23MAT124
ENGINEERING MATHEMATICS-1

INTRODUCTION TO MATRICES AND VECTORS

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Matrices

- A matrix is a rectangular array of numbers.
- The horizontal lines of numbers form rows and the vertical lines of numbers form columns. A matrix with m rows and n columns is said to be an m×n matrix.
- The entries of an m × n matrix are indexed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn-1} & a_{mn} \end{bmatrix}$$

Order of a matrix

- No: of rows and columns

5 Columns

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 9 & 8 & -6 & 7 & -3 \end{bmatrix}$$

2 Columns

Order of A = 2 × 5

Types of matrices

Row matrix

One row and any no: of columns.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}_{1\times n}$$

Column matrix

One column and any no: of rows

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$$

Square matrix

No: of rows = No: of columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 5 & 8 & 2 \\ 4 & 0 & -1 \end{bmatrix}$$

Rectangular matrix

No: of rows$\neq$ No: of columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 4 & 5 \end{bmatrix}$$

Types of matrices

Zero or null matrix

All elements are zero

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_{ij} = 0$ for all i, j

Diagonal matrix

Elements other than those occurring in the principal diagonal are zero

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for all $i = j$

Scalar matrix

Main diagonal elements are equal to the same scalar

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$a_{ij} = 0$ for all $i \neq j$

$a_{ij} = a$ for all $i = j$

Unit or Identity Matrix

All of its elements in the principal diagonal are unity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for all $i = j$

Types of matrices

Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$a_{ij} = 0$ for all $i > j$

Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

$a_{ij} = 0$ for all $i < j$

Operations on matrices

Equality of matrices

Two matrices are said to be equal only when all corresponding elements are equal. Therefore their size or dimensions are equal as well.

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 6 & -1 & 10 \\ 8 & 9 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & 3 \\ 6 & -1 & 10 \\ 8 & 9 & 7 \end{bmatrix}$$

$$\Rightarrow A = B$$

If $\mathbf{A} = \mathbf{B}$ then $a_{ij} = b_{ij}$

Let's try...

1. Find the value of x such that A=B

$$A=\begin{bmatrix} \sin x & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \cos x \end{bmatrix}_{2\times 2} \qquad B=\begin{bmatrix} \frac{1}{\sqrt{2}} & \sin x \\ \cos x & \cos x \end{bmatrix}_{2\times 2}$$

2. Find the values of a and b if

$$2\begin{bmatrix} 1 & 3 \\ 0 & a \end{bmatrix}+\begin{bmatrix} b & 0 \\ 1 & 7 \end{bmatrix}=\begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Operations on matrices

Addition of matrices

The sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices.

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 8 & 2 \\ 4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 8 & 0 \\ 3 & 10 & -2 \\ 9 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 10 & 3 \\ 8 & 18 & 0 \\ 13 & 1 & 0 \end{bmatrix}$$

Operations on matrices

Subtraction of matrices

The difference of two matrices is a matrix obtained by subtracting the corresponding elements of the given matrices.

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 8 & 2 \\ 4 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 8 & 0 \\ 3 & 10 & -2 \\ 9 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -6 & 3 \\ 2 & -2 & 4 \\ -5 & -1 & -2 \end{bmatrix}$$

Properties of matrix addition

- $A + B = B + A$ (Commutative law)
- $(A + B) + C = A + (B + C)$ (Associative law)
- $A + O = O + A = A$ (Existence of additive identity)

  O is the additive identity for matrix addition
- $A + (-A) = (-A) + A = O$ (Existence of additive inverse)

  -A is the additive inverse of A

Operations on matrices

Scalar multiplication of matrices

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then $\boldsymbol{kA} = \boldsymbol{Ak}$.

