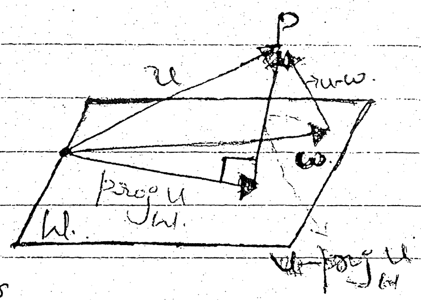
6.4: Best Approximations : Least Squares

We shall show how orthogonal projections can be used to solve certain approximation problems.

Thm-6.4.1: If 'W' is a finite-dimensional subspace of an IP space 'V' and if 'u' is any vector in V, then $\text{proj}_W u$ is the best approximation to 'u' from 'W' in the sense that

$$\|u - \text{proj}_W u\| < \|u - w\|$$

for every vector 'w' in 'W' that is different from $\text{proj}_W u$.



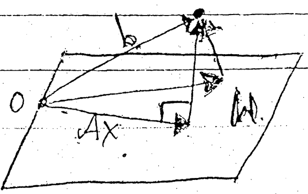
NOTE: The point 'Q' in W is close to P. or
Among all the vectors 'w' in W, the vector $w = \text{proj}_W u$ minimizes the distance $\|u-w\|$.

Least Square problem:
Given a linear System $AX=b$ of m-eqns in n-unknowns, find a vector 'X', if possible, that minimizes $\|AX-b\|$ w.r.t the Euclidean IP on R^m. Such a vector 'X' is called a least square solution of $AX=b$.)
(approximate sol.on)

Thm-6.4.2:
For any linear System $Ax=b$, the associated normal system : $\boxed{A^TAX = A^Tb}$ is consistent and all sol.ons of the normal system are least squares sol.on of $AX=b$.
Moreover, if W is the column space of 'A', and 'X' is any least squares sol.on of $AX=b$, then the orthogonal projection of 'b' on W is

do prob (1)

$$\boxed{\text{proj}_W b = AX}$$



W = Column space of 'A'
(least square soln 'X' produces the vector 'AX' in W closest to 'b')

Uniqueness of least Squares Solutions:

The conditions under which a linear system is guaranteed to have a unique solution is given by the following theorems.

Thm-6.4.3:

proof (02)

If A is an $m\times n$ matrix, then the following are equivalent.

(a) A has linearly independent column vectors.

(b) A^TA is invertible.

NOTE: [$A \to$ invertible (refer Thm-6.2.7)

$\Rightarrow$ $AX=b$ has exactly one solun, $|A| \neq 0$, A has rank n, Column vectors of A are L.I (lly row vectors), A has nullity 0 . . . .]

Thm-6.4.4: If A is an $m\times n$ matrix with L.I column vectors, then for every $m\times 1$ matrix b, the linear system $AX=b$ has a unique least squares solun. This solu.on is given by $\boxed{X=(A^TA)^{-1}A^Tb}$, ~~moreover~~ moreover if W is the column space of A, then the orthogonal projection of b on W is

proof (03) (06)

$$\boxed{\text{proj}_W b = AX = A(A^TA)^{-1}A^Tb}$$

Defn: If W is a subspace of R^m, then the transformation $P: R^m \longrightarrow W$ that maps each vector X in R^m into its orthogonal projection $\text{proj}_W X$ is called the orthogonal projection of R^m on W.

NOTE: The standard matrix for the orthogonal projection of R^m on W is $\boxed{[P] = A(A^TA)^{-1}A^T}$, where A is constructed using any basis for W as its column vectors.

Exercise - 6.4:

01) Find the normal system associated with the given linear system. (after Thm-6.4.2)

ⓐ $\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Solve: The associated normal system is given by,

$A^TA\,X = A^Tb$. $\quad A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}$

$A^TA = \underset{2\times3}{\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}} = \begin{bmatrix} 1+4+16 & -1+6+20 \\ -1+6+20 & 1+9+25 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$.

$A^Tb = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2-2+20 \\ -2-3+25 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$

∴ The normal system is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$.

ⓑ $\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}$

02) In each part find $\det(A^TA)$, and apply Thm-6.4.3 to determine whether A has linearly independent column vectors.

ⓐ $A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ Solve:- $A^T = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1+4+0 & -3+2+0 & -2+6+0 \\ -3+2+0 & 9+1+1 & 6+3+1 \\ -2+6+0 & 6+3+1 & 4+9+1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}$.

$|A^TA| = 5(154-100)+1(-14-40)+4(-10-44) = 270-54-216.$

$= 0$.

∴ No L.I column vectors.

(b) $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \\ 4 & -5 & 3 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & 0 & -1 & 4 \\ -1 & 1 & 0 & -5 \\ 3 & 1 & -2 & 3 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 4+0+1+16 & -2+0+0-20 & 6+0+2+12 \\ -2+0+0-20 & 1+1+0+25 & -3+1+0-15 \\ 6+0+2+12 & -3+1+0-15 & 9+1+4+9 \end{bmatrix}$$

$A^TA = \begin{bmatrix} 21 & -22 & 20 \\ -22 & 27 & -17 \\ 20 & -17 & 23 \end{bmatrix}$. $|A^TA| = 0$, Column vectors of 'A' are not L.I.

Q3) Find the least square soln of the linear system $AX=b$, and find the orthogonal projection of 'b' on the column space of 'A'. (Find soln of $(A^TA)X = A^Tb$. (by regular methods))

(a) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}$; $b = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$ 'A' has L.I column vectors.
∴ 'A' has a unique solun.

Soln: Given $AX = b$,

$$A^TA = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

∴ The normal system is $(A^TA)X = A^Tb$

is $\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$.

Solving this system yields the least squares solution
$x_1 = 5$ & $x_2 = 1/2$.

