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8.1: Linear Transformations

In this chapter, we shall define and study linear transformations from an arbitrary vector space V to another arbitrary vector space W. The results we obtain here have important applications in physics, engineering, and various branches of mathematics.

Defn:

If $T: V \longrightarrow W$ is a function from a vector space V into a vectorspace W, then T is called a linear transformation from V to W if for all vectors u and v in V and all scalars C;

(a) $T(u+v) = T(u) + T(v)$

(b) $T(cu) = cT(u)$.

Note. In the special case where $V = W$, the linear transformation $T: V \rightarrow V$ is called a linear operator on V.

Ex-8.1:

Examples: Let $T: R^3 \longrightarrow R^3$ is defined by $T(x,y,z) = (x+y, x-y, 2x-z)$. Show that T is a linear transformation.

Solu: Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any two vectors of R^3. Consider,

$$T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$$
$$= (x_1+x_2+y_1+y_2,\ x_1+x_2-y_1-y_2,\ 2x_1+2x_2-z_1-z_2)$$
$$= ((x_1+y_1)+(x_2+y_2),\ (x_1-y_1)+(x_2-y_2),\ 2x_1-z_1+(2x_2-z_2))$$
$$= ((x_1+y_1), x_1-y_1, 2x_1-z_1) + (x_2+y_2, x_2-y_2, 2x_2-z_2)$$
$$= T(u) + T(v)$$

$\therefore T(u+v) = T(u) + T(v)$

Consider, $c \in R^3$,

$$T(cu) = T(cx_1, cy_1, cz_1) = (cx_1+cy_1,\ cx_1-cy_1,\ 2cx_1-cz_1)$$
$$= c(x_1+y_1,\ x_1-y_1,\ 2x_1-z_1) = cT(u)$$

$\therefore$ T is a linear transformation.

Ex-8.1

02) Use the definition of a linear transformation given in this section to show that the function $T: R^3 \to R^2$ given by the formula, $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 - 4x_3)$ is a linear transformation.

Solun: $T(u+v) = T[(x_1, x_2, x_3) + (y_1, y_2, y_3)]$

$= T[(x_1+y_1, x_2+y_2, x_3+y_3)]$

$= [2(x_1+y_1) - (x_2+y_2) + (x_3+y_3), (x_2+y_2) - 4(x_3+y_3)]$

$= [2x_1 - x_2 + x_3, x_2 - 4x_3] + [2y_1 - y_2 + y_3, y_2 - 4y_3]$

$= T(u) + T(v).$

and

$T(cu) = T[cx_1, cx_2, cx_3] = (2cx_1 - cx_2 + cx_3, cx_2 - 4cx_3)$

$= c[2x_1 - x_2 + x_3, x_2 - 4x_3] = cT(u)$

$\therefore$ T is a linear transformation.

Eg-2: The mapping $T: V \to W$ such that $T(v) = 0$ for every v in V is a linear transformation called the zero transformation.

$\because T(u+v) = 0, T(u) = 0, T(v) = 0, T(ku) = 0.$

Eg-3: The mapping $I: V \Rightarrow V$ defined by ~~$I(v) = v$~~ is called identity operator on V.

Eg-(4) to Eg-(13).

Note: $T[c_1v_1 + c_2v_2 + \cdots + c_nv_n] = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$

Properties of L.T:

Thm-8.1.1: If $T: V \to W$ is a L.T, then

(a) $T(0) = 0$ (b) $T(-v) = -T(v)$, for all v in V

(c) $T(v-w) = T(v) - T(w)$ for all v and w in V.

Ex-8.1:

In Ex-3-10, Determine whether the function is a linear transformation. Justify your answer.

(3) $T: V \to R$, where V is an innerproduct space, and $T(u) = \|u\|$.

Solu: Let $u = (u_1, u_2)$ $v = (v_1, v_2)$. $\|u+v\| < \|u\| + \|v\|$

$$T(u+v) = \|u+v\|$$
$$= \|(u_1+v_1, u_2+v_2)\|$$
$$< \|(u_1, u_2)\| + \|(v_1, v_2)\|$$
$$< \|u\| + \|v\|.$$

Non-linear.

(4) $T: R^3 \to R^3$, where 'v_0' is a fixed vector in R^3 and $T(u) = u \times v_0$.

Solu: It is a linear transformation.

(5) $T: M_{22} \to M_{33}$ where 'B' is a fixed 2x3 matrix and $T(A) = AB$.

Solu: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$.

~~$\therefore T(A+B) = (A+B)$~~

$$C = \begin{bmatrix} m & n \\ p & q \end{bmatrix}.$$

$$\therefore T[A+C] = (A+C)B$$
$$= AB + CB.$$
$$= T(A) + T(C)$$

& $T(kA) = (kA)B = k(AB) = kT(A)$.

$\therefore$ T is a linear transformation.

06) $T: M_{nn} \to R$, where $T(A) = tr(A)$.

Solu $T(A+B) = tr(A+B)$

$= tr(A) + tr(B)$

$= T(A) + T(B)$

&

$T(cA) = tr(cA) = c\, tr(A)$

$\therefore T$ is a linear transformation.

(7) $F: M_{mn} \to M_{nm}$ where $F(A) = A^T$.

Solu $F(A+B) = (A+B)^T$

$=$

(8) $T: M_{22} \to R$ where (a) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$

(b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$.

Solu (a) ~~Let~~ $u = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ & $v = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be two vectors of M_{22}.

$T(u+v) = T\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

$= 3(a_1+a_2) - 4(b_1+b_2) + (c_1+c_2) - (d_1+d_2)$

$= (3a_1 - 4b_1 + c_1 - d_1) + (3a_2 - 4b_2 + c_2 - d_2)$

$= T(u) + T(v)$

&

$T(cu) = T\begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} = 3(ca_1) - 4(cb_1) + (cc_1) - (cd_1)$

$= c[3a_1 - 4b_1 + c_1 - d_1)$

$= c\, T(u)$

$\therefore T$ is a linear transformation.

(b) $T(u+v) = T\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} = (a_1+a_2)^2 + (b_1+b_2)^2$

$= (a_1^2 + a_2^2 + 2a_1a_2) + (b_1^2 + b_2^2 + 2b_1b_2)$

$\neq T(u) + T(v)$.

$T(cu) = T\begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} = (ca_1)^2 + (cb_1)^2 \neq c^2(a_1^2 + b_1^2)$.

$\therefore$ T is not a linear transformation.

(9) $T: P_2 \to P_2$, where.

(a) $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x+1) + a_2(x+1)^2$

(b) $T(a_0 + a_1x + a_2x^2) = (a_0 + a_1) + (a_1+1)x + (a_2+1)x^2$

Solⁿ (a) $u = a_0 + a_1x + a_2x^2$, $v = b_0 + b_1x + b_2x^2$.

$u+v = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$.

$T(u+v) = (a_0+b_0) + (a_1+b_1)(x+1) + (a_2+b_2)(x+1)^2$

$= [a_0 + a_1(x+1) + a_2(x+1)^2] + [b_0 + b_1(x+1) + b_2(x+1)^2]$

$= T(u) + T(v)$.

