, $\hat{j}$ and $\hat{k}$
linear

Ex-5.2:

07) Which of the following are linear combinations of $\vec{u} = (0, -2, 2)$ and $\vec{v} = (1, 3, -1)$?

a) $w = (2, 2, 2)$ Yes b) $(3, 1, 5)$ Yes c) $(0, 4, 5)$ No d) $(0, 0, 0)$ Yes.

08) Express the following as a linear combination of $\vec{u} = (2, 1, 4)$, $\vec{v} = (1, -1, 3)$ and $\vec{w} = (3, 2, 5)$.

$\vec{u}$ & $\vec{v}$
f $\vec{u}$ & $\vec{v}$.

a) $z = (-9, -7, -15)$

$z = c_1 u_1 + c_2 v_2 + c_3 w$

$(-9, -7, -15) = c_1(2, 1, 4) + c_2(1, -1, 3) + c_3(3, 2, 5)$

$$\left.\begin{array}{l} 2c_1 + c_2 + 3c_3 = -9 \\ c_1 - c_2 + 2c_3 = -7 \end{array}\right\} \Rightarrow 3c_1 + 5c_3 = -16$$

$$4c_1 + 3c_2 + 5c_3 = -15$$

-7.

$\Rightarrow z = -2u + v - 2w$

b) $(6, 11, 6) \Longrightarrow z = 4u - 5v + w$.

c) $(0, 0, 0) \Rightarrow 0u + 0v + 0w$ d) $(7, 8, 9) = 0u - 2v + 3w$

09) Express the following as linear combination of, $p_1 = 2 + x + 4x^2$, $p_2 = 1 - x + 3x^2$ & $p_3 = 3 + 2x + 5x^2$

a) $-9 - 7x - 15x^2$.

$z = c_1 p_1 + c_2 p_2 + c_3 p_3$.

$2c_2 = 8$

$-9 - 7x - 15x^2 = c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(3 + 2x + 5x^2)$

Equating the constants : $-9 = 2c_1 + c_2 + 3c_3$

Equating the coefficient of x : $-7 = c_1 - c_2 + 2c_3$

Equating the " " x^2 : $-15 = 4c_1 + 3c_2 + 5c_3$.

$-10 + 6$

$c_1 = -2$; $c_2 = 1$; $c_3 = -2$.

$7 - 4$
$7 - 1$

$\therefore z = -2p_1 + p_2 - 2p_3$.

b) $6+11x+6x^2$ c) 0 d) $7+8x+9x^2$

Sol: $4p_1-5p_2+p_3$ $0p_1+0p_2+0p_3$ $0p_1-2p_2+3p_3$.

10) Which of the following are linear Combinations of,

$$A=\begin{bmatrix}4 & 0\\-2 & -2\end{bmatrix},\quad B=\begin{bmatrix}1 & -1\\2 & 3\end{bmatrix},\quad C=\begin{bmatrix}0 & 2\\1 & 4\end{bmatrix}\ ?$$

a) $P=\begin{bmatrix}6 & -8\\-1 & -8\end{bmatrix}$ Yes   b) $P=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ Yes   c) $P=\begin{bmatrix}6 & 0\\3 & 8\end{bmatrix}$ Yes   d) $P=\begin{bmatrix}-1 & 5\\7 & 1\end{bmatrix}$ No

$c_1=1,\ c_2=2,\ c_3=-3$

Spanning:

Thm: If $v_1, v_2, \ldots, v_r$ are vectors in a vector space V, then

a) The set W of all linear combinations of $v_1, v_2, \ldots, v_r$ is a subspace of V.

b) W is a smallest subspace of V that contains $v_1, v_2, \ldots, v_r$ in the sense that every other subspace of V that contains $v_1, v_2, \ldots, v_r$ must contain W.

Defn:

If $S=\{v_1, v_2, \ldots, v_r\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of linear Combinations of the vectors in S is called the space spanned by $v_1, v_2, \ldots, v_r$, and we say that the vectors $v_1, v_2, \ldots, v_r$ span W.

To indicate that W is the subspace spanned by the vectors in $S=\{v_1, v_2, \ldots, v_r\}$, we write

$$W=\text{span}(S) \quad \text{or} \quad W=\text{Span}\{v_1, v_2, \ldots, v_r\}$$

Eg-2:

Describe the span of each of the following sets of vectors.

a) $v_1=(1,0,0)$ and $v_2=(0,1,0)$

b) $v_1=(1,0,1,0)$ and $v_2=(0,1,0,-1)$

Solⁿ: (a) Span $\{v_1, v_2\}$, is the set of all linear combinations, And we can write the general linear combination for these two vectors,

$$a v_1 + b v_2 = (a,0,0) + (0,b,0) = (a,b,0)$$

So, span $\{v_1, v_2\}$ will be all of the vectors from R^3 that are in the form $(a,b,0)$ for any choice of 'a' & 'b'.

(B)

$$a v_1 + b v_2 = (a,0,a,0) + (0,b,0,-b) = (a,b,a,-b)$$

So, span $\{v_1, v_2\}$ will be all the vectors from R^4 of the form $(a,b,a,-b)$ for any choices of 'a' and 'b'.

Eg-3:

Qⁿ ~~Describe~~ Determine a set of vectors that will exactly span each of the following vector spaces.

a) R^n:

We need to determine a possible set of spanning vectors show that every vector from the vector space is in the span of set of vectors. Next we need to show that each vector in the spanning set will also be in the vector space.

⇒ Any vector from R^n can be written as a linear combination of the standard basis $e_1 = (1,0,\dots,0)$ $e_2 = (0,1,0,\dots,0), \dots, e_n$ and so the span of the standard basis vectors will contain all of R^n. However since any linear combination of the standard basis vectors is in R^n, we can say that R^n must also contain the span of the standard basis vectors.

b) P_n :

We choose set of vectors (polynomials) as

$v_0 = 1,\ v_1 = x,\ v_2 = x^2 \ldots,\ v_n = x^n$; such that a linear combination of these is a polynomial of deg n or less and so will be in P_n.

$\therefore$ The span $\{1, x, x^2 \ldots, x^n\}$ will be contained on P_n.

Illy we can write a general polynomial of degree n or less,

$$p = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n .$$

as the following linear Combination

$$p = a_0 v_0 + a_1 v_1 + \cdots + a_n v_n .$$

$\therefore$ P_n is contained on the span of these vectors.

$$\therefore \quad P_n = \text{Span}\{1, x, \ldots, x^n\}.$$

Ex-4:

Determine if the following sets of vectors will span R^3.

a) $v_1 = (2,0,1)$, $v_2 = (-1,3,4)$ and $v_3 = (1,1,-2)$.

Solu: We must determine whether an arbitrary vector $\vec{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear Combi.

$\vec{b} = c_1 v_1 + c_2 v_2 + c_3 v_3$ of the vectors v_1, v_2 & v_3.

Expressing this vector in components gives,

$$\begin{aligned} c_1 + c_2 + 2c_3 &= b_1 \\ c_1 + 0c_2 + c_3 &= b_2 \\ 2c_1 + c_2 + 3c_3 &= b_3 . \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

[Note: a system of eqn $Ax = B$ is consistent iff det $(A) \neq 0$ (determinant of coefficient matrix $\neq 0$)]. Invertibility condn.

$\det(A) = -24 \neq 0$.

$\therefore$ This system will have a soln (consistent).
for every choice of $\vec{b} = (b_1, b_2, b_3)$.

Hence $\text{Span}\{v_1, v_2, v_3\} = R^3$.

[Note: spanning sets need not be unique]
[we can have more than one set of vectors span the same vector space]

b) $v_1 = (1, 2, -1)$, $v_2 = (3, -1, 1)$ and $v_3 = (-3, 8, -5)$

Soln: Let $\vec{b} = (b_1, b_2, b_3)$ any vector $\in R^3$, can be expressed as a linear combination.

$$\vec{b} = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\Rightarrow \begin{aligned} c_1 + 3c_2 - 3c_3 &= b_1 \\ 2c_1 - c_2 + 8c_3 &= b_2 \\ -c_1 + c_2 - 5c_3 &= b_3 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 3 & -3 \\ 2 & -1 & 8 \\ -1 & 1 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\Rightarrow \det(A) = 0$.

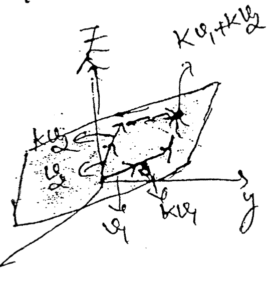
$\Rightarrow$ The system will not have a solution (inconsistent) and so $\vec{b} = (b_1, b_2, b_3)$ cannot be written as a linear combination of these three vectors.

Note:-

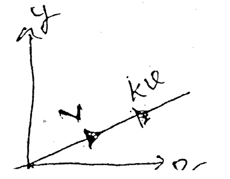
We can't just write down any set of three vectors and expect to get those three vectors to span R^3.

Ex-10: Spaces spanned by one or two vectors.

If v_1 & v_2 are noncollinear vectors in R^3 with their initial points at the origin, then span$\{v_1, v_2\}$ which consists of all linear combinations $k_1 v_1 + k_2 v_2$, is the plane determined by v_1 & v_2



|||ly if 'v' is a non-zero vector in R^2 or R^3 then span$\{v\}$, which is the set of all scalar multiples



Thm: 5.2.4

If $S=\{v_1, v_2, \dots, v_r\}$ and $S'=\{w_1, w_2, \dots, w_k\}$ are two set of vectors in a vector space V, then

$$\text{Span}\{v_1, v_2, \dots, v_r\} = \text{Span}\{w_1, w_2, \dots, w_k\}$$

if and only if each vector in S is a linear combination of those in S' and each vector in S' is a " " of " in S.

Exercise - 5.2

11) In each part determine whether the given vectors span R^3.

a) $V_1=(2,2,2)$, $V_2=(0,0,3)$, $V_3=(0,1,1)$.

Solⁿ: Consider $\vec{b}=c_1 v_1 + c_2 v_2 + c_3 v_3$.

$$\Rightarrow \begin{aligned} 2c_1 + 0c_2 + 0c_3 &= b_1 \\ 2c_1 + 0c_2 + c_3 &= b_2 \\ 2c_1 + 3c_2 + c_3 &= b_3 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2(-3) - 0 + 0 = -6 \neq 0.$$

$\therefore$ The system will have a solution for any choice of $\vec{b}$. and hence $\text{Span}\{v_1, v_2, v_3\} = R^3$.

b) $v_1=(2,-1,3)$, $v_2=(4,1,2)$, $v_3=(8,-1,8)$

$\det(A)=0 \Rightarrow$ The vectors do not span R^3

c) $v_1=(3,1,4)$, $v_2=(2,-3,5)$, $v_3=(5,-2,9)$, $v_4=(1,4,-1)$

Solⁿ: Consider $\vec{b}=c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$.

$$\begin{bmatrix} 3 & 2 & 5 & 1 \\ 1 & -3 & -2 & 4 \\ 4 & 5 & 9 & -1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} \multicolumn{4}{c|}{A} & B \\ 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 3 & 2 & 5 & 1 & b_1 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 11 & 11 & -11 & b_1 - 3b_2 \\ 0 & 17 & 17 & -17 & b_3 - 4b_2 \end{array}\right]$$

~~$R_2 \to R_2/11$~~ $\quad R_3 \to 11R_3 - 17R_2$

$$\left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 11 & 11 & -11 & 17(b_1 - 3b_2) \\ 0 & 0 & 0 & 0 & 11(b_3 - 4b_2) \end{array}\right].$$

A $\qquad\qquad$ B.

