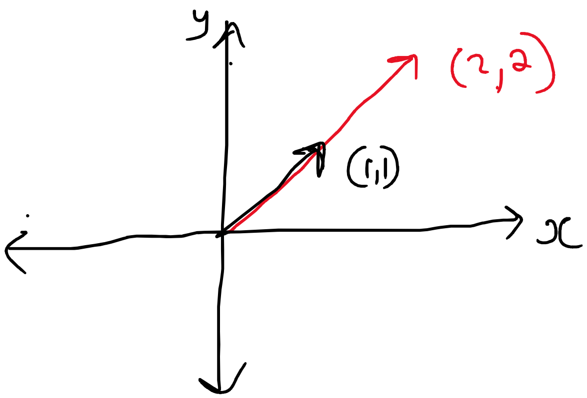
23MAT117
Linear Algebra

Linear independence - Intuition

- $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}$

Linearly independent



$$\begin{bmatrix}2\\2\end{bmatrix} = 2 \cdot \begin{bmatrix}1\\1\end{bmatrix}$$

$$\Rightarrow \vec{v_2} = 2\vec{v_1}$$

$$\Rightarrow 2\vec{v_1} - \vec{v_2} = 0$$

Linearly relation between vectors

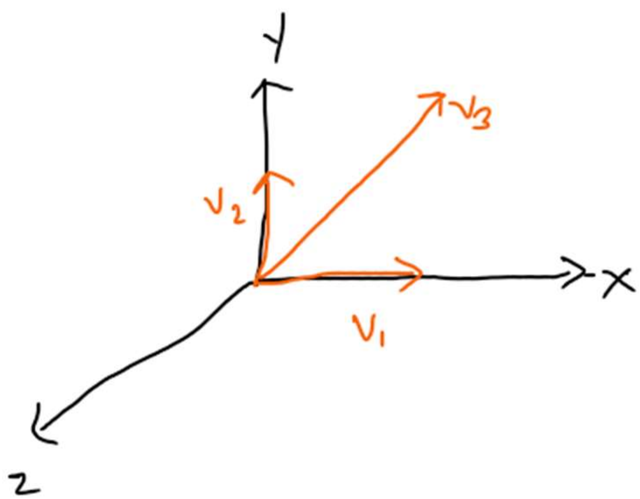
Linear independence - Intuition

- $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}$

$3\vec{v_1} + 4\vec{v_2} = \vec{v_3}$

$3\vec{v_1} + 4\vec{v_2} - \vec{v_3} = 0$

$\Rightarrow$ $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly dependent.



Linear combinations of $\vec{v_1}$ and $\vec{v_2}$ will give xy plane (all points); $\vec{v_3} \in$ xy plane.

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ are in the same plane

$\Rightarrow$ They are linearly dependent.

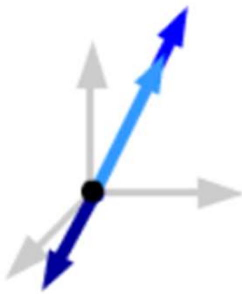
Geometrical Representation

Suppose we have a set of two non-zero vectors. One is a linear combination of the other whenever it is a scalar multiple of the other, i.e. whenever it is parallel to the other. Thus:

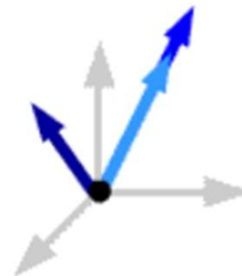
> A set of two vectors is linearly dependent if one is parallel to the other,
>
> 
>
> and linearly independent if they are not parallel.
>
> 

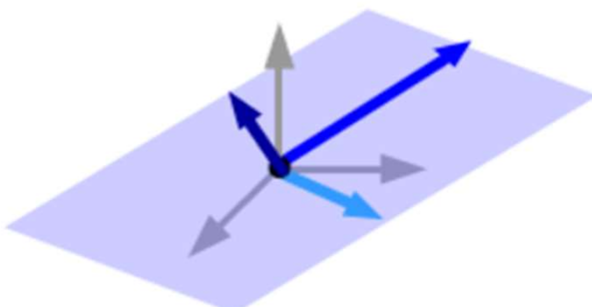
(This is true in either 2-space or 3-space.)

