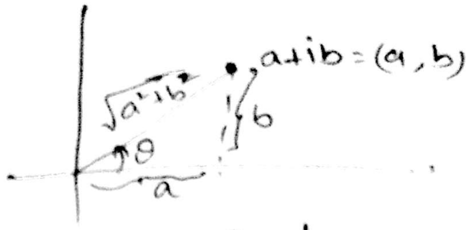

$\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$

Complex variable :(cartesian co-ordinates)

$z = x+iy$ where x, y are real variables is ca

complex variable.

$z + \bar{z} = 2x$ (real)

$z - \bar{z} = 2iy$ (imaginary)

mplex variable in polar co-ordinates :- (r, θ)

$x^2+y^2=r^2$

$x = r\cos\theta$; $y = r\sin\theta$

$z = x+iy = r\cos\theta + ir\sin\theta$

$z = re^{i\theta}$

Neighbourhood of a complex number :- The nei

of the complex number $z_0 = x_0+iy_0$ is the

ll points such that $|z-z_0| < \delta$ it is circ

$\Rightarrow (x-x_0)^2 + (y-y_0)^2 < \delta$

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$z = x+iy$ where x, y are real variables ...

complex variable.

$z + \bar{z} = 2x$ (real)

$z - \bar{z} = 2iy$ (imaginary)

Complex variable in polar co-ordinates: ...

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→ $|z - z_0| = r$ is a circle eq. with centre z_0 & radius r.

~~Differentiability~~ o

Complex function :- $w = f(z)$ where z is independent variable
dep. variable

is called a complex fn

eg:- $w = f(z) = z^2 + 3z + 1$

where $z = x + iy$

$\Rightarrow w = (x+iy)^2 + 3(x+iy) + 1$

$= \underbrace{(x^2 - y^2 + 3x + 1)}_{u(x,y)} + i\underbrace{(2xy + 3y)}_{v(x,y)}$

$\Rightarrow w = f(z) = u(x,y) + iv(x,y)$

$f(z) = u + iv$

Differentiation & differentiability :-

The complex fn $f(z)$ is said to be differentia

at z_0 if $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ exists.

The limit value is called derivative of the $f(z)$ at z_0 and it is denoted by $f'(z_0)$ he

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Analatic function :- A The function $f(z)$ is said to be analytic at the point z_0 if $f(z)$ is differentiable at every point of some neighbourhood of z_0

The function $f(z)$ is said to be analatic in a domain D, if it is analatic at every point of the domain D

Entire function:- A function which is analatic every where is known as entire function.

Ex:-

i) $f(z) = \dfrac{1}{\sin z}$ is not analatic at $z = \pm n\pi$; $n = 0, 1, 2, \ldots$

ii) $f(z) = e^{2z}$ is analatic every where except $z = \pm 1$

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1) Which of the following fns is entire function

a) $f(z) = \dfrac{z^2+4}{z \sin z}$ d) $f(z) = (z^2+5z+5)e^{2z}$

b) $f(z) = \cancel{z}\sqrt{z} + 2$

c) $f(z) = \frac{1}{z} + \sin\frac{1}{z}$

A)

a) is not analatic at $z=0$; $z = \pm n\pi$ so not entire fn

b) " " " at $z=0$

c) " " " " "

d) is entire fn.

Note:-

(i) If $f(z)$, $g(z)$ are any two analatic fns then

$f(z) \pm g(x)$; $f(z) \cdot g(z)$; $\dfrac{f(z)}{g(z) \neq 0}$ are also analatic functions.

(ii) If $f(z)$ is analatic then its derivative i.e $f'(z)$ is also analatic.

So that, if $f(z)$ is analatic the $f'(z), f''(z), \ldots$ analatic fns.

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Let $f(z) = u+iv$ be any complex function then

(i) C-R eqns in cartesian co-ordinates : (x, y)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(ii) C-R eqns in polar co-ordinates : (r, θ)

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Note:-

i) C-R equations are necessary for a function to be analatic but not sufficient.

(ii) If the C-R equations are not true for a fn $f(z)$ then it is not analatic

(iii) If $f(z)$ is analatic then its derivative $f'(z)$ can be calculated from one of the equations $f'(z) = u_x + iv_x$ or $f'(z) = v_y - iu_y$

(iv) If $f(z) = u+iv$ is analatic function then the family of curves $u(x,y) = C_1$ & $v(x,y) = C_2$ are mutually orthogonal.

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1) Which of the following fns is entire function

a) $f(z) = \dfrac{z^2+4}{z\sin z}$

d) $f(z) = (z^2+5z+5)e^{2z}$

b) $f(z) = \cancel{z}\sqrt{z}+2$

c) $f(z) = \dfrac{1}{z} + \sin\dfrac{1}{z}$

A)

a) is not analatic at $z=0$; $z=\pm n\pi$ so not entire fn

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Note:-

(i) If $f(z)$, $g(z)$ are any two analatic fns then $f(z)\pm g(x)$; $f(z)\cdot g(z)$; $\dfrac{f(z)}{g(z)}$ $g(z)\neq 0$ are also analatic functions.

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So that, if $f(z)$ is analatic the $f'(z), f''(z), \ldots$ analatic fns.

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Theorem:- Cauchy-Riemann equations

Let $f(z) = u + iv$ be any complex function

(i) C-R eqns in cartesian co-ordinates

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(ii) C-R eqns in polar co-ordinates

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \; ; \; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Note:-

i) C-R equations are necessary for a function to be analatic but not sufficient.

(ii) If the C-R equations are not true for a function then it is not analatic

(iii) If $f(z)$ is analatic then its derivative $f'(z)$ can be calculated from one of the equations $f'(z) = u_x + iv_x$ or $f'(z) = v_y - iu_y$

(iv) If $f(z) = u + iv$ is analatic function then the family of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ are mutually orthogonal.

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$x+y-L=c$

Construction of analatic function.

Method I :- Milne - Thompson Method (Milne's Method)

Step I :- When u is given take $f(z)$ is $f'(z) = u_x - iu_y$

" v is given " $f'(z) = v_y + iv_x$

Step II :- On the right hand side of above equation replace x by z and y by 0

Step III :- Integrating on both sides of above equation we get required analatic fn.

Q1) Find analatic fn $f(z)$ whose real part is $u = x^2 - y^2$

A) $f'(z) = u_x - iu_y$

$\frac{\partial u}{\partial x} = 2x \quad u_y = -2y \quad \Rightarrow f'(z) = 2x + i2y$

$f'(z) = 2z$

$f(z) = z^2 + c$

2) Find analatic fn $f(z)$ whose imaginary part is $v = e^x \sin y$

A) $f'(z) = v_x + iv_y$

$\frac{\partial v}{\partial x} = e^x \sin y \; ; \; \frac{\partial v}{\partial y} = e^x \cos y$

$f'(z) = e^x \cancel{\sin y}\,\cos y + i(e^x \cancel{\cos y}\,\sin y)$

put $x = z \quad y = 0$

$f'(z) = (e^z \cancel{\cos y})$

$f(z) = e^z + c$

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Method 2:

Harmonic function: A function which satisfies laplace equation is known as harmonic fn

i.e the fn $\phi(x,y)$ is harmonic if $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

Conjugate harmonic function: If $f(z) = u+iv$ is analatic function then u, v are harmonic functions. u is called conjugate harmonic fn of v and vice-versa

Let $f(z) = u+iv$ is analatic

then C-R eqns $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x \partial y} \quad ; \quad \frac{\partial^2 u}{\partial^2 y} = -\frac{\partial v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} = 0 \Rightarrow u \text{ is harmonic}$$

||ly $v(x,y)$ is harmonic

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$$= \lim_{\Delta z \to 0} \frac{z_0 \overline{\Delta z} + \Delta z \overline{z_0} + \Delta z \overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{z_0} + \frac{z_0 \overline{\Delta z}}{\Delta z} + \overline{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{z_0} + z_0 \left(\frac{\overline{\Delta z}}{\Delta z}\right) + \overline{\Delta z}$$

$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ is not unique

i.e along $\begin{matrix}\Delta x \to 0 \\ \Delta y \to 0\end{matrix}$ $\frac{\overline{\Delta z}}{\Delta z} = -1$

along $\begin{matrix}\Delta y \to 0 \\ \Delta x \to 0\end{matrix}$ $= 1$

The limit value is unique only at $z_0 = 0$

Hence the function $f(z) = |z|^2$ is differentiable only at $z = 0$ that is $z_0 = 0$

Q) Show that the function $f(z) = z^3$ is analytic everywhere

A)
$$f(z) = z^3 = (x+iy)^3$$

$$= x^3 - iy^3 - 3xy^2 + 3ix^2y$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$= x(x^2 - 3y^2) + iy(3x^2 - y^2)$$

$$u = x^3 - 3xy^2 \, ; \; v = 3x^2y - y^3$$

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Ex:- Is the fn f(z) [illegible] continuous at 0

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy} \quad \lim_{\substack{y\to mx\\ x\to 0}} \frac{x-i[illegible]}{x+i[illegible]} = 1$$

Along different paths limit values is not [illegible]

Limit does not exist Hence function is not continous.

Ex:- The fn $|z|^2$ is continous every where and no where it is differentiable except at [illegible]

Sol:- $f(z) = |z|^2 = z\bar{z} = x^2+y^2$. The function value at any point of z that is x, y is unique

Hence it is continous every where

differentiability:-

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{|z_0+\Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z_0+\Delta z)(\overline{z_0+\Delta z}) - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z_0+\Delta z)(\bar{z}_0+\overline{\Delta z}) - z_0\bar{z}_0}{\Delta z}$$

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$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$; $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

equal

$\frac{\partial u}{\partial y} = -6xy$ $\qquad \frac{\partial v}{\partial x} = 6yx$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

i) $\therefore$ For any value of z, $z(x,y)$ the above 4 partial derivatives exist and they are continuous

ii) CR Equns. are also true for Any z. Hence given function is Analytic every where.

2) Find whether the function $f(z) = z + 2\bar{z}$ is Analytic any where in the complex plane.