∴ The orthogonal projection of 'b' on the column space of 'A' is

$$\text{proj}_{W} b = AX = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -9/2 \\ -4 \end{bmatrix}$$

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(b) $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$ $\quad b = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Solution: Given $AX = \mathbf{b}$

$A^TA = \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}$ $\quad A^Tb = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$

$\therefore$ The normalised system is $(A^TA)x = A^Tb$.

$$\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$\therefore$ Least square Solutions are $x_1 = +\frac{3}{7}$ $\quad x_2 = -\frac{2}{3}$

The orthogonal projection of 'b' on the column space of 'A' is.

$$\text{proj}_W(b) = AX = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3/7 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 46/21 \\ -5/21 \\ 13/21 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$

Solu. $x_1 = 12,\ x_2 = -3,\ x_3 = 9$; $\text{proj}_W(b) = (3, 3, 9, 0)$

(d) $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix}$ Solu. $x_1 = 14$, $x_2 = 30$, $x_3 = 26$ $\quad \text{proj}_W b = \begin{bmatrix} 2 \\ 6 \\ -2 \\ 4 \end{bmatrix}$

(4) Find the orthogonal projection of 'u' on the subspace of R^3, Spanned by the vectors v_1 and v_2.

Solu. (a) $u = (2, 1, 3)$; $v_1 = (1, 1, 0)$, $v_2 = (1, 2, 1)$ $\quad A_{m\times n}$ $(A^TA)X = A^Tu$

Let $Ax = u \Rightarrow$ $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, $x_1 = -1$, $\frac{5}{3}$.

Since 'A' has L.I column vectors

$(A^TA)X = A^Tu$, find $X = (A^TA)^{-1}A^Tu$. $\quad \therefore \text{proj}_W u = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 5/3 \end{bmatrix}$ $= AX = \begin{bmatrix} 2/3 \\ 7/3 \\ 5/3 \end{bmatrix}$

(b) $u = (1, -6, 1)$; $v_1 = (-1, 2, 1)$; $v_2 = (2, 2, 4)$.

Solⁿ: The subspace ~~W~~ of R^3 spanned by v_1 and v_2 is the column space of matrix

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}$$

The column vectors of A are linearly independent.

$\therefore AX = u \Rightarrow \cancel{A^TAX = A^Tu}$ or $X = (A^TA)^{-1}A^Tu$ or

$$A^TAX = A^Tu \Rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} X = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 \\ 6 & 24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix}$$

$$x_1 = \frac{-7}{3}, \; \frac{1}{3}$$

$\therefore$ Orthogonal projection of $u = \text{proj}_W u = AX$

$$= \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$$

$x = A(A^TA)^{-1}A^Tu$

(5) Find the orthogonal projection of u on the subspace of R^4 spanned by v_1, v_2, v_3.

(a) $u = (6, 3, 9, 6)$; $v_1 = (2, 1, 1, 1)$, $v_2 = (1, 0, 1, 1)$, $v_3 = (-2, -1, 0, -1)$

Solⁿ: $A = [v_1 \; v_2 \; v_3] = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$ $\quad A^Tu = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$

$(A^TA)X = A^Tu \Rightarrow x_1 = +6, \; 3, \; +4$

$\therefore$ Orthogonal projection of $u = \text{proj}_W u = AX = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} +6 \\ 3 \\ +4 \end{bmatrix}$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

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(b) $u=(-2,0,2,4)$; $v_1=(1,1,3,0)$; $v_2=(-2,-1,-2,1)$; $v_3=(-3,-1,1,3)$

Solu:- $A=\begin{bmatrix}1 & -2 & -3\\ 1 & -1 & -1\\ 3 & -2 & 1\\ 0 & 1 & 3\end{bmatrix}$ $A^T=\begin{bmatrix}1 & 1 & 3 & 0\\ -2 & -1 & -2 & 1\\ -3 & -1 & 1 & 3\end{bmatrix}$

$A^TA=\begin{bmatrix}11 & -9 & -1\\ -9 & 10 & 8\\ -1 & 8 & 20\end{bmatrix}$ $A^Tu=\begin{bmatrix}4\\4\\20\end{bmatrix}$ $(A^TA)X=A^Tu$

$\Rightarrow x_1=\frac{-4}{5}$; $x_2=\frac{-8}{5}$; $x_3=\frac{8}{5}$.

$\therefore$ orthogonal projection of $u=\text{proj}_W u = AX = \begin{bmatrix}1 & -2 & -3\\ 1 & -1 & -1\\ 3 & -2 & 1\\ 0 & 1 & 3\end{bmatrix}\begin{bmatrix}-4/5\\-8/5\\8/5\end{bmatrix}$

$=\begin{bmatrix}-12/5\\-4/5\\12/5\\16/5\end{bmatrix}$

6). Find the orthogonal projection of $u=(5,6,7,2)$ on the solution R^4 space of the homogeneous linear system $x_1+x_2+x_3=0$
$2x_2+x_3+x_4=0$

Solu: $\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=0$

Find the bases for the system and arrange them in Columns

$A=\begin{bmatrix}1 & 0\\ 1 & 2\\ 1 & 1\\ 1 & 1\end{bmatrix}_{4\times 2}$ $A^T=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\end{bmatrix}_{2\times 4}$

~~$A^TA=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\end{bmatrix}$~~ ~~$A^TA=\begin{bmatrix}1 & 0\\ 1 & 2\\ 1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\end{bmatrix}$~~ ~~$A^Tu=\begin{bmatrix}1 & 0\\ 1 & 2\\ 1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}5\\6\\7\\2\end{bmatrix}$~~

$A^TA=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 2\\ 1 & 1\\ 1 & 1\end{bmatrix}$ ~~$=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 5 & 3 & 3\\ 1 & 3 & 2 & 2\\ 1 & 3 & 2 & 2\end{bmatrix}$~~

$=\begin{bmatrix}4 & 4\\ 4 & 6\end{bmatrix}$

$A^Tu=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 1 & 1\end{bmatrix}\begin{bmatrix}5\\6\\7\\2\end{bmatrix} = \begin{bmatrix}20\\21\end{bmatrix}$

$\therefore (A^TA)X=A^Tu \Rightarrow \begin{bmatrix}4 & 4\\ 4 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}20\\21\end{bmatrix}$

$\Rightarrow x_1=9/2$; $x_2=1/2$

$\therefore \text{proj}_W u=\begin{bmatrix}1 & 0\\ 1 & 2\\ \vdots\end{bmatrix}\begin{bmatrix}9/2\\1/2\end{bmatrix}=$

⑥ Find the orthogonal projection of $u = (5, 6, 7, 2)$ on the soln. space of homogeneous linear system

