Quiz 9

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***Abstract*—This document contains the solution of the question from NCERT 9th standard chapter 10 exercise 10.4 problem 4**

1 Exercise 10.4

1) If a line intersects two concentric circles with centre O at A,B,C and D, prove that AB=CD.

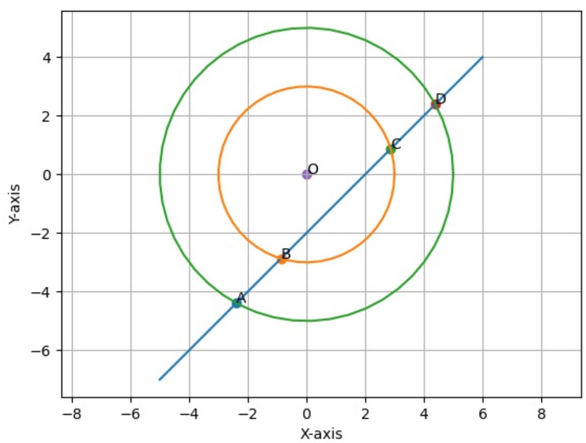

Fig. 1: Circle

Let the inner circle be of radius 3 and the outer circle be of radius 5. Then the equation of the two circles are,

$$\|\mathbf{x}\| = 9 \tag{1.0.1}$$
$$\|\mathbf{x}\| = 25 \tag{1.0.2}$$

Let the equation of the line be,

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{1.0.3}$$
$$\tag{1.0.4}$$

The equation of the conic section is,

$$g(\mathbf{x}) = \mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{1.0.5}$$

Then the parameter μ for the points of intersection of the line with the conic is given by,

$$\mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) + g(\mathbf{h}) = 0 \tag{1.0.6}$$

Two have 2 points of intersection the above quadratic equation must have positive discriminant.

$$4\left(\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right)^2 - 4\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(g(h)\right) > 0 \tag{1.0.7}$$

As the circles are concentric we can say that if the line intersects the inner circle at two points it will intersect the outter circle at two points.

$$\mathbf{V} = \mathbf{I},\ \ \mathbf{u} = \mathbf{O},\ \ f = -9 \tag{1.0.8}$$

The line we have considered is,

$$\mathbf{x} = \begin{pmatrix}0\\-2\end{pmatrix} + \mu\begin{pmatrix}1\\1\end{pmatrix} \tag{1.0.9}$$
$$\mathbf{m} = \begin{pmatrix}1\\1\end{pmatrix} \tag{1.0.10}$$
$$\mathbf{h} = \begin{pmatrix}0\\-2\end{pmatrix} \tag{1.0.11}$$
$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = 2 \tag{1.0.12}$$
$$\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) = -2 \tag{1.0.13}$$
$$g(\mathbf{h}) = -5 \tag{1.0.14}$$
$$\implies 4\left(\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right)^2 - 4\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(g(h)\right) \tag{1.0.15}$$
$$= 4\left(-2\right)^2 - 4\left(2\right)\left(-5\right) \tag{1.0.16}$$
$$= 56 > 0 \tag{1.0.17}$$

The point of intersection of the line and the inner circle is,

$$2\mu^2 - 4\mu - 5 = 0 \tag{1.0.18}$$
$$\mu_b = \frac{4 - \sqrt{56}}{4} = 1 - \sqrt{\frac{7}{2}} \tag{1.0.19}$$
$$\mu_c = \frac{4 + \sqrt{56}}{4} = 1 + \sqrt{\frac{7}{2}} \tag{1.0.20}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.0.21}$$
$$\mathbf{x} = \begin{pmatrix} \lambda \\ \lambda - 2 \end{pmatrix} \tag{1.0.22}$$

The point of intersection of the line and the outter circle is,

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -25 \tag{1.0.23}$$
$$g(\mathbf{h}) = 4 - 25 = -21 \tag{1.0.24}$$
$$\tag{1.0.25}$$

The remaining terms remains same as in the previous case.

$$2\mu^2 - 4\mu - 21 = 0 \tag{1.0.26}$$
$$\mu_a = \frac{4 - \sqrt{184}}{4} = 1 - \sqrt{\frac{23}{2}} \tag{1.0.27}$$
$$\mu_d = \frac{4 + \sqrt{184}}{4} = 1 + \sqrt{\frac{23}{2}} \tag{1.0.28}$$
$$\tag{1.0.29}$$

Hence the points A,B,C and D are,

$$\mathbf{x_a} = \begin{pmatrix} 1 - \sqrt{\frac{23}{2}} \\ -1 - \sqrt{\frac{23}{2}} \end{pmatrix} \tag{1.0.30}$$
$$\mathbf{x_b} = \begin{pmatrix} 1 - \sqrt{\frac{7}{2}} \\ -1 - \sqrt{\frac{7}{2}} \end{pmatrix} \tag{1.0.31}$$
$$\mathbf{x_c} = \begin{pmatrix} 1 + \sqrt{\frac{7}{2}} \\ -1 + \sqrt{\frac{7}{2}} \end{pmatrix} \tag{1.0.32}$$
$$\mathbf{x_a} = \begin{pmatrix} 1 + \sqrt{\frac{23}{2}} \\ -1 + \sqrt{\frac{23}{2}} \end{pmatrix} \tag{1.0.33}$$
$$\tag{1.0.34}$$

The distance between points A and B is,

$$AB = \|\mathbf{x_a} - \mathbf{x_b}\| = \left\| \begin{pmatrix} \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \\ \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \end{pmatrix} \right\| = \sqrt{23} - \sqrt{7} \tag{1.0.35}$$
$$CD = \|\mathbf{x_c} - \mathbf{x_d}\| = \left\| \begin{pmatrix} \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \\ \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \end{pmatrix} \right\| = \sqrt{23} - \sqrt{7} \tag{1.0.36}$$
$$\implies AB = CD \tag{1.0.37}$$