& $T(cu) = ca_0 + ca_1(x+1) + ca_2(x+1)^2$

$= c[a_0 + a_1(x+1) + a_2(x+1)^2] = cT(u)$.

$\therefore$ T is a L.T

(b) $T(u+v) = (a_0+b_0+1) + (a_1+b_1+1)x + (a_2+b_2+1)x^2$

$\neq T(u) + T(v)$.

$T(cu) = (ca_0+1) + (ca_1+1)x + (ca_2+1)x^2$

$\neq cT(u)$.

$\therefore$ T is not a linear transformation.

$a - 4b + c - d$

vectors of

$d_2)$

$2)$

$-(d_1)$

$)$

(10) $T: F(-\infty,\infty) \longrightarrow F(-\infty,\infty)$, where

(a) $T(f(x)) = 1 + f(x)$ (b) $T(f(x)) = f(x+1)$

Solu: (a) $T(f(x)+g(x)) = 1 + f(x) + g(x) \neq T(f(x)) + T(g(x))$

$T(cf(x)) = 1 + cf(x) \neq cT(f(x))$

$\therefore$ T is not a linear transformation.

(b) $T[f(x)+g(x)] = f(x+1) + g(x+1)$
$= T(f(x)) + T(g(x))$.

$T(cf(x)) = cf(x+1)$

$\therefore$ T is a linear transformation.

Example-13: Let $T: M_{nn} \rightarrow R$ be the transformation that maps an nxn matrix into its determinant; i.e. $T(A) = \det(A)$. S.T T is a L.T.

Solu: If $n>1$, then this transformation does not satisfy either of the properties required of a linear transformation.

$\det(A_1+A_2) \neq \det(A_1) + \det(A_2)$ in general.

Moreover, $\det(cA) = c^n \det(A)$, so

$\det(cA) \neq c\det(A)$ in general.

Thus, T is not a linear transformation.

Finding Linear Transformations from Images of Basis vectors.

If $T: V \rightarrow W$ is a linear transformation, and if $\{v_1, v_2, ---, v_n\}$ is any basis for V, then the image $T(v)$ of any vector 'v' in V can be calculated from the images $T(v_1), T(v_2), ---, T(v_n)$ of the basis vectors.

This can be done by first expressing V as a linear combination of the basis vectors, say

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$$V = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

and then using formula (1) to write

$$T(V) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n).$$

In words, a linear transformation is completely determined by its images of any bases vectors.

Example-14: Computing with Images of Bases vectors:

Consider the basis $S = \{v_1, v_2, v_3\}$ for R^3, where

$v_1 = (1, 1, 1)$

(x+1)

(x)) + T[g(x)]

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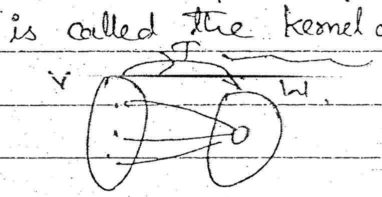
Ex-8.2: KERNEL AND RANGE

We shall discuss some basic properties of linear transformations that generalize properties of matrix transformations obtained earlier in the text.

If A is an $m \times n$ matrix, then the nullspace of A consists of all vectors x in R^n such that $Ax = 0$, and also the column space of A consists of all vectors b in R^m for which there is atleast one vector x in R^n such that $Ax = b$.

ie The nullspace of A consists of all vectors in R^n that multiplication by A maps into $\vec{0}$, and the column space of A consists of all vectors in R^m that are images of at least one vector in R^n under multiplication by A.

Defn:
If $T: V \to W$ is a linear transformation, then the set of vectors in V that T maps into 0 is called the kernel of T; it is ~~denoted~~ by $\ker(T)$.



ie $\ker(T) = \{ v \in V \,/\, T(v) = 0 \}$

Defn: The set of all vectors in W that are images under T of atleast one vector in V is called the range of T and it is denoted by $R(T)$.

ie $R(T) = \{ w \in W \,/\, T(v) = w \}$ for T.

Eg-1: Kernel and Range of a matrix transformation:
If $T_A : R^n \to R^m$ is multiplication by the $m \times n$ matrix A, then Kernel of (T_A) is the nullspace of A, and the range of (T_A) is the column space of A.

Eg-2: Kernel and Range of the zero transformation:

Let $T: V \to W$ be zero transformation i.e. $T(v) = 0$
($\because$ T maps every vector in V into $\vec{0}$,
$Ker(T) = V$, $\because$ $\vec{0}$ is the only image under T of vectors in V, we have $R(T) = \{0\}$.

Ex-3: Let $I: V \to V$ be the identity operator defined by $I(v) = v$ for all 'v' in V.
$\therefore$ $Ker(I) = \{0\}$ ($\because$ only '0' is mapped to 0)
$R(I) = V$.

Properties of Kernel and Range:

Thm-8.2.1: If $T: V \to W$ is a linear transformation, then
(a) the Kernel of T is a subspace of V.
(b) the Range of T is a subspace of W.

Ex-8.2

(1) Let $T: R^2 \to R^2$ be the linear operator given by the formula $T(x,y) = (2x-y, -8x+4y)$.
(a) Which of the following vectors are in $R(T)$?
(a) $(1,-4)$ (b) $(5,0)$ (c) $(-3,12)$

(a)
~~$T(x,y) = (1,-4)$~~
$2x - y = 1$
$-8x + 4y = -4$
$2x - y = 1$
$y = 2x - 1$.
$\forall (x,y)$ in R^2 we have an image in R^2.
$\therefore$ $R(T) = R^2$.
$\therefore$ $(1,-4) \in R^2$.

(b)
$2x - y = 5$
$-8x + 4y = 0$
$2x - y = 5$
$\Rightarrow 2x - y = 0$
$5 \neq 0$.
contradiction.
$(5,0) \notin R(T)$

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(c) $2x - y = -3$

$-8x + 4y = 12 \Rightarrow 2x - 4y = -3$.

(2) Let $T: R^2 \to R^2$ be the L. operator in Ex(1) which of the following are in Ker(T)?

(a) $(5, 10)$

$T(5,10) = 0$

$(10-10, -40+40) = 0$

$\therefore (5,10) \in K(T)$

(b) $(3,2)$

$T(3,2) = 0$

$(6-2, -24+8) \neq 0$

$\therefore (3,2) \notin K(T)$

(c) $(1,1)$. $T(1,1) = (2-1, -8+4) \neq 0$.

$\therefore (1,1) \notin K(T)$

(3) & (4) Assgn

(5) Let $T: P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = x\,p(x)$. Which of the following are in Ker(T)?

Solu: $T(p(x)) = x\,p(x) = 0$.

(a) x^2

$T(x^2) = x(x)^2 = x^3 \neq 0$. $x^2 \notin Ker(T)$

(b) 0

$T(0) = x(0) = 0 \quad \therefore 0 \in Ker(T)$

(c) $(1+x)$ $T(1+x) = x(1+x) = x^2 + x \neq 0 \quad \therefore (1+x) \notin Ker(T)$

(6) Let $T: P_2 \to P_3$ be the L.T in Exercise (5), which of the following are in R(T)?