$\text{Rank } A = 2 \neq \text{Rank}(A \vdots B) = 3.$

Inconsistent system.

$\therefore$ The vectors do not span R^3

d) $v_1 = (1, 2, 6)$, $v_2 = (3, 4, 1)$, $v_3 = (4, 3, 1)$, $v_4 = (3, 3, 1)$

Soln.

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 3 & b_1 \\ 2 & 4 & 3 & 3 & b_2 \\ 6 & 1 & 1 & 1 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & 4 & 3 & b_1 \\ 0 & -2 & -5 & -3 & b_2 - 2b_1 \\ 0 & -17 & -23 & -17 & b_3 - 6b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_3 - 17R_2} \left[\begin{array}{cccc|c} 1 & 3 & 4 & 3 & b_1 \\ 0 & -2 & -5 & -3 & b_2 - 2b_1 \\ 0 & 0 & 39 & 17 & 2b_3 - 17b_2 + 22b_1 \end{array}\right].$$

$\text{Rank}(A) = 3 = \text{Rank}(A \vdots B).$

$\therefore$ The vectors span R^3.

12) Let $f = \cos^2 x$, $g = \sin^2 x$, which of the following lie in the space spanned by 'f' and g?

a) $\cos(2x)$ $\qquad$ (b) $3 + x^2$ $\qquad$ (c) 1 $\qquad$ (d) $\sin x$ $\qquad$ (e) 0

a) $\cos(2x) \Downarrow \cos^2 x - \sin^2 x.$ lies (Yes)

(b) No.

(c) $1 = \cos^2 x + \sin^2 x$ (Yes)

(d) No

(e) $0 = 0.\sin^2 x + 0.\cos^2 x = 0$

13) Determine whether the following polynomials span P_2.

$p_1 = 1 - x + 2x^2$, $p_2 = 3 + x$, $p_3 = 5 - x + 4x^2$, $p_4 = -2 - 2x + 2x^2$.

Soln:- General polynomial for $P_2 = a_0 + a_1 x + a_2 x^2$

$$P_2 = c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 .$$

$$c_1[1 - x + 2x^2] + c_2[3 + x] + c_3[5 - x + 4x^2] + c_4[-2 - 2x + 2x^2]$$
$$= a_0 + a_1 x + a_2 x^2$$

$$\begin{array}{l} \text{Equating constants} \\ \text{equating coefficient of } x \\ \text{equating coefficient of } x^2 \end{array} \begin{bmatrix} 1 & 3 & 5 & -2 \\ -1 & 1 & -1 & -2 \\ 2 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{array}\right]. \qquad AX = B ..$$

Rank of $A \neq$ Rank $[A : B]$ inconsistent.

$\therefore$ The given set of polynomials do not span P_2 ..

14) Let $v_1 = (2, 1, 0, 3)$ $v_2 = (3, -1, 5, 2)$ and $v_3 = (-1, 0, 2, 1)$ which of the following vectors are in span $\{v_1, v_2, v_3\}$?

a) $(2, 3, -7, 3)$ — Yes

b) $(0, 0, 0, 0)$ — Yes

c) $(1, 1, 1, 1)$ — No

(d) $(-4, 6, -13, 4)$ — Yes.

20 & 21 excluded.

5.3 → Linear Independence:

We have learned that a set of vectors $S = \{v_1, v_2, ---, v_r\}$ spans a given vector space V if every vector in V is expressible as a linear combination of the vectors in S.

Note:

There may be more than one way to express a vector in V as a linear combination of vectors in a spanning set.

Now we shall study conditions under which each vector in V is expressible as a linear combination of the spanning vectors in exactly one way.

[OR, we will like to start looking at when it will be possible to express a given vector from a vector space as exactly one linear combination of the set S.

Defn:

Suppose $S = \{v_1, v_2, ---, v_n\}$ is a nonempty set of vectors and form the vector equation,

$$c_1 v_1 + c_2 v_2 + ---+ c_n v_n = 0.$$

This equation has atleast one solution, namely,

$c_1 = 0,\ c_2 = 0,\ ---,\ c_n = 0$, called as trivial solution.

If the trivial solution is the only solution to this equation then the vectors in the set S are called linearly independent and set S is called a linearly independent set.

If there is another solution then the vectors in the set S are called linearly dependent and the set S is called linearly dependent set.

Eg - 1, 2, 3, 4, 5 (Assignments)

Eg-1: In terms of components the vector equation in $k_1v_1 + k_2v_2 + k_3v_3 = \vec{0}$

$v_1 = (2,-1,0,3)$, $v_2 = (1,2,5,-1)$ & $v_3 = (7,-1,5,8)$
$\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 5 & 5 \\ 3 & -1 & 8 \end{bmatrix}$

Soln: $S = \{v_1, v_2, v_3\}$ is linearly dependent since $3v_1 + v_2 - v_3 = 0$
$\sim \begin{bmatrix} 2 & 1 & 7 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{bmatrix}$

$c_3 = -1, c_2 = 1, c_1 = 3$

Eg-2:
$p_1 = 1 - x$, $p_2 = 5 + 3x - 2x^2$ & $p_3 = 1 + 3x - x^2$, forms a linearly dependent set in P_2 $\because 3p_1 - p_2 + 2p_3 = 0$.
$\begin{vmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix} \cancel{\neq 0} = \text{zero}$

Eg-3:
The vectors i, j and k of R^3 is linearly independent $\because$
$c_1\hat{i} + c_2\hat{j} + c_3\hat{k} = 0 \Rightarrow c_1 = c_2 = c_3 = 0$.
$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

Eg-4:
$v_1 = (1,-2,3)$, $v_2 = (5,6,-1)$ & $v_3 = (3,2,1)$ is L.D $\because$
$k_1 = -t/2$, $k_2 = -t/2$, $k_3 = t$.
$\begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 0$

Eg-5:

$1, x, x^2, ---, x^n$ forms a L.I set of vectors in P_n.
$c_1 + c_2x + c_3x^2 + -- + c_nx^n = 0 + 0x + 0x^2 + -- + 0x^n$

$a_0p_0 + a_1p_1 + a_2p_2 + -- + a_np_n = 0$.
$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 0, ---, c_n = 0$.

$\Longrightarrow a_0 = a_1 = --- = a_n = 0$.

[$\because$ a nonzero polynomial of deg 'n' has at most 'n' distinct roots ($a_n \neq 0$), otherwise it would follow that
$a_0 + a_1x + -- + a_nx^n$ is a nonzero poly with infinitely many roots].

5.3.1:
A set S with two or more vectors is :
a) L.D iff atleast one of the vectors in S is expressible as a linear combination of the other vectors in S.

b) L.I iff no vector in S is expressible as a linear combination of the other vectors in S.

Eg - 6, 7 (Ass) Let $v_1 = (2,-1,0,3)$, $v_2 = (1,2,5,-1)$, $v_3 = (7,-1,5,8)$ form a L.D set. It follows from Theorem 5.3.1 that atleast one of the vectors is expressible as a linear combination of the other two. In this example each

Theorem – 5.3.2:

a) A finite set of vectors that contains the zero vector is linearly dependent.

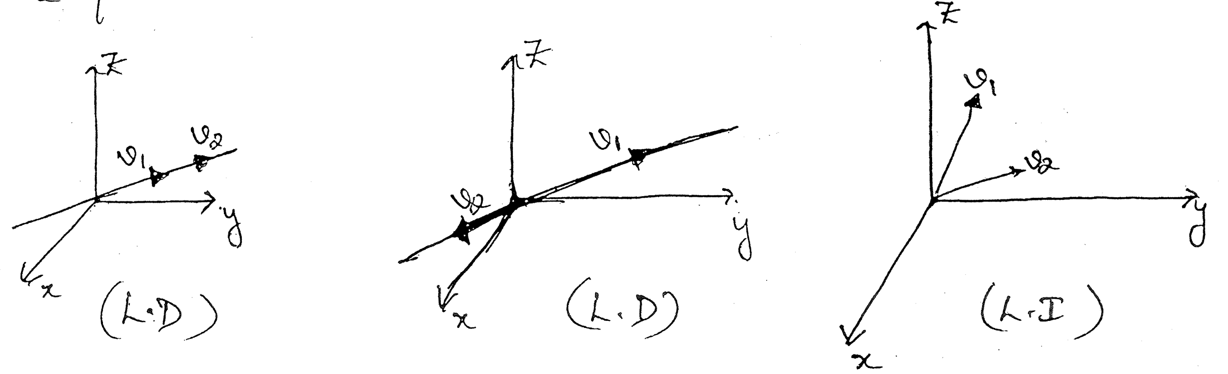
b) A set with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other.

Eg-8:

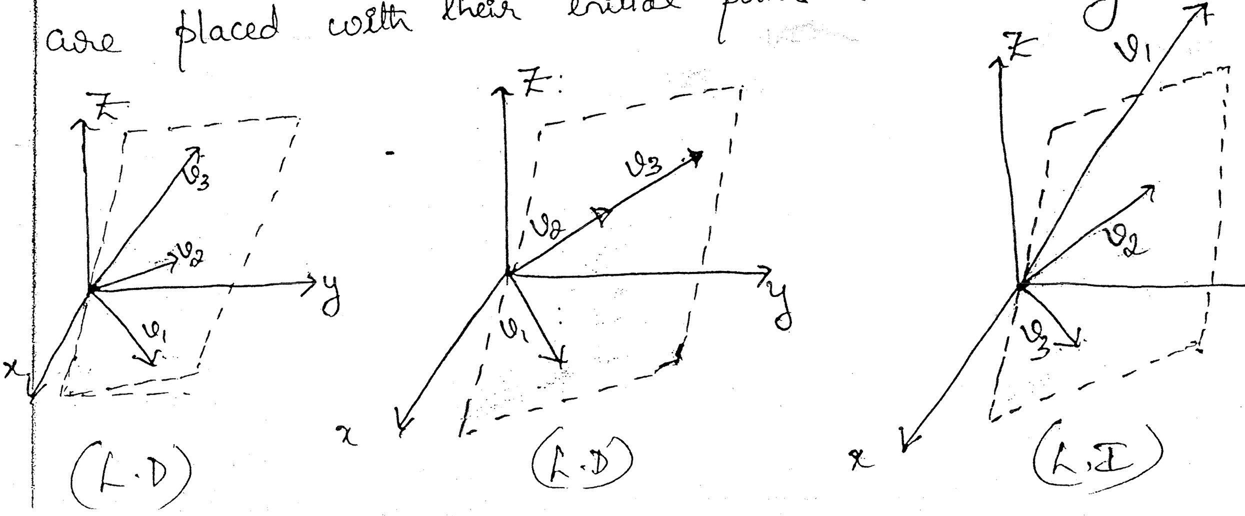
The functions $f_1 = x$ and $f_2 = \sin x$ form a L.I set of vectors in $F(-\infty, \infty)$, since neither function is a constant multiple of the other.

