

Quiz 9

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Abstract—This document contains the solution of the question from NCERT 9th standard chapter 10 exercise 10.4 problem 4

1 EXERCISE 10.4

- 1) If a line intersects two concentric circles with centre O at A,B,C and D, prove that AB=CD.

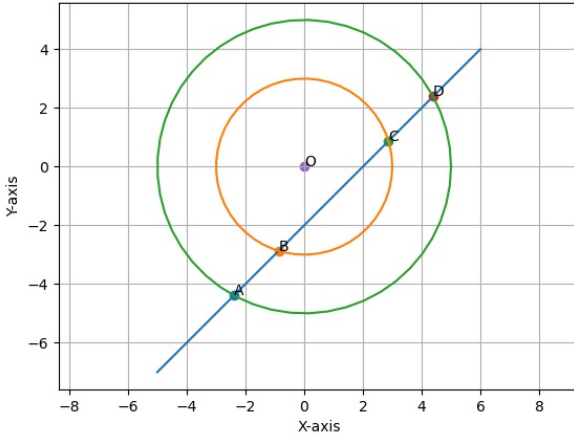


Fig. 1: Circle

Let the inner circle be of radius 3 and the outer circle be of radius 5. Then the equation of the two circles are,

$$\|\mathbf{x}\| = 9 \quad (1.0.1)$$

$$\|\mathbf{x}\| = 25 \quad (1.0.2)$$

Let the equation of the line be,

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad (1.0.3)$$

$$(1.0.4)$$

The equation of the conic section is,

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.0.5)$$

Then the parameter μ for the points of intersection of the line with the conic is given by,

$$\mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \quad (1.0.6)$$

Two have 2 points of intersection the above quadratic equation must have positive discriminant.

$$4(\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}))^2 - 4(\mathbf{m}^T \mathbf{V} \mathbf{m})(g(\mathbf{h})) > 0 \quad (1.0.7)$$

As the circles are concentric we can say that if the line intersects the inner circle at two points it will intersect the outer circle at two points.

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -9 \quad (1.0.8)$$

The line we have considered is,

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.0.9)$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.0.10)$$

$$\mathbf{h} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad (1.0.11)$$

$$\mathbf{m}^T \mathbf{V} \mathbf{m} = 2 \quad (1.0.12)$$

$$\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) = -2 \quad (1.0.13)$$

$$g(\mathbf{h}) = -5 \quad (1.0.14)$$

$$\Rightarrow 4(\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}))^2 - 4(\mathbf{m}^T \mathbf{V} \mathbf{m})(g(\mathbf{h})) \quad (1.0.15)$$

$$= 4(-2)^2 - 4(2)(-5) \quad (1.0.16)$$

$$= 56 > 0 \quad (1.0.17)$$

The point of intersection of the line and the inner circle is,

$$2\mu^2 - 4\mu - 5 = 0 \quad (1.0.18)$$

$$\mu_b = \frac{4 - \sqrt{56}}{4} = 1 - \sqrt{\frac{7}{2}} \quad (1.0.19)$$

$$\mu_c = \frac{4 + \sqrt{56}}{4} = 1 + \sqrt{\frac{7}{2}} \quad (1.0.20)$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.0.21)$$

$$\mathbf{x} = \begin{pmatrix} \lambda \\ \lambda - 2 \end{pmatrix} \quad (1.0.22)$$

The point of intersection of the line and the outer circle is,

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -25 \quad (1.0.23)$$

$$g(\mathbf{h}) = 4 - 25 = -21 \quad (1.0.24)$$

$$(1.0.25)$$

The remaining terms remains same as in the previous case.

$$2\mu^2 - 4\mu - 21 = 0 \quad (1.0.26)$$

$$\mu_a = \frac{4 - \sqrt{184}}{4} = 1 - \sqrt{\frac{23}{2}} \quad (1.0.27)$$

$$\mu_d = \frac{4 + \sqrt{184}}{4} = 1 + \sqrt{\frac{23}{2}} \quad (1.0.28)$$

$$(1.0.29)$$

Hence the points A,B,C and D are,

$$\mathbf{x}_a = \begin{pmatrix} 1 - \sqrt{\frac{23}{2}} \\ -1 - \sqrt{\frac{23}{2}} \end{pmatrix} \quad (1.0.30)$$

$$\mathbf{x}_b = \begin{pmatrix} 1 - \sqrt{\frac{7}{2}} \\ -1 - \sqrt{\frac{7}{2}} \end{pmatrix} \quad (1.0.31)$$

$$\mathbf{x}_c = \begin{pmatrix} 1 + \sqrt{\frac{7}{2}} \\ -1 + \sqrt{\frac{7}{2}} \end{pmatrix} \quad (1.0.32)$$

$$\mathbf{x}_d = \begin{pmatrix} 1 + \sqrt{\frac{23}{2}} \\ -1 + \sqrt{\frac{23}{2}} \end{pmatrix} \quad (1.0.33)$$

$$(1.0.34)$$

The distance between points A and B is,

$$AB = \|\mathbf{x}_a - \mathbf{x}_b\| = \left\| \begin{pmatrix} \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \\ \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \end{pmatrix} \right\| = \sqrt{23} - \sqrt{7} \quad (1.0.35)$$

$$CD = \|\mathbf{x}_c - \mathbf{x}_d\| = \left\| \begin{pmatrix} \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \\ \frac{\sqrt{7} - \sqrt{23}}{\sqrt{2}} \end{pmatrix} \right\| = \sqrt{23} - \sqrt{7} \quad (1.0.36)$$

$$\Rightarrow AB = CD \quad (1.0.37)$$