Solu. (a) $x + x^2$

$T(p(x)) = x(x+x^2)$

$= x^2 + x^3$.

(b) $(1+x)$

$T(1+x) = x(1+x) = x + x^2$

(a) $x + x^2$

$T(p(x)) = x\,p(x) = x + x^2$

$x(1+x)$.

$\therefore p(x) = (1+x) \in P_2$

(c) $3 - x^2$.

$T(3-x^2) = x(3-x^2) = 3x - x^3$.

(b) $xp(x) = 1 + x \ldots$

$p(x)$ does not exist in P_2

(c) $xp(x) = 3 - x^2$

$\nexists\, p(x)$ in P_2.

Defn: If $T: V \to W$ is a L.T then the dimension of the range of T is called the rank of T and is denoted by rank(T); the dimension of the kernel is called the nullity of T and is denoted by nullity (T).

Thm-8.2.2 If A is an $m \times n$ matrix and $T_A: R^n \to R^m$ is multiplication by A, Then

(a) nullity (T_A) = nullity (A) (b) Rank (T_A) = rank (A)

Thm-8.2.3 Dimension Theorem for L.T.

If $T: V \to W$ is a L.T from an n-dimensional vectorspace V to a vectorspace W then $\boxed{\text{rank}(T) + \text{nullity}(T) = n}$ → dimension of the domain

(7) Find a basis for the kernel of.

(a) the linear operator is $T: R^2 \to R^2$ $T(x,y) = (2x - y, -8x + 4y)$

$T(x,y) = 0$.

$2x - y = 0$ $\quad y = 2x$

$-8x + 4y = 0$. $\quad x = 1$

$\quad y = 2$

∴ Null space is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t$.

∴ Basis for the kernel is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Nullity (T) = dimension of the kernel

$= 1$.

$$\begin{bmatrix} 4 & 1 & -2 & -3 \\ 0 & -1 & -4 & 5 \\ 0 & -6 & -24 & 54 \end{bmatrix} \sim \begin{bmatrix} 4 & 1 & -2 & -3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & -24 \end{bmatrix}$$

$\Rightarrow x_4 = 0$ & $x_2 + 4x_3 = 0$

$4x_1 \ldots x_3 = 0$

If $x_3 = t$, $x_2 = -4t$, $x_1 = 3t/2$.

(b) Linear transformation in (3) $T: R^4 \to R^3$.

$T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4)$

$$\begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$\therefore$ Null space is   Nullity$(T) = 1$.

Basis $(3/2, -4, 1, 0)$

(c) Linear Transformation in (5) $T: P_2 \to P_3$, $T(p(x)) = x\,p(x)$

Soln: $x\,p(x) = 0$.

$x[a_0 + a_1 x + a_2 x^2] = 0$.

$$[a_0 \ a_1 \ a_2] \begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix} = 0.$$

P_2 $p(x)$ $\vec{0}$ Nullity$(T) = 0$

No basis exists.

(8) Find a basis for the range of

(a) the linear operator $T: R^2 \to R^2$,

$T(x, y) = (2x - y, -8x + 4y)$

$$\begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$2c_1 - c_2 = b_1 \Rightarrow c_2 = 2c_1 - b_1 \to (1)$

$-8c_1 + 4c_2 = b_2 \to (2)$

Substituting (1) in (2) $-8c_1 + 4(2c_1 - b_1) = b_2$

$-4b_1 = b_2$

Let $b_1 = k$ $b_2 = -4k$.

$\therefore$ $\vec{b} = k\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is in the column space of A.

$\therefore$ The basis for the $R(T)$ is $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Rank$(T) = 1$.

Margin notes (left): nension of ... of T. ... nension of ... and is ... $R^n \to R^m$ is ... ank(A) ... nensional ... k(T)+ ... $= (2x-y, 8x+4y)$ ... sion of the kernel

(a) Basis for range of (T) is the column space ~~of A~~

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 11 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19/11 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix}$.

Basis for range of T is $\left\{ \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix} \right\}$

(c) Rank (T) = 2 ; Nullity (T) = 1.

(11) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(a)

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-2+2$

$\rightarrow C_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ The basis for column space is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(b) $AX = 0$

$2x_1 - x_3 = 0$ $\quad x_3 = 2x_1, \quad x_2 = b.$

$4x_1 - 2x_3 = 0$ $\quad$ let: $x_1 = a \quad x_2 = b \quad x_3 = 2a$

$$X = \begin{bmatrix} a \\ b \\ 2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ The basis for null space of A is $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Rank (T) = 1 ; Nullity (A) = 2.

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8-35-12.

space

(12) $A = \begin{bmatrix} 4 & 1 & 5 & 2 \\ 1 & 2 & 3 & 0 \end{bmatrix}$

(a) $\rightarrow \begin{bmatrix} 4 & 1 & 5 & 2 \\ 0 & -9 & -7 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 1/4 & 5/4 & 1/2 \\ 0 & 1 & 7/9 & -2/9 \end{bmatrix}$

$\rightarrow \quad c_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$c_1 = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} \quad c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(b) $Ax = 0$.

$$x_1 + \frac{x_2}{4} + \frac{5x_3}{4} + \frac{x_4}{2} = 0$$

$$x_2 + \frac{7x_3}{9} - \frac{2x_4}{9} = 0.$$

$$x_2 = \frac{-7x_3}{9} + \frac{2x_4}{9}.$$

$x_3 = t \quad x_4 = r \quad x_2 = \frac{-7}{9}t + \frac{2}{9}r.$

$$x_1 + \frac{-7}{36}t + \frac{2}{36}r + \frac{5}{4}t + \frac{1}{2}r = 0$$

$$x_1 + t\left[\frac{-7}{36} + \frac{5}{4}\right] + r\left[\frac{2}{36} + \frac{1}{2}\right] = 0.$$

$$x_1 = \frac{-13}{36}t - \frac{15}{36}r$$

$$x = \begin{bmatrix} \frac{-13}{36}t - \frac{15}{36}r \\ \frac{-7}{9}t + \frac{2}{9}r \\ t \\ r \end{bmatrix} = \begin{bmatrix} -13/36 \\ -7/9 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -15/36 \\ 2/9 \\ 0 \end{bmatrix}$$

$\frac{-7+20}{36}$

$\frac{13}{36}$

$\frac{2+13}{36}$

$\frac{15}{36}$

-2+2

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$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

13-8

2.

$$x_1 = \frac{-1}{4}\left[\frac{-7}{9}t + \frac{2}{9}r\right] - \frac{5}{4}[t] - \frac{1}{2}(r)$$

$$x_1 = t\left[\frac{7}{36} - \frac{5}{4}\right] + r\left[\frac{-2}{36} - \frac{1}{2}\right]$$

$$x_1 = \frac{-13}{36}t - r\left[\frac{+15}{36}\right]$$

7 − 20

−13 − 2

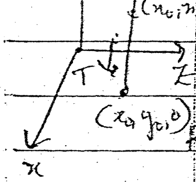
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{-13t}{36} - \frac{15}{36}r \\ -7t/9 + \frac{2r}{9} \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -13/36 \\ -7/9 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -15/36 \\ 2/9 \\ 0 \\ 1 \end{bmatrix}$$