Geometric interpretation of L.I:

1) In R^2 or R^3, a set of two vectors is linearly independent iff the vectors do not lie on the same line when they are placed with their initial points at the origin.



02) In R^3, a set of three vectors is linearly independent iff the vectors do not lie in the same plane when they are placed with their initial points at the origin.



Thm - 5.3.3 :

Let $S = \{v_1, v_2, \dots, v_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

Q) Determine if each of the following sets of vectors are linearly independent or linearly dependent.

a) $v_1 = (1,-3)$, $v_2 = (-2,2)$ and $v_3 = (4,-1)$.

Soln. Vector eqn, $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$.

$\Rightarrow$ The system of eqn,
$$c_1 - 2c_2 + 4c_3 = 0$$
$$-3c_1 + 2c_2 - c_3 = 0.$$

$\Rightarrow c_1 = \frac{3}{2}t$; $c_2 = \frac{11}{4}t$; $c_3 = t$.

[If a system of eqns has more unknowns than eqns then we will have infinitely many solutions].

b) $v_1 = (-2,1)$, $v_2 = (-1,-3)$ and $v_3 = (4,-2)$

Soln: Lin combo, $c_1(-2,1) + c_2(-1,-3) + c_3(4,-2) = 0$.
$$-2c_1 - c_2 + 4c_3 = 0.$$
$$c_1 - 3c_2 - 2c_3 = 0.$$

System has infinitely many solns (so more than the trivial soln)

$\therefore$ The vectors are linearly dependent.

c) $v_1 = (2,-2,4)$, $v_2 = (3,-5,4)$ and $v_3 = (0,1,1)$.

Soln:
$$2c_1 + 3c_2 = 0$$
$$-2c_1 - 5c_2 + c_3 = 0.$$
$$4c_1 + 4c_2 + c_3 = 0.$$

$\Rightarrow c_1 = \frac{-3t}{4}$, $c_2 = \frac{t}{2}$ $c_3 = t$. or. $\det(A) = 0$ (if coefficient matrix is square) $\Rightarrow$ infinitely many solns

$\Rightarrow$ L.D $\Rightarrow$ L.D.

d) $v_1 =$ … and …

Soln: Tl…

$\Rightarrow$

$\Rightarrow$

e) $v_1 =$

Soln

$\Rightarrow$

Exe…

01) Expl… vecto…

a) $u_1 =$

b) $u_1 =$

c) $p_1 =$

d) $A =$

d) $v_1 = (1, 1, -1, 2)$, $v_2 = (2, -2, 0, 2)$ and $v_3 = (2, -8, 3, -1)$.

Soln: The system of eqns are.

$$c_1 + 2c_2 + 2c_3 = 0$$
$$c_1 - 2c_2 - 8c_3 = 0.$$
$$-c_1 + 3c_3 = 0$$
$$2c_1 + 2c_2 - c_3 = 0.$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & -8 \\ -1 & 0 & 3 \\ 2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 10 \\ 0 & 2 & 5 \\ 0 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(determinant method cannot be applied)

$\Rightarrow c_1 = 3t$, $c_2 = -\frac{5t}{2}$ $c_3 = t$.

$\Rightarrow$ L.D.

e) $v_1 = (1, -2, 3, -4)$, $v_2 = (-1, 3, 4, 2)$ and $v_3 = (1, 1, -2, -2)$.

Soln $\Rightarrow$

$$c_1 - c_2 + c_3 = 0$$
$$-2c_1 + 3c_2 + c_3 = 0$$
$$3c_1 + 4c_2 - 2c_3 = 0$$
$$-4c_1 + 2c_2 - 2c_3 = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & 1 \\ 3 & 4 & -2 \\ -4 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 26 \\ 0 & 0 & 8 \end{bmatrix}$$

$\Rightarrow$ only trivial solution $\Rightarrow$ vectors are linearly independent.

Exercise - 5.3

01) Explain why the following are linearly dependent sets of vectors. (Solve this problem by inspection).

a) $u_1 = (-1, 2, 4)$ & $u_2 = (5, -10, -20)$ in R^3. $u_2 = -5u_1$ (u_2 is a scalar mle of u_1)

b) $u_1 = (3, -1)$, $u_2 = (4, 5)$, $u_3 = (-4, 7)$ in R^2 no of vectors > The no of coordinates $3 > 2 \Rightarrow$ L.D.

If $c_3 = 19$, $c_2 = -17$, $c_1 = 14.4$

c) $P_1 = (3 - 2x + x^2$, $P_2 = 6 - 4x + 2x^2$ in P_2 (P_2 is a scalar mul of P_1) $P_2 = 2P_1 \Rightarrow$ L.D

d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22} $B = -A \Rightarrow$ L.D (B is a scalar mul of A).

v is R^2

rs

rs then

then

coefficient mx is square) by solving

02) Which of the following sets of vectors in R^3 are linearly dependent?

a) $(4,-1,2)$, $(-4,10,2)$ L.I

b) $(-3,0,4)$, $(5,-1,2)$, $(1,1,3)$ L.I

c) $(8,-1,3)$, $(4,0,1)$ L.I

d) $(-2,0,1)$, $(3,2,5)$, $(6,-1,1)$, $(7,0,-2)$ (L.D) $\rightarrow$ 5.3.3.

03) Which of the following sets of vectors in R^4 are linearly dependent?

a) $(3,8,7,-3)$, $(1,5,3,-1)$, $(2,-1,2,6)$, $(1,4,0,3)$ L.I

b) $(0,0,2,2)$, $(3,3,0,0)$, $(1,1,0,-1)$ L.I

c) $(0,3,-3,-6)$, $(-2,0,0,-6)$, $(0,-4,-2,-2)$, $(0,-8,4,-4)$ L.I

d) $(3,0,-3,6)$, $(0,2,3,1)$, $(0,-2,-2,0)$, $(-2,1,2,1)$ L.I

04) Determine if the following sets of vectors are linearly independent or linearly dependent.

a) $v_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ & $v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Solu:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\Rightarrow \begin{bmatrix} c_1 & 0 & c_2 \\ 0 & c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0.$$

$\Rightarrow$ lin Ind.

b) $v_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$

$$c_1 v_1 + c_2 v_2 = 0 \Rightarrow \begin{aligned} c_1 + 4c_2 &= 0 \\ 2c_1 + c_2 &= 0 \\ -c_1 - 3c_2 &= 0. \end{aligned}$$

$$\Rightarrow \boxed{c_1 = c_2 = 0} \Rightarrow \text{L.I.}$$

c) $v_1 = \begin{bmatrix} 8 & -2 \\ 10 & 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} -12 & 3 \\ -15 & 0 \end{bmatrix}$

Solun:

$$c_1 v_1 + c_2 v_2 = 0 \Rightarrow \begin{array}{l} 8c_1 - 12c_2 = 0 \\ -2c_1 + 3c_2 = 0 \\ 10c_1 - 15c_2 = 0 \end{array}$$

$\Rightarrow \quad c_1 = \frac{3}{2} t$ & $c_2 = t$

$\Rightarrow$ L.D.

Q. Determine if the following sets of vectors are linearly independent or linearly dependent:

a) $p_1 = 1$, $p_2 = x$ and $p_3 = x^2$ in P_2.

Solun:- $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$

$\Rightarrow \quad c_1(1) + c_2 x + c_3 x^2 = 0c_1 + 0x + 0x^2$

$\Rightarrow \quad c_1 = c_2 = c_3 = 0 \Rightarrow$ L.I.

b) $p_1 = x - 3$, $p_2 = x^2 + 2x$ & $p_3 = x^2 + 1$ in P_2.

Solun: $c_1(x-3) + c_2(x^2+2x) + c_3(x^2+1) = 0 + 0x + 0x^2$

$(c_2 + c_3)x^2 + (c_1 + 2c_2)x + (-3c_1 + c_3) = 0 + 0x + 0x^2$

$$\Rightarrow \left.\begin{array}{l} c_2 + c_3 = 0 \\ c_1 + 2c_2 = 0 \\ -3c_1 + c_3 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = c_3 = 0.$$

$\Rightarrow$ L.I.

c) $p_1 = 2x^2 - x + 7$, $p_2 = x^2 + 4x + 2$ & $p_3 = x^2 - 2x + 4$ in P_2.

Solu: $c_1[2x^2 - x + 7] + c_2[x^2 + 4x + 2] + c_3[x^2 - 2x + 4] = 0.$

$\Rightarrow [2c_1 + c_2 + c_3]x^2 + [-c_1 + 4c_2 - 2c_3]x + [7c_1 + 2c_2 + 4c_3] = 0$

$$\Rightarrow \begin{array}{l} 2c_1 + c_2 + c_3 = 0 \\ -c_1 + 4c_2 - 2c_3 = 0 \\ 7c_1 + 2c_2 + 4c_3 = 0. \end{array}$$

$\Rightarrow \quad c_1 = \frac{-2}{3} t, \quad c_2 = \frac{t}{2}, \quad c_3 = t$

Ex-5.3:

04) Which of the following sets of vectors in P_2 are linearly dependent?

a) $2-x+4x^2$; $3+6x+2x^2$; $2+10x-4x^2$ (L.I)

b) $3+x+x^2$; $2-x+5x^2$; $4-3x^2$ (L.I)

c) $6-x^2$; $1+x+4x^2$ (L.I)

d) $1+3x+3x^2$; $x+4x^2$; $5+6x+3x^2$; $7+2x-x^2$ (L.D)

05) Assume that v_1, v_2, and v_3 are vectors in R^3 that have their initial points at the origin. In each part determine whether the three vectors lie in a plane.

a) $v_1=(2,-2,0)$, $v_2=(6,1,4)$, $v_3=(2,0,-4)$ They do not lie in a plane (L.I)

b) $v_1=(-6,7,2)$, $v_2=(3,2,4)$, $v_3=(4,-1,2)$ → They do lie in a plane, lies in a plane (L.D)

06) Assume that v_1, v_2, v_3 are vectors in R^3 that have their initial points at the origin. In each part determine whether the three vectors lie on the same line.

a) $v_1=(-1,2,3)$, $v_2=(2,-4,-6)$, $v_3=(-3,6,0)$ → (do not lie) They do not lie on the same line. Solu: $v_2=-2v_1$, $v_3\neq kv_1$ or kv_2.

b) $v_1=(2,-1,4)$, $v_2=(4,2,3)$, $v_3=(2,7,-6)$ → (do not lie)

c) $v_1=(4,6,8)$, $v_2=(2,3,4)$, $v_3=(-2,-3,-4)$

$\boxed{\text{(lie on a line)} \quad v_1=2v_2 \quad v_3=-v_2}$

07) a) Show that the vectors $v_1=(0,3,1,-1)$, $v_2=(6,0,5,1)$ and $v_3=(4,-7,1,3)$ form a linearly dependent set in R^4.

b) Express each vector as a linear combination of the other two. $c_1=7, c_2=-2, c_3=3$

$v_1=\frac{2}{7}v_2-\frac{3}{7}v_3$, $v_2=\frac{7}{2}v_1+\frac{3}{2}v_3$, $v_3=\frac{-7}{3}v_1+\frac{2}{3}v_2$.