$x_1 + x_2 + x_3 = 0$

$2x_2 + x_3 + x_4 = 0$

Soln. $W \rightarrow$ soln. space of homogeneous linear system.
Find the bases for the system of :

$$x_1 = -(x_2 + x_3) \implies x_1 = -\left[\frac{-x_3}{2} + x_3 - \frac{x_4}{2}\right]$$

$$x_2 = \frac{-1}{2}(x_3 + x_4)$$

$$\implies x_1 = -\left[\frac{x_3}{2} - \frac{x_4}{2}\right]$$

Let $x_3 = t$, $x_4 = r$.

$$x_1 = \frac{x_4}{2} - \frac{x_3}{2}$$

$\therefore x_2 = -\frac{1}{2}[t + r] \quad x_1 = \frac{1}{2}[r - t]$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(r-t) \\ -\frac{1}{2}(t+r) \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

$\therefore W \rightarrow$ ~~span~~ bases : $\{X_1, X_2\} = \left\{\left(\frac{-1}{2}, \frac{-1}{2}, 1, 0\right), \left(\frac{1}{2}, \frac{-1}{2}, 0, 1\right)\right\}$

$\therefore AX = u$

where: $A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}.$

$$A^TA = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} \quad A^Tu = \begin{bmatrix} \frac{-1}{2} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$

$$\therefore (A^TA)X = (A^Tu) \implies \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}.$$

$\therefore$ Soln is $x_1 = 1$; $x_2 = 1$.

$$\therefore \operatorname{proj}_W u = AX = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

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(6,7,2)
ear system

system,

$\left[\frac{x_3}{} - \frac{x_4}{2}\right]$

$-\frac{x_4}{2}\right]$

$-\frac{x_3}{2}$

$\begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$.

$\left(\frac{1}{2}, \frac{-1}{2}, 0, 1\right)\}$

$\begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$

07) Use formula $[P] = A(A^TA)^{-1}A^T$ and the method of Example-3 to find the standard matrix for the orthogonal projection $P: R^2 \to R^2$ onto

(a) the x-axes (b) the y-axes.

Solu:-

(a) Take the unit vector along x-axes as a bases. for x-axis

i.e. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $A^T = [1 \ 0]$.

$P = AA^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \ 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) Take the unit vector along y-axes as a bases

i.e. $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $A^T = [0 \ 1]$

$P = AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

08) Use formula $[P] = A(A^TA)^{-1}A^T$ and the method of Example-3 to find the standard matrix for the orthogonal projection $P: R^3 \to R^3$ onto

(a) the xz-plane.

Solu. Take the unit vectors along x and z-axis (Bases for XZ-plane)

$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$P = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) the yz-plane $\longrightarrow$ $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

09) Let W be the plane with equation $5x-3y+z=0$.

ⓐ Find a basis for W

ⓑ Use formula $[P]=A(A^TA)^{-1}A^T$ to find the standard matrix for the orthogonal projection onto W.

ⓒ Use the matrix obtained ⓑ to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ on W.

ⓓ Find the distance b/n the point $P_0(1,-2,4)$ and the plane W, and check your result using Th 3.5.2

Solu

ⓐ $x = \frac{1}{5}[3y - z] = \frac{1}{5}[3t - r]$

$$X = \begin{bmatrix} \frac{3t}{5} - \frac{r}{5} \\ t \\ r \end{bmatrix} = t\begin{bmatrix} 3/5 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1/5 \\ 0 \\ 1 \end{bmatrix} \text{ or}$$

or $z = 3y - 5x$ $\therefore X = \begin{bmatrix} s \\ t \\ 3t-5s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$\therefore X_1 = (1, 0, -5)$ $X_2 = (0, 1, 3)$

$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}$

ⓑ $[P] = A(A^TA)^{-1}A^T$

$$= \begin{bmatrix} 10/35 & 15/35 & -5/35 \\ 15/35 & 26/35 & 3/35 \\ -5/35 & 3/35 & 34/35 \end{bmatrix}$$

ⓒ Orthogonal projection of P_0 is $= P.P_0$

$$= \frac{1}{35}\begin{bmatrix} 10x_0 + 15y_0 - 5z_0 \\ 15x_0 + 26y_0 + 3z_0 \\ -5x_0 + 3y_0 + 34z_0 \end{bmatrix}$$

ⓓ $\|P_0 - PP_0\| =$

(10) Let

ⓐ Fin

ⓑ Use

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Ans Eq

$5x-3y+z=0$

the standard

onto W.

the orthogonal

W.

$P_0(1,-2,4)$

result using

$$PP_0 = \frac{1}{35}\begin{bmatrix}-40\\-25\\125\end{bmatrix}$$

$$\therefore \|P_0 - PP_0\| = \left\|\begin{bmatrix}1\\-2\\4\end{bmatrix} - PP_0\right\| = \left\|\begin{bmatrix}15/7\\-9/7\\3/7\end{bmatrix}\right\|$$

$$= \sqrt{\left(\frac{15}{7}\right)^2 + \left(-9/7\right)^2 + \left(3/7\right)^2} = 3\sqrt{35}/7 = 15/\sqrt{35}$$

$\mathfrak{R}$

$s\begin{bmatrix}1\\0\\-5\end{bmatrix} + t\begin{bmatrix}0\\1\\3\end{bmatrix}$

(10) Let W be the line with parametric eqns,
$x=2t$, $y=-t$, $z=4t$ $(-\infty < t < \infty)$

(a) Find a basis for W $v_1 = (2,-1,4)$

(b) Use formula $[P] = A(A^TA)^{-1}A^T$ to find the standard matrix for orthogonal projection onto W.

$$P = \frac{1}{21}\begin{bmatrix}4 & -2 & 8\\-2 & 1 & -4\\8 & -4 & 16\end{bmatrix}$$

(c) Use the matrix obtained in (b) to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ on W. PP_0

(d) Find the distance b/w the point $P_0(2,1,-3)$ and the line W. $\|P_0 - PP_0\| = \frac{\sqrt{497}}{7}$

Exclude (11-15).

Ans Eg (1-4).

? P_0 !

Orthogonal Matrices : Change of Basis.