For example,

$$2\begin{bmatrix} 1 & 2 & 3 \\ 5 & 8 & 2 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 10 & 16 & 4 \\ 8 & 0 & -2 \end{bmatrix}$$

Properties of scalar multiplication of a matrix

- $k(A + B) = kA + kB$
- $(k + l)A = kA + lA$

Operations on matrices

Multiplication of matrices

The product of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B.

$$\begin{array}{ccccc} \boldsymbol{A} & \times & \boldsymbol{B} & = & \boldsymbol{AB} \\ \downarrow & & \downarrow & & \downarrow \\ m \times n & & n \times p & & m \times p \end{array}$$

Matrix multiplication

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} (a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) & (a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) \\ (a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) & (a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) \end{bmatrix}$$

2× 3 **3× 2** **2× 2**

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1 \times 4) + (2 \times 6) + (3 \times 5) & (1 \times 8) + (2 \times 2) + (3 \times 3) \\ (4 \times 4) + (2 \times 6) + (7 \times 5) & (4 \times 8) + (2 \times 2) + (7 \times 3) \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Properties of matrix multiplication

Assuming that matrices A, B, and C are conformable for the operations indicated, the following are true:

- $A(BC) = (AB)C = ABC$ $(Associative\ law)$
- $A(B+C) = AB + AC$ $(First\ distributive\ law)$
- $(A+B)C = AC + BC$ $(Second\ distributive\ law)$
- $AI = IA = A$ $(Existence\ of\ multiplicative\ identity)$

Non-commutativity of multiplication of matrices

Even if AB and BA are both defined, it is not necessary that $AB = BA$.

For example,

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

$$AB \neq BA$$

Zero matrices as the product of two non zero matrices

For real numbers, if $ab = 0$, then either $a = 0$ or $b = 0$. This need not be true for matrices.

For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Let's try...

1) Find the product of the matrices if possible.

a) $A = \begin{bmatrix} -6 & -3 & -1 \end{bmatrix}$

$B = \begin{bmatrix} -3 & 6 \\ -2 & -6 \\ 6 & -1 \end{bmatrix}$

b) $A = \begin{bmatrix} 4 & -12 & 1 \\ 1 & 5 & 0 \end{bmatrix}$

$B = \begin{bmatrix} -4 & 1 & 9 \\ -3 & 7 & -6 \end{bmatrix}$

c) $A = \begin{bmatrix} 9 & 2 & 7 \\ 5 & 8 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 2 & 13 \\ -3 & 0 & 7 \\ -4 & -1 & 12 \end{bmatrix}$

2) Find the value of $A^{2025} - A^{2020}$, if A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Transpose of a matrix

Matrix obtained by interchanging rows and columns is called the transpose of the matrix.

$$A = \begin{bmatrix} 8 & 6 & 0 \\ 5 & 4 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 8 & 5 \\ 6 & 4 \\ 0 & 2 \end{bmatrix}$$

Properties of transpose of matrices

- $(A')' = A$
- $(kA)' = kA'$, where k is any constant
- $(A + B)' = A' + B'$
- $(AB)' = B'A'$

Trace of a matrix

Let A be an $n\times n$ matrix. The trace of A, denoted by tr(A) is the sum of the diagonal elements of A .

$$A = \begin{bmatrix} a_{11} & a_{12}\ldots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22}\ldots & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn-1} & a_{nn} \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22} + \ldots + a_{nn}$$

Let's try...

Find the trace of A, B, C and I_4 where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 8 & 1 \\ -2 & 7 & -5 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Properties of the Matrix Trace

Let A and B be n×n matrices.

- $tr(A + B) = tr(A) + tr(B)$
- $tr(A - B) = tr(A) - tr(B)$
- $tr(kA) = k \cdot tr(A)$
- $tr(AB) = tr(BA)$
- $tr(A') = tr(A)$

Symmetric matrices

A square matrix A is said to be symmetric if $A^T = A$.

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}, A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

For example

$$\begin{bmatrix} 8 & 9 \\ 9 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Skew- symmetric matrices

A square matrix A is said to be skew-symmetric if $A^T = -A$.

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, A^T = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

For example

$$\begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$$

Skew-symmetric matrices

- For any square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew-symmetric matrix.

- Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

$$\mathbf{A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')}$$

Let's try…

Express the following matrices as a sum of symmetric and skew-symmetric matrices

a) A= $\begin{bmatrix} -1 & 6 \\ 2 & 0 \end{bmatrix}$

b) B= $\begin{bmatrix} 0 & 8 & -3 \\ 4 & 1 & 7 \\ -1 & 2 & 5 \end{bmatrix}$

Determinant

Each square matrix A has a unit scalar value called the determinant of A, denoted by det A or |A|.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\, a_{22} - a_{21} a_{12}$$

Let’s try…

Find the determinant of the following matrices.

$$A = \begin{bmatrix} 7 & -10 & 0 \\ 8 & -1 & 4 \\ 3 & 6 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 5 & 0 & 5 \\ 7 & 0 & 3 & 0 \\ 4 & -3 & 5 & -3 \\ -2 & 8 & -8 & 8 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Minors

- If A is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of A.
- The determinant of such a submatrix is called a minor of A and is designated by M_{ij} , where *i* and *j* correspond to the deleted row and column, respectively.
- M_{ij} is the minor of the element a_{ij} in A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$\boxed{\textit{Sum of a row number } i \textit{ and column } j = \begin{cases} \textit{even, then } C_{ij} = M_{ij} \\ \textit{odd, then } C_{ij} = -M_{ij} \end{cases}}$$

$$C_{11}(i=1, j=1) = (-1)^{1+1} M_{11} = M_{11}$$

$$C_{12}(i=1, j=2) = (-1)^{1+2} M_{12} = -M_{12}$$

$$C_{13}(i=1, j=3) = (-1)^{1+3} M_{13} = M_{13}$$

Adjoint of a matrix

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix.

Easy way to find the adjoint of a 2x2 matrix

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, then\ adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Change sign

Interchange

Let's try…

Find the minors, cofactors, and adjoint of the given matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

Inverse of a matrix using adjoint

$$A^{-1} = \frac{1}{|A|} adj A, \ |A| \neq 0$$

Let's try…

Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$.

Representation of Vectors

- A **Vector** is an ordered finite list of numbers, usually written as vertical arrays in square or curved brackets:

$$\boldsymbol{a} = \begin{bmatrix} 2 \\ -1 \\ 6 \\ 9 \end{bmatrix}, \ \boldsymbol{b} = \begin{pmatrix} -3.1 \\ 2.7 \\ -9.2 \end{pmatrix}$$

- Sometimes vectors are also represented in a row form with each element separated by commas in a parenthesis like $\boldsymbol{a}$ = (-2, -1, 6, 9), $\boldsymbol{b}$ = (-3.1, 2.7, -9.2).
- Some books also represent vectors as $\boldsymbol{a}$ = $(-2, -1, 6, 9)^T$ or $\boldsymbol{b}$ = $(-3.1, 2.7, -9.2)^T$. Here T represents Transpose.
- The **elements / entries / components** of a vector are the values in the array.
- The **size / dimension / length** of the vector is the number of elements in the vector.
- A vector of unit magnitude is called a **unit vector**.
- A standard unit vector is a vector with one component '1' and all other elements '0'.

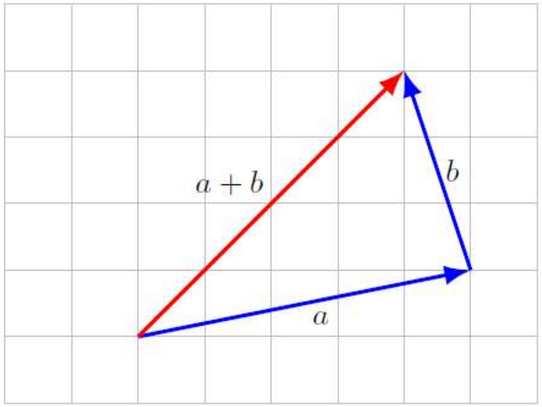
$$\hat{\imath} = \widehat{e_1} = (1,0,0), \quad \hat{\jmath} = \widehat{e_2} = (0,1,0), \quad \hat{k} = \widehat{e_3} = (0,0,1),$$
$$\widehat{e_1} = (1,0,0,\dots), \quad \widehat{e_2} = (0,1,0,0,\dots), \dots$$