08) For which real values of 'λ' do the following vectors form a linearly dependent set in R^3?

$v_1=\left(\lambda,-\frac{1}{2},-\frac{1}{2}\right)$, $v_2=\left(-\frac{1}{2},\lambda,-\frac{1}{2}\right)$, $v_3=\left(-\frac{1}{2},-\frac{1}{2},\lambda\right)$

By inspection $\lambda=-\frac{1}{2}$, $\lambda=1$.

$$\begin{vmatrix}\lambda & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \lambda & -\frac{1}{2}\\ \cdots \end{vmatrix}=0 \Rightarrow 4\lambda^3-3\lambda-1=0$$

$\lambda=1, -\frac{1}{2}$

Prob. 19) Thm - If ... on the ... function ... funct ... (n-1) ... C

Ex-9: S = vectors ... Hence

Ex-10 S.T ... Solu. Wh...

Ex-5.3: 19) Use a... of the ... 6, ... (c). 1, Si... (e) ~~Cos~~ ... (e) (3-x...

problems 9–18, 21, 22 excluded.

19) Thm – 5.3.4: Linear Independence of functions:

If the functions $f_1, f_2, \ldots, f_n$ have $(n-1)$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty,\infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty,\infty)$.

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Ex-9:

S.T the functions $f_1 = x$ & $f_2 = \sin x$ form a L.I set of vectors in $C^1(-\infty,\infty)$.

Solu:- $W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0 \ \forall\ x \in (-\infty,\infty)$.

Hence the given set of functions form a L.I set.

Ex-10:

S.T $f_1 = 1$, $f_2 = e^x$, & $f_3 = e^{2x}$ form a L.I set of vectors in C^2

Solu. $W(x) = 2e^{3x} \neq 0 \ \ \forall x \in (-\infty,\infty) \Rightarrow$ L.I.

Ex-5.3:

19) Use appropriate identities, where required, to determine which of the following are L.D. in $F(-\infty,\infty)$.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$.
L.D.

(b) $x,\ \cos x$
L.I

(c) $1,\ \sin x,\ \sin(2x)$
L.I

(d) $\cos(2x),\ \sin^2 x,\ \cos^2 x$
L.D.

~~(e) $\cos 2x, \sin x$,~~

(e) $(3-x)^2,\ x^2 - 6x,\ 5$
L.D.

(f) $0,\ \cos^3(\pi x),\ \sin^5(3\pi x)$.
L.D.

20) Use the Wronskian to show that the following sets of vectors are linearly independent.

ⓐ $1, x, e^{x}$ ⓑ $\sin x, \cos x, x\sin x$ ⓒ $e^{x}, xe^{x}, x^{2}e^{x}$

ⓓ $1, x, x^{2}$.

A basis of a vector space is defined as a subset of vectors that are linearly independent and span vector space.

5.4: Bases and Dimension

We usually think of a line as being 1-D, a plane as 2-D and the space around us as 3-D. In this section we make the ~~static~~ intuitive notion of "dimension" more precise.

The scale of measurement along the co-ordinate axes are essential ingredients of any coordinate system. Usually, one tries to use the same scale on each axis and have the integer points on the axes spaced 1 unit of distance apart. However, this is not always practical or appropriate. Unequal scales, or scales in which the integral points are more or less than 1 unit apart, may be required to fit a particular graph on a printed page or to represent physical quantities with diverse units in the same co-ordinate system [time in seconds on one axis and temperature in hundreds of degree on another, for eg].

When a co-ordinate system is specified by a set of bases vectors, then the lengths of those vectors correspond to the distance b/n successive integer points on the co-ordinate axes. Thus, it is the direction of the bases vectors that define the directions of co-ordi axes and the length of bases vectors that establish the scales of measurement.

Defⁿ:

If V is any vector space and $S = \{v_1, v_2, ---, v_n\}$ is a set of vectors in V, then S is called a basis of V, if the following two conditions hold:

1) S is linearly independent
2) S spans V.

Thm-5.4.1: Uniqueness of Basis Representation:

If $S = \{v_1, v_2, ---, v_n\}$ is a basis for a vector space V, then every vector 'u' in V can be expressed in the form $u = c_1 v_1 + c_2 v_2 + ---+ c_n v_n$ in exactly one way.

Co-ordinates Relative to a basis:

If $S = \{v_1, v_2, ---, v_n\}$ is a basis for a vector space V, and $u = c_1 v_1 + c_2 v_2 + --- + c_n v_n$ is the expression for a vector u in terms of the basis S, then the scalars $c_1, c_2, ---, c_n$ are called the co-ordinates of u relative to the basis S. The vector $(c_1, c_2, ---, c_n)$ in R^n constructed from these co-ordinates is called the co-ordinate vector of u relative to S; it is denoted by

$$(u)_S = (c_1, c_2, ---, c_n).$$

Note:

→ The Co-ordinate vectors depend not only on the basis S but also on the order in which the basis vectors are written.

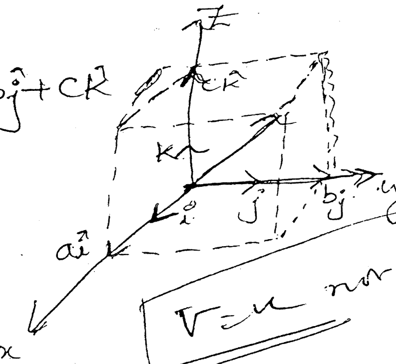
→ A change in the order of the basis vectors results in a corresponding change of order for the entries in the co-ordinate vectors.

Ex-1: Standard basis for R^3:

$S = \{i, j, k\}$, then i, j, k are linearly independent and also i, j, k spans R^3 & $\because$

$u = (a, b, c) \in R^3 \Rightarrow (a, b, c) = a\hat{i} + b\hat{j} + c\hat{k}$

$\therefore (u)_s = (a, b, c) = u$



$V = u$ nor a vector space matrix

Ex-2: Standard basis for R^n is

$S = \{e_1, e_2, \ldots, e_n\}$, $e_1 = (\underbrace{1, 0, 0, \ldots, 0}_{n\text{-comp}})$.

if $V = v_1, v_2, \ldots, v_n \in R^n$ then

$(V)_s = (v_1, v_2, \ldots, v_n)$.

Ex-3; Ex-4 (Assignment)

Eg: 1

Determine if each of the sets of vectors will be a basis for R^3.

a) $v_1 = (1, -1, 1)$, $v_2 = (0, 1, 2)$ and $v_3 = (3, 0, -1)$.

Solu: To show that the set S spans R^3, we must show that an arbitrary vector $b = (b_1, b_2, b_3)$ can be expressed as a linear combination of the vectors in S.

is $b = c_1 v_1 + c_2 v_2 + c_3 v_3$.

Expressing this equation in terms of components gives

$$c_1(1, -1, 1) + c_2(0, 1, 2) + c_3(3, 0, -1) = (b_1, b_2, b_3)$$

$$\Rightarrow \left.\begin{aligned} c_1 + 0c_2 + 3c_3 &= b_1 \\ -c_1 + c_2 + 0c_3 &= b_2 \\ c_1 + 2c_2 - c_3 &= b_3 \end{aligned}\right\} \rightarrow (1)$$

to show that S spans R^3, we should prove that eqn (1) has a ~~unique~~ solu^n for all choices of $b = (b_1, b_2, b_3)$.

To prove that S is linearly independent, we must show that the only solu^n of $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ is $c_1 = c_2 = c_3 = 0$.

In terms of components

$$\left.\begin{aligned} c_1 + 0c_2 + 3c_3 &= b_1\ 0 \\ -c_1 + c_2 + 0c_3 &= b_2\ 0 \\ c_1 + 2c_2 - c_3 &= b_3\ 0 \end{aligned}\right\} \rightarrow ②$$

observe that systems ① & ② have the same coefficient matrix.

In order to prove that S is linearly independent and it spans R^3, we have to show that the matrix of coefficients has a nonzero determinant.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \qquad \det(A) = -10 \neq 0 .$$

$\Rightarrow$ These set of vectors form a bases of R^3.

b) $v_1 = (1,1,0)$ and $v_2 = (-1,0,0)$

solu.

$$c_1 v_1 + c_2 v_2 = b$$

$$c_1(1,1,0) + c_2(-1,0,0) = (b_1, b_2, b_3)$$

The 3rd component of the each of these vectors is zero and hence the linear combination will never have any non-zero third component.

$\therefore$ If we choose $u = (u_1, u_2, u_3)$ to be any vector in R^3 with $u_3 \neq 0$, then we will not be able to find scalars c_1 and c_2 to satisfy the above eqn.

$\therefore$ These two vectors do not span R^3 and hence cannot be a basis for R^3.

Note:

These two vectors are linearly independent.

c) $v_1 = (1,-1,1)$, $v_2 = (-1,2,-2)$ and $v_3 = (-1,4,-4)$

To S.T the set S spans R^3, we must S.T an arbitrary vector $u = (u_1, u_2, u_3)$ can be expressed as a linear combination of the vectors in S

Solu:- i.e. $c_1 v_1 + c_2 v_2 + c_3 v_3 = u$

$$c_1(1,-1,1) + c_2(-1,2,-2) + c_3(-1,4,-4) = (u_1, u_2, u_3)$$

To see that S is L.I, assume that $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$.

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 4 \\ 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \cancel{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}$$

$\Rightarrow \det(A) = 0$.

$\Rightarrow$ Three vectors do not span R^3 and are not linearly independent.

$\Rightarrow$ They are not form a basis ~~of~~ for R^3.

Ex-5: Standard basis for P_n is $S = \{1, x, x^2, ---, x^n\}$. Yes

Ex-6: Standard basis for M_{22} is

$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ Yes

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4$ # To see that S is L.I, assume that $c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = 0$. It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ S is L.I

Def^n: A nonzero vector space V is called finite-dimensional if it contains a finite set of vectors $\{v_1, v_2, ---, v_n\}$ that forms a basis.

If no such set exists, V is called infinite dimensional

Note: The zero vector space is always finite dimensional.

By examples 2, 5 and 6 R^n, P_n and M_{nn} are finite dimensional. The vector spaces $F(-\infty,\infty)$, $C(-\infty,\infty)$, $C^m(-\infty,\infty)$ and $C^\infty(-\infty,\infty)$ are infinite dimensional.

3) Let $v_1 = (1,2,1)$, $v_2 = (2,9,0)$, and $v_3 = (3,3,4)$. Show that the set $S = \{v_1, v_2, v_3\}$ is a basis for R^3.

| 1 2 3 | ∴ the system will have a soln for all choices of b.

Theorem - 5.4.2:

Let V be a finite dimensional vector space, and let $\{v_1, v_2, ---, v_n\}$ be any basis.

a) If a set has more than 'n' vectors, then it is linearly dependent.

b) If a set has fewer than 'n' vectors, then it does not span V.

Theorem - 5.4.3:

All bases for a finite dimensional vector space have the same no. of vectors.

Defn:

The dimension of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the no of vectors in a basis for V.

Note:

Zero vector space have dimension zero.