Defn:
A square matrix A is said to be orthogonal matrix if $A^{-1} = A^T \Rightarrow AA^T = A^TA = I$.

Eg.-2: A Rotation matrix:
The standard matrix for the clockwise rotation of R^2 through an angle θ is,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix is orthogonal.

Thm-6.5.1: The following are equivalent for an $n\times n$ matrix A,

ⓐ A is orthogonal ⓚ

ⓑ The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.

ⓒ The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Ex-6.5 ① ⓐ Show that the matrix $A = \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix}$ is orthogonal in three ways: by calculating A^TA, by using part ⓑ of Thm-6.5.1, and by using part ⓒ of Thm-6.5.1.

Soln: Case(i):

$$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Case(ii): [Row vectors of A form an orthonormal set in R^3 w.r.t Euclidean IP].

$u_1 = \left(4/5, 0, -3/5\right)$ $\quad u_2 = \left(\frac{-9}{25}, \frac{4}{5}, \frac{-12}{25}\right)$ $\quad u_3 = \left(\frac{12}{25}, \frac{3}{5}, \frac{16}{25}\right)$

$\|u_1\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{-3}{5}\right)^2}$; $\|u_2\| = 1$ $\quad \|u_3\| = 1$.

$= \sqrt{\frac{16+9}{25}} = 1$

$\langle u_1, u_2 \rangle = \frac{-36}{125} + 0 + \frac{36}{125} = 0$; $\langle u_2, u_3 \rangle = \frac{-108}{(25)^2} + \frac{12}{25} - \frac{192}{(25)^2}$

$= 0$.

$\langle u_3, u_1 \rangle = 0$.

$\therefore$ The row vectors of 'A' are orthonormal & hence the matrix 'A' is orthogonal.

III'ly for case(iii) : [Prove it with column vectors]

(1)(b) Find the inverse of matrix 'A' in part(a).

Theorem - 6.5.2 :

(a) The inverse of an orthogonal matrix is orthogonal.

(b) A product of orthogonal matrices is orthogonal.

(c) If 'A' is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

Eg: 3 : $\det[A] = \pm 1$ for an orthogonal matrix 'A':

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Since the row vectors of 'A' form orthonormal sets in R^2, 'A' is orthogonal. $\det(A) = 1$.

Interchanging the rows produces an orthogonal matrix for which $\det(A) = -1$.

Theorem - 6.5.3 :

If 'A' is an $n \times n$ matrix, then the following are equivalent,

(a) 'A' is orthogonal (b) $\|Ax\| = \|x\|$ $\forall$ x in R^n.

(c) $Ax \cdot Ay = x \cdot y$ for all x and y in R^n.

Co-ordinate Matrices:

Let $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then each vector 'v' in V can be uniquely expressed as $v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$ [Linear Combination].

Then the co-ordinate matrix of vector 'v' relative to S is

$$[v]_S = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

~~Change of Bases: If we change the bases for a vector space V, from some old bases B to some new bases B', How is the old co-ordinate matrix $[v]_B$ of~~

Change of Bases:

If we change the bases for a vector space V from some old bases $B = \{u_1, u_2, ---, u_n\}$ to some new bases $B' = \{u_1', u_2', ---, u_n'\}$, then the old co-ordinate matrix $[v]_B$ of a vector v is related to the new coordinate matrix $[v]_{B'}$ of the same vector v by the equation

$$[v]_B = P[v]_{B'},$$

where the columns of P are the coordinate matrices of the new bases vectors relative to the old bases, i.e the column vectors of P are

$$[u_1']_B, [u_2']_B, \ldots, [u_n']_B.$$

Note: The matrix P is called the transition matrix from B' to B : $P = \left[[u_1']_B \,\middle|\, [u_2']_B \,\middle|\, --- \,\middle|\, [u_n']_B \right]$.

Example-4: Finding a transition Matrix:

Consider the bases $B = \{u_1, u_2\}$ and $B' = \{u_1', u_2'\}$ for R^2, where $u_1 = (1, 0)$; $u_2 = (0, 1)$; $u_1' = (1, 1)$; $u_2' = (2, 1)$.

(a) Find the transition matrix from B' to B.

(b) Use, $[v]_B = P[v]_{B'}$ to find $[v_B]$ if

$$[v]_{B'} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Solu: (a) First we must find the coordinate matrices for the new basis vectors u_1' and u_2' relative to the old basis B.

By inspection, $u_1' = u_1 + u_2$; $u_2' = 2u_1 + u_2$.

so that $[u_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[u_2']_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Thus, the transition matrix from B' to B is

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b) $[v]_B = P[v]_{B'} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

[Verification: $-3u_1' + 5u_2' = 7u_1 + 2u_2 = v = (7, 2)$]

Theorem - 6.5.4:

If 'P' is the transition matrix from B' to B for a finite-dimensional vector space V, then (a) 'P' is invertible (b) P^{-1} is the transition matrix from B to B'.

Thm - 6.5.5:

If 'P' is the transition matrix from one orthonormal basis to another orthonormal basis for an IP space V then 'P' is an orthonormal matrix, i.e. $P^{-1} = P^T$.

Excercise - 6.5.

2(a) Show that the matrix $A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$ is orthogonal.

(b) Let $T: R^3 \to R^3$ be multiplication by the matrix 'A' in part (a). Find $T(x)$ for the vector $x = (-2, 3, 5)$. Using the Euclidean inner product on R^3, verify that $\|T(x)\| = \|x\|$.

Soln: (a) $A^T A = I$. or

(b) $T(x) = AX$.

$$= \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 14/3 \\ -5/3 \\ 11/3 \end{bmatrix}$$

& $\|T(x)\| = \|x\| = \sqrt{38}$.

03) Determine which of the following matrices are orthogonal. For those that are orthogonal, find the inverse ($A^{-1} = A^T$).

ⓐ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Orthogonal

ⓑ $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ orthogonal

ⓒ $\begin{bmatrix} 0 & 1 & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$

ⓓ $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ Orthogonal

ⓔ $\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -5/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & -5/6 \\ 1/2 & 1/6 & -5/6 & 1/6 \end{bmatrix}$

ⓕ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

Co-ordinate Matrix :

~~ⓑ Find the inverse of the matrix A = 4/5~~

⑤ Find the Co-ordinate matrix for w relative to the bases $S = \{u_1, u_2\}$ for R^2.