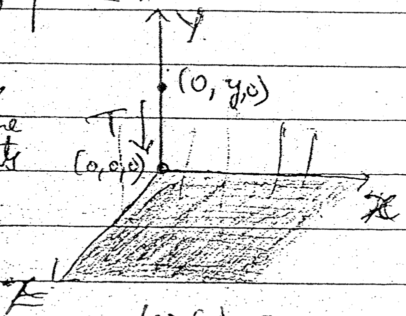
$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4/7 \\ 2/7 \\ 0 \\ 1 \end{bmatrix}$$

(13) $A = \begin{bmatrix} 1 & 4 & 5 & 0 & 9 \\ 3 & -2 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 2 & 3 & 5 & 1 & 8 \end{bmatrix}$

Take 'A' → Reduce to Row echelon form. Take the corresponding columns vectors in 'A'.

(14) Describe the kernel and range of
(a) the orthogonal projection on the xz-plane.





The kernel of T is the set of points that T maps into $\vec{O} = (0,0,0)$. These points are the z on y-axis. Since T maps every point in R^3 into the xz-plane, the range of T must be some subset of this plane.

R(T). Ker(T) is the y-axis

But every point $(x_0, 0, y_0)$ in the xz plane is the image under T of some point; i.e. it is the image of all pts on the vertical line

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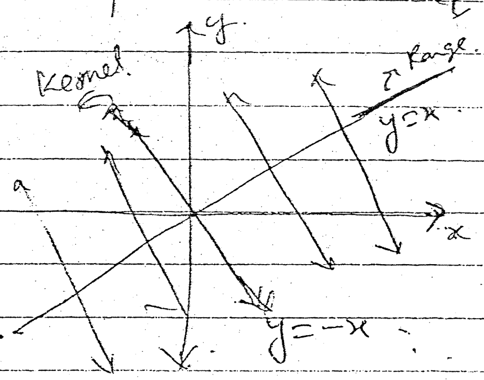
that passes through $(x_0, 0, y_0)$. Thus $R(T)$ is the entire xz-plane.

(b) The orthogonal projection on yz-plane.

Solu → Kernel $(T) = x$-axis Range $(T) = yz$ plane.

(c) The orthogonal projection on the plane with the eqⁿ $y = x$.



The kernel of T is the set of points that T maps onto $\vec{0} = (0,0,0)$

these are the points of a line passing through the origin given by $y = -x$

and whose parametric eqⁿ is

$$x = -t \;,\; y = t \;,\; z = 0.$$

Range is : line $y = x$.

(15) Let V be any vectorspace, and let $T : V \to V$ be defined by $T(v) = 3v$.

(a) what is the kernel of T.

$T(v) = 0 \Rightarrow 3v = 0 \Rightarrow v = 0$.

$\therefore K(T) = \{0\}$

(b) what is the range of T

$R(T) = V$

(16) Find the nullity of T.

(a) $T: \underset{V}{R^5} \to \underset{R}{R^7}$ has rank 3.

$R(T) = 3$. $n = 5 = \dim(V)$ (from Thm. 8.2.3 dimension Thm)

$\therefore$ Nullity $(T) = 5 - 3 = 2$.

(b) $T: P_4 \to P_3$ has rank 1.

$\dim(P_4) = 4+1 = 5 = n$ Rank $(T) = 1$.

$\therefore$ Nullity $(T) = 5 - 1 = 4$.

(c) The range of $T: R^6 \to R^3$ is R^3.

~~$R(T) = 3$~~ Rank (T) = dimension of range of $T = 3$.

$n = 6$.

$\therefore$ Nullity $(T) = 6 - 3 = 3$.

(d) $T: M_{22} \to M_{22}$ has rank $\{3\}$

$\dim(M_{mn}) = m \times n = 4 = n$.

$\therefore$ Rank $(T) = 3$.

$\therefore$ Nullity $= n -$ Rank $(T) = 1$

(17) Let A be a 7×6 matrix such that $Ax = 0$ has only the trivial sol.on, and let $T: R^6 \to R^7$ be multiplication by A. Find the rank and nullity of T.

$T: R^6 \to R^7$ is a linear transformation by multiplication of A. $AX = 0$

has only trivial solun $\Rightarrow$ null space basis $= \{0\}$

$\therefore$ Nullity $(T) = 0$.

dimension of $V = R^6 = 6 = n$.

$\therefore$ Rank $(T) = n -$ Nullity $(T) = 6 - 0 = 6$.

(18) Let A be 5×7 matrix with rank 4.

(a) What is the dimension of the solution space of $AX = 0$

(b) Is $Ax = b$ consistent for all vectors b in R^5? Explain.

?.3

Thm)

(a) Given $R(T) = 4$, ~~$n = \dim(V) = 5$~~

~~Nullity (T)~~ = dimension of Range of (T) =

$T_A : R^n \to R^m$ is multiplication by the $A_{m \times n}$
5 x 7.

$T_A : R^7 \to R^5$

$\therefore n = \dim(V) = 7$.

$\therefore$ Nullity $(T) = n - \text{Rank}(T) = 7 - 4 = 3$.

T = 3.

(b) Is $AX = b$ consistent for all vectors 'b' in R^5? Explain:

No, In order for $AX=b$ to be consistent for all 'b' in R^5, we must have $R(T) = R^5$.

But $R(T) \neq R^5$, $\because$ given $R(T)$ = dim of range of ~~T~~ $= 4$.

(21) Let $T: R^3 \to R^3$ be multiplication by $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix}$

(a) Show that the Kernel of T is a line through the origin, and find parametric eqns for it.

x=0 has
be multiplicat
T.
multiplication

Soln. $AX = 0$.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ -2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 \\ 0 & -5 & -5 \\ 0 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

basis = {0}.

$$\to \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 + 4x_3 = 0 \Rightarrow x_1 - 3x_3 + 4x_3 = 0$
$x_1 + x_3 =$

$x_2 + x_3 = 0$

$x_2 = -x_3$

$x_1 = -x_3$ $\boxed{x_3 = t}$

$$\therefore X = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

of $AX=0$

R^5? Explain.

Basis for Nullspace is $\{-1, -1, 1\}$ & parametric eqn for it is $x = -t$; $y = -t$; $z = t$.

(b) S.T the range of 'T' is a plane through the origin, find an eqn for it.