Ex-9: ① $\dim(R^n) = n$ (Ex-7) ② $\dim(P_n) = n+1$ (Eg-5)

③ $\dim(M_{mn}) = mn$ eg.(6) (Standard bases)

Theorem - 5.4.5:-

If V is a n-dimensional vector space and if 'S' is a set in V with exactly 'n' vectors, then 'S' is a bases for V if either 'S' spans V & 'S' is L.I.

Ex-11:

a) S.T $v_1 = (-3, 7)$, and $v_2 = (5, 5)$ form a basis for R^2

Soln: $v_1 \neq k_1 v_2$ or $v_2 \neq k_2 v_1 \Rightarrow$ L.I $\Rightarrow$ Base (Thm-5.4.5)

S.T $v_1 = (2, 0, -1)$, $v_2 = (4, 0, 7)$, $v_3 = (-1, 1, 4)$ form a basis for R^3 (by inspection).

$n = 3$ Set has three vectors

$v_1 \neq k_1 v_2$ or $v_2 \neq k v_1 \Rightarrow$ L.I and v_3 does not ~~lie~~ lie in XZ-plane

$$\begin{bmatrix} 2 & 4 & -1 \\ 0 & 0 & 1 \\ -1 & 7 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 \\ 0 & 0 & 1 \\ 0 & 18 & 7 \end{bmatrix}$$

$\Rightarrow v_1, v_2, v_3$ are L.I

$\sim \begin{bmatrix} 2 & 4 & -1 \\ 0 & 18 & 7 \end{bmatrix}$ ...

Exercise - 5.4

01) Explain why the following set of vectors are not bases for the indicated vector spaces (By inspection).

a) $u_1 = (1, 2)$, $u_2 = (0, 3)$, $u_3 = (2, 7)$ for R^2.

Solu: A basis for R^2 has two L.I vectors (Thm-5.4.2, if set has more than 'n' vectors, then S is L.D) Do not span R^2

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & -3 \end{bmatrix}, \quad n \neq 2$$

b) $u_1 = (-1, 3, 2)$, $u_2 = (6, 1, 1)$ for R^3.

Solu: If a set has less than 'n' vectors, they do not span V (Thm-5.4.2). R^3 has 3 L.I vectors. Do not span R^3

$$\begin{bmatrix} -1 & 6 \\ 3 & 1 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 6 \\ 0 & 19 \\ 0 & 13 \end{bmatrix}, \quad n = 3$$

c) $p_1 = 1 + x + x^2$, $p_2 = x - 1$ for P_2.

Solu: A basis for P_2 has three L.I vectors. Do not span P_2

d) $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$ $E = \begin{bmatrix} 7 & 1 \\ 2 & 9 \end{bmatrix}$ for M_{22}.

~~span M_{22}~~ Do not span M_{22}

Solu: A basis for M_{22} has four L.I for M_{22}. (but 5 is given)

02) Which of the following sets of vectors are basis for R^2.

a) $(2, 1)$ $(3, 0)$

Solu: no of vectors = dimension = 2

and the given vectors are linearly independent.

$\therefore$ from (Thm-5.4.5) they form a basis.

b) $(4, 1)$, $(-7, -8)$.

Solu: form as basis (from Thm-5.4.5)

c) $(0, 0)$, $(1, 3)$

only many solutions is $(c_1(0,0) + c_2(1,3)) = 0$. $c_1 = a$, $c_2 = 0$.

L.D $\Rightarrow$ Not a basis

d) $(3, 9)$, $(-4, -12)$ Solu: $3c_1 - 4c_2 = 0$, $9c_1 - 12c_2 = 0$ only many solu L.D.

$\Rightarrow$ Not a basis

03) Which vectors

a) $(1, 0, 0)$

b) $(3, 1, -$

c) $(2, -3, 1$

04) Which

a) $1 - 3x -$

Check

$\Rightarrow$ L

b) $4 + 6x +$

$\Rightarrow$ L

c) $1 + x +$

$\Rightarrow$ L

d) $-4 + x -$

$\Rightarrow$ L

05) Show

$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$

Solu: L.I

06) Let V

$v_3 =$ C

a) S.T

Sol L.I

b) Find

A

not bases

5.4.2, if.

not spans V

$\begin{bmatrix} \ & 1 \\ \ & 2 \end{bmatrix}$ $E=\begin{bmatrix} 7 & 1 \\ 2 & 9 \end{bmatrix}$

is for R^2.

lent.

$c_1=a$, $c_2=0$.

ly many solns
L.D.

03) Which of the following sets of vectors are bases for R^3.

Sol a) $(1,0,0)$, $(2,2,0)$, $(3,3,3) \Rightarrow$ bases. (Thm 5.4.5)

b) $(3,1,-4)$, $(2,5,6)$, $(1,4,8)$ (basis)

Sol c) $(2,-3,1)$, $(4,1,1)$, $(0,-7,1)$ Not a basis

04) Which of the following sets of vectors are basis for P_2?

a) $1-3x+2x^2$; $1+x+4x^2$; $1-7x$.

Sol Check for L.I (Thm 5.4.5)

$\Rightarrow$ L.D $\Rightarrow$ Not a basis

$$\begin{vmatrix} 1 & -3 & 2 \\ 1 & 1 & 4 \\ 1 & -7 & 0 \end{vmatrix} = 1(0+28)+3(0-4)+2(-7-1)$$
$$= 28-12-16 = \underline{0}$$

b) $4+6x+x^2$; $-1+4x+2x^2$, $5+2x-x^2$

$\Rightarrow$ L.D $\Rightarrow$ Not a basis

c) $1+x+x^2$, $x+x^2$, x^2,

$\Rightarrow$ L.I $\Rightarrow$ basis

d) $-4+x+3x^2$, $6+5x+2x^2$, $8+4x+x^2$.

Sol $\Rightarrow$ L.I $\Rightarrow$ bases.

05) Show that the following set of vectors is a basis of M_{22}?

Sol $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.

Soln. L.I $\Rightarrow$ basis (Thm - 5.4.5)

06) Let V be the space spanned by $v_1=\cos^2 x$, $v_2=\sin^2 x$, $v_3=\cos(2x)$.

a) S.T $S=\{v_1, v_2, v_3\}$ is not a basis.

Sol L.D $\because$ $-\cos^2 x+\sin^2 x+\cos(2x)$ $\boxed{c_1=-1 \quad c_2=c_3=1}$

$= -\cos^2 x+\sin^2 x+(\cos^2 x-\sin^2 x)=0$

b) Find a basis.

A... two vectors from S will be a basis.

07) Find the co-ordinate vector of W relative to the basis $S=\{u_1, u_2\}$ for R^2.

a) $u_1=(1,0)$, $u_2=(0,1)$, $w=(3,-7)$

Soln

$$w = c_1u_1 + c_2u_2$$

$$(3,-7) = c_1(1,0) + c_2(0,1)$$

$$\Rightarrow c_1=3 , c_2=-7.$$

∴ The co-ordinate vector of w relative to the basis S is $(W)_S = (c_1, c_2) = (3,-7)$

b) $u_1=(2,-4)$; $u_2=(3,8)$; $w=(1,1)$

Soln $w=c_1u_1+c_2u_2 \Rightarrow c_1 = 5/28$, $c_2 = 3/14$.

∴ $(W)_S = (c_1, c_2) = (5/28 , 3/14)$

c) $u_1=(1,1)$, $u_2=(0,2)$; $w=(a,b)$

Soln $(w_S) = \left(a, \frac{b-a}{2}\right)$.

08) Find the co-ordinate vector of V relative to the basis $S=\{v_1, v_2, v_3\}$.

a) $v=(2,-1,3)$; $v_1=(1,0,0)$, $v_2=(2,2,0)$, $v_3=(3,3,3)$.

Soln $v = c_1v_1 + c_2v_2 + c_3v_3$.

$$\Rightarrow c_1=3 ; c_2=-2 ; c_3=1.$$

∴ $(V)_S = (3,-2,1)$

b) $v=(5,-12,3)$; $v_1=(1,2,3)$, $v_2=(-4,5,6)$, $v_3=(7,-8,9)$

Soln $c_1=-2$; $c_2=0$; $c_3=1$.

∴ $(V)_S=(-2,0,1)$.

09) Find the
relative
a) $p=4-3x$
Soln $c_1=4$
b) $p=2-x+$
$c_1=0$
10) Find the
$S=\{A_1,$
$A=\begin{bmatrix}2 & 0\\ -1 & 3\end{bmatrix}$
Soln $(A)_S$
11) Determine
space of
13) x_1-4x_2
$2x_1-8x_2$
Soln. Both
$\Rightarrow x_1$
let
Soln
$v_1=(4,$
a basis

to the

$\mathcal{C}$

09) Find the Co-ordinate vector of p relative to the bases $S=\{p_1, p_2, p_3\}$.

a) $p=4-3x+x^2$; $p_1=1$, $p_2=x$, $p_3=x^2$

Soln $c_1=4$, $c_2=-3$, $c_3=1 \Rightarrow (p)_s=(4,-3,1)$

b) $p=2-x+x^2$; $p_1=1+x$, $p_2=1+x^2$, $p_3=x+x^2$

$c_1=0$; $c_2=2$; $c_3=-1$ $(p)_s=(0,2,-1)$

$\mathcal{C}$

10) Find the Co-ordinate vector of A relative to the basis. $S=\{A_1, A_2, A_3, A_4\}$.

$)$

$A=\begin{bmatrix}2&0\\-1&3\end{bmatrix}$; $A_1=\begin{bmatrix}-1&1\\0&0\end{bmatrix}$, $A_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $A_3=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ $A_4=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

Soln. $(A)_s=(-1,1,-1,3)$

11) Determine the dimension ~~of~~ and a basis for the solution space of the system.

13) $x_1-4x_2+3x_3-x_4=0$

$2x_1-8x_2+6x_3-2x_4=0.$

to the.

Soln. Both the eqns are same.

$\Rightarrow$ $x_1=4x_2-3x_3+x_4$

$v_3=(3,3,3).$

let $x_2=r$, $x_3=s$ $x_4=t$ $\Rightarrow x_1=4r-3s+t$

Soln $X=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}4r-3s+t\\r\\s\\t\end{bmatrix}=r\begin{bmatrix}4\\1\\0\\0\end{bmatrix}+s\begin{bmatrix}-3\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

$_3=1.$

$=rv_1+sv_2+tv_3.$

$), v_3=(7,-8,9)$

$v_1=(4,1,0,0)$; $v_2=(-3,0,1,0)$, $v_3=(1,0,0,1)$ forms a basis for the solution and the dimension is 3.

14) $x_1 - 3x_2 + x_3 = 0$

$2x_1 - 6x_2 + 2x_3 = 0$

$3x_1 - 9x_2 + 3x_3 = 0$

Soln: all the 3 eqns are identical.

$x_1 = 3x_2 - x_3$

let $x_2 = t$; $x_3 = s$ $x_1 = 3t - s$

$$X = \begin{bmatrix} 3t-s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$S = \{(3,1,0), (-1,0,1)\}$ is a basis for the soln and the dimension is '2'.

15) $2x_1 + x_2 + 3x_3 = 0$

$x_1 + 5x_3 = 0$

$x_2 + x_3 = 0.$

No bases ; dimension $= 0$.