ⓐ $u_1 = (1,0)$, $u_2 = (0,1)$; $w = (3,-7)$.

Solu: $(w)_s = k_1 u_1 + k_2 (u_2)$

$\Rightarrow (3,-7) = k_1 (1,0) + k_2 (0,1)$

$k_1 = 3, \quad k_2 = -7.$

$\therefore$ Co-ordinate matrix for $w = (w)_s = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

ⓑ $u_1 = (2,-4)$; $u_2 = (3,8)$; $w = (1,1)$

$(w)_s = k_1 u_1 + k_2 u_2 \Rightarrow (1,1) = k_1 (2,-4) + k_2 (3,8)$

$2k_1 + 3k_2 = 1$; $-4k_1 + 8k_2 = 1$

$k_1 = 5/28 \qquad k_2 = 3/14$

$k_1 = 0.1785 \qquad = 0.214$

$\therefore [w]_s = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 5/28 \\ 3/14 \end{bmatrix}$

ⓒ $u_1 = (1,1)$; $u_2 = (0,2)$; $w = (a,b)$.

$w = k_1 u_1 + k_2 u_2 \Rightarrow (a,b) = k_1 (1,1) + k_2 (0,2)$ $\quad k_1 + 2k_2 = b$

$2k_2 = b -$

$\boxed{k_1 = a} \quad k_2 = \frac{b-a}{2}$

$\therefore [w]_s = \begin{bmatrix} a \\ \frac{b-a}{2} \end{bmatrix}$

06) Fin-
S=
ⓐ v_1 =
ⓑ v =
(v)
07) Fina
ⓐ p =
Solu.
ⓑ p =
Solu:
08) Fin-
S' =
A.
Solu:

06) Find the co-ordinate matrix for 'v' relative to $S = \{v_1, v_2, v_3\}$.

ⓐ $v = (2,-1,3)$; $v_1 = (1,0,0)$; $v_2 = (2,2,0)$, $v_3 = (3,3,3)$

$(v)_S = (3,-2,1)$ $\quad [v]_S = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

ⓑ $v = (5,-12,3)$; $v_1 = (1,2,3)$, $v_2 = (-4,5,6)$, $v_3 = (7,-8,9)$

$(v)_S = (-2,0,1)$ $\quad [v]_S = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

07) Find the co-ordinate matrix for 'P' relative to $S = \{p_1, p_2, p_3\}$.

ⓐ $p = 4 - 3x + x^2$; $p_1 = 1$; $p_2 = x$; $p_3 = x^2$.

Solu: $[P]_S = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$

ⓑ $p = 2 - x + x^2$; $p_1 = 1 + x$, $p_2 = 1 + x^2$, $p_3 = x + x^2$.

Solu: $\therefore [P]_S = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

08) Find the co-ordinate matrix for 'A' relative to $S = \{A_1, A_2, A_3, A_4\}$:

$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solu: $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = k_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$2 = -k_1 + k_2 \qquad -1 = k_3$

$0 = k_2 + k_1 \qquad 3 = k_4$

$\Rightarrow k_2 = 1$; $k_1 = -1$.

$\therefore [A]_S = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

$k_1 + 2k_2 = 6$

$2k_2 = 5$

(09). Consider the co-ordinate matrices

$[w]_S = \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix}$; $[q]_S = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$; $[B]_S = \begin{bmatrix} -8 \\ 7 \\ 6 \\ 3 \end{bmatrix}$

(a) Find w if S is the bases $v_1(1,0,0)$, $v_2(2,2,0)$, $v_3(3,3,3)$.

Solu. $w = k_1 v_1 + k_2 v_2 + k_3 v_3$

$= 6(1,0,0) + (-1)(2,2,0) + 4(3,3,3)$

$= (16, 10, 12)$

(b) Find q if S is the bases: $p_1 = 1$, $p_2 = x$, $p_3 = x^2$

Solu: $q = k_1 p_1 + k_2 p_2 + k_3 p_3$

$q = 3(1) + 0(x) + 4(x^2) \Rightarrow q = 3 + 4x^2$.

(c) Find B if S is the bases,

$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$; $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solu: $B = k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4$.

$= -8\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + 7\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore B = \begin{bmatrix} 15 & -1 \\ 6 & 3 \end{bmatrix}$

(10) Consider the Bases $B = \{u_1, u_2\}$ & $B' = \{v_1, v_2\}$ for R^2; where $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

(a) Find a transition matrix from B' to B.

(b) Find the transition matrix from B to B'.

(c) Compute the co-ordinate matrix $[w]_B$ where $w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and use $[w]_{B'} = P^{-1}[w]_B$ to compute $[w]_{B'}$ directly.

Solu. (

(b)

Sc

In

(c)

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(from B' to B)

Solu (a) By inspection, $v_1 = 2u_1 + u_2$; $v_2 = -3u_1 + 4u_2$

∴ so that $[v_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $[v_2]_B = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Thus, the transition matrix from B' to B is,

$$P = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}.$$

(b) from B to B'.

$u_1 = k_1 v_1 + k_2 v_2$; $u_2 = k_1' v_1 + k_2'(v_2)$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$; $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = k_1' \begin{bmatrix} 2 \\ 1 \end{bmatrix} + k_2' \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

$1 = 2k_1 - 3k_2$, $0 = k_1 + 4k_2$; $0 = 2k_1' - 3k_2'$

$k_1 = 4/11$, $k_2 = -1/11$; $1 = k_1' + 4k_2'$.

$\therefore u_1 = \frac{4}{11} v_1 - \frac{1}{11} v_2$; $k_1' = 3/11$ $k_2' = 2/11$.

So that $[u_1]_{B'} = \begin{bmatrix} 4/11 \\ -1/11 \end{bmatrix}$ $[u_2]_{B'} = \begin{bmatrix} 3/11 \\ 2/11 \end{bmatrix}$.

Thus, the transition matrix from B to B' is,

$$P' = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} 1/11.$$

(c) Given $w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, find $[w]_B$.

$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow k_1 = 3$ $k_2 = -5$.