Geometrical Representation



If any two of the vectors are parallel, then one is a scalar multiple of the other. A scalar multiple is a linear combination, so the vectors are **linearly dependent**. (Notice that all three vectors also lie in a plane.)

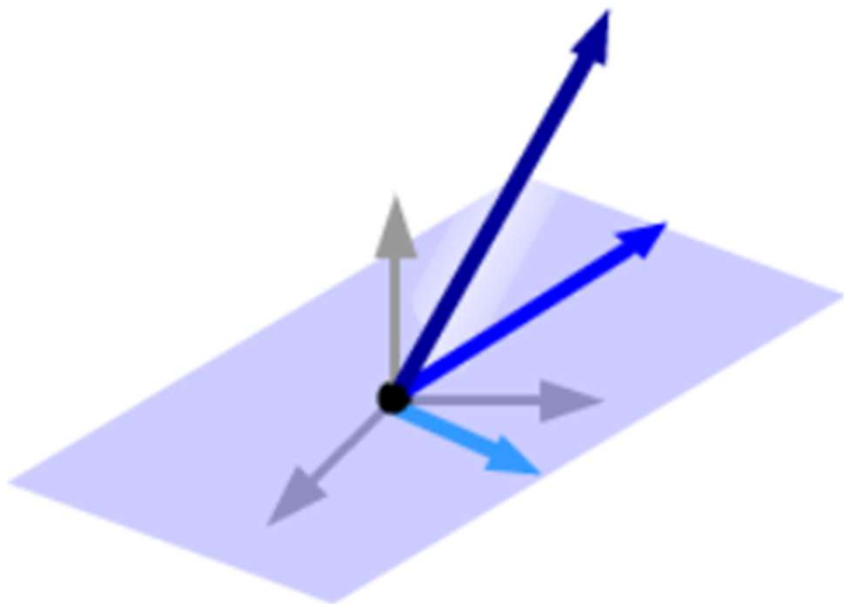




If no two of the vectors are parallel but all three lie in a plane, then any two of those vectors span that plane. The third vector is a linear combination of the first two, since it also lies in this plane, so the vectors are **linearly dependent**.

Geometrical Representation

If the three vectors don't all lie in some plane through the origin, none is in the span of the other two, so none is a linear combination of the other two. The three vectors are **linearly independent.**



Example

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad Let \text{ us try } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ c_2 \end{pmatrix}$$