A) $f(z) = x + iy + 2(x - iy)$

$u = 3x \qquad v = -y$

$$\left.\begin{array}{ll} \frac{\partial u}{\partial x} = 3 & \frac{\partial v}{\partial y} = -1 \\ \frac{\partial u}{\partial y} = 0 & \frac{\partial v}{\partial x} = 0 \end{array}\right\} \text{Not Analytic.}$$

Hence CR equations fails for any z

$\therefore$ function is not Analytic any where in the complex plane.

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→ Sufficient conditions for a function to be Analytic

1) The single valued function $f(z)$ is said to be Analytic in a domain D if the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and continous.

2) u, v satisfy CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ must be true.

Note:-
Limit of a function of two variables exist if the limit value along any path or diff paths is unique, that is

ex:- $\lim_{z \to 0} f(z)$ exist

if $\lim_{(x,y) \to (0,0)} f(z)$ is same as

$\lim_{\substack{x \to 0 \\ y \to 0}} f(z)$ is same as $\lim_{\substack{y \to 0 \\ x \to 0}} f(z)$ or let $y = m$

then $\lim_{z \to 0} f(z$

$= \lim_{\substack{y = mx \\ x \to 0}} f(x)$ all are sa

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By Millne's methods

$$f'(z) = u_x - i u_y$$

$$= e^{2x}((1+2x)\cos 2y - 2y\sin 2y)$$

$$\cancel{- i(2e^{2x}(x\cos 2y - y\sin 2y))}$$

$$i(-(2x+1)\sin 2y - 2y\cos 2y)$$

replace x by z & y by 0

$$f'(z) = e^{2z}((1+2z)) - i(\cancel{2e^{2z}(z)})$$

$$= e^{2z}(1+2z)$$

Integrating w.r.t z on both sides we have.

$$f(z) = \int (1+2z)e^{2z}\,dz + c$$

$$= (1+2z)\frac{e^{2z}}{2} - (2)\frac{e^{2z}}{4} + c$$

$$= e^{2z}\left(\frac{1}{2} + z - \frac{1}{2}\right) + c$$

$$= ze^{2z} + c$$

To find conjugate harmonic function:-

Consider $f(z) = u + iv$ be an analytic fn then u & v satisfies CR equations i.e

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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If $f(z) = u + iv$ is analytic function whose imaginary part is $\tan^{-1}(y/x)$ then find $f(z)$

$v = \tan^{-1}(y/x)$

$f'(z) = v_y + i v_x$

$= \frac{\partial}{\partial y}\left(\tan^{-1}(y/x)\right) + i \frac{\partial}{\partial x}\left(\tan^{-1}(y/x)\right)$

$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} + i\left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\right) y\left(-\frac{1}{x^2}\right)$

$f'(z) = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$

$f'(z) = \frac{z}{z^2} + i\left(\frac{-0}{z^2}\right) = \frac{1}{z}$

$f(z) = \log z + c$

3) Determine the analytic fn whose real part is

$u = e^{2x}(x\cos 2y - y\sin 2y)$

$\frac{\partial u}{\partial x} = e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y - y\sin 2y)$

$= e^{2x}((1+2x)\cos 2y - 2y\sin 2y)$

$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$

$= e^{2x}(-(2x+1)\sin 2y - 2y\cos 2y)$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since the fn $u(x,y)$ satisfies Laplace equation therefore $u(x,y)$ is harmonic fn.

To find conjugate harmonic fn. $v(x,y)$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\Rightarrow dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy \quad \text{(by CR eqns)}$$

~~$dv = -2e^{-2xy}($~~

$$dv = \underbrace{(2y\cos(x^2-y^2) + 2x\sin(x^2-y^2))e^{-2xy}}_{M}dx$$

$$+ \underbrace{(2x\cos(x^2-y^2) - 2y\sin(x^2-y^2))e^{-2xy}}_{N}dy$$

is exact.

Its solution is given by

$$\int dv = \underset{(y \text{ is const})}{\int M\,dx} + \int \left(\begin{matrix}\text{terms of } N \\ \text{not involving } x\end{matrix}\right) dy + C$$

$$\Rightarrow v = \int \left(2y\cos(x^2-y^2) + 2x\sin(x^2-y^2)\right)e^{-2xy}dx + 0 + C$$

$$= \int -d(e^{-2xy}\cos(x^2-y^2)) + C$$

$$v = -e^{-2xy}\cos(x^2-y^2) + C$$

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1) Show that the fn.

$$u = e^{-2xy}\sin(x^2-y^2)$$ is harmonic. Hence f...

its conjugate harmonic fn.

A)

$$\frac{\partial u}{\partial x} = e^{-2xy}\cos(x^2-y^2)\times 2x + (-2y)e^{-2xy}\sin(x^2-y^2)$$

$$= 2e^{-2xy}(x\cos(x^2-y^2) - y\sin(x^2-y^2))$$

$$\frac{\partial^2 u}{\partial x^2} = 2(-2y)e^{-2xy}(x\cos(x^2-y^2) - y\sin(x^2-y^2))$$

$$+ 2e^{-2xy}(\cos(x^2-y^2) - 2x^2\sin(x^2-y^2) - 2xy \ldots$$

$$\frac{\partial u}{\partial y} = e^{-2xy}\cos(x^2-y^2)(-2y) - 2x\, e^{-2xy}\sin(x^2-y^2)$$

$$= e^{-2xy}(-2y\cos(x^2-y^2) - 2x\sin(x^2-y^2))$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-2xy}(-2\cos(x^2-y^2) - 4y^2\sin(x^2-y^2) + 4xy\cos(x^2 \ldots$$

$$+ (-2x)\, e^{-2xy}(-2y\cos(x^2-y^2) - 2x\sin(x^2-y^2))$$

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(i) When u is given to find its conjugate harmonic fn v.

by chain rule

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

(is exact)

by using CR eqns.

$$dv = \underbrace{\left(-\frac{\partial u}{\partial y}\right)}_{M}dx + \underbrace{\left(\frac{\partial u}{\partial x}\right)}_{N}dy \quad \text{(is exact)}$$

its G.S is

$$v = \underset{(y \text{ is const.})}{\int M\,dx} + \int (\text{terms of } N \text{ not involving } x)\,dy + c$$

(ii) When v is given to find its conjugate harmonic fn u.

by chain rule

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

(is exact)

by using CR eqns.

$$du = \underbrace{\left(\frac{\partial v}{\partial y}\right)}_{M}dx + \underbrace{\left(-\frac{\partial v}{\partial x}\right)}_{N}dy \quad \text{(is exact)}$$

its G.S is

$$u = \underset{(y \text{ is const.})}{\int M\,dx} + \int (\text{terms of } N \text{ not involving } x)\,dy + c$$

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$$= 0 + 0 + 2|f'(z)|^2 + 2|f'(z)|^2$$

$$= 4|f'(z)|^2 \quad \because f'(z) = u_x + iv_x$$

4) If $f(z)$ is analytic fn of z then show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re f(z)|^2 = 2|f'(z)|^2$$

A) $Re f(z) = u$

$$\Rightarrow \frac{\partial^2}{\partial x^2}u^2 + \frac{\partial^2 u^2}{\partial y^2}$$

$$= 2u u_{xx} + 2(u_x)^2 + 2u u_{yy} + 2(u_y)^2$$

$$= 2u(u_{xx} + u_{yy}) + 2((u_x)^2 + (u_y)^2)$$

from C.R eqns $u_y = -v_x$

as $f(z)$ is analytic u, v are harmonic

$$\Rightarrow 2u(0) + 2((u_x)^2 + (-v_x)^2)$$

$$= 2|f'(z)|^2 \quad [\because f'(z) = u_x + iv_x]$$

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$(1+i)f(z) = (1+i)e^z + c$

$f(z) = e^z + \underbrace{\frac{c}{1+i}}_{= k}$

Q) Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ where $f(z) = u + iv$ is analytic fn.

A) $|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2 + v^2$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$

$\frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(u^2) + \frac{\partial^2}{\partial y^2}(v^2)$

$\frac{\partial}{\partial x}(2u\,u_x)$

$= (2u\,u_{xx} + 2u_x^2) + (2v\,v_{xx} + 2v_x^2) + (2u\,u_{yy} + 2(u_y)^2)$

$+ (2v\,v_{yy} + 2v_y^2)$

put $= 2u(u_{xx} + u_{yy}) + 2v(v_{xx} + v_{yy}) + 2(u_x^2 + v_x^2)$

$+ 2(u_y^2 + v_y^2)$

Since $f(z)$ is analytic $\therefore$ u, v are harmonic and u, v

satisfies laplace eq.

i.e $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

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2) Find analytic fn $f(z) = u + iv$ where $u - v = e^x(\cos y - \sin y)$

A) $f(z) = u + iv$

$if(z) = -v + iu$

$(1+i)f(z) = (u-v) + i(u+v)$

$\because f(z)$ is analytic

$\therefore (1+i)f(z)$ is also analytic.

$$\underbrace{(1+i)f(z)}_{F(z)} = \underbrace{(u-v)}_{U} + i\underbrace{(u+v)}_{V}$$

$U = u - v = e^x(\cos y - \sin y)$

$\Rightarrow U = e^x(\cos y - \sin y)$

$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$; $\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$

by Millne's method we have

$F'(z) = U_x - iU_y$

$= e^x(\cos y - \sin y) + i(e^x(\sin y + \cos y))$

replacing x by z & y by 0 we get

$F'(z) = e^z(1) + i(e^z)$

$= e^z(1+i)$

$F(z) = e^z(1+i) + C$

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Along path II.

$$f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}}$$

$$f'(z) = \lim_{\delta y \to 0} \left[\frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i\,\frac{v(x, y+\delta y) - v(x,y)}{i\delta y} \right]$$

$$f'(z) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{— (2)}$$

Since $f(z)$ is differentiable $\therefore$ the limit is unique along any path hence (1) = (2)

i.e

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

eq. real & img. parts.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(C-R eqns)

Sufficient conditions:-

Let $f(z) = u + iv$ be a single valued fn possessing the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in the region R and they are also continous

Let u, v satisfies C-R eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

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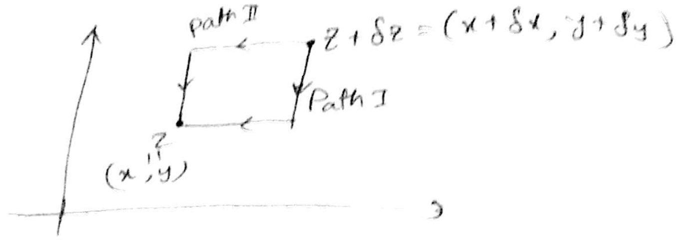
putting these in above equation

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{[u(x+\delta x, y+\delta y) + i\,v(x+\delta x, y+\delta y)] - (u, \dots}{\delta x + i\,\delta y}$$



along path I: $\delta y = 0,\ \delta x = 0$

$$f'(z) = \underset{\substack{\delta y \to 0 \\ \delta x \to 0}}{\text{Lt}}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \left[\frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i\,\frac{v(x+\delta x, y) - v(x,y)}{\delta x} \right]$$

$$\frac{\partial u}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{u(x+\delta x, y) - u(x,y)}{\delta x}$$

$$\frac{\partial u}{\partial y} = \underset{\delta y \to 0}{\text{Lt}} \frac{u(x, y+\delta y) - u(x,y)}{\delta y}$$

$$\Rightarrow \boxed{f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}} \quad (1)$$

Necessary and sufficient conditions for a fn to be analytic.

