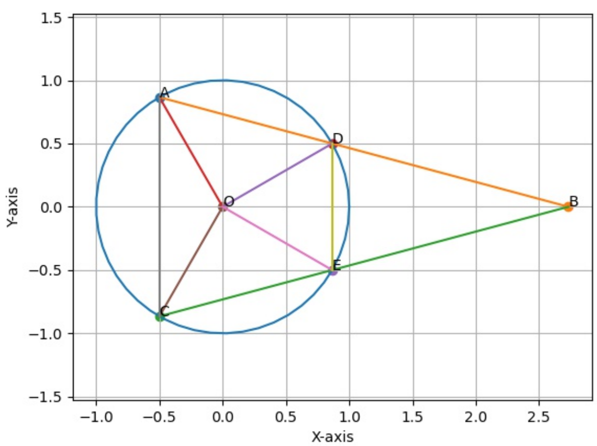

Fig. 1: Circle

Let the circle be unit circle centred at origin,

$$\|\mathbf{x}\|^2 = 1 \tag{1.0.1}$$

Let the points A, B, C be such that,

$$A = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} \tag{1.0.2}$$

$$C = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \tag{1.0.3}$$

$$D = \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \tag{1.0.4}$$

$$E = \begin{pmatrix} \cos\theta_4 \\ \sin\theta_4 \end{pmatrix} \tag{1.0.5}$$

The angle subtended by chord AC is,

$$\phi_1 = |\theta_1 - \theta_2| \tag{1.0.6}$$

The angle subtended by chord DE is,

$$\phi_2 = |\theta_3 - \theta_4| \tag{1.0.7}$$

The direction vector of the line passing through the points A and D is,

$$\mathbf{m_1} = \begin{pmatrix} \cos\theta_3 - \cos\theta_1 \\ \sin\theta_3 - \sin\theta_1 \end{pmatrix} \tag{1.0.8}$$

$$= \begin{pmatrix} -2\sin\left(\frac{\theta_1+\theta_3}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right) \\ 2\cos\left(\frac{\theta_3+\theta_1}{2}\right)\sin\left(\frac{\theta_3-\theta_1}{2}\right) \end{pmatrix} \tag{1.0.9}$$

$$\mathbf{m_1} = -2\sin\left(\frac{\theta_3-\theta_1}{2}\right)\begin{pmatrix} \sin\left(\frac{\theta_1+\theta_3}{2}\right) \\ -\cos\left(\frac{\theta_1+\theta_3}{2}\right) \end{pmatrix} \tag{1.0.10}$$

$$\|\mathbf{m_1}\| = 2\sin\left(\frac{\theta_3-\theta_1}{2}\right) \tag{1.0.11}$$

The direction vector of the line passing through the points C and E is,

$$\mathbf{m_2} = \begin{pmatrix} \cos\theta_4 - \cos\theta_2 \\ \sin\theta_4 - \sin\theta_2 \end{pmatrix} \tag{1.0.12}$$

$$= \begin{pmatrix} -2\sin\left(\frac{\theta_2+\theta_4}{2}\right)\sin\left(\frac{\theta_4-\theta_2}{2}\right) \\ 2\cos\left(\frac{\theta_4+\theta_2}{2}\right)\sin\left(\frac{\theta_4-\theta_2}{2}\right) \end{pmatrix} \tag{1.0.13}$$

$$\mathbf{m_1} = -2\sin\left(\frac{\theta_4-\theta_2}{2}\right)\begin{pmatrix} \sin\left(\frac{\theta_2+\theta_4}{2}\right) \\ -\cos\left(\frac{\theta_2+\theta_4}{2}\right) \end{pmatrix} \tag{1.0.14}$$

$$\|\mathbf{m_2}\| = 2\sin\left(\frac{\theta_4-\theta_2}{2}\right) \tag{1.0.15}$$

Then,

$$\cos \angle ABC = \frac{\mathbf{m_1}^\top \mathbf{m_2}}{\|\mathbf{m_1}\| \|\mathbf{m_2}\|\|} \tag{1.0.16}$$

$$= \begin{pmatrix} \sin\left(\frac{\theta_1+\theta_3}{2}\right) \\ -\cos\left(\frac{\theta_1+\theta_3}{2}\right) \end{pmatrix}^\top \begin{pmatrix} \sin\left(\frac{\theta_2+\theta_4}{2}\right) \\ -\cos\left(\frac{\theta_2+\theta_4}{2}\right) \end{pmatrix} \tag{1.0.17}$$

$$= \cos\left(\frac{\theta_1 + \theta_3 - \theta_2 - \theta_4}{2}\right) \tag{1.0.18}$$

$$= \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \tag{1.0.19}$$

$$\angle ABC = \frac{\phi_1 - \phi_2}{2} \tag{1.0.20}$$

Hence, $\angle ABC$ is equal to half of the angle subtended by chords AD and CE.
Example, Let us take,

$$\theta_1 = \frac{2\pi}{3} \tag{1.0.21}$$

$$\theta_2 = \frac{4\pi}{3} \tag{1.0.22}$$

$$\theta_3 = \frac{\pi}{6} \tag{1.0.23}$$

$$\theta_4 = \frac{\pi}{6} \tag{1.0.24}$$

$$\tag{1.0.25}$$

Then the point of intersection B is,

$$B = \begin{pmatrix} 1 + \sqrt{3} \\ 0 \end{pmatrix} \tag{1.0.26}$$

Then the angle subtended by chord AC at the centre is,

$$\phi_1 = \frac{4\pi}{3} - \frac{2\pi}{3} = \frac{2\pi}{3} \tag{1.0.27}$$

and the angle subtended by chord DE at the centre is,

$$\phi_2 = \frac{\pi}{6} - \frac{-\pi}{6} = \frac{\pi}{3} \tag{1.0.28}$$

Then the angle ABC is,

$$\cos \angle ABC = \frac{\mathbf{BA}^\top \mathbf{BC}}{\|BA\| \|BC\|} \tag{1.0.29}$$

$$\mathbf{BA} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 1 + \sqrt{3} \\ 0 \end{pmatrix} \tag{1.0.30}$$

$$= \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{1.0.31}$$

$$\mathbf{BC} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 1 + \sqrt{3} \\ 0 \end{pmatrix} \tag{1.0.32}$$

$$= \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{1.0.33}$$

$$\mathbf{BA}^\top \mathbf{BC} = \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ \frac{\sqrt{3}}{2} \end{pmatrix}^\top \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \tag{1.0.34}$$

$$= \frac{9}{2} + 3\sqrt{3} \tag{1.0.35}$$

$$\|\mathbf{BA}\| = \|\mathbf{BC}\| = \sqrt{6 + 3\sqrt{3}} \tag{1.0.36}$$

$$\cos \angle ABC = \frac{\frac{9}{2} + 3\sqrt{3}}{6 + 3\sqrt{3}} \tag{1.0.37}$$

$$= \frac{\sqrt{3}}{2} \tag{1.0.38}$$

$$\angle ABC = \frac{\pi}{3} \tag{1.0.39}$$

$$\angle ABC = \phi_1 - \phi_2 \tag{1.0.40}$$

Parameter	Value	Description
θ_1	$\frac{2\pi}{3}$	Phase of A
θ_2	$\frac{4\pi}{3}$	Phase of C
θ_3	$\frac{\pi}{6}$	Phase of D
θ_4	$-\frac{\pi}{6}$	Phase of E

TABLE 1