P346: DIY PROJECT
Solving Laplace equation in electrostatics

Niti Singh

November 24th, 2022

Abstract

The Laplace Equation is a valuable approach to the determination of the electric potentials in free space or region. The Laplace equation is named after the physicist Pierre-Simon Laplace and can be used for various aspects of physics. In this report, the theoretical approach to the numerical solution of the Laplace equation by using the finite difference method and its algorithm is discussed, which is later exemplified using the problem of Charged square walls and Parallel-plate capacitors.

1 Introduction

The Laplace equation is encountered in electrostatics, where the electric potential is related to the electric field as a direct consequence of Gauss's law in the absence of a charge density. In the presence of volume charge density, the relation is given by the inhomogeneous version of the equation and is known as Poisson's equation. We begin with:

$$\nabla E = \frac{\rho_v}{\epsilon} \; and \; E = -\nabla V \tag{1}$$

$$\nabla . \nabla V = \nabla^2 V = -\frac{\rho_v}{\epsilon} \tag{2}$$

This equation 2 is Poisson's equation which in a density-free region becomes the Laplace equation:

$$\nabla^2 V = 0 \tag{3}$$

Since it is a boundary value problem, the solution of this equation in a domain requires the specification of the conditions that the unknown function must satisfy at the boundary of the domain. When the function is specified on any part of the boundary, we call that part the Dirichlet boundary; when the normal derivative of the function is specified on any part of the boundary, we call that part the Neumann boundary. It follows the uniqueness theorem, and specific analytical solutions can be determined.

The equation is also used in gravity, where the gravitational potential is related to the gravitational field; in thermal physics, with potential playing the role of temperature; and in fluid mechanics, with a potential for the velocity field of an incompressible fluid. The application of this equation is wide, but this report covers an electrostatic view.

Since Laplace is a form of an elliptic, second-order partial differential equation, various methods can be used to solve them. However, the Finite difference approximation method is used in this report considering its efficiency. This can be done by discretization, which is a process of using the discrete form of the differential equation instead of the continuous form. The calculus problem here is transformed into a matrix algebra algorithm which is much more feasible to code and converges readily due simplification of the steps involved. The finite difference method is a numerical technique

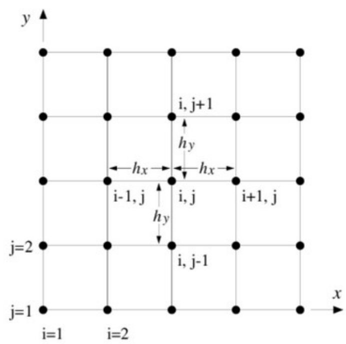

Figure 1: Mesh grid in the given domain (x,y)

where the first and second-order partial derivatives are approximated using finite differences. Considering a 2-D region where the function V(x,y) is defined. This 2-D domain is divided into regular rectangular grids of height h_y and width h_x. The points of intersection of these points are called mesh points or nodal points. Each nodal point is designated by a numbering scheme i and j, where i indicates the x increment and j indicates the y increment, as shown in Figure 2. In a case study on Electric potential distribution, the potential at each nodal point (x_i, y_j) is the average potential of the surrounding hatched region.

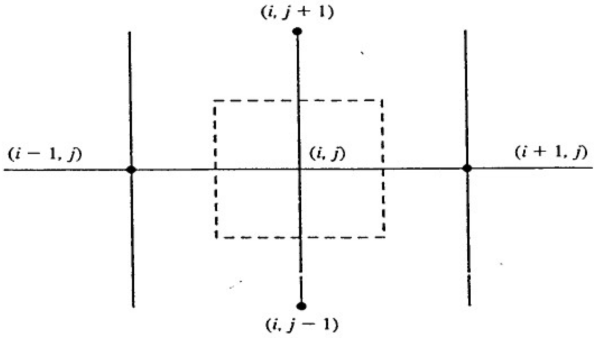

Figure 2: Five-point grid around an arbitrary node

By using Taylor series expansion for any function f, we get:

$$f(x+\delta x) = f(x) + \delta x f'(x) + \frac{1}{2}\delta x^2 f''(x) + \frac{1}{6}\delta x^3 f'''(x) + O(\delta x^4)$$

$$f(x-\delta x) = f(x) - \delta x f'(x) + \frac{1}{2}\delta x^2 f''(x) - \frac{1}{6}\delta x^3 f'''(x) + O(\delta x^4)$$

where O is the truncation error. Thus, for the first-order derivative, we get:

$$f'(x) = \frac{f(x+\delta x) - f(x)}{\delta x} + O(\delta x)$$

Thus, by subtracting the Taylor expansions, we get a second-order central finite difference for the first derivative:

$$f'(x) = \frac{f(x+\delta x) - f(x-\delta x)}{2\delta x} + O(\delta x^2)$$

Similarly, we get approximations for second-order central finite difference for the second derivative:

$$f''(x) \approx \frac{f(x+\delta x) - f(x-\delta x) - 2f(x)}{\delta x^2} + O(\delta x^2)$$

Now for the Laplace equation in a 2-D region cartesian coordinates, we have:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

So we deduce the central finite difference for the Laplace equation as:

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_{(i,j)} = \frac{V_{(i+1,j)} + V_{(i-1,j)} - 2V_{(i,j)}}{{h_x}^2} + O({h_x}^2)$$

$$\left(\frac{\partial^2 V}{\partial y^2}\right)_{(i,j)} = \frac{V_{(i,j+1)} + V_{(i,j-1)} - 2V_{(i,j)}}{{h_y}^2} + O({h_y}^2)$$