Solu: Basis of Column space of A

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 3 & 4 & 7 & b_2 \\ -2 & 2 & 0 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & -5 & -5 & b_2-3b_1 \\ 0 & 8 & 8 & b_3+2b_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & -5 & -5 & b_2-3b_1 \\ 0 & 0 & 0 & +5(b_3+2b_1)+8(b_2-3b_1) \end{array}\right]$$

$+10b_1$
$-24b_1$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & -5 & -5 & b_2-3b_1 \\ 0 & 0 & 0 & -14b_1+8b_2+5b_3 \end{array}\right]$$

$$\Rightarrow -14b_1+8b_2+5b_3=0.$$

or

$$14b_1-8b_2-5b_3=0.$$

or The eqn of the plane through the origin is

$$14x-8y-5z=0.$$

Exclude: problems 19, 20, 22-24.

(25) Let $J: P_1 \to R$ be the integration transformation $J(p)=\int_{-1}^{1} p(x)\,dx$. Describe the kernel of 'J'.

$J(p)=\int_{-1}^{1} p(x)\,dx=0 \Rightarrow$ all polynomials of the form kx.

(26) Let $D: V\to W$ be differentiation transformation $D(f)=f'(x)$, where $V=C^3(-\infty,\infty)$ & $W=F(-\infty,\infty)$. Describe kernels of $D\circ D$ and $D\circ D\circ D$.

Solu: $\ker(D\circ D)$ consists of all functions of the form $ax+b$;
$\ker(D\circ D\circ D)$ consists of all functions of the form ax^2+bx+c.

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the origin,

8.3:- Inverse Linear Transformation :

In this section we discuss some of the properties of linear transformations in general (like us functions)

A linear transformation $T: V \to W$ is said to be one-to-one if 'T' maps distinct vectors in V into distinct vectors in W.

+10e.b,
-24b,

Thm-8.3.1: If $T: V \to W$ is a linear transformation then the following are equivalent.

(a) 'T' is one-to-one

(b) The kernel of 'T' contains only the zero vector; ie, $\ker(T) = \{0\}$.

(c) Nullity $(T) = 0$.

origin is

Thm-8.3.2: If V is a finite-dimensional vector space, and $T: V \to V$ is a linear operator, then the following are equivalent.

(a) 'T' is one-to-one (b) $\ker(T) = \{0\}$

(c) Nullity $(T) = 0$ (d) The range of 'T' is V ie $R(T) = V$.

Exercise-8.3

ition

01) In each part, find $\ker(T)$, and determine whether the linear transformation 'T' is one-to-one.

the form

(a) $T: R^2 \to R^2$, $T(x,y) = (y,x)$

$T(x,y) = 0$

$(y,x) = 0 \Rightarrow x=0, y=0$ $\ker(T) = \{0\}$

$D(f) = f'(x)$

kernels of

(b) $T: R^2 \to R^2$, $T(x,y) = (0, 2x+3y)$

T is 1-1 $\Rightarrow T(x,y) = 0$ $2x+3y=0$

ax+b;

$3y = \cancel{2x}$ $2x = -3y$

m

$x = -\frac{3}{2}y$.

$y = k \Rightarrow x = -\frac{3}{2}k$.

(a) $T: R^3 \to R^2$, where $T(x,y,z) = (x+y+z,\ x-y-z)$

$T(x,y) = 0 \Rightarrow x+y+z=0,\ x-y-z=0 \Rightarrow x=0.$

$Ker(T) = \{k(0,1,-1)\}$; T is not one-to-one

(c) $T: R^2 \to R^2$; $T(x,y) = (x+y,\ x-y)$

$T(x,y) = 0 \Rightarrow x+y=0$
$\qquad\qquad\quad x-y=0.$

Solving
$x=0$
$y=0.$

$\Rightarrow T$ is 1-1

(d) $T: R^2 \to R^3$; $T(x,y) = (x, y, x+y)$

$x=0$; $y=0$ $\quad x+y=0. \Rightarrow T$ is 1-1

(e) $T: R^2 \to R^3$; where $T(x,y) = (x-y,\ y-x,\ 2x-2y)$

$x-y=0 \quad y-x=0. \quad 2x-2y=0$

$x=y=k.$ ($\cancel{x=y=0}$)

$k \neq 0$ also

$Ker(T)\{k(1,1)\}$

T is not 1-1 $\because Ker(T) \neq 0 \Rightarrow T$ is not 1-1

(2) In each part let $T: R^2 \to R^2$ be multiplication by A. Determine whether T has an inverse; if so, find

Note: $T_A : R^n \to R^n$ is matrix multiplication by A, then T_A is 1-1 iff A is invertible.

(2) In each part let $T: R^2 \to R^2$ be matrix multiplication by A. Determine whether T has an inverse, if so, find. $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(a) $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ $\qquad \det(A) = 1 \neq 0 \quad \therefore A$ has an inverse.

$A^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \frac{1}{1}$

$\therefore T^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}$

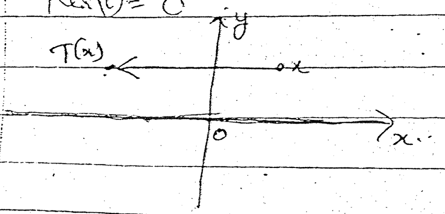
(b) Is T one to one? Justify your Conclusion.

Not $\because$ Nullity $(T) = 1$ $y = -x$, $y = -k$, $x = k$, $k\begin{pmatrix}+1\\-1\end{pmatrix}$

(6) As indicated in the accompanying figure, $T: R^2 \to R^2$ be the linear operator that reflects each point about the y-axes.

(a) Find the Kernel of T

$Ker(t) = 0$



(b) Is T one-to-one? Justify your conclusion.

1-1.

$\because T\{0,0\} = 0$

$Ker(t) = 0$

$\Rightarrow T$ is 1-1

(7) In each part use the given information to find whether the L.T is 1-1.

(a) $T: R^n \to R^n$, nullity $(T) = 0$. $\Rightarrow$ 1-1

(b) $T: R^n \to R^n$; rank $(T) = n-1 \Rightarrow$ No (even if $R(T) = R^n$ $N(T) = 1$)

(c) $T: R^m \to R^n$, $n < m$ No $n = \dim(v)$

(d) $T: R^n \to R^n$, $R(T) = R^n$ (by Rank-Nullity Thm of transformation) 1-1

(8) In each part determine whether the L.T T is 1-1.

(a) $T: P_2 \to P_3$, $T(a_0 + a_1x + a_2x^2) = x(a_0 + a_1x + a_2x^2)$

(1-1)

(b) $T: P_2 \to P_2$, where $T(p(x)) = p(x+1)$

(1-1) $x \to x+1$

(9) Ass

Ex (7) Let $T: R^3 \to R^3$ be the linear operator defined by the formula

$T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 - 4x_2 + 3x_3, 5x_1 + 4x_2 - 2x_3)$

Determine whether T is 1-1; if so, find $T^{-1}(x_1, x_2, x_3)$.

Sol: The matrix for T is

$$[T] = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \quad |T| \neq 0 \Rightarrow [T]^{-1} \text{ exists}$$

The matrix for $[T^{-1}]$ is

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = [T^{-1}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 2x_2 - 3x_3 \\ -11x_1 + 6x_2 + 9x_3 \end{bmatrix}$$

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(b) $\begin{bmatrix} 6 & -3 \\ 4 & -2 \end{bmatrix}$ $\det(A) = -12 + 12 = 0$.