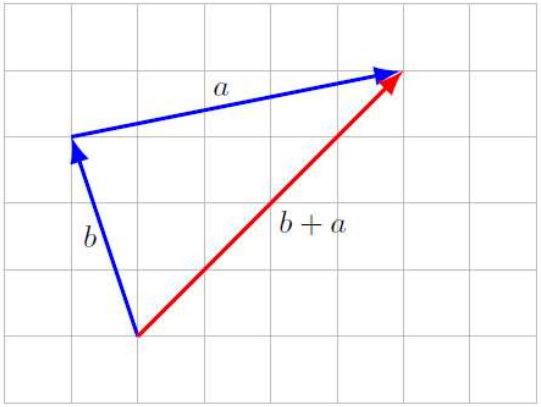
- Vectors can also be represented using the standard unit vectors:

$$\boldsymbol{a} = 2\,\widehat{e_1} - \widehat{e_2} + 6\widehat{e_3} + 9\widehat{e_4}\,, \quad \boldsymbol{b} = -3.1\hat{\imath} + 2.7\hat{\jmath} - 9.2\hat{k}$$

- A real vector is a vector with all elements as real numbers.
- Just like **R** represents the set of real numbers, the set $\boldsymbol{R}^{\boldsymbol{n}}$ represents the set of all n-dimensional vectors. $\boldsymbol{a} \in \boldsymbol{R}^4$ and $\boldsymbol{b} \in \boldsymbol{R}^3$

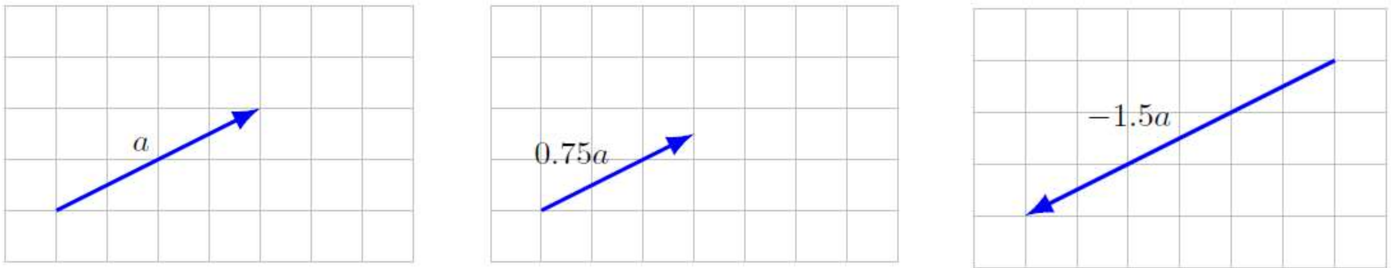
Vector Addition





- Vector addition is *commutative*: $a + b = b + a$.
- Vector addition is *associative*: $(a + b) + c = a + (b + c)$. We can therefore write both as $a + b + c$.
- $a + 0 = 0 + a = a$. Adding the zero vector to a vector has no effect. (This is an example where the size of the zero vector follows from the context: It must be the same as the size of a.)
- $a - a = 0$. Subtracting a vector from itself yields the zero vector. (Here too the size of 0 is the size of a.)

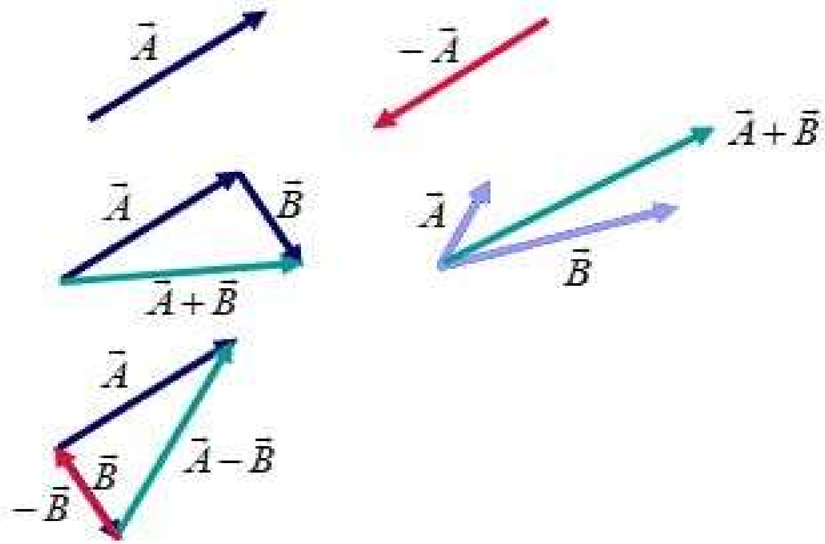
Scalar Multiplication



- The vector 0:75a represents the displacement in the direction of the displacement **a**, with magnitude scaled by 0.75
- -(1.5)**a** represents the displacement in the opposite direction, with magnitude scaled by 1.5.