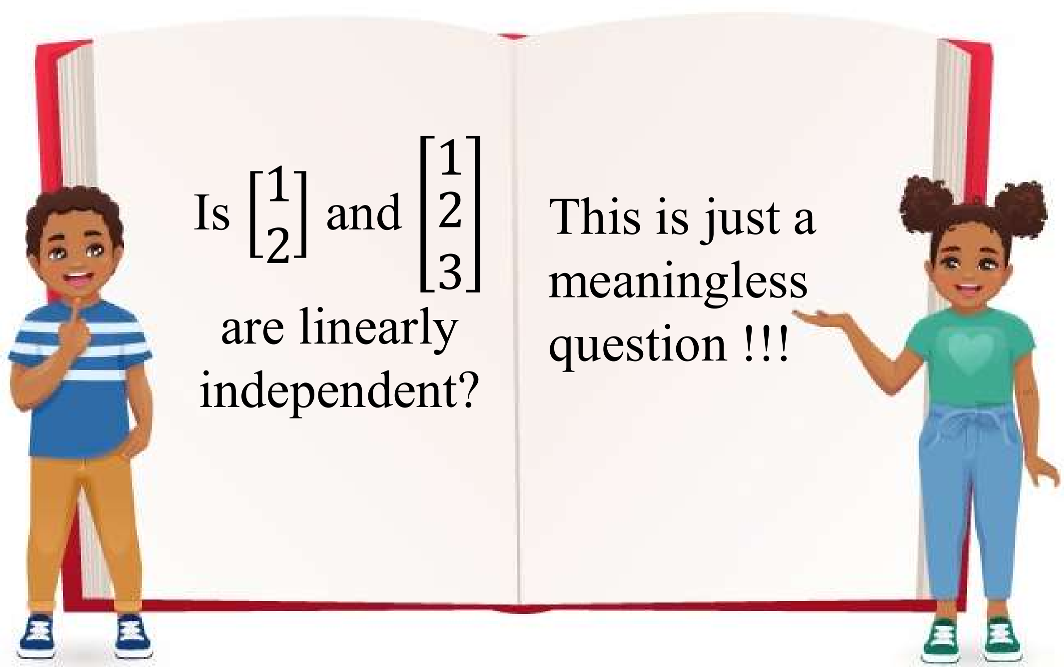
$$What\text{ever be the value of } c_1, c_2 \text{ we can't make } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Similiarly \text{ there is no } c_1, c_2 \text{ such that } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Similiarly \text{ there is no } c_1, c_2 \text{ such that } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

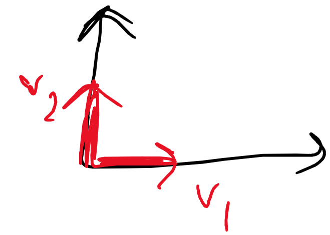
First level definition of linear independence

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ of same tuple size form an independent set if none of the vector in the set can be expressed as linear combination of remaining vectors in the set.



Examples

- $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ $\longrightarrow$ Linearly independent



- $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ $\longrightarrow$ Linearly independent

- 2 vectors are independent if they are not in the same plane.
- 3 vectors are independent if they are not in the same plane.

- $\left\{\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}1\\-2\end{bmatrix}\right\}$ $\longrightarrow$ $\begin{bmatrix}2\\3\end{bmatrix} = a\begin{bmatrix}1\\-2\end{bmatrix}$ can you get a ?

No $\Rightarrow$ Linearly independent.

Example

Example. Are the vectors [2, 3], [3, 4] and [1, 1] in 2-space linearly independent or linearly dependent?

Check if any of them is a linear combination of the others, i.e. check whether any of the following equations has a solution:

$$[2, 3] = a[3, 4] + b[1, 1]$$

$$[3, 4] = a[2, 3] + b[1, 1]$$

$$[1, 1] = a[2, 3] + b[3, 4]$$

It's easy to see that the first equation has the solution $a = 1$, $b = -1$, so

$$[2, 3] = (1)[3, 4] + (-1)[1, 1] .$$

The three vectors are thus linearly dependent.

Example

Example. Are the vectors [1, 2, 3], [4, 5, 6] and [1, 0, 1] in 3-space linearly independent or linearly dependent?

Check if any of them is a linear combination of the others, i.e. check if any of the following equations has a solution.

$$[1, 2, 3] = a[4, 5, 6] + b[1, 0, 1]$$

$$[4, 5, 6] = a[1, 2, 3] + b[1, 0, 1]$$

$$[1, 0, 1] = a[1, 2, 3] + b[4, 5, 6].$$

Each equation is equivalent to a linear system of three equations in two variables. All three systems turn out to have no solution, i.e. none of the three vector equations has a solution, so the vectors are linearly independent.

Linear independence and dependence

A set of vectors $\{u_1, u_2\ , u_3\ \ldots\ u_m\}$ is said to be linearly independent if

$$c_1u_1 + c_2\ u_2 + c_3u_3\ \ldots\ +c_mu_m = 0$$ has at least one solution, namely

$$c_1 = c_2 = c_3 = \ldots = c_m = 0.$$

Otherwise the set $\{u_1, u_2\ , u_3\ \ldots\ u_m\}$ are said to be linearly dependent.