Statement:-

The necessary & sufficient condition for the ~~fcnt~~ fn $w = f(z) = u + iv$ to be analytic in a region 'R' are

i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continous fns of x and y in the region R

ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Proof

Necessary condition: Let $w = f(z) = u(x,y) + i\,v(x,y)$ be analytic in a region 'R' then $f'(z)$ exists and it is unique at every point of the region R.

Let $\delta x, \delta y$ be the increments in x and y respec.

then $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$

Now $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

$f(z+\delta z) = f((x+\delta x) + i(y+\delta y)) = u(x+\delta x, y+\delta y) +$
$i\,v(x+\delta x, y+\delta y$

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differentiating the above fns partially w.r.t r

we have $f'(re^{i\theta})e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$ - (2)

Diff. eq. (1) partially w.r.t θ we get

$$f'(re^{i\theta})\cancel{r}e^{i\theta} \times i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow ir(e^{i\theta}f'(re^{i\theta})) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

from (2)

$$ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

equating real and imaginary parts on both side

$$\boxed{-r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} \; ; \; r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \; ; \; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

CR eqns in polar co-ordinates

Note:- To find analytic fns $f(z)$ in polar co-ordinates when one of u, v is given

$$e^{i\theta}f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

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$$\frac{f(z+\delta x)-f(z)}{\delta x} \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

but $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial x}$ are exists and continuous

$\therefore$ $f'(z)$ is also exists

Hence the fn is differentiable i.e. $f(z)$ is

analytic.

Cauchy-Riemann equations in polar co-ordinates

Let $f(z) = u + iv$ be analytic fn then

$x = r\cos\theta$

$y = r\sin\theta$

$z = x + iy = r\cos\theta + ir\sin\theta = r(e^{i\theta})$

$z = re^{i\theta}$

then $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

$\Rightarrow f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ — (1)

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Now we have to show that the fn $f(z)$ is analytic i.e $f'(z)$ exists at every point of the region R

By tailor's theorem for fn of 2 variables we h

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + i v(x+\delta x, y+\delta y)$$

$$= \left(u(x,y) + \left(\delta x \frac{\partial u}{\partial x} + \delta y \frac{\partial u}{\partial y}\right) + \ldots\right)$$

$$+ i\left(v(x,y) + \left(\delta x \frac{\partial v}{\partial x} + \delta y \frac{\partial v}{\partial y}\right) + \ldots\right)$$

since δx & δy are very small positive real num

∴ neglecting higher order terms i.e $(\delta x)^2, (\delta x)^3$
$(\delta y)^2, (\delta y)^3$

in the above eq.

we have

$$f(z+\delta z) \simeq \cancel{u(x,y)} + \frac{\partial u}{\partial} \quad u(x,y) + i v(x,y)$$

$$\delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

by CR eqns $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\cancel{\Rightarrow f(z+\delta z) = u(x,y) + i v(x,y)}$$

$$\Rightarrow f(z+\delta z) \simeq f(z) + \delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \delta y\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right.$$

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5) $du = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta \quad \left(\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r} \right)$

$du = \left(\frac{1}{r} \frac{\partial u}{\partial \theta} \right) dr + \left(-r \frac{\partial u}{\partial r} \right) d\theta$ is exact.

6) $f'(re^{i\theta}) = \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right) e^{-i\theta}$

$= \left(\frac{\partial u}{\partial r} + i \left(-\frac{1}{r} \frac{\partial u}{\partial \theta} \right) \right) e^{-i\theta}$

r by z & θ by 0

$f'(z) = \left(\frac{\partial u}{\partial r} - \frac{i}{r} \frac{\partial u}{\partial \theta} \right) \Big|_{\substack{r = z \\ \theta = 0}}$

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1) ~~If~~ Find of $f(z) = u + iv$ be an analytic fn of z and if $u - v = (x-y)(x^2+4xy+y^2)$ then find $f(z)$

Hint: $f(z) = u + iv$

$if(z) = -v + iu$

$(1+i) f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

2) if $v = x^3 - 3xy^2$ find analytic fn.

3) P.T $u = e^{-x}[(x^2-y^2)\cos y + 2xy \sin y)$ is harmonic.

4) Find the conjugate harmonic fn whose real part is $u = e^{-x}(x\cos y + y \sin y)$

5) Find its conjugate harmonic fn.

$v = (r - \frac{1}{r})\sin\theta \;;\; r \neq 0$

6) Find analytic fn $f(z)$ where

$u = r^2\cos 2\theta - r\cos\theta + 2$

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$$f'(re^{i\theta}) = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)e^{-i\theta}$$

replacing r by z & θ by 0

we have $f'(z) = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\Big|_{(r=z,\,\theta=0)}$

when u is given

$$f'(z) = \left(\frac{\partial u}{\partial r} - i\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\Big|_{\substack{r=z\\ \theta=0}}$$

when v is given

$$f'(z) = \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial r}\right)\Big|_{\substack{r=z\\ \theta=0}}$$

1) Find analytic fn $f(z)$ whose real part is

$u(r,\theta) = -r^3\sin 3\theta$

*) $\frac{\partial u}{\partial r} = -3r^2\sin 3\theta \qquad \frac{\partial u}{\partial \theta} = -3r^3\cos 3\theta$

$$f'(z) = \left(-3r^2\sin 3\theta + i\times\frac{1}{\cancel{r}}3r^{\cancel{3}\,2}\cos 3\theta\right)$$

$r = z$
$\theta = 0$

$\Rightarrow f'(z) = 0 + i3z^2$

$f(z) = iz^3 + C$

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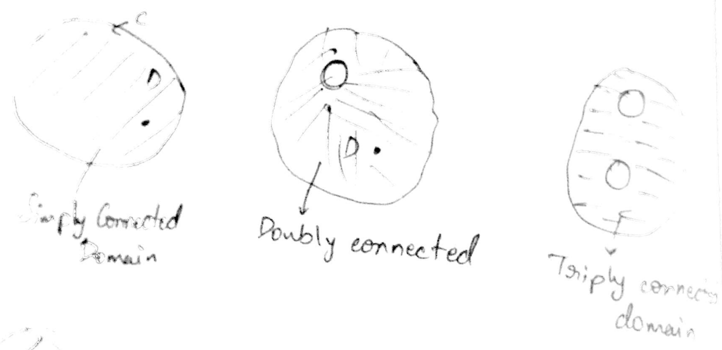
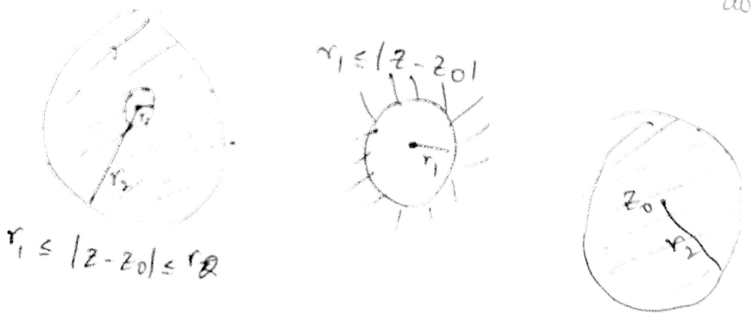
eg: the boundaries of circles, squares, triangles are closed contours.

Simply connected domain: A domain D in which every closed path contains points of the same domain D is called simply connected domain.

Multiply Connected Domain: A Domain which is not simply connected is called Multiply connected domain.





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Evaluate the integral [illegible] from $(1,1)$ to $(2,4)$ [illegible] parabola $y = x^2$ [illegible]

$z = x + iy$

$\bar{z} = x - iy$

$dz = dx + i\,dy$

$\int_C (z - \bar{z})\,dz = \int_C 2iy\,(dx + i\,dy)$

$y = x^2$

$dy = 2x\,dx \qquad dx = \frac{1}{2\sqrt{y}}\,dy$

$= \int_1^4 2iy\left(\frac{1}{2\sqrt{y}}\,dy + i\,dy\right)$

$i\int_1^4 \sqrt{y}\,dy - \int 2y\,dy$

$= i\,\frac{y^{3/2}}{3/2}\Big|_1^4 - y^2\Big|_1^4$

$= \left(\frac{16}{3} - \frac{2}{3}\right)i - 15$

$= -15 + \frac{14i}{3}$

3) <u>Contour</u> :- Contour is a continous chain of smoo[th]

arcs.