16) $x + y + z = 0$

$3x + 2y - 2z = 0$

$4x + 3y - z = 0$

$6x + 5y + z = 0$

Bases $= (4, -5, 1)$, dim $= 1$

$8x_1 + 2x_3 = 0$

11) $x_1 + x_2 - x_3 = 0$

$-2x_1 - x_2 + 2x_3 = 0$

$-x_1 + x_3 = 0.$

Basis $= (1,0,1)$ dim $= 1$

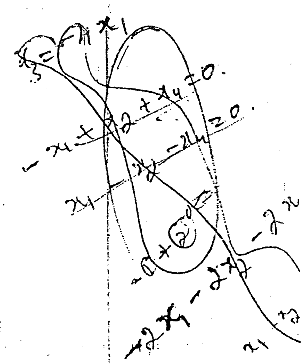
12) $3x_1 + x_2 + x_3 + x_4 = 0.$

$5x_1 - x_2 + x_3 - x_4 = 0.$

soln. Bases $= \{(-\frac{1}{4}, -\frac{1}{4}, 1, 0)$

$(0, -1, 0, 1)\}$.

dim $= 2$.



17. Determine bases for the following subspaces of R^3.

a) The plane $3x - 2y + 5z = 0$

$(\frac{2}{3}, 1, 0)$, $(-\frac{5}{3}, 0, 1)$

b) The plane $x - y = 0$.

$(1,1,0)$, $(0,0,1)$

c) The line $x = 2t$, $y = -t$, $z = 4t$

$(2, -1, 4)$

d) all the vectors of the form (a, b, c) where $b = a + c$

$(1,1,0)$, $(0,1,1)$

$\Rightarrow \begin{bmatrix} a \\ a+c \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

18) Determine
Subspaces
a) all vecto
b) all
$c = a - b$.
c) all "
$\rightarrow$ 1-di

19) Determine
consisting
for wl

; the solu

sion $=0$

$-x_3=0$

$+2x_3=0$

$x_3=0$.

1, 0, 1) dim = 1

$-x_3+x_4=0$.

$+x_3-x_4=0$.

$\{(-\frac{1}{4}, -\frac{1}{4}, 1, 0)$

$(0,-1,0,1)\}$.

2.

R^3.

$x-y=0$.

, (0, 0, 1)

ectors of the

) where $b=a+c$

18) Determine dimensions for following subspaces of R^4.

a) all vectors of the form $(a, b, c, 0) \rightarrow$ 3 dimension.

b) all " " " " (a, b, c, d), where $d=a+b$ & $c=a-b$. $\rightarrow$ 2-dimension.

c) all " of the form (a,b,c,d) where $a=b=c=d$ $\rightarrow$ 1-dimensional.

19) Determine the dimension of the subspace of P_3 consisting of all polynomials $a_0+a_1x+a_2x^2+a_3x^3$ for which $a_0=0$. 3-dimensional Base $\{x, x^2, x^3\}$.

(a) $\begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} a \\ b \\ a-b \\ a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(19) $\therefore a_0 + a_1x + a_2x^2 + a_3x^3 = a_1\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}x + a_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}x^2 + a_3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}x^3$

$= a_1\begin{bmatrix} 0 \\ x \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 0 \\ x^2 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 0 \\ x^3 \end{bmatrix}$

5.5 :

Row space, Column space & Null space

Defn:-

For $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ the

vectors $r_1 = [a_{11}\ a_{12} \cdots a_{1n}] \ldots\ r_m = [a_{m1}\ a_{m2} \cdots a_{mn}]$ in R^n formed from the rows of A are called the row vectors of A and the vectors $c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

in R^m formed from the columns of A are called column vectors of A.

Note:- 1. Elements in row ~~space~~ depending on columns.
2. Elements in column space depending on rows.

Ex-1:

$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$ Row vectors of A: $r_1 = [2\ 1\ 0]$ $r_2 = [3\ -1\ 4]$ in R^n (n=3)

Column vectors of A: $c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ in R^m (m=2)

→ (Do problem-1 (Exer))

Defn:

If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the row space of A and the subspace of R^m spanned by the column vectors of A is called the column space of A.

The solution space of homogeneous system of eqn $Ax = 0$, which is a subspace of R^n, is called the nullspace of A.

Thm-5...

A syste...
is consis...

Ex-2: A...

(1) S.T $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

Soln $x_1 = 2$

$\Rightarrow x_1$

Thm-5.5...

If x_0 ...
system A...
the nullsp...
homogeneo...
$Ax = b$ a...

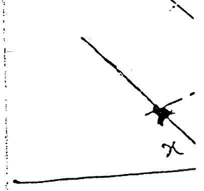
$x =$

and conve...
the vecto...

Note:

The v...
The gener...
$c_1 v_1 + c_2 v_2$...

Geometrical...



ace

$$A/b \begin{bmatrix} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 0 & 17 & 51 \end{bmatrix}$$

the $\Rightarrow x_3 = 3$, $x_2 = -1$, $x_1 = 2$

on

$n_2 - - - a_{mn}]$.

called the

$\ldots, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

called column

$r_2 = [3 \ -1 \ 4]$

$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

bspace of R^m

the row space

by the column

of A.

system of eq.

lled the

Thm-5.5.1:

"A system of linear equations $Ax=b$ is consistent if and only if 'b' is in the column space of A"

$\rightarrow$ (Exercise problems $\rightarrow$ (2) a, b, c, ... , (3) (c), (b))

Ex-2: Assign

(1) S.T ~~Let $Ax=b$ be the linear system~~

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$ ~~is consistent.~~

S.T b is in the column space of A, and express b as a linear combination of column vectors of A.

Solu: $x_1 = 2,\ x_2 = -1,\ x_3 = 3$ (by Gauss elimination). Since the system is consistent, $\Rightarrow b$ is in the column space of A

$\Rightarrow x_1 c_1 + x_2 c_2 + x_3 c_3 = b$ i.e.,

$$\Rightarrow 2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Thm-5.5.2:

If 'x_0' denotes any single solution of a consistent linear system $AX=b$, and if $v_1, v_2, \ldots, v_k$ form a basis for the nullspace of 'A', that is, the solution space of the homogeneous system $AX=0$, then every solution of $Ax=b$ can be expressed in the form,

$$X = x_0 + c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \longrightarrow (*)$$

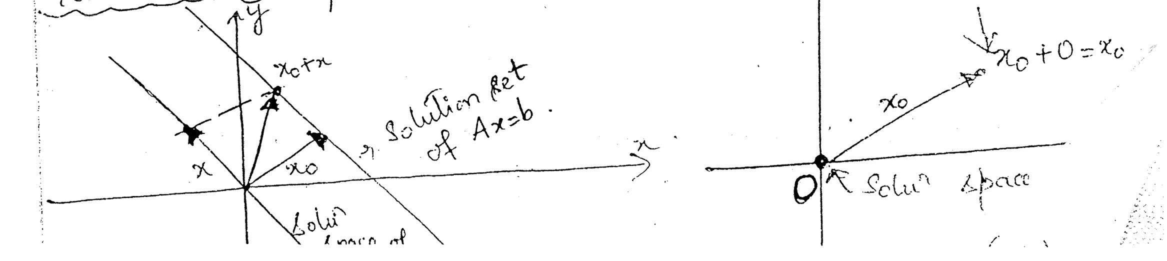
and conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$ the vector 'X' in this formula is a solution of $AX=b$.

($\rightarrow$ Pr $\rightarrow$ (4), (5))

Note:

The vector 'x_0' is the particular solution of $Ax=b$. The general solution of $Ax=b$ is (*) and. The general solution of $Ax=0$. $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ is the general solution of $Ax=0$.

Geometrical interpretation: (Adding x_0 to each vector 'x' in the solution space of $Ax=0$ translates the solution space).



Bases for Row spaces, column spaces and Nullspaces.

Thm - 5.5.2:

Elementary row operations do not change the row space of a matrix.

Thm - 5.5.3

Elementary row operations do not change the nullspace of a matrix.

* **Thm - 5.5.6:**

If a matrix R is in row-echelon form, then the row vectors with the leading 1's (i.e the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

*

Thm - 5.5.5. (State it first)

Example - 6:

Find bases for the row and column spaces of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Sol. Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis for the row space of any row-echelon form of A.

Reducing A to row-echelon form we obtain

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases for Row and column spaces: The matrix $R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row-echelon form.

From Theorem 5.5.6 the vectors $r_1 = [1\ -2\ 5\ 0\ 3]$, $r_2 = [0\ 1\ 3\ 0\ 0]$, $r_3 = [0\ 0\ 0\ 1\ 0]$ form a basis for the row space of R, and the vectors

$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$, $c_2 = \begin{bmatrix} -2 \\ 1 \\ \vdots \end{bmatrix}$ $c_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$ form a bases for the Column space

---

By Thm
vectors
The row
basis f

∴ The

Note:
A and
find a
Column

From Th
of R th
then the
a basis

∴ The
1's of
$c_1' =$
the colum
vectors o
$c_1 =$

Thm-5.5.
If
a) A given
the corre
b) A given
column sp
form a

ullspaces.

By Thm - 5.5.6 The nonzero row vectors of R form a basis for the row space of R, and hence a basis for the row space of 'A'.

ange the

$\therefore$ The basis vectors are
$r_1 = [1 \ -3 \ 4 \ -2 \ 5 \ 4]$
$r_2 = [0 \ 0 \ 1 \ 3 \ -2 \ -6]$
$r_3 = [0 \ 0 \ 0 \ 0 \ 1 \ 5]$

ullspace of a matrix.

Note:

Keeping in mind, 'A' and 'R' may have different column spaces, we cannot find a basis for the column space of 'A' directly from Column vectors of 'R'.

rm, Then the nonzero pace of R, ng 1's of column space

From Thm - 5.5.5b; if we can find a set of column vectors of R that forms a basis for the column space of 'R', then the corresponding column vectors of 'A' will form a basis for the column space of 'A'.

$\therefore$ The 1^{st}, 3^{rd} and 5^{th} columns of R contain the leading 1's of the row vectors, so

of

$$c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of 'R'; thus the corresponding column vectors of 'A', namely.

ange the is for the the row space

$$\therefore \quad c_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \quad c_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \quad c_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for column space of 'A'.

Thm-5.5.5:

If 'A' and B are row equivalent matrices, then

a) A given set of column vectors of 'A' is linearly independent iff the corresponding column vectors of 'B' are L.I.

b) A given set of column vectors of 'A' forms a basis for the column space of 'A' iff the corresponding column vectors of 'B' form a basis for the column space of 'B'.

Ex-7: [Basis for a vector space using row operations].
Find a basis for the space spanned by the vectors

Solⁿ: $v_1 = (1, -2, 0, 0, 3)$, $v_2 = (2, -5, -3, -2, 6)$ $v_3 = (0, 5, 15, 10, 0)$
$v_4 = (2, 6, 18, 8, 6)$.

Solⁿ:- Note: Except for a variation in notation, the space spanned by these vectors is the row space of the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Reducing this matrix to row-echelon form, we obtain

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors in this matrix are
$w_1 = (1, -2, 0, 0, 3)$, $w_2 = (0, 1, 3, 2, 0)$, $w_3 = (0, 0, 1, 1, 0)$

These vectors form a basis for the row space and consequently form a basis for the subspace of R^5 spanned by v_1, v_2, v_3 and v_4.