$\therefore$ 'A' has no inverse.

(c) $\begin{bmatrix} 4 & 7 \\ -1 & 3 \end{bmatrix}$ $\det(A) = 12 + 7 = 19$.

$$A^{-1} = \frac{1}{19}\begin{bmatrix} 3 & -7 \\ 1 & 4 \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/19 & -7/19 \\ 1/19 & 4/19 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{19}x_1 - \frac{7}{19}x_2 \\ \frac{1}{19}x_1 + \frac{4}{19}x_2 \end{bmatrix}$$

03) In each part let $T: R^3 \to R^3$ be multiplication by A. Determine whether T has an inverse; if so, find $T^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

(a) $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ No inverse

$\det(A) = -1 - 5 + 2(1+2)$
$= -6 + 6 = 0$.

(b) $A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$\det(A) = -8$.

$$A^{-1} = \begin{bmatrix} 1/8 & 1/8 & -3/4 \\ 1/8 & 1/8 & 1/4 \\ -3/8 & 5/8 & 1/4 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(4) In each part determine whether multiplication by A is one-to-one linear transformation.

(a) $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$ Not

(b) $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & -1 & 2 & 4 \\ -1 & 3 & 0 & 0 \end{bmatrix}$ Not

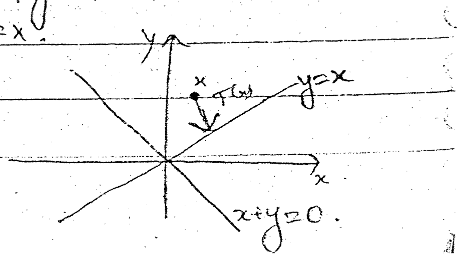
(c) $A = \begin{bmatrix} 4 & -2 \\ 1 & 5 \\ 5 & 3 \end{bmatrix}$ Yes

$\text{Nullity}(A) = 0 \Rightarrow A$ is 1-1.

(5) As indicated in the accompanying figure, let $T: R^2 \to R^2$ be the orthogonal projection on the line $y = x$.

(a) Find the kernel of 'T'

Set of points on $x+y=0$ are projected onto $y=x$. Bases $= \left\{ k\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

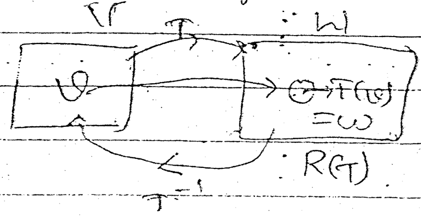


$\Rightarrow T^{-1}(x_1, x_2, x_3) = (4x_1 - 2x_2 - 3x_3, -11x_1 + 6x_2 + 9x_3, -12x_1 + 7x_2 + 10x_3)$

Inverse Linear Transformation :.

Let $T: V \to W$ be a 1-1 L.T, then the inverse linear transformation is inverse of T is denoted by T^{-1} and is given by $T^{-1}: R(T) \to V$, defined by

$T^{-1}[T(v)] = T^{-1}(w) = v$.



Note: If $R(T) = W$ Then $T^{-1}: W \to V$.

Thm-8.3.3: If $T_1 : U \to V$; $T_2 : V \to W$ are 1-1 linear transformations, then: (a) $T_2 \circ T_1$ is 1-1 (b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

(10) In each part determine whether the linear operator $T: R^n \to R^n$ is 1-1, if so, find $T^{-1}(x_1, x_2, ---, x_n)$.

(a) $T(x_1, x_2, x_3, ---, x_n) = (0, x_1, x_2, ---, x_{n-1})$

~~Sol~~ T.P $\Rightarrow$ T is 1-1

$T(x_1, x_2, ---, x_n) = 0$

$\Rightarrow (0, x_1, x_2, --, x_{n-1}) = 0$.

$0 = 0 \quad x_1 = 0, x_2 = 0 \; --- \; x_{n-1} = 0$ ~~but x_n need~~ not be zero.

$\therefore T$ is not 1-1

(b) $T(x_1, x_2, ---, x_n) = (x_n, x_{n-1}, ---, x_2, x_1)$

Solve $T(x_1, x_2, ---, x_n) = 0$

$\Rightarrow (x_n, x_{n-1}, ---, x_2, x_1) = 0$

$x_n = 0, \; x_{n-1} = 0, \; ---, \; x_2 = 0, \; x_1 = 0$

$\therefore T$ is 1-1.

and hence $T^{-1}(x_n, x_{n-1}, --, x_2, x_1) = (x_1, x_2, ---, x_n)$

but $T^{-1}(x_1, x_2, --, x_n) = (x_n, x_{n-1}, x_{n-2}, ---, x_1)$

(c) $T(x_1, x_2, ..., x_n) = (x_2, x_3, ---, x_n, x_1)$.

$\Rightarrow T(x, y) = 0 \Rightarrow (x_2, x_3, ---, x_n, x_1) = 0$

$\Rightarrow x_n = 0$ #n

and hence $T^{-1}(x_1, x_2, x_3, ---, x_n) = (x_n, x_1, x_2, ---, x_{n-1})$

Note: $[A] =$ $T(x_1, x_2, x_3) = AX$

$T^{-1}[X] = T^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (A^{-1})X = .$

(11) Let $T: R^n \to R^n$ be the linear operator defined by the formula $T(x_1, x_2, \dots, x_n) = (a_1 x_1, a_2 x_2, \dots a_n x_n)$

(a) Under what conditions will T have an inverse?

(b) Assuming that the conditions determined in part (a) are satisfied, find a formula for $T^{-1}(x_1, x_2, \dots, x_n)$.

Solu: $[a_1, a_2 \dots a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$ or $\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & & & & a_n \end{bmatrix}$

If $a_i \neq 0$, $i = 1, 2, \dots, n$.

(b) $T^{-1}(x_1, x_2, x_3, \dots, x_n) = \left(\frac{1}{a_1}x_1, \frac{1}{a_2}x_2, \dots \frac{1}{a_n}x_n\right)$.

$T^{-1}(a_1 x_1) = x_1$

$T^{-1}(x_1) = \frac{1}{a_1}x_1$

(12) Let $T_1: R^2 \to R^2$ & $T_2: R^2 \to R^2$ be the linear operators given by the formulas $T_1(x, y) = (x+y, x-y)$,
$T_2(x, y) = (2x+y, x-2y)$

(a) Show that T_1 and T_2 are 1 to 1.

Solu: $\ker\{T_1, T_2\} = 0$

given $T_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $T_2 = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

$\det(A) \neq 0 \Rightarrow A$ is invertible $\Rightarrow T$ is 1-1.

(b) Find the formulas for $T_1^{-1}(x, y)$, $T_2^{-1}(x, y)$ & $(T_2 \circ T_1)^{-1}(x, y)$.