<u>Closed Contour</u> :- (Simple closed Curve)

A Closed curve without points of ~~self in~~

self intersection is called closed countou[r]

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Ex:-
1) Evaluate the line integral

$\int_c z^2 dz$ where c is a st. line in the x-y plane from $z=0$ to $z=1+2i$

Soln:-

$$z^2 dz = (x+iy)^2 (dx + i dy) \quad \text{--- (1)}$$

st. line c: from $z=0$ to $z=1+2i$

$$(0,0) \qquad (1,2)$$

$$y = 2x$$

$$y = 2x$$
$$dy = 2dx$$

put these values in equation (1) we have

$$z^2 dz = (x+i2x)^2 (dx + i2dx)$$
$$= x^2 (1+2i)^3 dx$$

$(0,0)$ $(1,2)$ $x: 0$ to 1

$$\int z^2 dz = \int_0^1 (1+2i)^3 (x^2)\, dx$$

$$= (1+2i)^3 \frac{x^3}{3}\Big|_0^1 = \frac{(1+2i)^3}{3}$$

(or)

$$\int_c z^2 dz = \int_0^{1+2i} z^2 dz = \frac{z^3}{3}\Big|_0^{1+2i} = \frac{(1+2i)^3}{3}$$

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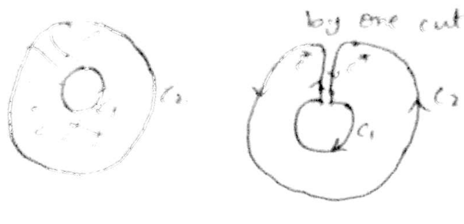
Cauchy's Integral theorem for multiply connected Domains:

A multiply connected domain can be made simply connected domain in particular, a doubly connected domain can be made simply connected domain "by one cut".



Let $f(z)$ be analytic in a doubly connected domain then apply cauchy's theorem for a simply connected domain along the oriented boundary.

Now apply cauchy's theorem along " " B which is combination of C_1, C^*, C_2, C^* we have

$$\oint_B f(z)\,dz = 0$$

$$\Rightarrow \int_{C_2} f(z)\,dz + \int_{C^*} f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{C^*} f(z)\,dz = 0$$

Since C^*, C^* are in opposite directions therefore one is positive and another one is -ve. Hence in the above equation. The second and 4th integral gets cancelled.

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i.e $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$I_1 = \oint_C u\,dx - v\,dy$

$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$ (By Green's theorem)

by CR eq's $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$\Rightarrow I_1 = 0$

$I_2 = i\oint_C v\,dx + u\,dy$

$= i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$ (By Green's theorem)

by CR eqns $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow I_2 = 0$

$\because I = I_1 + I_2$

$\therefore I = 0$

$$\boxed{\oint_C f(z)\,dz = 0}$$

Cauchy's integral theorem

(Cauchy's theorem)

Statement: Let $f(z)$ be analytic function in a simply connected domain D then $\oint_C f(z)\,dz = 0$ where C is closed curve lying entirely within the D.

Proof: Let $f(z) = u + iv$ be a single valued function which is continous within and on simple closed curve 'C' and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continous.

Now

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (udx - vdy) + i(vdx + udy)$$

$$= \underbrace{\oint_C (udx - vdy)}_{I_1} + \underbrace{i\oint (vdx + udy)}_{I_2}$$

Since $f(z)$ is analytic

$\therefore$ u, v satisfies C-R eqns.

Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,d$$

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Hence as $r_0 \to 0$ we have

$$\int_{C_1} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta$$

$$= i\,f(a)\int_0^{2\pi} d\theta$$

$$\boxed{\int_{C_1} \frac{f(z)}{z-a} = 2\pi i\, f(a)}$$

Putting this value in above eq. we have

$$\boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)}$$

Generalized Cauchy's integral formula:- (Cauchy's integral formula for derivatives)

Statement:- Let $f(z)$ be an analytic function everywhere within and on a simple closed curve C enclosing a point $z=a$ then

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = 2\pi i\,\frac{f^n(a)}{\lfloor n}$$

where $n = 1, 2, 3, \ldots, 0$

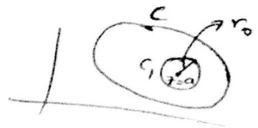
Proof:- By the cauchy's integral formula we have

$$\int_C \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a)$$

Differentiating the above equation partially w.r.t a withing the integral sign we have

Now $f(z)$ is analytic in doubly connected domain 38
i.e in the shaded portion.

By Cauchy's theorem for multiply connected domains we have $\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$ since $\frac{f(z)}{z-a}$ is analytic in shaded portion.



to evaluate $\int_{C_1} \frac{f(z)}{(z-a)}\,dz$

The eq. of circle in polar co-ordinates is

$$(z-z_0) = re^{i\theta}$$

On the circle C_1

$$\text{Let } z-a = r_0 e^{i\theta}$$

$$\text{then } z = a + r_0 e^{i\theta}$$

$$dz = r_0 e^{i\theta} i\,d\theta$$

Limits : 0 to 2π

putting these values in the above integral we have

$$\int_{C_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+r_0e^{i\theta})}{\cancel{r_0 e^{i\theta}}}\,\cancel{r_0 e^{i\theta}}\, i\,d\theta$$

$$= i\int_0^{2\pi} f(a+r_0e^{i\theta})\,d\theta$$

As $r_0 \to 0$ the circle C_1 shrinks to the point a

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$\Rightarrow \oint \int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0$

$\Rightarrow \int_{C_2} f(z)dz = -\int_{C_1} f(z)dz$

By neglecting - sign the direction along the cu... C_1 can be made into anticlockwise direction.

i.e $\int_{C_2} f(z)dz = \int_{C_1} f(z)dz$

anticlockwise

Cauchy's Integral formula :- Let

Statement :- Let $f(z)$ be an analytic function everywhere within and on a simple closed curve C enclosing a point ~~A~~ $z=a$, then

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

where ~~C is~~ the integral is taken in +ve sense along 'C' i.e anticlockwise direction.

Proof :- Let $f(z)$ be analytic within the closed contour C and let $z=a$ be a point within C. Draw a circle with centre at 'a' and radius r_0.

Now i.e By we ir

3) Evaluate $\oint_C \frac{z^2+1}{z^2+3z+2}\,dz$ when c is $|z+i|=2$

A) The fn is not analytic at

$$z^2+3z+2=0$$
$$(z+2)(z+1)=0$$
$$z=-2,-1$$

put $z=-2$

$|-2+i| = \sqrt{5} > 2$ outside c

put $z=-1$

$|-1+i| = \sqrt{2} < 2$ inside c

$$\Rightarrow \int_C \frac{z^2+1}{z^2+3z+2}\,dz = \oint_C \frac{z^2+1/z+2}{(z+1)}\,dz = 2\pi i \left.\frac{z^2+1}{z+2}\right|_{z=-1}$$

$$= 2\pi i \frac{2}{1} = 4\pi i$$

4) $\int_C \frac{e^{2z}}{(z-2)^3}\,dz$; c : $|z+i\cancel{+}i| = 16$

A) Not analytic at $z=2$

put $z=2$ in c

$|3+i| = \sqrt{10} < 16$

point is inside

we know

$$\int \frac{f(z)}{(z-a)^{n+1}} = 2\pi i \frac{f^n(a)}{n!}$$

$$\Rightarrow \int \frac{e^{2z}}{(z-2)^{\cancel{2}+1}} = \cancel{2}\pi i \frac{(e^{2\cancel{z}})''}{\cancel{2!}} = 4\pi i e^{\cancel{2z}4}$$

$a=2$ at $z=2$

2) Evaluate the integral $\oint_c \tan z \int \frac{\sin 2z}{z^2+9} dz$ where 3) E.

$c = |z-3i| = 1$

A) The fn $\frac{\sin 2z}{z^2+9}$ is not analytic at the A)

Points $\rightarrow z^2+9=0$

$z = \pm 3i$

$c : |z-3i| = 1$

put $z=3i \Rightarrow c: 0<1$

$\Rightarrow z=3i$ is within c

$\hookrightarrow$ factor is $(z-3i)$

put $z=-3i \Rightarrow |-3i-3i| = |-6i| = 6 > 1$

$\therefore z=-3i$ is ~~outt~~side c

$\hookrightarrow$ factor is $z+3i$

Now, $\frac{\sin 2z}{z^2+9} = \frac{\sin 2z}{(z-3i)(z+3i)} = \frac{\sin 2z / z+3i}{z-3i} = \frac{f(z)}{z-a}$

$f(z) = \frac{\sin 2z}{z+3i}$

$a = 3i$

by Cauchy's integral formul

$$\oint_c \frac{\sin 2z}{z^2+9} = \oint_c \frac{\sin 2z / z+3i}{z-3i}$$

$$= 2\pi i f(a)$$

$$= 2\pi i \left. \frac{\sin 2z}{z+3i} \right|_{z=3i}$$

$$= \frac{\pi}{3} \sin 6i$$

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$$\int_c \frac{-f(z)}{(z-a)^2}(-1)\,dz = 2\pi i f'(a)$$

$$\Rightarrow \int_c \frac{f(z)}{(z-a)^2}\,dz = 2\pi i f'(a)$$

again differentiating w.r.t a we have

$$\int_c f(z)\left(\frac{-2}{(z-a)^3}(-1)\right) = 2\pi i f''(a)$$

$$\Rightarrow \oint_c \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i\, f''(a)}{\lfloor 2}$$

lly differentiating n times we have $\oint_c \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i f^{n}(a)}{n!}$

1) Evaluate the integral $\int_c \frac{z^2+2z}{(z-1)}\,dz$ where C is $|z-1| = 2$ in counter colock wise direction.