Results

- ✓ Two vectors are independent if one is not multiple of other.
- ✓ A collection that contain repeated vector is dependent.
- ✓ The empty set is linearly independent.
- ✓ The set 0 is linearly dependent.
- ✓ A nonzero single-ton set is linearly independent.
- ✓ The set of vectors $\{v_{1,} v_{2,} \ldots, v_n\}$is dependent if any one of the v_i s is zero or any of the v_i is a linear combination of some other vectors.
- ✓ If a set of vectors is linearly independent, then any rearrangement of the vectors is also linearly independent.

Example

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

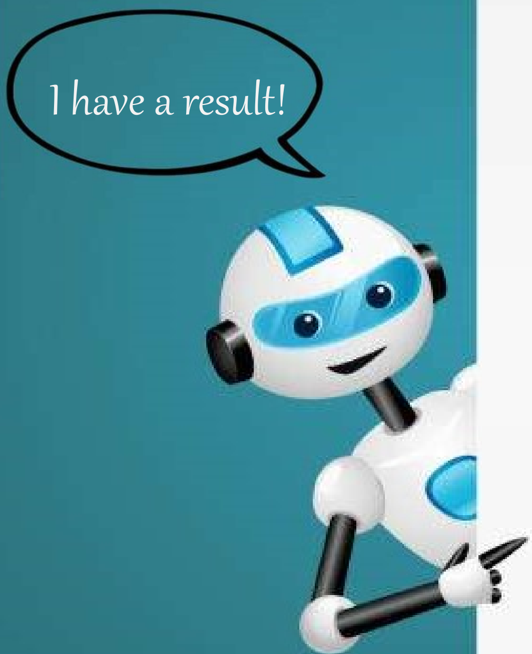
$$\quad v_1 \qquad v_2 \qquad v_3$$

Sol: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow$

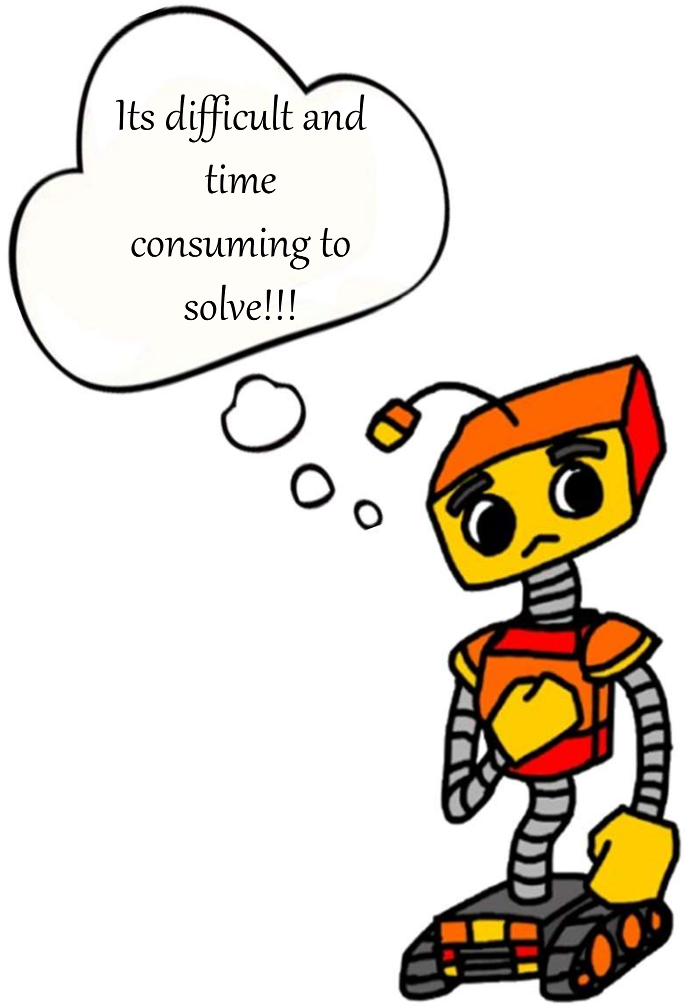
$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \ c_2 + \qquad &= 0 \\ 3c_1 + 2c_2 + \ c_3 &= 0 \end{aligned}$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$



A set of n vectors is linearly independent if the matrix with these vectors as columns has a non-zero determinant and the set is dependent if the determinant is zero.