Vector Math

- Vector Inverse
  - Just switch direction
- Vector Addition
  - Use head-tail method, or parallelogram method
- Vector Subtraction
  - Use inverse, then add
- Vector Multiplication
  - Two kinds!
  - Scalar, or dot product
  - Vector, or cross product



Vector Addition by Components

$$\bar{A}+\bar{B}=(A_x+B_x)\hat{i}+(A_y+B_y)\hat{j}+(A_z+B_z)\hat{k}$$

Vector dot product

Illustration of dot product:

If $\underline{A}$ and $\underline{B}$ are two vectors of form,

$$\underline{A} = A_1 i + A_2 j + A_3 k$$

$$\underline{B} = B_1 i + B_2 j + B_3 k$$

Then the dot product of A and B is,

$$\underline{A}.\underline{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

The dot relationship of unit vectors along three axes :

$$i \, . \, j = j \, . \, k = k \, . \, i = 0$$

and $i \, . \, i = j \, . \, j = k \, . \, k = 1$

The dot relati
axes :
i . j
and i . i

Illustration of cross product:

If $\underline{A}$ and $\underline{B}$ are two vectors of form

$$\underline{A} = A_1 i + A_2 j + A_3 k$$

$$\underline{B} = B_1 i + B_2 j + B_3 k$$

Then the cross Product of $\underline{A}$ and $\underline{B}$ is,

$$\underline{A} \times \underline{B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

The cross relationship of unit vectors along three axes are:

$$i \times i = j \times j = k \times k = 0$$

$$i \times j = k \quad \& \quad j \times i = -k$$

$$j \times k = i \quad \& \quad k \times j = -i$$

$$k \times i = j \quad \& \quad i \times k = -j$$

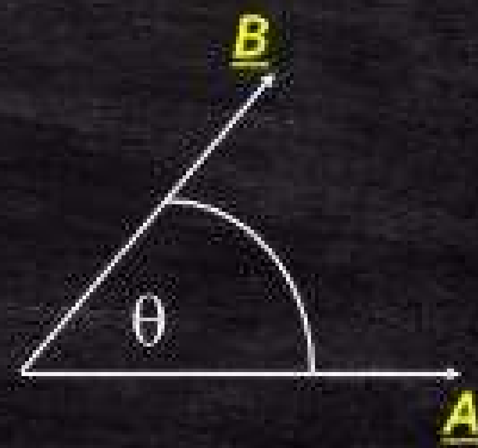
Applications of dot product:

1. Finding angle between two vectors:

$$\underline{A}.\underline{B} = |\underline{A}||\underline{B}| \cos\theta$$

$$\cos\theta = \frac{A.B}{AB}$$

$$\theta = \cos^{-1}\left(\frac{A.B}{AB}\right)$$



Laws of Operations

- Commutative

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

- Associative with respect to scalar multiplication

$$a(\mathbf{A} \cdot \mathbf{B}) = (a\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (a\mathbf{B})$$