$\therefore [w]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. [Usual formula is $[v]_B = P[v]_{B'}$!]

[P^{-1} is the transition matrix from B to B'!]

$[w]_{B'} = P^{-1} [w_B]$

$= \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} \frac{1}{11} \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$[w]_{B'} = \frac{1}{11} \begin{bmatrix} -3 \\ -13 \end{bmatrix}$

(11) Repeat the directions of Exercise-10 with the same vector w but with

$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $u_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(a) $\begin{bmatrix} 13/10 & -1/2 \\ -2/5 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix}$ (c) $[w]_B = \begin{bmatrix} -17/10 \\ 8/5 \end{bmatrix}$. $[w]_{B'} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

(12) Consider the bases $B = \{u_1, u_2, u_3\}$ and $B = \{v_1, v_2, v_3\}$ for R^3, where

$u_1 = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix}$ $v_3 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$

(a) Find the transition matrix from B to B'.

(b) Compute the co-ordinate matrix $[w]_B$, where $w = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$ and use (11) to compute $[w]_{B'}$.

(c) Check your work by computing $[w]_{B'}$ directly.

solu

(a) $u_1 = k_1 v_1 + k_2 v_2 + k_3 v_3$

$$\begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} = k_1 \begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

$-3 = -6k_1 - 2k_2 - 2k_3$

$0 = -6k_1 - 6k_2 - 3k_3$

$-3 = 4k_2 + 7k_3$.

$k_1 = \frac{3}{4}$, $k_2 = -\frac{3}{4}$, $k_3 = 0$.

$\therefore [u_1]_{B'} = \begin{bmatrix} 3/4 \\ -3/4 \\ 0 \end{bmatrix}$

III^{ly} $[u_2]_{B'} = \begin{bmatrix} 3/4 \\ -17/12 \\ 2/3 \end{bmatrix}$ & $[u_3]_{B'} = \begin{bmatrix} 1/12 \\ -17/12 \\ 2/3 \end{bmatrix}$

$\therefore$ The transition matrix from B to B' is

$$P = \begin{bmatrix} 3/4 & 3/4 & 1/12 \\ -3/4 & -17/12 & -17/12 \\ 0 & 2/3 & 2/3 \end{bmatrix}.$$

(b)

(c)

13 Repe... vec... $u_1 =$ ... (a)

with the

$=1$
-1
$=\begin{bmatrix}-4\\-7\end{bmatrix}$
$B=\{u_1,u_2,u_3\}$

$\begin{bmatrix}-2\\-6\\4\end{bmatrix}$ $v_3=\begin{bmatrix}-2\\-3\\7\end{bmatrix}$

, B'.

where

$[w]_{B'}$

directly.

(b) $[w_B] = P[w]_{B'}$ where P is the transition matrix from B' to B.

$$w=\begin{bmatrix}-5\\8\\-5\end{bmatrix} = k_1\begin{bmatrix}-3\\0\\-3\end{bmatrix} + k_2\begin{bmatrix}-3\\2\\-1\end{bmatrix} + k_3\begin{bmatrix}1\\6\\-1\end{bmatrix}.$$

$$[w]_B = \begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

$$[w]_{B'} = P^{-1}[w]_B$$ where P^{-1} is the transition matrix from B to B'.

$$= \begin{bmatrix}3/4 & 3/4 & 1/12\\ -3/4 & -17/12 & -17/12\\ 0 & 2/3 & 2/3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}19/12\\-43/12\\4/3\end{bmatrix}.$$

(c) $$[w]_{B'} = \begin{bmatrix}-5\\8\\-5\end{bmatrix} = k_1\begin{bmatrix}-6\\-6\\0\end{bmatrix} + k_2\begin{bmatrix}-2\\6\\4\end{bmatrix} + k_3\begin{bmatrix}-2\\-3\\7\end{bmatrix}$$

$$\Rightarrow [w]_{B'} = \begin{bmatrix}19/12\\-43/12\\4/3\end{bmatrix}.$$

(13) Repeat the directions of Ex-12 with the same vector w, but with

$u_1=\begin{bmatrix}2\\1\\1\end{bmatrix}, u_2=\begin{bmatrix}2\\-1\\1\end{bmatrix}, u_3=\begin{bmatrix}1\\2\\1\end{bmatrix}$; $v_1=\begin{bmatrix}3\\1\\-5\end{bmatrix}, v_2=\begin{bmatrix}1\\1\\-3\end{bmatrix}, v_3=\begin{bmatrix}-1\\0\\2\end{bmatrix}$

(a) $P=\begin{bmatrix}3 & 2 & 5/2\\ -2 & -3 & -1/2\\ 5 & 1 & 6\end{bmatrix}$ (b) $\begin{bmatrix}-7/2\\23/2\\6\end{bmatrix}$.

(14) Consider the bases $B = \{p_1, p_2\}$ and $B' = \{q_1, q_2\}$ for P_1, where,

$$p_1 = 6+3x, \quad p_2 = 10+2x, \quad q_1 = 2, \quad q_2 = 3+2x.$$

(a) Find the transition matrix from B' to B.

(b) Find the transition matrix from B to B'.

(c) Compute the co-ordinate matrix $[p]_B$, where $p = -4+x$, and use (11) to compute $[p]_{B'}$.

(d) Check your work by computing $[p]_{B'}$ directly.

(a) $\begin{bmatrix} -2/9 & 7/9 \\ 1/3 & -1/6 \end{bmatrix}$ (b) $\begin{bmatrix} 3/4 & 7/2 \\ 3/2 & 1 \end{bmatrix}$ (c) $[p]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[p]_{B'} = \begin{bmatrix} -11/4 \\ 1/2 \end{bmatrix}$

(15) Let V be the space spanned by $f_1 = \sin x$ and $f_2 = \cos x$.

(a) Show that $g_1 = 2\sin x + \cos x$ and $g_2 = 3\cos x$ form a basis for V

(b) Find the transition matrix from $B' = \{g_1, g_2\}$ to $B = \{f_1, f_2\}$

(c) Find the transition matrix from B to B'.

(d) Compute the co-ordinate matrix $[h]_B$, where $h = 2\sin x - 5\cos x$, and use (11) to obtain $[h]_{B'}$.