WORKING PROCEDURE:

Given set of vectors $S = \{v_1, v_2, \ldots, v_k\}$ in R^n, the following procedure produces a subset of these vectors that form a basis for span(S) and express those vectors of S that are not in the basis as linear combination of basis vectors.

Step-1:

Form the matrix 'A' having $v_1, v_2, \ldots, v_k$ as its column vectors.

Step-2: Reduce the matrix 'A' to its reduced row-echelon form R, and let $w_1, w_2, \ldots, w_k$ be the column vectors of 'R'.

Step-3:
Identify ...
leading ...
column ...
Span(S).

Step-4:
Express e...
contain a ...
proceeding ...
this by ...
eqns inv...
-ding eq...
vectors n...
basis vect...

Eg, 3, 4, 5...

Exercise - ...

1. List the ...

2. Express th...
column vec...

a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ [...

c) $\begin{bmatrix} -3 & 6 & \ldots \\ 5 & -4 & \ldots \\ 2 & 3 & \ldots \\ 1 & 8 & \ldots \end{bmatrix}$

$-1c_1 + 2c_2 \ldots$

s] .
e vectors .
$v_3 = (0,5,15,10,0)$
e space
f the matrix
e obtain .
are
$v_3 = (0,0,1,1,0)$
ce and
of R^5 spanned
in R^n,
of these vectors
xpress those
as linear
..., v_k as its
ced row-echelon
n vectors of

Step-3:
Identify the columns that contain leading 1's in 'R'. The corresponding column vectors of 'A' are the basis vectors for Span(S).

Step-4:
Express each column vectors of 'R' that does not contain a leading 1 as a linear combination of the proceeding column vectors that do contain 1's (Do this by inspection). This yields a set of dependency eqns involving the column vectors of 'R'. The correspon-ding eqns for the column vectors of 'A' express the vectors not in the basis as linear combination of the basis vectors.

Eg, 3, 4, 5, 6, 7, 8, 9.

Exercise - 5.5 :

1. List the row, column vectors of the matrix $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$

2. Express the product Ax as a linear combination of the column vectors of 'A'

a) $\begin{array}{cc} c_1 & c_2 \end{array}$ $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $1c_1 + 2c_2$ column vectors

$\begin{bmatrix} 8 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow c_1 = 1, c_2 = 2$

b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$ $-2c_1 + 3c_2 + 5c_3$

c) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

$-1c_1 + 2c_2 + 5c_3$

d) $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$

$3c_1 + 0c_2 - 5c_3$

3. Determine whether 'b' is in the column space of 'A' and if so, express 'b' as a linear combination of the column vectors of 'A'.

a) $A=\begin{bmatrix}1 & 3\\ 4 & -6\end{bmatrix}$, $b=\begin{bmatrix}-2\\ 10\end{bmatrix}$

b) $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 3\end{bmatrix}$; $b=\begin{bmatrix}-1\\ 0\\ 2\end{bmatrix}$

Solⁿ Consider $k_1C_1+k_2C_2=b$

$$k_1\begin{bmatrix}1\\ 4\end{bmatrix}+k_2\begin{bmatrix}3\\ -6\end{bmatrix}=\begin{bmatrix}-2\\ 10\end{bmatrix}$$

$\Rightarrow k_1=1 \quad k_2=-1.$

$\therefore b=C_1-C_2.$

Consider

$k_1C_1+k_2C_2+k_3C_3=b.$

Not a possible.

c) $A=\begin{bmatrix}1 & -1 & 1\\ 9 & 3 & 1\\ 1 & 1 & 1\end{bmatrix}$; $b=\begin{bmatrix}5\\ 1\\ -1\end{bmatrix}$

d) $A=\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{bmatrix}$ $b=\begin{bmatrix}2\\ 0\\ 0\end{bmatrix}$

$C_1+(t-1)C_2+tC_3=b.$

Solⁿ: $C_1-3C_2+C_3=b$.

e) $A=\begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 2 & 1\\ 1 & 2 & 1 & 3\\ 0 & 1 & 2 & 2\end{bmatrix}$; $b=\begin{bmatrix}4\\ 3\\ 5\\ 7\end{bmatrix}$

4. Suppose that $x_1=-1$; $x_2=2$, $x_3=4$, $x_4=-3$ is a solution of a nonhomogeneous linear system $AX=b$ and that the solution set of the homogeneous system $AX=0$ is given by the formulas,

$x_1=-3r+4s$; $x_2=r-s$; $x_3=r$, $x_4=s$.

Solⁿ: Given $X_0=\begin{bmatrix}-1\\ 2\\ 4\\ -3\end{bmatrix}$. & Vector form of the G-S of $AX=0$,

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=r\begin{bmatrix}-3\\ 1\\ 1\\ 0\end{bmatrix}+s\begin{bmatrix}4\\ -1\\ 0\\ 1\end{bmatrix}.$$

The vector form of the General solution of $Ax=b$ is given by $X=X_0+c_1v_1+c_2v_2+c_3v_3 \cdots + c_nv_n$.

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ 4\\ -3\end{bmatrix}+r\begin{bmatrix}-3\\ 1\\ 1\\ 0\end{bmatrix}+s\begin{bmatrix}4\\ -1\\ 0\\ 1\end{bmatrix}.$$

05). Find the
the given
the vector

a) $x_1-3x_2=$
$2x_1-6x_2=$

Solⁿ:- $X=\begin{bmatrix}1\\ 0\end{bmatrix}$

$X=t\begin{bmatrix}3\\ 1\end{bmatrix}$ i

b) x_1+x_2+2
$x_1+\quad+$
$2x_1+x_2+$

Solⁿ $X=\begin{bmatrix}-2\\ 7\\ 0\end{bmatrix}+$

d) x_1+2x_2-
$-2x_1+x_2+$
$-x_1+3x_2-$
$4x_1-7x_2$

06) Find a b

a) $A=\begin{bmatrix}1 & -1\\ 5 & -4\\ 7 & -6\end{bmatrix}$

Solⁿ: $Ax=0$ $X=$

d), e) are

07) In each
By inspec
of A

a) $A=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$

ce of 'A' and the column

$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$; $b=\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

sider

$+k_2c_2+k_3c_3=b$

a possible

$=\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$c_2+tc_3=b$.

. a solution of that the solution iven by the

s.

. the G.S of $Ax=0$.

$+s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

$Ax=b$ is

$+c_n u_n$.

05). Find the vector form of the G.S of the given system $Ax=b$; use it to find the vector form of the general sol.on of $Ax=0$.

a) $x_1 - 3x_2 = 1$

$2x_1 - 6x_2 = 2$

Solu: $X=\begin{bmatrix} 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the vector form of the G.S of $Ax=b$.

$X=t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the vector form of the G.S of $Ax=0$.

b) $x_1+x_2+2x_3=5$

$x_1 + \quad +x_3=-2$

$2x_1+x_2+3x_3=3$

Solu: $X=\begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix}+t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

(c) $x_1-2x_2+x_3+2x_4=-1$

$2x_1-4x_2+2x_3+4x_4=-2$

$-x_1+2x_2-x_3-2x_4=1$

$3x_1-6x_2+3x_3+6x_4=-3$

Solu: $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}+r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}+s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(d) $x_1+2x_2-3x_3+x_4=4$

$-2x_1+x_2+2x_3+x_4=-1$

$-x_1+3x_2-x_3+2x_4=3$

$4x_1-7x_2 \qquad -5x_4=-5$

Solu: $\begin{bmatrix} 6/5 \\ 7/5 \\ 0 \\ 0 \end{bmatrix}+s\begin{bmatrix} 7/5 \\ 4/5 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} 1/5 \\ -3/5 \\ 0 \\ 1 \end{bmatrix}$.

06) Find a basis for the nullspace of A.

a) $A=\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

Solu: $Ax=0$ $X=t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$

b) $A=\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$Ax=0$ $X=t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}+s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

c) $A=\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -2 \\ -4 \\ 0 \\ 7 \end{bmatrix}$

d), e) assgn.

07) In each part, a matrix in row-echelon form is given. By inspection, find bases for the row and column spaces of 'A'

(a) $A=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ r_1, r_2 c_1, c_3

(b) r_1, r_2 c_1, c_2 $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ r_1, r_2 r_3, r_4 c_1, c_2 c_3, c_4

d) $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow r_1, r_2, r_3, r_4$

c_1, c_2, c_3, c_4.

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08) For the matrices in excercise(6), find a basis for the row space of A by reducing the matrix to row-echelon form.

a) $(1,-1,3)$, $(0,1,-19)$ b) $(1,0,-\frac{1}{2})$ c) $(1,4,5,2)$ $(0,1,1,4/7)$

d) $\rightarrow (1,4,5,6,9)$, $(0,1,1,1,2)$

e) $(1,-3,2,2,1)$, $(0,1,2,0,-1)$, $(0,0,1,0,\frac{-5}{12})$

09) For the matrices in Ex-6, find a basis for the column space of A.

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \xrightarrow[R_3 \to 7R_1 - R_3]{R_2 \to 5R_1 - R_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & 19 \\ 0 & -1 & 19 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

Identify the column vectors containing the leading 1's in R. The corresponding column vectors of A are the basis for the column space of A:

$\therefore$ Basis $= \left\{ c_1' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, c_2' = \begin{bmatrix} -1 \\ -4 \\ -6 \end{bmatrix} \right\}$.

b) $B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Solu. $c_1' = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$,

c) $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

Solu. $c_1' = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $c_2' = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

Solu. $\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix}$

e) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$ Solu: $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 3 \\ -3 \\ -6 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \\ -2 \\ 0 \\ 2 \end{bmatrix}$.

~~f(-x) = f(x)~~

~~f(-x)~~

10. For the matrices in Exercise 6, find a basis for the row space of A consisting entirely of vectors of A.

(a) $r_1 = (1\ -1\ 3)$, $r_2 = (5, -4, -4)$

(b) $r_1 = (2, 0, -1)$ (c) $(1,4,5,2), (2,1,3,0)$.

(d) $(1,4,5,6,9)$, $(3,-2,1,4,-1)$ (e) $(1,-3,2,2,1), (0,3,6,0,-3)$, $(2,-3,-2,4,4)$

11) Find a basis for the subspaces of R^4 spanned by the given vectors.

a) $(1,1,-4,-3), (2,0,2,-2), (2,-1,3,2)$

Soln: $$A = \begin{bmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & -2 & 10 & 4 \\ 0 & -3 & 11 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & -3 & 11 & 8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & -4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\tfrac{1}{2} \end{bmatrix}.$$

v_1, v_2, v_3 are rows in $A \simeq R$ (Echelon form)

$\therefore$ Basis for the subspaces of R^4 is

$\{(1,1,-4,-3), (0,1,-5,-2), (0,0,1,-\tfrac{1}{2})\}$.

Note: not based on the working procedure.

b) $(-1,1,-2,0), (3,3,6,0), (9,0,0,3)$.

Soln $(1,-1,2,0)$, $(0,1,0,0)$; $(0,0,1,-\tfrac{1}{6})$.

c) $(1,1,0,0)$, $(0,0,1,1)$, $(-2,0,2,2)$, $(0,-3,0,3)$.

