Amrita School of Engineering, Bengluru-35
23MAT117-Linear Algebra
Lab Practice Sheet-2

(Left Inverse, Right Inverse, Pseudo inverse, Solution to Linear System)

Left Inverse of a matrix: If all columns of matrix A of size $m \times n$ are independent (or $rank(A) = n$), then the matrix has a left inverse such that

$$A_{LI} \times A = I_{n\times n}$$

$$A_{LI} \equiv Left \text{ inverse of A}$$

$$A_{LI} = \left(A^T A\right)^{-1} A^T$$

If all columns are independent, you may notice that 'm' should be greater than or equal to 'n'. If $m = n$, then it is a square matrix and left inverse is same as original inverse.

Perform the following experiments:

a) Create a random 4 by 3 matrices
b) Check whether $rank\ (A) = n = 3$
c) Find Left inverse of A. **Ans:** $\boldsymbol{A_{LI} = inv(A' * A) * A'}$
d) Check whether $A_{LI} * A$ produces an identity matrix.

Right Inverse of a matrix: If all rows of matrix A of size $m \times n$ are independent (or $rank(A)$ = m), then the matrix has a right inverse such that

$$A \times A_{RI} = I_{m\times m}$$

$$A_{RI} \equiv Right \text{ inverse of A}$$

$$A_{RI} = A^T \left(AA^T\right)^{-1}$$

Create such matrices and verify the result.

If all rows are independent, you may notice that 'm' should be less than or equal to 'n'. If $m = n$, then it is a square matrix and right inverse is same as original inverse.

Perform the following experiments:

a) Create a random 5 by 7 matrices
b) Check whether $rank\ (A) = m = 5$
c) Find right inverse of A. **Ans:** $\mathbf{A_{RI} = A' * inv(A * A')}$
d) Check whether $A * A_{LI}$ produces an identity matrix.

Pseudoinverse of an $m \times n$ matrix, A

Every matrix $A_{\boldsymbol{m\times n}}$ has a pseudoinverse $\boldsymbol{A^+}$ or $\boldsymbol{pinv(A)}$

- If $m = n = rank(A)$, then A is invertible and $\boldsymbol{A^{+}} = \boldsymbol{A^{-1}}$
- If $rank(A) = n$, (all columns of A are independent), then $\boldsymbol{A^{+}} = (\boldsymbol{A^{T}A})^{-1}\,\boldsymbol{A^{T}} = \boldsymbol{Left\ Inverse\ of\ A},$ and $\boldsymbol{A^{+}A} = \boldsymbol{I_n}$.
- If $rank(A) = m$, (all rows of A are independent), then $\boldsymbol{A^{+}} = \boldsymbol{A^{T}}(\boldsymbol{A^{T}A})^{-1} = \boldsymbol{Right\ Inverse\ of\ A},$ and $\boldsymbol{A^{+}A} = \boldsymbol{I_m}$.
- Pseudoinverse $\boldsymbol{A^{+}}$ of any matrix A can be obtained in MATLAB using the command '$\boldsymbol{pinv(A)}$'.

a) Generate a 9×9 random integer matrix and verify that $inv(A) = pinv(A)$.
b) Generate a 5×4 random integer matrix and verify that $pinv(A)$ is same as the left inverse of A.
c) Generate a 3×7 random integer matrix and verify that $pinv(A)$ is same as the right inverse of A.

Solutions to Linear System $\boldsymbol{AX = B}$:

Solves the system AX=b if A is invertible. Here if inv is replaced with pinv we will get the same solution.

- A = [1 2 3; 4 5 6; 9 1 3]; b = [6;15;13]; X = A\b
- A = [1 2 3; 4 5 6; 9 1 3]; b = [6;15;13]; X = inv(A)*b

- **A = [1 2 3; 4 5 6; 5 7 9]; b = [6;15;21]; pinv(A)*b**

With 'pinv' we can obtain a solution of the infinitely many solutions of a system (When A is not invertible)

Practice questions

1. Consider the matrices given below: A, B, C, D and E.

a) Which of the matrices have inverse? Find if it exists.
b) Which of the matrices will have a left inverse? Find if it exists.
c) Which of the matrices will have a right inverse? Find if it exists.
d) Find pseudo inverse of all these matrices using the command 'pinv'. Compare the answers with the answers obtained in (a), (b), (c) and (d).

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 1 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 1 & 4 & 1 \\ 0 & 0 & 8 & 1 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & 8 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

2. Solve the given linear system using MATLAB.

   $$x + y = 6;\ 2x - y = 9.$$

3. Solve the system using pinv command

   $$x + y + z = 6;\ 2x - y + z = 3;\ 3x + 2y + z = 10.$$

4. Solve the given system using pinv command.

   $$x + y + z = 6;\ 2x + y + z = 7;\ 3x + 2y + 2z = 13.$$

5. Solve the following systems of linear equations (rref command can be used in MATLAB to get the row reduced echelon form). Mention what the solution geometrically represents (a point or a line or a plane or a hyperplane).
   a) $3x+3y-z=4;\ 3x-8y+6z=7;\ x+y+10z=22$
   b) $4x-3y+2z+5w=10;\ 9x-2y-3z+6w=7;\ 2x+11y+3z-6w=13;\ 8x-3y+5z-w=14$
   c) $x-3y+2z+5w= 3;\ 2x-2y+3z+6w=11;\ 2x+11y-3z-6w=40;\ 5x+6y+2z+5w=54$
   d) $4x-3y+2z+5w=10;\ 9x-2y-3z+6w=7;\ 5x+1y-5z+w=13;\ 8x-6y+4z+10w=20$
   e) $x+y-z=7;\ 2x-2y+3z=9;\ 3x+2y-5z=10$
   f) $x-3y+2z+5w= 0;\ 2x-2y+3z+6w=0;\ 2x+11y-3z-6w=0;\ 5x+6y+2z+5w=0$
   g) $4x-3y+2z+5w=0;\ 9x-2y-3z+6w=0;\ 5x+1y-5z+w=0;\ 8x-6y+4z+10w=0$
   h) $x+y-2z=0;\ 2x-3y+z=0;\ 3x-2y-z=0$
   i) $x+y-5z+3w=0;\ 2x-3y-10z+4w=0;\ x-9y-5z+w=0;\ 4x-11y-20z+8w=0$