(e) check your work by computing $[h]_{B'}$ directly.

Solution (a)
$$W = \begin{vmatrix} g_1 & g_2 \\ g_1' & g_2' \end{vmatrix}$$

$$= \begin{vmatrix} 2\sin x + \cos x & 3\cos x \\ 2\cos x - \sin x & -3\sin x \end{vmatrix} = -6\sin^2 x - 3\sin x\cos x - [6\cos^2 x - 3\cos x\sin x]$$

$$= -6[\sin^2 x + \cos^2 x] + 0$$

$$= \underline{\underline{-6}} \neq 0,$$

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$B' = \{q_1, q_2\}$

$, q_2 = 3+2x$

B.

B'.

, where

B'.

directly:

$] \ \& [P]_{B'} = \begin{bmatrix} -11/4 \\ 1/2 \end{bmatrix}$

sinx and

cosx form

$\{g_1, g_2\}$

to B'.

obtain

directly.

nx cosx

-3 cosx sinx

to

$\therefore$ g_1 and g_2 are linear independent and thus form a basis.

(b) $g_1 = k_1 f_1 + k_2 f_2$ $\qquad B' = \{g_1, \ldots$

$2\sin x + \cos x = k_1 \sin x + k_2 \cos x$.

Eq^{n}g the coefficients of $\sin x \Rightarrow 2 = k_1$

" " " , $\cos x \Rightarrow 1 = k_2$

$$\therefore [g_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

lll^y $g_2 = 3\cos x = k_1 \sin x + k_2 \cos x$.

$$\Rightarrow [g_2]_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$\therefore$ The transition matrix from B' to B.

$$P = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

(c) $f_1 = k_1 g_1 + k_2 g_2 \Rightarrow \sin x = (2\sin x + \cos x) k_1 + 3k_2 \cos x$

$\Rightarrow 1 = 2k_1 \qquad 0 = k_1 + 3k_2$

$k_1 = 1/2 \qquad k_2 = -1/6$.

$f_2 = k_1 g_1 + k_2 g_2 \Rightarrow \cos x = k_1(2\sin x + \cos x) + k_2(3\cos x)$

$\Rightarrow k_1 = 0 \quad ; \quad 1 = k_1 + 3k_2$

$\Rightarrow k_2 = 1/3$.

$\therefore$ The transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/6 & 1/3 \end{bmatrix}.$$

(d) $2\sin x - 5\cos x = k_1(\sin x) + k_2(\cos x)$

$\Rightarrow k_1 = 2 \quad ; \quad k_2 = -5$

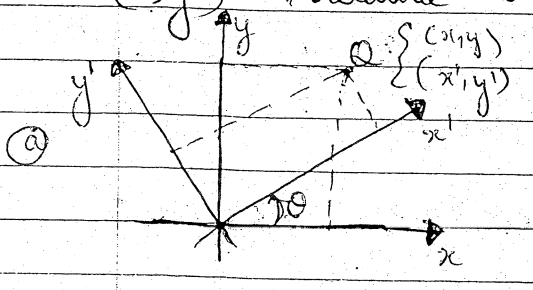
$$[h]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \quad [h]_{B'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

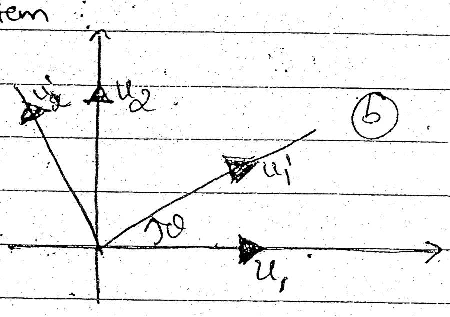
Example-6: Application to rotation of axes:

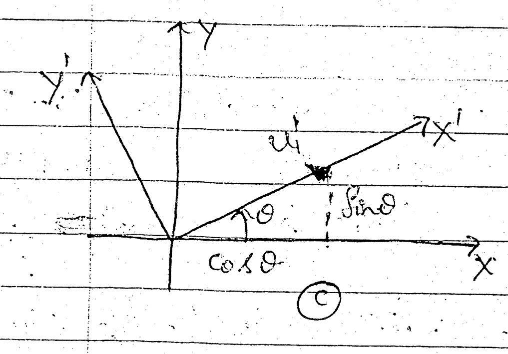
Suppose a Rectangular xy-co-ordinate system is given and a new $x'y'$-coordinate system is obtained by rotating the xy-system counter clockwise about the origin through an angle θ. Then each point Q in the plane has two sets of co-ordinates:

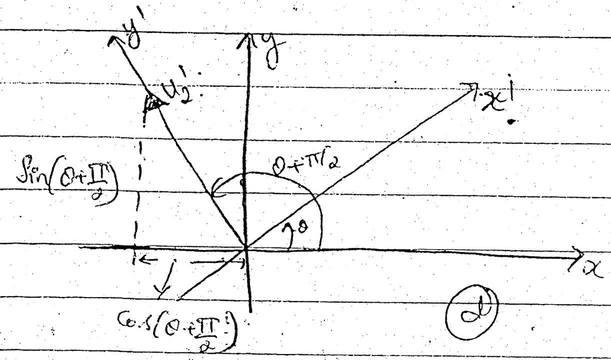
$(x, y) \to$ relative to xy-system and

$(x', y') \to$ relative to $x'y'$-system.









By introducing unit vectors u_1 and u_2 along the +ve x and y axes and unit vectors u_1' & u_2' along the positive x' and y' axes, we can regard this rotation as a change from an old bases $B = \{u_1, u_2\}$ to a new bases $B' = \{u_1', u_2'\}$ (fig ⓑ). Thus, the new co-ordinates (x', y') and the old co-ordinates (x, y) of a point Q will be related by $\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$, where

P is the transition from B' to B.

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To find 'P' we must determine the co-ordinate matrices of new vectors u_1' and u_2' relative to the old bases. As indicated in (fig c), the components of u_1' in the old bases are '$\cos\theta$' and '$\sin\theta$', so that

$$[u_1']_B = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Clearly, from (fig. d), we see that the components of u_2' in the old bases are $\cos(\theta + \pi/2) = -\sin\theta$ and $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$, so.

$$[u_2']_B = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus the transition matrix from B' to B is

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Observe 'P' is orthogonal matrix $\Rightarrow P^{-1} = P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

$$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\Rightarrow \quad x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

Ex:

(16) Let a rectangular $x'y'$-co-ordinate system to be obtained by rotating a rectangular xy-coordinate system counter clockwise through an angle $\theta = \frac{3\pi}{4}$.

