

# Quiz 12

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**Abstract**—This document contains the solution of the question from NCERT 9th standard chapter 10 exercise 10.6 problem 4

## 1 EXERCISE 10.6

- 1) Let the vertex of an angle  $ABC$  be located outside a circle and let the sides of the angle intersect the chords  $AD$  and  $CE$  with the circle. Prove that  $\angle ABC$  is equal to half the difference of the angles subtended by the chords  $AC$  and  $DE$  at the centre.

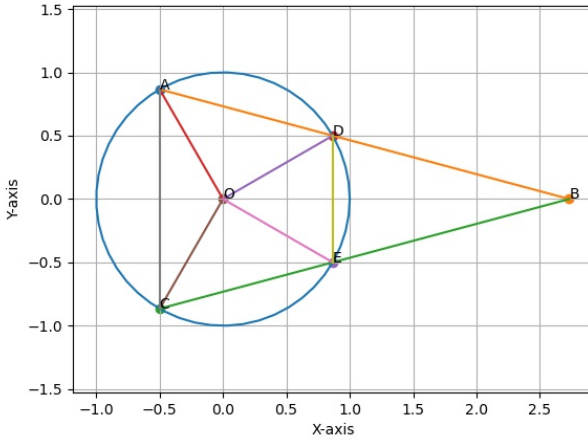


Fig. 1: Circle

Let the circle be unit circle centred at origin,

$$\|\mathbf{x}\|^2 = 1 \quad (1.0.1)$$

Let the points A, B, C be such that,

$$A = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} \quad (1.0.2)$$

$$C = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix} \quad (1.0.3)$$

$$D = \begin{pmatrix} \cos \theta_3 \\ \sin \theta_3 \end{pmatrix} \quad (1.0.4)$$

$$E = \begin{pmatrix} \cos \theta_4 \\ \sin \theta_4 \end{pmatrix} \quad (1.0.5)$$

The angle subtended by chord AC is,

$$\phi_1 = |\theta_1 - \theta_2| \quad (1.0.6)$$

The angle subtended by chord DE is,

$$\phi_2 = |\theta_3 - \theta_4| \quad (1.0.7)$$

The direction vector of the line passing through the points A and D is,

$$\mathbf{m}_1 = \begin{pmatrix} \cos \theta_3 - \cos \theta_1 \\ \sin \theta_3 - \sin \theta_1 \end{pmatrix} \quad (1.0.8)$$

$$= \begin{pmatrix} -2 \sin\left(\frac{\theta_1 + \theta_3}{2}\right) \sin\left(\frac{\theta_3 - \theta_1}{2}\right) \\ 2 \cos\left(\frac{\theta_3 + \theta_1}{2}\right) \sin\left(\frac{\theta_3 - \theta_1}{2}\right) \end{pmatrix} \quad (1.0.9)$$

$$\mathbf{m}_1 = -2 \sin\left(\frac{\theta_3 - \theta_1}{2}\right) \begin{pmatrix} \sin\left(\frac{\theta_1 + \theta_3}{2}\right) \\ -\cos\left(\frac{\theta_1 + \theta_3}{2}\right) \end{pmatrix} \quad (1.0.10)$$

$$\|\mathbf{m}_1\| = 2 \sin\left(\frac{\theta_3 - \theta_1}{2}\right) \quad (1.0.11)$$

The direction vector of the line passing through the points C and E is,

$$\mathbf{m}_2 = \begin{pmatrix} \cos \theta_4 - \cos \theta_2 \\ \sin \theta_4 - \sin \theta_2 \end{pmatrix} \quad (1.0.12)$$

$$= \begin{pmatrix} -2 \sin\left(\frac{\theta_2 + \theta_4}{2}\right) \sin\left(\frac{\theta_4 - \theta_2}{2}\right) \\ 2 \cos\left(\frac{\theta_4 + \theta_2}{2}\right) \sin\left(\frac{\theta_4 - \theta_2}{2}\right) \end{pmatrix} \quad (1.0.13)$$

$$\mathbf{m}_2 = -2 \sin\left(\frac{\theta_4 - \theta_2}{2}\right) \begin{pmatrix} \sin\left(\frac{\theta_2 + \theta_4}{2}\right) \\ -\cos\left(\frac{\theta_2 + \theta_4}{2}\right) \end{pmatrix} \quad (1.0.14)$$

$$\|\mathbf{m}_2\| = 2 \sin\left(\frac{\theta_4 - \theta_2}{2}\right) \quad (1.0.15)$$