Example

Are the vectors(1,2) and (-5,3) linearly independent?

The vectors (1,2) and (-5,3) are linearly independent since the matrix $\begin{vmatrix} 1 & -5 \\ 2 & 3 \end{vmatrix}$ has a non-zero determinant.

Are the vectors (2,-1,1) , (3,-4,-2) and (5,-10 ,-8) linearly independent?

The vectors $u = (2, -1, 1)$, $v = (3, -4, -2)$, and $w = (5, -10, -8)$ are dependent since the $\begin{vmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{vmatrix}$ determinant is zero.

Practice Questions

1. $\left\{\begin{bmatrix}6\\7\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$ ➡ **Linearly Independent**

2. $\left\{\begin{bmatrix}2\\0\\0\end{bmatrix},\begin{bmatrix}0\\4\\0\end{bmatrix},\begin{bmatrix}0\\0\\9\end{bmatrix}\right\}$ ➡ **Linearly Independent**

3. $\left\{\begin{bmatrix}9\\7\\8\end{bmatrix},\begin{bmatrix}-9\\-7\\-8\end{bmatrix}\right\}$ ➡ **Linearly Dependent**

4. $\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}5\\6\\7\\0\end{bmatrix},\begin{bmatrix}8\\9\\0\\0\end{bmatrix},\begin{bmatrix}10\\0\\0\\0\end{bmatrix}\right\}$ ➡

$$k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3 + k_4\bar{v}_4 = \bar{0}$$

$$k_1 + 5k_2 + 8k_3 + 10k_4 = 0 \qquad k_4 = 0$$
$$2k_1 + 6k_2 + 9k_3 = 0 \qquad k_3 = 0$$
$$3k_1 + 7k_2 = 0 \qquad k_2 = 0$$
$$4k_1 = 0 \longrightarrow k_1 = 0$$

only soln. $\Rightarrow$ L.I

5. $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}4\\5\end{bmatrix}\right\}$

$$c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} + c_3\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_1 + c_2 + 4c_3 = 0 \longrightarrow (1)$$
$$c_2 + 5c_3 = 0 \Rightarrow c_3 = \frac{-c_2}{5} \rightarrow (2)$$

Sub in (1);
$$c_1 + c_2 - \frac{4c_2}{5} = 0$$
$$c_1 + \frac{c_2}{5} = 0$$
$$c_2 = -5c_1 \longrightarrow (3)$$

Let $c_1 = a$. Then $c_2 = -5a$ and $c_3 = a$.

$\Rightarrow$ Linearly dependent

Practice Questions

1. Write an example of 6 linearly independent vectors.

$$\left\{ \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \right\}$$

2. Are the row vectors of any unit matrix linearly independent? Yes
3. Are the column vectors of any unit matrix linearly independent? Yes
4. Are the row vectors of a triangular square matrix of order three linearly independent? Yes

eg:

$$\begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\2 & 8 & 0\\3 & 4 & 5\end{bmatrix}$$

5. How many independent rows does the matrix A have?

$$A = \begin{bmatrix}2 & 3 & 1\\0 & 4 & 5\\0 & 0 & 6\\0 & 0 & 0\end{bmatrix} \begin{array}{l}\to \vec{r_1}\\ \to \vec{r_2}\\ \to \vec{r_3}\\ \to \vec{r_4}\end{array}$$

$\{\vec{r_1}, \vec{r_2}, \vec{r_3}\}$ are 3 independent rows

Answer = 3


Thank You