Soln $(1,1,0,0)$, $(0,1,1,1)$, $(0,0,1,1)$, $(0,0,0,1)$.

12) Find a subset of the vectors that forms a basis for the space spanned by the vectors; then express each vector that is not in the basis as a linear combination of the basis vectors.

a) $v_1 = (1,0,1,1)$; $v_2 = (-3,3,7,1)$, $v_3 = (-1,3,9,3)$,
$v_4 = (-5,3,5,-1)$.

Soln

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -3 & 3 & 7 & 1 \\ -1 & 3 & 9 & 3 \\ -5 & 3 & 5 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 10 & 4 \\ 0 & 3 & 10 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \cancel{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 4 & -2 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 10/3 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis : $\{ v_1 , v_2 \}$.

$v_3 = c_1 v_1 + c_2 v_2$

$(-1,3,9,3) = c_1(1,0,1,1) + c_2(-3,3,7,1)$.

$c_1 - 3c_2 = -1$; $c_1 + 7c_2 = 9$

$3c_2 = 3$; $c_1 + c_2 = 3$

$\boxed{c_2 = 1}$ $\Rightarrow \boxed{c_1 = 2}$

$\boxed{\therefore v_3 = 2v_1 + v_2}$

$v_4 = k_1 v_1 + k_2 v_2$

$(-5,3,5,-1) = k_1(1,0,1,1) + k_2(-3,3,7,1)$

$k_1 - 3k_2 = -5$

$3k_2 = 3$ $\boxed{k_2 = 1}$ $\boxed{\therefore v_4 = -2v_1 + v_2}$

$k_1 + 7k_1 = 5$

$k_1 + k_2 = -1$ $\boxed{k_1 = -2}$

basis for
press each
near

),

0	1	1	
3	1	$\cancel{6}$	$\cancel{4}$
3	10	4	
0	0	0	

13

$V_1 + V_2$

b) $v_1 = (1, -2, 0, 3)$, $v_2 = (2, -4, 0, 6)$,
$v_3 = (-1, 1, 2, 0)$, $v_4 = (0, -1, 2, 3)$

Soln. Basis $= \{v_1, v_3\}$, $v_2 = 2v_1 + 0v_3$,
$v_4 = v_1 + v_3$.

c) $v_1 = (1, -1, 5, 2)$, $v_2 = (-2, 3, 1, 0)$, $v_3 = (4, -5, 9, 4)$,
$v_4 = (0, 4, 2, -3)$ $v_5 = (-7, 18, 2, -8)$.

Soln $\{v_1, v_2, v_4\}$; $v_3 = 2v_1 - v_2$, $v_5 = -v_1 + 3v_2 + 2v_4$.

Ass 14)

5.6:

RANK AND Nullity:

We studied about the relationships between systems of linear equations and the row space, column space and nullspace of the coefficient matrix.

In this section we will discuss about the relationships b/n the dimensions of the row space, column space, and nullspace of a matrix and its transpose.

→ Four fundamental matrix spaces associated with "A" are,

i) Row space of A
ii) Column space of A
iii) Nullspace of A
iv) nullspace of AT.

Theorem 5.6.1:

If "A" is any matrix, then the row space and column space of "A" have the same dimension.

Defn:

The common dimension of the row space and column space of a matrix A is called the rank of A and is denoted by rank(A);

The dimension of the null space of 'A' called the nullity of 'A' and it is denoted by nullity(A).

Ex-1: Rank and Nullity of a 4X6 Matrix.

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

1+15
4-9

Soln.

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow (1)$$

Since there are two non-zero rows (two leading 1's), the row space and column space are both 2-Dimensional, so rank$(A) = 2$.

Nullity: we must find the dimension of the solution space of the linear system $AX = 0$.

From (1)
$$x_1 - 2x_2 + 0x_3 - 4x_4 - 5x_5 + 3x_6 = 0.$$
$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0.$$

$$\Rightarrow x_1 = 2x_2 + 4x_4 + 5x_5 - 3x_6$$
$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$$

let $x_3 = r,\ x_4 = s,\ x_5 = t,\ x_6 = u$

$x_1 = 4r + 28s + 37t - 13u$

$x_2 = 2r + 12s + 16t - 5u$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

∴ The four vectors on the right side form a bases for the solution space $(AX=0)$, so that

nullity $(A) = 4$.

Thm - 5.6.2:

If 'A' is any matrix, then rank $(A) =$ rank (A^T).

Problem - 1:

Verify the rank $(A) =$ rank (A^T), given $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$

Soln: Row reduced echelon form leading 1's in non-zero rows, Rank of $(A) =$ Rank $(A^T) = 2$

Thm - 5.6.3:

If 'A' is a matrix with n-columns, then

$\boxed{\text{rank}(A) + \text{nullity}(A) = n}$.

Ex-5.6:

Q2) Find the rank and nullity of the matrix, then verify the Theorem (5.6.3). (i.e Rank + nullity = no of Column vectors).

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ Nullity $= 1$, Rank $= 2$, $n = 3$.

(Echelon form), find the nullspace.

b) $A=\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $N=2,\ R=1,\ n=3.$

c) $A=\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ $N=2,\ R=2,\ n=4.$

d) $A=\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$ $N=3,\ r=2,\ n=5$ (e) Assign

(13) Are there values of 'r' and 's' for which the rank of

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$ is one or two? If so, find those values.

$\rightarrow$ Rank $=2$, if $r=2, s=1$.

Rank is never 1.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (r-2) & 2 \\ 0 & 0 & 3 \\ 0 & 0 & (r^2-4)-2(s-1) \end{bmatrix}$

Note: $A_{m\times n}$.

Fundamental Space	Dimension (known)
Row space of A	r (rank)
Column space of A	r
Nullspace of A	$n-r$
Nullspace of A^T	$m-r$.

Ex-5.6:

04) In each part use the information in the table to find the dimension of the row space of A, column space of A, the null space of A and nullspace of A^T. ($A_{m\times n}$)

	(a)	(b)	(c)	(d) $A^T=9\times5$	(e) $A^T=5\times9$	(f)	(g) $A^T=2\times6$
Size of A	$\overset{m}{3}\times\overset{n}{3}$	3×3	3×3	5×9	9×5	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
	↓	↓	↓	↓	↓	↓	↓

Solun Dim(A)

Row space →	3	2	1	2	2	0	2
Column space →	3	2	1	2	2	0	2
Null space(A) →	$3-3=0$	1	2	7	3	4	0
Null space (A^T) →	0	1	2	3	7	4	4.

(43) 05) In poss... smal... of ... a) A ... Solu: Rank= ... Note 06) If value... its ... Solu: Rank ... Theo: If n-unk... a) Ax. c) The [A ... Theor If a) Ax b) I c) A eve...

Assgn

and 'ε' for

If so, find

$r=2, s=1$.

never 1.

0

2) 2

3

(s^2-4)

$2(s-1)$

able to

, column

ullspace

$A^T = 2\times 6$.

$\times 2$

05) In each part, find the largest possible values for rank of 'A' and smallest possible values for the nullity of 'A'.

a) A is 4×4. (b) A is 3×5 (c) A is 5×3.

Solu: Rank $=4$, Nullity $=0$ Rank $=3$, nullity $=2$ Rank $=3$, nullity $=0$.

Note: $\text{Rank}(A) \le \min(m,n)$

06) If 'A' is $m\times n$ matrix, what is the largest possible value for its rank and the smallest possible value for its nullity.

Solu: $\text{Rank} = \min(m,n)$, $\text{nullity} = n - \min(m,n)$.

Theorem - 5.6.5:

If $Ax=b$ is a linear system of 'm' equations in n-unknowns, then the following are equivalent.

a) $Ax=b$ is consistent b) 'b' is in Column space of 'A'

c) The co-efficient matrix 'A' and the augmented matrix $[A|b]$ have the same rank.

Theorem - 5.6.8:

If 'A' is $m\times n$ matrix, then the following are equivalent

a) $Ax=0$ has only the trivial solution.

b) The column vectors of 'A' are L.I.

c) $Ax=b$ has at most one solution (none or one) for every $m\times 1$ matrix b.

problems - 5.6:

7) & 8. In each part determine whether the linear System $AX = b$ is consistent. If so, (state the number of parameters in its general solution) find the nullity of A:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank [A|b]	3	3	1	2	3	0	2
Consistent →	Yes	No	Yes	Yes	No	Yes	Yes
Nullity	0	1	2 (n-rank = (3-1))	7	7	4	0

9) What conditions must be satisfied by b_1, b_2, b_3, b_4, b_5 for over determined linear system.

$$x_1 - 3x_2 = b_1,$$
$$x_1 - 2x_2 = b_2$$
$$x_1 + x_2 = b_3$$
$$x_1 - 4x_2 = b_4$$
$$x_1 + 5x_2 = b_5$$

to be consistent.

Solu: The given system is not consistent for all possible values of b_1, b_2, b_3, b_4, b_5.

Exact conditions under which the system is consistent can be obtained by solving the system by Gauss-Jordan elimination

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 1 & -2 & b_2 \\ 1 & 1 & b_3 \\ 1 & -4 & b_4 \\ 1 & 5 & b_5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 4 & b_3 - b_1 \\ 0 & -1 & b_4 - b_1 \\ 0 & 8 & b_5 - b_1 \end{array}\right]$$

stem $AX=b$
rameters in

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & 0 & b_3-4b_2+3b_1 \\ 0 & 0 & b_4+b_2 \cancel{\mp} 2b_1 \\ 0 & 0 & b_5-8b_2+7b_1 \end{array}\right]$$

(g) 6×2

2

2

Yes

0.

The system is consistent iff $b_1, b_2, \ldots, b_5$ satisfy the conditions.

$$b_3-4b_2+3b_1=0 \qquad b_3=4b_2-3b_1$$
$$-2b_1+b_4+b_2=0 \qquad b_4=2b_1-b_2$$
$$b_5-8b_2+7b_1=0 \qquad b_5=8b_2-7b_1$$

$, b_2, b_3, b_4, b_5$

$\Rightarrow$ Let $b_1=r$, $b_2=s$,

$\Rightarrow$ $b_3=4s-3r$, $b_4=2r-s$,

$b_5=8s-7r$.

11. Suppose that A is a 3×3 matrix whose nullspace is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

~~$b_4-b_1-b_2+b_1$~~

~~$b_4-b_2-2b_1$~~

all

Soln: No.

b_2 ~~$-b_1-b_2$~~

12. Discuss how the rank of A varies with "t".

is

System

a) $A=\begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \longrightarrow R\begin{bmatrix} 1 & 1 & t \\ 0 & (t-1) & (1-t) \\ 0 & 0 & (1-t)(t+2) \end{bmatrix}$

Rank	if	t
1	if	$t=1$
2	if	$t=-2$
3	if	$t\neq 1,-2$, ~~...~~

(b) $A=\begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

Rank	if	t
2		$t=1, 3/2$
3	if	$t\neq 1, 3/2$

13) Are there values of r and s for which the rank of

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow$ be one or two? If so, find those values.

~~$1(t-1)-1(1-t)+t(1-t^2)=0$~~