

Optimization

S Nithish

Abstract—This document contains the solution of the question from NCERT 11th standard chapter 10 exercise 10.4 problem 4

1 EXERCISE 10.4

- 1) What are the points on y axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.
The given line is,

$$(4 \ 3)\mathbf{x} = 12 \quad (1.0.1)$$

Let the required point on y axis be $\begin{pmatrix} 0 \\ y \end{pmatrix}$, then the distance of this point from the given line is,

$$d = \frac{\left| (4 \ 3) \begin{pmatrix} 0 \\ y \end{pmatrix} - 12 \right|}{\sqrt{3^2 + 4^2}} \quad (1.0.2)$$

$$d = \frac{|3y - 12|}{5} \quad (1.0.3)$$

$$d = 4 \implies \frac{|3y - 12|}{5} = 4 \quad (1.0.4)$$

$$|3y - 12| = 20 \quad (1.0.5)$$

$$y = 4 + \frac{20}{3} = \frac{32}{3} \text{ or } y = 4 - \frac{20}{3} = -\frac{8}{3} \quad (1.0.6)$$

$$(1.0.7)$$

The foot of perpendicular to the line from the point $(0, \frac{32}{3})$ is,

$$\mathbf{x}_0 = \min_{\mathbf{x}} \left\| \mathbf{x} - \begin{pmatrix} 0 \\ \frac{32}{3} \end{pmatrix} \right\|^2 \quad (1.0.8)$$

$$\text{s.t } (4 \ 3)\mathbf{x} = 12 \quad (1.0.9)$$

This can be converted into an unconstrained optimization problem,

$$\mathbf{x}_0 = \min_{\lambda} \left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{32}{3} \end{pmatrix} \right\|^2 \quad (1.0.10)$$

$$(1.0.11)$$

$$\left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{32}{3} \end{pmatrix} \right\|^2 \quad (1.0.12)$$

$$= \left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{20}{3} \end{pmatrix} \right\|^2 \quad (1.0.13)$$

$$= 25\lambda^2 + 2\lambda \left(\frac{80}{3} \right) + \left(\frac{20}{3} \right)^2 \quad (1.0.14)$$

A numerical solution for this can be obtained with,

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \quad (1.0.15)$$

Here λ_0 is the initial guess of lambda and α is the step size in gradient descent.

$$f'(\lambda_n) = 50\lambda_n + \frac{160}{3} \quad (1.0.16)$$

$$(1.0.17)$$

So we get,

$$\lambda_{n+1} = \lambda_n - \alpha \left(50\lambda_n + \frac{160}{3} \right) \quad (1.0.18)$$

We get the optimal λ to be,

$$\lambda^* = -1.0667 \quad (1.0.19)$$

$$\mathbf{x}_0 = \begin{pmatrix} 3\lambda \\ 4(1 - \lambda) \end{pmatrix} \quad (1.0.20)$$

$$= \begin{pmatrix} -3.2 \\ 8.2667 \end{pmatrix} \quad (1.0.21)$$

The foot of perpendicular to the line from the point $(0, -\frac{8}{3})$ is,

Parameter	Value	Description
λ_0	1	Initial guess
α	0.01	step size
N	10000	Number of iterations
ϵ	10^{-7}	Tolerance in λ

TABLE 1

$$\mathbf{x}_1 = \begin{pmatrix} 3.2 \\ -0.2667 \end{pmatrix} \quad (1.0.34)$$

Parameter	Value	Description
λ_0	2	Initial guess
α	0.01	step size
N	10000	Number of iterations
ϵ	10^{-7}	Tolerance in λ

TABLE 1

$$\mathbf{x}_0 = \min_{\mathbf{x}} \left\| \mathbf{x} - \begin{pmatrix} 0 \\ -\frac{8}{3} \end{pmatrix} \right\|^2 \quad (1.0.22)$$

$$\text{s.t } \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = 12 \quad (1.0.23)$$

This can be covered into an unconstrained optimization problem,

$$\mathbf{x}_0 = \min_{\lambda} \left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{8}{3} \end{pmatrix} \right\|^2 \quad (1.0.24)$$

$$(1.0.25)$$

$$\left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{8}{3} \end{pmatrix} \right\|^2 \quad (1.0.26)$$

$$= \left\| \lambda \begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{20}{3} \end{pmatrix} \right\|^2 \quad (1.0.27)$$

$$= 25\lambda^2 - 2\lambda \left(\frac{80}{3} \right) + \left(\frac{20}{3} \right)^2 \quad (1.0.28)$$

A numerical solution for this can be obtained with,

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \quad (1.0.29)$$

Here λ_0 is the initial guess of lambda and α is the step size in gradient descent.

$$f'(\lambda_n) = 50\lambda_n - \frac{160}{3} \quad (1.0.30)$$

$$(1.0.31)$$

So we get,

$$\lambda_{n+1} = \lambda_n - \alpha \left(50\lambda - \frac{160}{3} \right) \quad (1.0.32)$$

We get the optimal λ to be,

$$\lambda^* = 1.0667 \quad (1.0.33)$$