Solu: $[T_1^{-1}] = [T_1]^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

or $T_1^{-1}(x, y) = T_1^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}(x+y, x-y)$

$-1-1 = -2$

$-4-1 = -5$

$9+1 = 10$

(c)

(B)

(a)

Solu

$P = T^{-1}(xP)$

$T^{-1}(P) = \frac{P}{x}$

(38)

$(T_2^{-1}) = (T_2)^{-1} = \frac{1}{-5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$.

$\therefore (T_2^{-1})(x,y) = [T_2]^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}(2x+y, x-2y)$

(c) Verify $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

$(T_2 \circ T_1)(x,y) = T_2(T_1(x,y)) = T_2(x+y, x-y)$

$= [2(x+y)+(x-y), (x+y)-2(x-y)]$

$= [3x+y, -x+3y]$

$= \underbrace{\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}}_{f}\begin{bmatrix} x \\ y \end{bmatrix}$.

$T_1^{-1} \circ T_2^{-1} = T_1^{-1}[T_2^{-1}(x,y)]$

$= T_1^{-1}[\frac{1}{5}(2x+y, x-2y)]$

$= \frac{1}{2}\times\frac{1}{5}(2x+y+x-2y, 2x+y-x+2y)$

$= \frac{1}{10}(3x-y, x+3y)$

$(f^{-1}) = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}[3x-y, x+3y]$

(B) Let $T_1 : P_2 \to P_3$ & $T_2 : P_3 \to P_3$ be the L.T's given by $T_1(p(x)) = xp(x)$, $T_2(p(x)) = p(x+1)$

(a) find $T_1^{-1}(p(x))$, $T_2^{-1}(p(x))$, $(T_2 \circ T_1)^{-1}(p(x))$.

Solun $T_1^{-1}(p(x)) = T^{-1}[p] = \frac{p}{x}$.

$x \quad x+1$

$x-1 \quad x$

$p = T^{-1}(xp)$

$T_2^{-1}[p(x)] = \cancel{B}[p(x-1)]$

$p(x-1) \quad p(x+1-1)$

$p(x)$

$T^{-1}(p) = \frac{p}{x}$

$(T_2 \circ T_1)^{-1}(p(x)) = ?$

$(T_2 \circ T_1)(p(x)) = T_2[T_1(p(x))] = T_2[x\,p(x)]$

$f = p(x+1)$.

$T_2 \circ T_1$

$xp(x)$ $xp(x)$ $p(x+1)$ $p(x)$

$9+1$
$=10$

$(T_2 \circ T_1)^{-1}(p(x)) = (\cancel{p(x+1)}) \frac{1}{x}p(x-1)$

(b) Verify that $[T_2 \circ T_1]^{-1} = T_1^{-1} \circ T_2^{-1}$.

(14)

Assign

(15) Let $T : P_1 \to R^2$ be the $T(p(x)) = (p(0), p(1))$
$= (a_0, a_0 + a_1)$

(a) Find $T(1-2x)$

$p_1(x) = a_0 + a_1 x$
$p(0) = a_0$
$p(1) = a_0 + a_1$

$T(1-2x)$

$1 - 2x = a_0 + a_1 x$
$a_0 = 1, \; a_1 = -2.$

$T(1-2x) = (1, 1-2) = (1, -1)$

(b) S.T 'T' is L.T.

(c) S.T 'T' is 1-1

(d) find $T^{-1}(2,3)$, sketch its graph.

(28)

~~$T(a_0 + a_1 x) = (2,3)$~~

$T(a_0, a_0 + a_1) = (2,3)$

$a_0 = 2, \; a_0 + a_1 = 3.$

$a_1 = 3 - 2 = 1$

Solve

$\therefore T(2,1) = (2,3)$

$\therefore (T^{-1})(2,3) = (2,1)$

ie $a_0 + a_1 x$

$= 2 + x \Rightarrow$ graph it

Exclude 16, 17, 19.

(18) Let $T : R^2 \to R^2$ be the linear operator given by
$T(x,y) = (x + ky, -y)$. S.T 'T' is 1 to 1 for every

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real value of 'k' & $T^{-1} = T$.

$Ker(T) = 0$. $T(x, y) = 0$

$x + ky = 0 \quad -y = 0$.

$\Rightarrow x = 0 \quad x \,\&\, y = 0$

$\therefore \Rightarrow$ T is 1 to 1 for every value of k.

$T = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} \quad T^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix} \quad -1$

$= \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$

$\Rightarrow T^{-1} = T$

(98) Let $J : P_1 \rightarrow R$ be the integration transformation $J(p) = \int_{-1}^{1} p(x)\,dx$. Determine whether J is 1 to 1.

Solu $Ker(J) = 0$

$\therefore \int_{-1}^{1} p(x)\,dx = 0$

$\int_{-1}^{1} [a_0 + a_1 x]\,dx = 0$.

$\left[a_0 x + a_1 \frac{x^2}{2}\right]_{-1}^{1} = 0$.

$x\left[a_0 + \frac{a_1}{2}x\right]_{-1}^{1} = 0$.

$a_0[1+1] + a_1[0-0] = 0$.

$\times$. $a_0 = 0$

all polynomials of the form Kx.

p(1)

+a1)

or given by

for every

(21) Let V be the vector space $C^1[0,1]$ & let $T: V \to R$ be defined by $T(f) = f(0) + 2f'(0) + 3f'(1)$. Verify that T is a Linear transformation. Determine whether T is 1 to 1.

Solu:

$\because f(x) = x^2(x-1)^2$ is in its kernel.

Not 1-1.

Ex-8.4:

Matrices of general Linear Transformations:

Let $T: V \to W$ be the linear transformation from n-dimensional vector space V into m-dimensional vector space W. Let B and B' are bases for V and W resp, then for each vector x in V, the coordinate matrix $[x]_B$ will be a vector in R^n & the co-ordinate matrix $[T(x)]_{B'}$ will be in R^m, Then

$$A[x]_B = [T(x)]_{B'} \quad \text{——} \quad (1)$$

where A is the matrix for T with respect to the bases B and B'. If $B = \{u_1, u_2, ---, u_n\}$ & $B' = \{v_1, v_2, ---, v_n\}$ then

$$A = \left[[T(u_1)]_{B'} \;\middle|\; [T(u_2)]_{B'} \;\middle|\; ---- \;\middle|\; [T(u_n)]_{B'} \right]$$

and A is denoted by $[T]_{B',B}$ & $\boxed{(1) \Rightarrow [T]_{B',B}[x]_B = [T(x)]_{B'}}$

Example-1: Matrix for a linear Transformation.

Let $T: P_1 \to P_2$ be the linear transformation defined by $T(p(x)) = xp(x)$. Find the matrix for T with respect to the standard bases,

$$B = \{u_1, u_2\} \quad \& \quad B' = \{v_1, v_2, v_3\}$$

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where $u_1 = 1$, $u_2 = x$; $v_1 = 1$; $v_2 = x$, $v_3 = x^2$

Solu: From the given formula for T, we obtain

$T(u_1) = T(1) = (x)(1) = x$; $T(u_2) = T(x) = (x)(x) = x^2$

we find the co-ordinate vector for $T(u_1)$ & $T(u_2)$ w.r.t the bases $B' = \{v_1, v_2, v_3\}$.