A) The function $\frac{z^2+2z}{(z-1)}$ is not analytic at $z=1$

put $z=1$ in C $|z-1|=2$ we have

$|1-1| = 0 < 2 \therefore z=1$ is within C

Now $$\frac{z^2+2z}{z-1} = \frac{f(z)}{z-a}$$

$$f(z) = z^2+2z$$

$$a = 1$$

Then by Cauchy's integral formula we have

$$\int_c \frac{z^2+2z}{z-1}\,dz = \oint_c \frac{dz}{z-a}\,dz = 2\pi i f(a)$$

$$= 2\pi i (z^2+2z)\Big|_{z=1}$$

$$= 6\pi i$$

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$$\Rightarrow -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\right]+\left[1+z+z^2+\dots\right]$$

(ii) $1<|z|<2$

$1<|z|$ & $|z|<2$

$\frac{1}{|z|}<1$ $\quad \frac{|z|}{2}<1$

$\frac{1}{|z|}-1<0 \Rightarrow 1-\frac{1}{|z|}>0$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1}$$

$$=\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

(iii) $|z|>2 \Rightarrow |z|>1$ & $|z|>2$

$\Rightarrow 1>\frac{1}{|z|}$ $\quad 1>\frac{2}{|z|}$

$\Rightarrow 1-\frac{1}{|z|}>0$ $\quad 1-\frac{2}{|z|}>0$

$$f(z)=\frac{1}{z-2}-\frac{1}{z-1}$$

$$=\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$\Rightarrow \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

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Laurent's Series

If $f(z)$ is analytic inside and boundary of ring shaped region R bounded by two concentric circles C_1 and C_2 of radii R_1 and R_2 ($R_1 > R_2$) respec. having center at A then for all z in R we have

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} \dots$$

Q) Expand $f(z) = \frac{1}{z^2-3z+2}$ in the region

i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $0<\dots$

A)

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = A(z-2)+B(z-1)$$

$$A=-1 \ ; \ B=1$$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{z-2}$$

$$|z|<1 \Rightarrow \left|\frac{z}{2}\right| < \frac{1}{2} < 1$$

$$\Rightarrow |z|<1 \ \& \ \frac{|z|}{2}<1$$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{-1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{(-1)(1-z)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

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Unit - III

Series of Complex Numbers

A series $(a_1+ib_1)+(a_2+ib_2)+\dots+(a_n+ib_n)\dots$ where $a_1, a_2, \dots, a_n$ are real numbers is called a series of terms and can be expressed as $\sum_{n=1}^{\infty} a_n + i\sum_{n=1}^{\infty} b_n$

A series of the form $a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots = \sum_{n=0}^{\infty} a_n(z-a)^n$ is called power series in $z-a$

Taylor series :-

Statement:

If $f(z)$ is analytic inside a circle 'C' with cent at 'a' then $\forall z$ inside C, we have

$$f(z) = f(k) + (z-a)f'(a) + \frac{(z-a)^2}{2L} f''(a) + \dots \quad — (1)$$

Note:

put $a=0$ in eq. (1) we get

$$f(z) = f(K) + z f'(0) + \frac{z^2 f''(0)}{2} \dots$$

this is known as Maclauran Series.

[illegible]

8) Expand $\cos z$ in a taylor series about $z = \pi/4$

Solution: $f(z) = \cos z$, $a = \pi/4$, $f(a) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f'(z) = -\sin z \Rightarrow f'(\pi/4) = -1/\sqrt{2}$

$f''(z) = -\cos z \Rightarrow f''(\pi/4) = -1/\sqrt{2}$

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

$$\cos z = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\left(-1/\sqrt{2}\right) + \left(z - \frac{\pi}{4}\right)^2 \left(\frac{-1}{\sqrt{2}}\right)^2 + \cdots$$

9) Expand the function $\frac{\sin z}{z - \pi}$ about $z = \pi$

A)

$z - \pi = t$

$$\frac{\sin z}{z-\pi} = \frac{\sin(\pi + t)}{t} = -\frac{\sin t}{t} = -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!}\right]$$

$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \cdots$$

$$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!}$$

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(iv) $0<|z-1|<1 \Rightarrow 0<|z-1|$ & $|z-1|<1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= \frac{1}{-1(1-(z-1))} - \frac{1}{z-1}$$

$$= (-1)\left[1-(z-1)\right]^{-1} - (1-z)^{-1}$$

7) Expand $\cos z$ in a taylor series about $z = \pi/4$

Solution:- $f(z) = \cos z$, $a = \pi/4$ $\quad f(a) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f'(z) = -\sin z \Rightarrow f'(\pi/4) = -1/\sqrt{2}$

$f''(z) = -\cos z \Rightarrow f''(\pi/4) = -1/\sqrt{2}$

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

$$\cos z = \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\left(-1/\sqrt{2}\right) + \left(z-\frac{\pi}{4}\right)^2 \left(\frac{-1}{\sqrt{2}}\right)^2 + \dots$$

8) Expand the function $\frac{\sin z}{z-\pi}$ about $z = \pi$

A)

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$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!}$$

$$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!}$$

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(iv)

$0<|z-1|<1 \Rightarrow 0<|z-1|$ & $|z-1|<1$

$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{|z-1|-1} - \frac{1}{z-1}$

$= \frac{-1}{(1-|z-1|)} - \frac{1}{z-1}$

$= (-1)\left[1-|z-1|\right]^{-1} + (1-z)^{-1}$

2) Expand $\cos z$ in a taylor series about $z = \pi/4$

Solution:-

$f(z) = \cos z$, $a = \pi/4$ $f(a) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f'(z) = -\sin z \Rightarrow f'(\pi/4) = -1/\sqrt{2}$

$f''(z) = -\cos z \Rightarrow f''(\pi/4) = 1/\sqrt{2}$

$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$

$\cos z = \frac{1}{\sqrt{2}} + (z-\frac{\pi}{4})(-1/\sqrt{2}) + (z-\frac{\pi}{4})^2 \left(\frac{-1}{\sqrt{2}}\right)^2 + \dots$

2) Expand the function $\frac{\sin z}{z-\pi}$ about $z = \pi$

A)

$z - \pi = t$

$\frac{\sin z}{z-\pi} = \frac{\sin(\pi+t)}{t} = -\frac{\sin t}{t} = -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right]$

$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \dots$

$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!}$

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$$= 1 - \frac{8}{z}\left[1 + \left(\frac{-3}{z}\right) + \left(\frac{-3}{z}\right)^2 + \dots\right] + \frac{3}{z}\left(1 + \left(\frac{-2}{z}\right) + \left(\frac{-2}{z}\right)^2 + \dots\right)\Big]$$

6) Expand $f(z) = \dfrac{(z-2)(z+3)}{(z+1)(z+4)}$ in the region (i) $|z| < 1$ (ii) $1 < |z| < 4$ (iii) $|z| > 4$

Definations :

i) Zero of an analytic function :- Zero of a Analytic function $f(z)$ is a value of z such that $f(z) = 0$ particularly a point 'a' is called a zero of an analytic function $f(z)$ if $f(a) = 0$.

2) Zero of m^{th} order :-

eg :-

i) $f(z) = (z-1)^3$ then $z = 1$ is a zero of ~~an~~ order 3 of $f(z)$

ii) $f(z) = \dfrac{1}{1-z}$ then $z = \infty$ is a simple zero of $f(z)$

iii) $f(z) = \sin z$ then $z = 0, \pm\pi, \pm 2\pi \dots$ are simple zero's of $f(z)$

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3) Expand $f(z) = \frac{z}{(z+1)(z+2)}$ about $z = 2$

A) $f(z) = \frac{t-2}{t(t-1)}$ $\qquad \frac{t-2}{t} \cdot \frac{1}{(1)(1-t)} \qquad \frac{t-2}{t}(1-t)^{-1}$

$= \frac{2-t}{t}(1-t)^{-1}$

$= \frac{2-t}{t}\left[1+t+t^2+t^3+\dots\right]$

$= (2-t)\left[\frac{1}{t}+1+t+t^2+\dots\right]$

4) Expand by laurent's series (i) $\frac{1}{z-2}$ for $|z|>2$

A) $f(z) = \frac{1}{z-2}$ $\qquad |z|>2 \Rightarrow 1 > \frac{2}{|z|}$

$= \frac{1}{z\left(1-\frac{2}{|z|}\right)} = \frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\dots\right)$

5) $f(z) = \frac{1}{z^2-4z+3}$ for $1<|z|<3$

$1<|z|$ & $|z|<3$

$1 < \frac{3}{|z|}$

$\frac{1}{(z-3)(z-1)} = A(z-1)+B(z-3)$

$A = \frac{1}{2}\,;\ B = -\frac{1}{2}$

$\therefore f(z) = \frac{1}{2(z-3)} - \frac{1}{2(z-1)}$

$= \frac{1}{2}\left[\frac{1}{z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\dots\right] + \left(1+z+z^2+\dots\right)\right]$

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(iii) $\frac{1}{z(z-1)(z-2)} = A(z-1)(z-2) + B(z)(z-2) + C(z)(z-1)$

$A = \frac{1}{2}$; $B = -1$; $C = \frac{1}{2}$

for $|z| > 2$

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$|z| > 2 \Rightarrow |z| > 0$ $\quad |z| > 1 \Rightarrow 1 > \frac{2}{|z|}$

$1 > \frac{1}{z}$

$$f(z) = \frac{1}{2z} - \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2z(1-\frac{2}{z})}$$

$$\Rightarrow f(z) = \frac{1}{2z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{2z}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right)$$

(iv) $\frac{z^2-1}{z^2+5z+6} = 1 + \frac{(-5z-7)}{z^2+5z+6}$

$z^2+5z+6)\; z^2-1\,(1$
$\quad -z^2+5z+6$
$\quad -5z-7$

$$f(z) = 1 + \frac{(-5z)-7}{z^2+5z+6}$$

$$\frac{-5z-7}{(z+3)(z+2)} = A(z+2) + B(z+3)$$

$B = 3 \quad A = -8$

$$f(z) = 1 + \frac{-8}{z-(-3)} + \frac{3}{z+2}$$

$|z| > 3 \Rightarrow |z| > 2$

$1 > \frac{3}{|z|}$ $\quad 1 > \frac{2}{|z|}$

$$f(z) = 1 - \frac{8}{z\left(1-\left(\frac{-3}{z}\right)\right)} + \frac{3}{z\left(1-\left(\frac{-2}{z}\right)\right)}$$

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(iii) Singular points : A singular points (or singularity) of a function $f(z)$ is the point at which the function $f(z)$ ceases to be analytic.

* Different types of singularities:

(1) Isolated singularity: A point $z=a$ is called an isolated [illegible] of an analytic function

if (i) $f(z)$ is not analytic at point $z=a$

(ii) $f(z)$ is analytic in the deleted [illegible] of $z=a$ i.e their exist a neighbourhood [illegible] point $z=a$ which contains no other singularity.

eg: a) if $f(z) = \frac{e^z}{z^2+1}$ then $z = \pm i$ are [illegible] isolated singular points of $f(z)$

b) if $f(z) = \frac{e}{\sin z}$ then $z = \pm n\pi$ are [illegible] no. of isolated singular points [illegible]

(iv) Poles of an analytic function : If $z=a$ is an isolated singular point of an analytic function $f(z)$ then $f(z)$ can be expanded in laurent series about the point $z=a$ i.e ~~f(z)~~

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$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

The series of -ve integral powers of $z-a$ namely $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is known as the principle part of the laurentz series of $f(z)$.