- Distributive with respect to vector addition

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{D})$$

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by

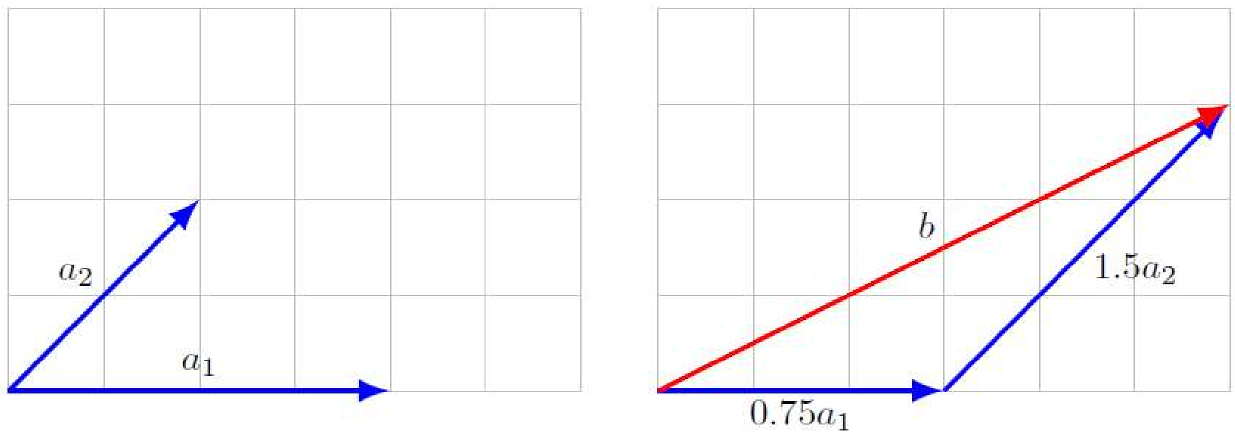
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Linear Combination



The linear combination $b = 0.75a_1 + 1.5a_2$.

Any vector can be written as a linear combination of unit vectors.

Linear combination of vectors

A linear combination of vectors is an expression formed by multiplying each vector by a scalar and then adding the results.

The linear combination of the vectors $u_1, u_2, u_3 \ldots u_m$ with scalars $a_1, a_2, a_3 \ldots a_m$ is the vector

$$a_1u_1 + a_2u_2 + a_3u_3 \ldots + a_mu_m$$

Example

Write (1,0) as the linear combination of (1,1) and (-1,2).

$$(1,0) = \boxed{1}(1,1) + \boxed{0}(-1,2)$$

$$= (1,1) + (0,0)$$

$$= (1,1) \quad \times$$

Application of linear combination of vectors

- **Audio mixing**:

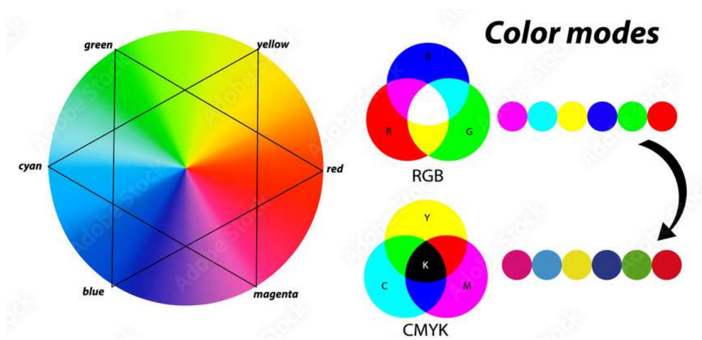
When $\mathbf{a}_1$, $\mathbf{a}_2$,… $\mathbf{a}_m$ are vectors representing audio signals (over the same period of time, for example, simultaneously recorded), they are called tracks. The linear combination $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_m\mathbf{a}_m$ is perceived as a mixture (also called a mix) of the audio tracks, with relative loudness given by $c_1, c_2, \ldots, c_m$. A producer in a studio, or a sound engineer at a live show, chooses values of $c_1, c_2, \ldots, c_m$ to give a good balance between the different instruments, vocals, and drums.

- **Colour mixing**:

$$\text{RGB} = \left[\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}, \begin{bmatrix} 4 & 7 \\ 3 & 8 \end{bmatrix}\right]$$

➢ Gray Scale = 0.2126R + 0.7152G + 0.0722B

➢ CMYK (Cyan, Magenta, Yellow, blacK) used in Commercial printers also uses a combination of these four colours to get different colour combinations



Application of linear combination of vectors

- **Points on a line segment**:

Given a line segment AB, having endpoints A with position vector **a** and B with position vector **b**. All points in the line segment can be represented as $(1\text{-}\alpha)\mathbf{a} + \alpha\mathbf{b}$, where $0 \leq \alpha \leq 1$.

Example:

Points between (1,0) and (5,10) can be written as:

$$(1\text{-}\alpha)(1{,}0) + \alpha(5{,}10) = (1 + 4\alpha, 10\alpha),\ 0 \leq \alpha \leq 1$$

Note:

If α can take any value then this linear combination will give all values on that particular line.


Thank you