$T(u_1) = x = c_1(1) + c_2(x) + c_3(x^2)$

$\Rightarrow c_2 = 1 \quad c_1 = c_3 = 0$.

$\therefore [T(u_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[T(u_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thus the matrix for T with respect to B and B' is

$$[T]_{B',B} = \left[[T(u_1)]_{B'} \;\middle|\; [T(u_2)]_{B'} \right] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example-2:

Let $T: P_1 \to P_2$ be the linear transformation in Example-1. Show that the matrix $[T]_{B',B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ satisfies $[T]_{B',B}[x]_B = [T(x)]_{B'}$ for every vector $X = a + bx$ in P_1.

Solu: $T(x) = x(p(x)) = ax + bx^2$.

The coordinate vector of X w.r.t bases B is

$[X]_B = [ax+b]_B = \begin{bmatrix} a \\ b \end{bmatrix}$ and.

The coordinate vector of $T(X)$ w.r.t bases B' is

$[T(X)]_{B'} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$

$\therefore \ [T]_{B',B}\,[X]_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = [T(X)]_{B'}$ ⓘ

Note: If $B = B'$, then $A = [T]_{B,B}$ and it is denoted by $[T]_B$, Then ⓘ $\Rightarrow$

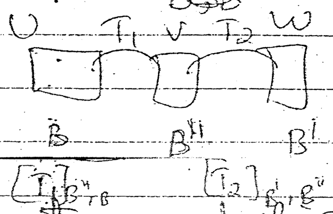
$$[T]_B\,[x]_B = [T(x)]_B$$

where $[T]_B = \Big[\, [T(u_1)]_B \;\big|\; [T(u_2)]_B \;\big|\; \cdots \;\big|\; [T(u_n)]_B \,\Big]$

Theorem - 8.4.1:
If $T : R^n \to R^m$ is a linear transformation and if B and B' are standard bases for R^n and R^m, respectively then $[T]_{B',B} = [T]_B$.

Thm - 8.4.2: If $T_1 : U \to V$ & $T_2 : V \to W$ are linear transformations and if B, B'' and B' are bases of U, V & W respectively then ~~$[T_2 \circ T_1]_{B',B}$~~

$$[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}\,[T_1]_{B'',B}.$$

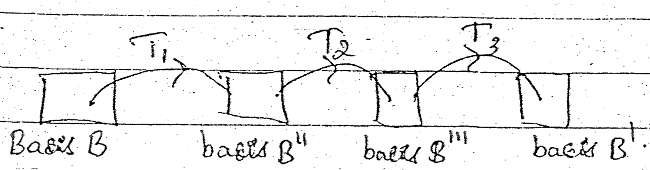


Thm - 8.4.3: If $T : V \to V$ is a linear operator, and if B is a basis for V, Then the following statement are equivalent.

ⓐ T is 1-1 ⓑ $[T]_B$ is invertible

Note: ① when ⓐ & ⓑ holds $[T^{-1}]_B = [T]_B^{-1}$

② If



then

$$[T_3 \circ T_2 \circ T_1]_{B',B} = [T_3]_{B',B'''}\,[T_2]_{B''',B''}\,[T_1]_{B'',B}$$

Exercise : 8.4

(1) Let $T: P_2 \to P_3$ be the linear trans defind by $T(p(x)) = x\,p(x)$.

(a) Find the matrix for T with respect to the standard bases $B=\{u_1, u_2, u_3\}$ and $B'=\{v_1, v_2, v_3, v_4\}$ where
$u_1 = 1$, $u_2 = x$, $u_3 = x^2$,
$v_1 = 1$, $v_2 = x$; $v_3 = x^2$, $v_4 = x^3$.

Solu: $T(u_1) = x$; $T(u_2) = x^2$; $T(u_3) = x^3$.
we find the co-ordinate vectors for $T(u_1), T(u_2)$ & $T(u_3)$ w.r.t the Bases $B'=\{v_1, v_2, v_3, v_4\}$.

$T(u_1) = x = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 \Rightarrow x = c_1 + c_2x + c_3x^2 + c_4x^3$
$\Rightarrow \boxed{c_2 = 1}$

$\therefore [T(u_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$T(u_2) = x^2 = c_1 + c_2x + c_3x^2 + c_4x^3 \Rightarrow [T(u_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$T(u_3) = x^3 \Rightarrow [T(u_3)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ The matrix for T w.r.t to B & B' is

$$[T]_{B',B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Verify that the matrix $[T]_{B',B}$ obtained in part (a) satisfies $[T]_{B',B}[x]_B = [T(x)]_{B'}$ for every vector $x = c_0 + c_1x + c_2x^2$ in P_2.

Solu $[X] = c_0 + c_1x + c_2x^2 \Rightarrow [X]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ &.

$T(X) = c_0x + c_1x^2 + c_2x^3 \Rightarrow [T(X)]_{B'} = \begin{bmatrix} 0 \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}$

$\therefore [T]_{B',B}[X]_B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}$

Assign ② Let $T: P_2 \to P_1$

③ Let $T: P_2 \to P_2$ be the linear operator defined by $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x-1) + a_2(x-1)^2$.

ⓐ find the matrix for T w.r.t standard basis $B = \{1, x, x^2\}$ for P_2.

ⓑ Verify that the matrix $[T]_B$ obtained in ⓐ satisfies $[T]_B [X]_B = [T(x)]_B$ for every vector $X = a_0 + a_1x + a_2x^2$ in B

Solu:- ⓐ $u_1 = 1$, $u_2 = x$ $u_3 = x^2$

~~$T(u_1) = B$~~ ~~$T(u_2) = a_0$~~

~~$T(1) = a_0$~~ $T(u_1) = 1$ $T(x) = (x-1)$

~~$1 = a_0$~~ $T(x^2) = (x-1)^2 = x^2 - 2x + 1$

$T(u_1) = 1 = c_1 + c_2x + c_3x^2$

$\Rightarrow c_1 = 1$ $c_2 = c_3 = 0$ $\Rightarrow [T(u_1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(u_2) = (x-1) = c_1 + c_2x + c_3x^2$

$\Rightarrow c_1 = -1$ $c_2 = 1$ $c_3 = 0$.

$\Rightarrow [T(u_2)]_B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$T(u_3) = x^2 - 2x + 1 = c_1 + c_2x + c_3x^2$

$\Rightarrow c_1 = 1$ $c_2 = -2$ $c_3 = 1$.

$[T(u_3)]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ The matrix for T is $[T]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

ⓑ $[X] = a_0 + a_1x + a_2x^2$ $[X]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$

$T(X) = a_0 + a_1(x-1) + a_2(x-1)^2 = a_0 + a_1(x-1) + a_2(x^2 - 2x + 1)$

$= (a_0 - a_1 + a_2) + (a_1 - 2a_2)x_2 + a_2x^2$

$[T(X)]_B = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_1 - 2a_2 \\ a_2 \end{bmatrix}$ $\therefore [T]_B [X]_B = [T(x)]_B$