If the principle part contains a finite no. of terms say m (i.e $b_n = 0$, $\forall n$ such that $n > m$) then the singular point $z = a$ is called a pole of order m of $f(z)$.

Simple pole is a pole of order 1.

eg:- if $f(z) = \frac{z^2}{(z-1)(z+2)^2}$ then $z = 1$ is a simple pole

and $z = -2$ is a pole of order 2.

- Essential Singularity :- If the principle part of $f(z)$ contains an infinite no. of terms i.e the series $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ contains " " " " then the point $z = a$ is called essential singularity of $f(z)$

eg:- $z = 0$ is an " " " of $e^{1/z}$

since the principle part of $e^{1/z}$ contains ∞ no. of terms containing -ve powers of $z-0$

$= 1 - \frac{8}{z}\left[1+\left(-\frac{3}{z}\right)+\left(-\frac{3}{z}\right)^2+\dots\right)\Big] + \frac{3}{z}\left(1+\left(\frac{-2}{z}\right)+\left(\frac{-2}{z}\right)^2+\dots\right)\Big]$ 43

Q) Expand $f(z) = \frac{(z-2)(z+3)}{(z+1)(z+4)}$ in the region (i) $|z|<1$

(ii) $1<|z|<4$ (iii) $|z|>4$

Definations :

i) Zero of an analytic function :- Zero of a Analytic function $f(z)$ is a value of z such that $f(z)=0$ particularly a point 'a' is called a zero of an analytic function $f(z)$ if $f(a)=0$.

ii) Zero of m^{th} order :-

eg :-

i) $f(z) = (z-1)^3$ then $z=1$ is a zero of ~~an~~ order 3 of $f(z)$

ii) $f(z) = \frac{1}{1-z}$ then $z=\infty$ is a simple zero of $f(z)$

iii) $f(z) = \sin z$ then $z = 0, \pm\pi, \pm 2\pi \dots$ are simple zero's of $f(z)$

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$\therefore \int f(z)\,dz = 2\pi i\,[R_1 + R_2 + \dots + R_n]$ 52

Note:-

(i) if $z=a$ is a simple pole then residue of $f(z)$ at $z=a$ is $\lim_{z\to a}\left[(z-a)f(z)\right]$

(ii) if $z=a$ is a pole of order m then residue of $f(z)$ at $z=a$ is $\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\lim_{z\to a}\left[(z-a)^m f(z)\right]$

1) Find the residues of the following function at ~~each~~ each of the poles.

i) $\frac{3z}{z^2+2z+5}$

A) Let $f(z) = \frac{3z}{z^2+2z+5} = \frac{3z}{(z+1)^2+4} = \frac{3z}{(z+1)^2-(2i)^2}$

$\Rightarrow f(z) = \frac{3z}{(z+1+2i)(z+1-2i)}$

the poles of $f(z)$ are

$z+1+2i=0$ $\qquad$ $z+1-2i=0$

$z = -(1+2i)$ $\qquad$ $z = -1+2i$

which are simple poles

$\therefore$ Residue of $f(z)$ at $z=-(1+2i)$ is

$\lim_{z\to -1-2i} (z+1+2i)\frac{3z}{(z+1+2i)(z+1-2i)}$

$R_1 = \frac{3(-1-2i)}{-4i} = \frac{3+6i}{4i} = \frac{-3i}{4}(1+2i)$

2)

+)

3) $\overline{\quad}$ $(z-$

A)

R

coefficient b_1 as given by $b_1 = \frac{1}{2\pi i} \int_C f(z) dz$

$\Rightarrow \int_C f(z) dz = 2\pi i b_1 = 2\pi i [\text{Residue of } f(z)]_{z=a}$

Cauchy Residue theorem: If $f(z)$ is analytic within and on a closed curve C ~~exp~~ except at a finite no. of poles $z_1, z_2, z_3, \ldots, z_n$ within C & $R_1, R_2, \ldots, R_n$ be the residues of $f(z)$ at these poles then $\int_C f(z) dz = 2\pi i [R_1 + R_2 + \ldots + R_n]$

Proof: Let $\gamma_1, \gamma_2, \ldots \gamma_n$ be the circles with centers at $z_1, z_2, \ldots, z_n$ respectively, and their radii so small that they lie entirely with in closed curve C ~~and~~ and donot overlap Now, $f(z)$ is analytic within the region enclosed by the curve C b/w these circles. $\therefore$ By cauchy's theorem for multiply connected regions

$$\int_C f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \ldots + \int_{\gamma_n} f(z) dz$$

By defination we have $\frac{1}{2\pi i} \int_{\gamma_j} f(z) dz = \text{Res}[f; z = z_j]$
$= R_j$ for $j = 1, 2, \ldots, n$]

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(vi) **Removable Singularity**: If the principle part of $f(z)$ contains no terms i.e if $b_n = 0 \; \forall n$, the singularity $z = a$ is called removable singularity of $f(z)$. In this case $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$

eg: if $f(z) = \frac{1-\cos z}{z}$ when $z = 0$ is a removable singularity.

(vii) **Singularities at infinity**:- Taking $z = \frac{1}{t}$ in $f(z)$ we obtain $f(\frac{1}{t}) = F(t)$ then the nature of the singularity at $z = \infty$ is defined to be the same as that of $F(t)$ at $t = 0$

eg:- (a) $f(z) = z^3$ has a pole of order 3 at $z = \infty$

since $f(\frac{1}{t}) = \frac{1}{t^3}$ has a pole of " " " $t = 0$

(b) $f(z) = e^z$ has an essential singularity at $z = \infty$

since $f(\frac{1}{t}) = e^{1/t}$ has an " " at $t = 0$

iii) **Residues**:- The co-efficient of $\frac{1}{z-a}$ in the expansion of $f(z)$ about the isolated singularity $z = a$ is called the residue of $f(z)$ at that point. Thus the residue of $f(z)$ at $z = a$ is ... from Laurentz series we know that the

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10) Evaluate $\int_C \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)}$, $|z| = 3$

A)

poles 1, 2

$R_1 = \lim_{z \to 1} \frac{\cos \pi z^2 + \sin \pi z^2}{z-2} = \frac{-1}{-1} = 1$

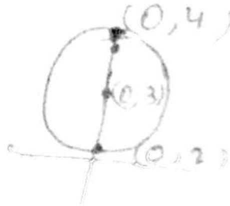
$R_2 = \lim_{z \to 2} \frac{\cos \pi z^2 + \sin \pi z^2}{z-1} = 1$

$\oint_C f(z) = 2\pi i (2) = 4\pi i$

11) $\int_C \frac{\sin z}{(z-1)^2 (z^2+9)} dz$; $C : |z - 3i| = 1$

A) $|x + iy - 3i| = 1 \Rightarrow |(x-0) + i(y-3)| = 1$

$(x-0)^2 + (y-3)^2 = 1$



poles 1, 3i, -3i

outside the circle

$R_1 = \lim_{z \to 3i} (z-3i) \frac{\sin z}{(z-1)^2 (z-3i)(z+3i)}$

$= \frac{\sin 3i}{6i(3i-1)^2}$

4) Determine the poles of the fn $f(z) = \frac{z^2}{z^4-1}$ & Residue at each pole. (Ans: $\frac{1}{4}, -\frac{1}{4}, \frac{i}{4}, -\frac{i}{4}$)

5) $\frac{2z+1}{z^2-z-2}$ [Ans: $\frac{1}{3}, \frac{5}{3}$]

6) $\frac{z+1}{z^2(z-2)}$ [Ans: $-\frac{3}{4}, \frac{3}{4}$]

7) $\frac{z^2}{(z-1)(z-2)^2}$ [Ans: 1, 0]

8) $\frac{e^z}{z^2+\pi^2}$ (Ans: $\frac{i}{2\pi}, -\frac{i}{2\pi}$)

9) Evaluate $\oint_C \frac{3z+4}{z(z-1)(z-2)}\, dz$ where C is a circle $|z| = \frac{3}{2}$.

A) The poles of $f(z)$ are given as

$z = 0, 1, 2$ which are simple

but given that $|z| = \frac{3}{2}$

$-\frac{3}{2} \le z \le \frac{3}{2}$

The poles $z = 0$ & 1 lies inside the circle and 2 lies outside C

R_1 = Residue of $f(z)$ at $z = 0$

$= \lim_{z \to 0} z \frac{3z+4}{z(z-1)(z-2)} = 2$

$R_2 = \lim_{z \to 1} \frac{3z+4}{z(z-2)} = -7$

$\oint_C \frac{3z+4}{z(z-1)(z-2)}\, dz = 2\pi i\,[R_1 + R_2] = 2i\pi(-5)$

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Residue of $z = -1+2i$ at $f(z)$

$$\lim_{z \to -1+2i} \cancel{(z+1-2i)} \frac{3z}{\cancel{(z+1-2i)}(z+1+2i)}$$

$$R_2 = \frac{3}{4}(2+i)$$

2) $\dfrac{4z-3}{z(z-1)(z-2)}$

A) poles are 0, 1, 2

$$R_1 = \lim_{z \to 0} \cancel{z} \frac{4z-3}{\cancel{z}(z-1)(z-2)} = \frac{-3}{2}$$

$$R_2 = \lim_{z \to 1} \frac{4z-3}{z(z-2)} = \frac{1}{-1} = -1$$

$$R_3 = \lim_{z \to 2} \frac{4z-3}{z(z-1)} = \frac{5}{2}$$

3) $\dfrac{z^2}{(z-1)^2(z+2)}$

A) poles 1, -2

$$R_1 = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} \frac{z^2}{(z+2)}$$

=

and c is the unit circle $|z| = 1$

Hence by residue theorem

$$\oint_c \phi(z)\,dz = 2\pi i \sum R_c$$

where $\sum R_c$ is the sum of the residues of ,

as it holds inside c.