Then,

$$\cos \angle ABC = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \quad (1.0.16)$$

$$= \begin{pmatrix} \sin\left(\frac{\theta_1+\theta_3}{2}\right) \\ -\cos\left(\frac{\theta_1+\theta_3}{2}\right) \end{pmatrix}^\top \begin{pmatrix} \sin\left(\frac{\theta_2+\theta_4}{2}\right) \\ -\cos\left(\frac{\theta_2+\theta_4}{2}\right) \end{pmatrix} \quad (1.0.17)$$

$$= \cos\left(\frac{\theta_1 + \theta_3 - \theta_2 - \theta_4}{2}\right) \quad (1.0.18)$$

$$= \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \quad (1.0.19)$$

$$\angle ABC = \frac{\phi_1 - \phi_2}{2} \quad (1.0.20)$$

Hence,  $\angle ABC$  is equal to half of the angle subtended by chords AD and CE.

Example, Let us take,

$$\theta_1 = \frac{2\pi}{3} \quad (1.0.21)$$

$$\theta_2 = \frac{4\pi}{3} \quad (1.0.22)$$

$$\theta_3 = \frac{\pi}{6} \quad (1.0.23)$$

$$\theta_4 = \frac{\pi}{6} \quad (1.0.24)$$

$$(1.0.25)$$

Then the point of intersection B is,

$$B = \begin{pmatrix} 1 + \sqrt{3} \\ 0 \end{pmatrix} \quad (1.0.26)$$

Then the angle subtended by chord AC at the centre is,

$$\phi_1 = \frac{4\pi}{3} - \frac{2\pi}{3} = \frac{2\pi}{3} \quad (1.0.27)$$

and the angle subtended by chord DE at the centre is,

$$\phi_2 = \frac{\pi}{6} - \frac{-\pi}{6} = \frac{\pi}{3} \quad (1.0.28)$$

Then the angle ABC is,

$$\cos \angle ABC = \frac{\mathbf{BA}^\top \mathbf{BC}}{\|\mathbf{BA}\| \|\mathbf{BC}\|} \quad (1.0.29)$$

$$BA = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 1 + \sqrt{3} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.0.30)$$

$$BC = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 1 + \sqrt{3} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.0.31)$$

$$\mathbf{BA}^\top \mathbf{BC} = \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ \frac{\sqrt{3}}{2} \end{pmatrix}^\top \begin{pmatrix} -\frac{3}{2} - \sqrt{3} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (1.0.32)$$

$$= \frac{9}{2} + 3\sqrt{3} \quad (1.0.33)$$

$$\|\mathbf{BA}\| = \|\mathbf{BC}\| = \sqrt{6 + 3\sqrt{3}} \quad (1.0.34)$$

$$\cos \angle ABC = \frac{\frac{9}{2} + 3\sqrt{3}}{6 + 3\sqrt{3}} \quad (1.0.35)$$

$$= \frac{\sqrt{3}}{2} \quad (1.0.36)$$

$$\angle ABC = \frac{\pi}{3} \quad (1.0.37)$$

$$\angle ABC = \phi_1 - \phi_2 \quad (1.0.38)$$

Parameter	Value	Description
$\theta_1$	$\frac{2\pi}{3}$	Phase of A
$\theta_2$	$\frac{4\pi}{3}$	Phase of C
$\theta_3$	$\frac{\pi}{6}$	Phase of D
$\theta_4$	$-\frac{\pi}{6}$	Phase of E

TABLE 1