(a) Find the $x'y'$-co-ordinates of the point whose xy-coordinates are $(-2, 6)$.

(b) Find the xy-coordinates of the point whose $x'y'$-coordinates are $(5, 2)$.

Solu: (a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$ where $P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x' \\ y' \end{bmatrix}$ $\quad P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}$$

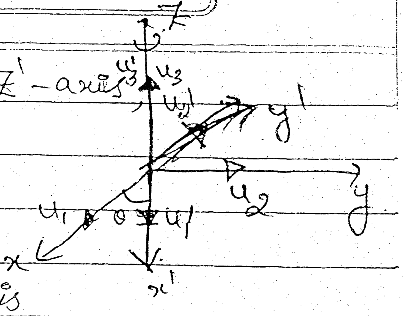
(17) Repeat (16) for $\theta = \pi/3$.

Example - 7 : Application to rotation of axes in 3-space.

Suppose that a rectangular xyz-coordinate system is rotated around its Z-axis counterclockwise through an angle θ. (fig). If we introduce unit vectors u_1, u_2 and u_3 along the +ve x, y & Z axes and unit vectors u_1', u_2' and u_3' along the +ve x', y', z' axes, we can regard the rotation as a change from basis $B = \{u_1, u_2, u_3\}$ to the new basis $B' = \{u_1', u_2', u_3'\}$ in the light of example (6). It should be evident that

$$[u_1']_B = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \text{ and } [u_2']_B = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

Since u_3' extends 1 unit up the +ve z'-axis



$$[u_3']_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The transition matrix from B' to B is

$$P = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \& \quad P^{-1} = P^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The new co-ordinates (x', y', z') of a point Q can be computed from its old co-ordinates (x, y, z) by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Note: If the system is rotated around its y-axis, then

$$P = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

and if it is x-axis, then $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$.

(18) Let a rectangular $x'y'z'$-coordinate system be obtained by rotating xyz-co-ordinate system counter clockwise about z-axis through an angle $\theta = \pi/4$.

(a) find the $x'y'z'$-coordinates of the point whose xyz-coordinates are $(-1, 2, 5)$.

(b) Find the xyz-coordinates of the point whose $x'y'z'$-coordinates are $(1, 6, -3)$.

Solu:-

(a)

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad P^{-1} = P^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ \frac{3}{\sqrt{2}} \\ 5 \end{bmatrix}$$

(b) Find the xyz-coordinates of the point whose $x'y'z'$-coordinates are $(1, 6, -3)$.

Soln

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -5\sqrt{2}/2 \\ 7\sqrt{2}/2 \\ -3 \end{bmatrix}$$

(19) Repeat (18) for a rotation $\theta = \pi/3$ counterclockwise about Y-axis

(a) $\left(\frac{-1}{2} - \frac{5\sqrt{3}}{2}, 2, \frac{5}{2} - \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{1}{2} - \frac{3\sqrt{3}}{2}, 6, \frac{-3-\sqrt{3}}{2}\right)$

(20) Repeat (18) for a rotation of $\theta = 3\pi/4$ counterclockwise about x-axis.

(a) $\left(-1, 3\sqrt{3}/2, -7\sqrt{2}/2\right)$ (b) $\left(1, -\cancel{3/\sqrt{2}}, 9\sqrt{3}/2\right)$

(21) Assg

(22) A rectangular $x''y''z''$ coordinate system is obtained by first rotating a rectangular xyz-coordinate system 60° counterclockwise about z-axis to obtain an $x'y'z'$ system, and then rotating $x'y'z'$-system 45° counterclockwise about y'-axis.

Find A such that $\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where

(x, y, z) & (x'', y'', z'') are the xyz & $x''y''z''$-coordinate of the same point.

Soln:

First, find the matrix P_1, by rotating a rectangular xyz-coordinate system 60° about z-axis t whose

$$= \begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1^{-1} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now 'P_2' about Y'-axis $\theta = 45^\circ$.

$$P_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

unterclockwise

$, \frac{-3-\sqrt{3}}{2})$

ckwise about

$$P_2^{-1} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$\sqrt{3}/2)$

$$\therefore \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = P_2^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = P_2^{-1} P_1^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

n is obtained coordinate
-axis to
ting $x'y'z'$-system

$$A = \begin{bmatrix} \sqrt{2}/4 & \sqrt{6}/4 & -\sqrt{2}/2 \\ -\sqrt{3}/4 & 1/2 & 0 \\ \sqrt{2}/4 & \sqrt{6}/4 & \sqrt{2}/2 \end{bmatrix}$$

ise

(23) What conditions must be 'a' & 'b' satisfy for the matrix $\begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$ to be orthogonal.

$$|A| = (a+b)^2 + (a-b)^2 = a^2+b^2+\cancel{2ab}+a^2+b^2-\cancel{2ab}$$
$$= 2a^2+2b^2$$

$y''z''$-coordinate

$|A| = 1 \Rightarrow$ orthogonal.

ting a
t z-axis

$$\therefore a^2+b^2 = \frac{1}{2}$$

Note: ① A matrix multiplication by a 2x2 orthogonal matrix is either a rotation or a rotation followed by a reflection about x-axis.

② If A is orthogonal then the multiplication by A is a rotation if $\det(A)=1$ and a rotation followed by a reflection if $\det(A)=-1$.

(26) Determine whether multiplication by A is a rotation or a rotation followed by reflection, about the x-axis. Find the angle of rotation.

ⓐ $A=\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \rightarrow |A|=1$

ⓑ $A=\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

Example: $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$|A|=1$, Rotation

$A=\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$|A|=-1$

reflection.

(27) Determine whether multiplication by A is rotation or a rotation followed by a reflection.

ⓐ $A=\begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$

$|A|=-1$

Reflection

ⓑ $A=\begin{bmatrix} 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$

$|A|=1$.

Rotation