c is the unit circle

poles of $\phi(z)$ are

$bz^2 + 2az$

$b(z^2$

b

Problems :-

Evaluate :-

① $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \quad a > b > 0$

A) We have $z = e^{i\theta}$

$$\Rightarrow \frac{dz}{iz} = d\theta$$

also $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Substitute these results in the given problem we get,

$$\oint_c \frac{dz/iz}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} = \frac{2}{i} \oint \frac{dz}{bz^2 + 2az + b}$$

$$= \cancel{\frac{2}{i}} \oint_c \phi(z)\,dz$$

where $\phi(z) = \frac{2}{i} \times \frac{1}{bz^2 + 2az + b}$

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Intt'Applications of Residue theorem

a) **TYPE ①** :- Integration round the unit circle :-

We consider the integral of the type

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

consider $z = e^{i\theta}$

$$dz = e^{i\theta} \cdot i \cdot d\theta$$

$$\Rightarrow \frac{dz}{iz} = d\theta$$

Also $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \frac{1}{2}\left(z + \frac{1}{z} \right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta} \right)$$

$$= \frac{1}{2i}\left(z - \frac{1}{z} \right)$$

Substitute these results in above we get,

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \frac{1}{i} \oint_C f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right] \frac{dz}{z}$$

$$= \oint_C \phi(z)\, dz$$

where $\phi(z) = \dfrac{1}{iz} f\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]$

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12) $\int_C \frac{e^z}{z^2+1}\,dz$, $|z|=2$ [Ans $2\pi i \sin 1$]

13) $f(z) = \frac{\sin z}{z - \cos z}$; $|z|=2$ [0]

14) $\oint_C \frac{e^z}{(z+1)^2}\,dz$, $|z-3|=3$ [0]

15) $\oint_C \frac{2z-1}{z(z+1)(z-3)}\,dz$, $|z|=2$ $\left[-\frac{5\pi i}{6}\right]$

16) $\oint \frac{z-3}{z^2+2z+5}\,dz$ (i) $|z|=1$ [0] (ii) $|z+1-i|=2$ $\downarrow$ $(\pi(i-2))$

(iii) $|z+1+i|=2$ $\downarrow$ $\pi(i+2)$

Inte
a) TYP
W
2π
∫
0
C
Also
1)
A)
Susbti

$$R = \lim_{z \to \alpha} \frac{2}{bi} (z-\alpha) \times \frac{1}{(z-\alpha)(z-\beta)} \qquad \alpha = -1+\sqrt{\ }$$

$$\beta = -1$$

$$R = \frac{2}{bi(\alpha-\beta)} = \frac{1}{i\sqrt{a^2-b^2}}$$

$$|\alpha| < 1$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = 2\pi i (R)$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

(2)

4)

$$\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} \quad (a^2<1)$$

we have $z = e^{i\theta}$

$$\frac{dz}{iz} = d\theta$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

substitute

$$\oint_C \frac{dz/iz}{1+\frac{a}{2}\left(z+\frac{1}{z}\right)} = \oint_C \frac{dz/iz}{\frac{2z+az^2+a}{2z}}$$

$$= \frac{1}{2i}\oint_C \frac{dz}{2z+a} = \frac{2}{ai}\oint_C \frac{dz}{z^2+\frac{2z}{a}+1}$$

$\phi(z)$ is $= \frac{2}{ai}$

Residue of $R = z^2 + \frac{2z}{a} + 1 = 0$

$$R = \frac{-\frac{2}{a} \pm \sqrt{\frac{4}{a^2}-4}}{2} = \frac{-1 \pm \sqrt{1-a^2}}{a}$$

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$|z| = 1$

Residues of

C is the unit circle $|z| = 1$ 59

poles of $\phi(z)$ are given by

$$bz^2 + 2az + b = 0$$

$$b\left[z^2 + \frac{2a}{b}z + 1\right] = 0$$

$$b \neq 0 \Rightarrow z^2 + \frac{2a}{b}z + 1 = 0$$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2} - 4}}{2}$$

$$= \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$a = \sqrt{a^2}$

Let $\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$

$\beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$

$\Rightarrow |\alpha| < 1$ & $|\alpha\beta| = 1$ & $|\beta| > 1$

since $a > 0$ $b > 0$

but $|z| = 1$

$\Rightarrow$ $|\alpha| < 1$ lies inside C

and $|\beta| > 1$ lies outside C

$\therefore$ Residue of $\phi(z)$ at $z = \alpha$ is given by

$$R = \lim_{z \to \alpha} (z - \alpha)\, \phi(z)$$

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and C is the unit circle $|z| = 1$

Hence by residue theorem

$$\oint_C \phi(z)\,dz = 2\pi i \sum R_c$$

where $\sum R_c$ is the sum of the residues of $\phi(z)$,

as it holds inside C.

<u>Problems</u>:-

<u>Evaluate</u>:-

① $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \qquad a > b > 0$

Sol) We have $z = e^{i\theta}$

$$\Rightarrow \frac{dz}{iz} = d\theta$$

also $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Substitute these results in the given problem

we get,

$$\oint_C \frac{dz/iz}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} = \frac{2}{i} \oint \frac{dz}{bz^2 + 2az + b}$$

$$= \frac{2}{i} \oint_C \phi(z)\,dz$$

where $\phi(z) = \frac{2}{i} \times \cancel{\frac{1}{bz^2 + 2az + b}}$

C is the unit circle

poles of $\phi(z)$ are

$bz^2 + 2az +$

$b(z^2 +$

$b \neq$

$$= -\frac{1}{ai} \times \frac{1}{\frac{1}{a} - a}$$

$$R = \frac{i}{a^2 - 1}$$

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = 2\pi i \times \frac{i}{a^2-1}$$

$$= \frac{2\pi}{1-a^2}$$

4) $$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$$

A)

$$z = e^{i\theta}$$

$$\frac{dz}{iz} = d\theta \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\int_0^{2\pi} \frac{dz/iz}{13 + 5 \times \frac{1}{2i}\frac{(z^2-1)}{z}} = \cancel{\frac{1}{i}} \int_0^{2\pi} \frac{dz \times 2}{26zi + 5z^2 - 5}$$

$$= \cancel{\frac{1}{i}} \int_0^{2\pi} \frac{dz \times 2}{5z^2 + 26iz - 5}$$

$$= \cancel{\frac{1}{i}} \int_0^{2\pi} \frac{dz \times 2}{5z^2 + 25i\cancel{z} + iz - 5}$$

$$= \cancel{\frac{1}{i}} \int_0^{2\pi} \frac{dz \times 2}{5z(z+5i) + i(z+5i)}$$

$$= \cancel{\frac{1}{i}} \int_0^{2\pi} \frac{dz \times 2}{(5z+i)(z+5i)}$$

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$$\phi = \frac{1}{-ai} \times \frac{1}{z^2 - \frac{z}{a} + (-a+1)}$$

~~residue is~~

R poles

$$z^2 - \frac{z}{a} + (-a+1) = 0$$

root $= \dfrac{+\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 4(-a+1)}}{2}$

$= 1 \pm \sqrt{4}$.

$$\phi = -\frac{1}{ai}\left(\frac{1}{z^2 - z\frac{(a^2+1)}{a} + 1}\right)$$

poles $= \dfrac{\frac{a^2+1}{a} \pm \sqrt{\frac{(a^2+1)^2}{a^2} - 4}}{2\left(\frac{a^2+1}{a}\right)}$

$=$

$a, \frac{1}{a}$

$$\phi = -\frac{1}{ai}\left(\frac{1}{z^2 - z\left(a+\frac{1}{a}\right)+1}\right)$$

roots ~~poles~~ are

$$(z-a)\left(z-\frac{1}{a}\right) = 0$$

poles

$z = a, \frac{1}{a}$

$1/a$ is inside

$$R = \lim_{R \to 1/a} -\frac{1}{ai} \times \frac{1}{(z-a)\cancel{\left(z-\frac{1}{a}\right)}} \times \cancel{\left(z-\frac{1}{a}\right)}$$

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$\alpha = \dfrac{-1+\sqrt{1-a^2}}{a}$

$\beta = \dfrac{-1-\sqrt{1-a^2}}{a}$

$|\alpha| < 1 \qquad |\beta| > 1$

~~so R =~~ α

$\cancel{\oint_C \dfrac{dz}{1+a\cos\theta}} R = \dfrac{2}{ai} \lim_{z\to\alpha} \cancel{(z-\alpha)} \times \dfrac{1}{\cancel{(z-\alpha)}(z-\beta)}$

$= \dfrac{2}{ai(\alpha-\beta)} = \dfrac{\cancel{2} \times \alpha}{ai(2\sqrt{1-a^2})} = \dfrac{1}{i(\sqrt{1-a^2})}$

$\int_0^{2\pi} \dfrac{d\theta}{1+a\cos\theta} = 2\pi \cancel{i} \dfrac{1}{\cancel{i}\sqrt{1-a^2}}$

$= \dfrac{2\pi}{\sqrt{1-a^2}}$

(2) $\int_0^{2\pi} \dfrac{d\theta}{1-2a\cos\theta+a^2} \qquad 0 < a < 1$

4) we have $z = e^{i\theta}$

$\dfrac{dz}{iz} = d\theta \qquad \cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$

$\oint_0^{2\pi} \dfrac{\frac{dz}{iz}}{1-2a\times\frac{1}{2}(z+\frac{1}{z})+a^2} = \dfrac{1}{i}\int_0^{2\pi} \dfrac{dz}{-az^2+z+a^2z-a}$

$\phi(z) = \dfrac{1}{i(-az^2+z+a^2z-a)}$

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Observe that the integrand has simple ... at $\pm i, \pm 3i$ but $z = i$ and $z = 3i$ are ... two poles which lie inside the semi ... of the countour C. ∴ By residue theorem ...

$$\int_C f(z)\,dz = 2\pi i \left[\text{Res. of } f(z) \text{ at } z = i + \text{Res. of } f(z) \text{ at } z = \ldots \right.$$

Res. of $f(z)$ at $z = i$

$$R_1 = \lim_{z \to i} \cancel{(z-i)} \times \frac{z^2 - z + 2}{(z+3i)(z-3i)(z+i)\cancel{(z-i)}}$$

$$= \frac{+1-i}{8 \times 2i} = \frac{-(i+1)}{16}$$

$$R_2 = \lim_{z \to 3i} \cancel{(z-3i)} \frac{z^2 - z + 2}{(z+3i)\cancel{(z-3i)}(z^2+1)}$$

$$= \frac{i(-9-3i+2)}{i \times 6i \times (-8)} = \frac{3-7i}{48}$$

$$\int_C f(z) = 2\pi i \left(\frac{-i-1}{16} + \frac{3-7i}{48} \right)$$

$$= 2\pi i \left[\frac{-3i - \cancel{3} + \cancel{3} - 7i}{48} \right.$$

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Integrals of the type $\int_{-\infty}^{+\infty} f(x)\,dx$

a) Integral

We consider the line integral over the region C $f(z)\,dz$ where C is the closed contour consisting of the semi-circle $C_R : |z| = R$, to together with the real axis $-R$ to R

If $f(z)$ has no singular point on the real axes, by residue theorem we have

$$\oint_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\,[\text{Sum of the residues at interior poles}]$$

1) Prove that $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} = \frac{5\pi}{12}$

A) To evaluate the general ~~ing~~ integral we consider

$$\oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,dz = \int_C \frac{z^2 - z + 2}{(z^2+9)(z^2+1)}\,dz = \int_C f(z)\,dz$$

where c is the countour consisting of the semicircle C_R of radius R together with the part of the real axis from $-R$ to $+R$

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$\alpha = \frac{-i}{5}$, $\beta = -5i$

$|\alpha| < 1$ $\quad |\beta| > 1$

$\int_0^{2\pi} \frac{d\theta}{13+5i}$ $\quad R = \lim_{z \to -\frac{i}{5}} \cancel{2} \times \frac{1}{\cancel{(5z+i)}(z+5i)}$

$= \cancel{2} \times \frac{1}{5i - \frac{i}{5}}$

$= \frac{5 \times 2}{2(24i)} = \frac{5}{\cancel{48}i}$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = 2\pi \cancel{i} \times \frac{5}{6\,\cancel{48}\cancel{i}} = \frac{5\pi}{24} \quad \frac{5\pi}{6}$$

~~Ans : $\frac{5\pi}{6}$~~

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$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \left[\frac{-\sqrt{3}\,i - 2i - 1 + \sqrt{3}\,i}{12} \right]$$

$$= \cancel{2}\pi i \left[\frac{-\cancel{\sqrt{3}} - i - 2i + \cancel{\sqrt{3}} - i}{\cancel{12}\ 6} \right]$$

$$= \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$\int_{0}^{\infty} f(z)\,dz = \frac{1}{2} \int_{-\infty}^{\infty} f(z)\,dz = \frac{\pi}{3}$$

3) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} \qquad a > b > 0$

A) consider

$$\int_{-\infty}^{\infty} \frac{dz}{(z^2+a^2)(z^2+b^2)^2} = \int_{-\infty}^{\infty} f(z)\,dz$$

~~Resid~~ poles are $z = \pm ai \rightarrow$ single pole

pole of order 2 $z = \pm bi$

$z = ai, bi$ lies withing the semicircle

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of these

Only $z = e^{\frac{\pi i}{6}}, e^{\frac{3\pi i}{6}}, e^{\frac{5\pi i}{6}}$ lies inside the

semi circle.

$\therefore$ Residue of $f(z)$ at $z = e^{\pi i/6}$ is

$$R_1 = \lim_{z \to e^{\pi i/6}} \left[(z - e^{\pi i/6}) \times \frac{1}{z^6+1} \right]$$

$$= \lim_{z \to e^{\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6 \times e^{5\pi i/6}}$$

$$= \frac{\cancel{2}}{\cancel{6}3 \times (-\sqrt{3}+i)} = \frac{-\sqrt{3}-i}{3(3+1)} = \frac{-\sqrt{3}}{12}$$

$\therefore$ Residue of $f(z)$ at $z = e^{3\pi i/6}$ is

$$R_2 = \lim_{z \to e^{3\pi i/6}} \left[(z - e^{3\pi i/6}) \times \frac{1}{z^6+1} \right]$$

$$= \frac{1}{6 \times e^{\frac{15 i\pi}{6}}}$$

$$= \frac{1}{6[0+i]} = \frac{-i}{6}$$

Residue of $f(z)$ at $z = e^{5\pi i/6}$

$$R_3 = \frac{1}{6 \times e^{\frac{25\pi i}{6}}}$$

$$= \frac{\cancel{2}}{\cancel{6}3[-1-i\sqrt{3}]} \quad \frac{1}{3[\sqrt{3}+i]}$$

$$= \frac{-1+\sqrt{3}i}{3[1+3]} = \frac{-1+\sqrt{3}i}{12}$$

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2) P.T $\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$

A) Since integrand is even fn.

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^\infty \frac{dx}{x^6+1}$$

Consider line integral over C

$$\int_C \frac{dz}{z^6+1} = \int_C f(z)\,dz$$

The poles of $f(z)$ ~~are~~ i.e $\frac{1}{z^6+1}$ are the roots of

The eq. $z^6+1=0$

$$\Rightarrow z^6 = -1 = e^{\pi i}$$

$$z = e^{\pi i/6}$$

$$z = (\cos\pi + i\sin\pi)^{1/6}$$

$$= \left(\cos(2n+1)\pi + i\sin(2n+1)\pi\right)^{1/6}$$

$$z = \cos(2n+1)\frac{\pi}{6} + i\sin(2n+1)\frac{\pi}{6}$$

$$\Rightarrow z = e^{i(2n+1)\frac{\pi}{6}}$$

where $n = 0, 1, 2, 3, 4, \ldots$

$\Rightarrow$ when $n=0$

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Residue at $z = -1$

$$R_1 = \lim_{z \to -1} (z+1) \frac{z^2}{(z+1)(z^2+1)^2}$$

$$= \frac{1}{4}$$

Residue at $z = +i$

$$R_2 = \lim_{z \to i} \frac{1}{(2-1)!} \frac{d}{dz} \left[\frac{z^2}{(z+1)(z^2+i)^2 \cancel{(z^2-i)^2}} \right]$$

$$= \lim_{z \to i} \frac{d}{dz} \left[\frac{z^2}{(z+1)(z^2+i)^2} \right]$$

$$= \lim_{z \to i} \left[\frac{2z(z+1)(z^2+i)^2 - z^2\left[(z^2+i)^2 + (z+1)\,2(z^2+i)\ldots\right.}{(z+1)^2(z^2+i)^4} \right.$$

$$= \frac{2i(i+1)(i-1)^2 - (-1)\left[(i-1)^2 + (i+1)(i-1)\ldots\right.}{(i+1)^2(i-1)^4}$$

$$= \frac{2i(-1-1)(i-1) - (-1)\left[(i-1)^2 + (-1-1)4i\right]}{(i+1)^2(i-1)^4}$$

$$= \frac{-4i(i-1) - (-1)\left[\cancel{-1}\cancel{+1} - 2i - 8i\right]}{(i+1)^2(i-1)^4}$$

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$$= \frac{1}{(a^2-b^2)(2bi)}\left[\frac{-4b^2+2a^2-2b^2}{(a^2-b^2)(+4b^2)}\right]$$

$$= \frac{\cancel{2}a^2-\cancel{3}b^2}{(a^2-b^2)^2\,4\cancel{8}b^3 i} = \frac{a^2-3b^2}{(a^2-b^2)^2\,4b^3 i}$$

$$\int_{-\infty}^{\infty} f(z)\,dz = \cancel{2}\pi\cancel{i}\left[\frac{1}{\cancel{2}a\cancel{i}(a^2-b^2)^2}+\frac{a^2-3b^2}{(a^2-b^2)^2\,\cancel{4}b^3\cancel{i}}\right]$$

$$= \pi\left[\frac{1}{a(a^2-b^2)^2}+\frac{a^2-3b^2}{(a^2-b^2)^2\,2b^3}\right]$$

4) $\int_{-\infty}^{\infty} \frac{x^2}{(x+1)(x^2+1)^2}\,dx$ (Ans: $\frac{-\pi i}{4}$)

5) $\int_{0}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx$ (Ans: $\frac{\pi}{200}$)

6) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$ [$\frac{3}{8}$

A)

4) $$\int_{-\infty}^{\infty} \frac{z^2}{(z+1)(z^2+1)^2}\,dx = \int_{-\infty}^{\infty} f(z)\,dz$$

poles are $z=-1 \rightarrow$ single pole

poles are $z=\pm i \rightarrow$ multiple pole

~~Residue at $z=i$~~

$$\cancel{R = \lim_{z\to i} \frac{1}{(2-1)!}\frac{d}{dz}\{}$$

pole at $z = ai$

$$R_1 = \underset{z \to ai}{Lt} \; (z - ai) \frac{1}{(z+ai)(z-ai)(z^2+b^2)^2}$$

$$= \frac{1}{2ai\,(b^2 - a^2)^2}$$

pole at $z = bi$

$$R_2 = \underset{z \to bi}{Lt} \; \frac{1}{(2-1)!} \frac{d}{dz} \left[\cancel{(z-bi)^2} \times \frac{1}{(z^2+a^2)\cancel{(z-bi)^2}(z+bi)^2} \right]$$

$$= \underset{z \to bi}{Lt} \; \frac{d}{dz} \left[(z^2+a^2)^{-1} (z+bi)^{-2} \right]$$

$$= \underset{z \to bi}{Lt} \left\{ (-1)(z^2+a^2)^{-2} \times 2z (z+bi)^{-2} + (-2)(z+bi)^{-3} (z^2+a^2)^{-1} \right.$$

$$= (-1)(a^2-b^2)^{-2} \times \cancel{2bi}\,(2bi)^{-2} + (-2)(2bi)^{-3}(a^2-b^2)^{-1}$$

$$= \frac{-1}{(a^2-b^2)^2(2bi)} + \frac{-2}{(2bi)^3(a^2-b^2)}$$

$$= \frac{1}{(a^2-b^2)(2bi)} \left[\frac{-1}{a^2-b^2} + \frac{+2}{+4b^2} \right)$$

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