Hence, we get:

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_{(i,j)} + \left(\frac{\partial^2 V}{\partial y^2}\right)_{(i,j)} = 0 =$$

$$\frac{V_{(i+1,j)} + V_{(i-1,j)} - 2V_{(i,j)}}{{h_x}^2} + \frac{V_{(i,j+1)} + V_{(i,j-1)} - 2V_{(i,j)}}{{h_y}^2}$$

Hence, we obtain:

$$V_{(i,j)} = \frac{1}{4}\left(V_{(i+1,j)} + V_{(i-1,j)} + V_{(i,j+1)} + V_{(i,j-1)}\right)$$

2 Pseudo-code

Below is the pseudo-code for the solution of the Laplace equation in two dimensions.

```
#Input:
#boundary values,
#Guess value for V,
grid seperation,
#Dimension for X and Y,
#Max iterations

define function with input values
# Input Set Dimension for X and Y
# Input Set grid
    X = grid for X
    Y = grid for Y
# Input Set array size and guess = V
# Input Set Boundary condition as
    V(ymax,x) = Vt
    V(0,x) = Vb
    V(y,xmax) = Vr
    V[y, 0] = Vl

# Iteration
    for loop in range(iteration){
    for loop in range(X){
    for loop in range(Y){
    #Descretize Equation
    V[i, j] = (1/4) * (V[i+1][j] + V[i-1][j] +
                V[i][j+1] + V[i][j-1])
    }}}

    plot contour of solution

return X{array} , Y{array}, potential
```

3 Illustrations

Using this method, the potential in a square box is plotted using three different boundary conditions, i.e., One wall charged, two walls charged, and all walls charged.

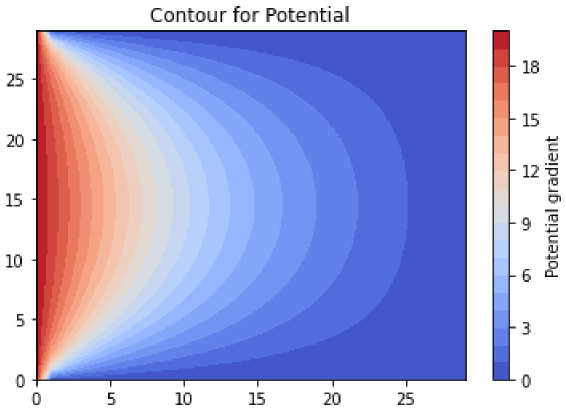

Figure 3: $V_{left} = 20$V; others $= 0$V

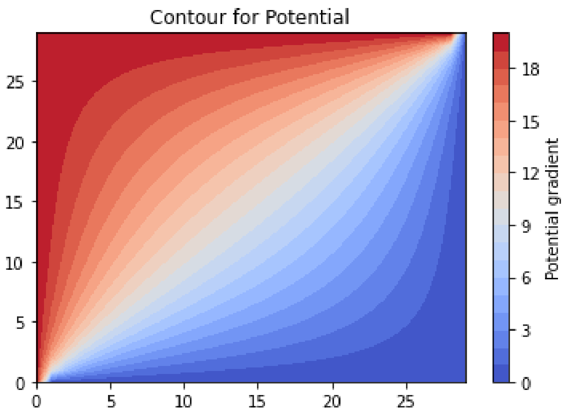

Figure 4: $V_{left} = V_{top} = 20$V; others $= 0$V

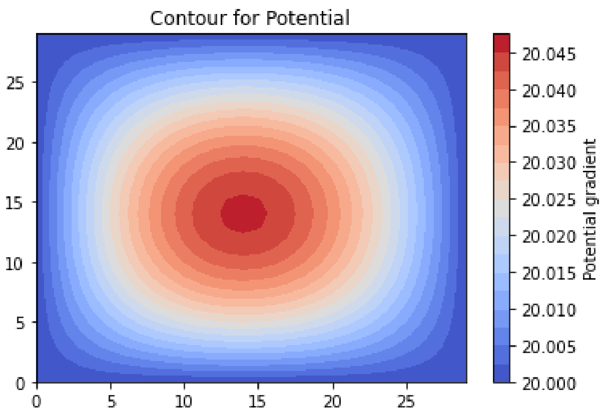

Figure 5: All walls $= 20$V

A potential gradient is formed in a parallel plate capacitor, and a 3D plot showing the equipotential surface is obtained.

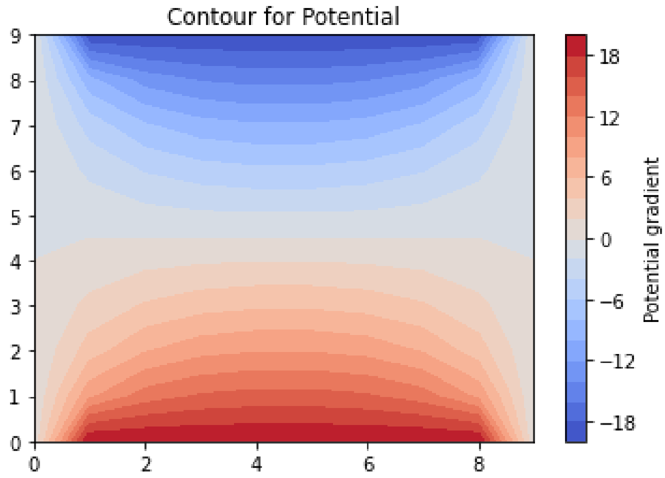

Figure 6: Potential gradient plot for Capacitor

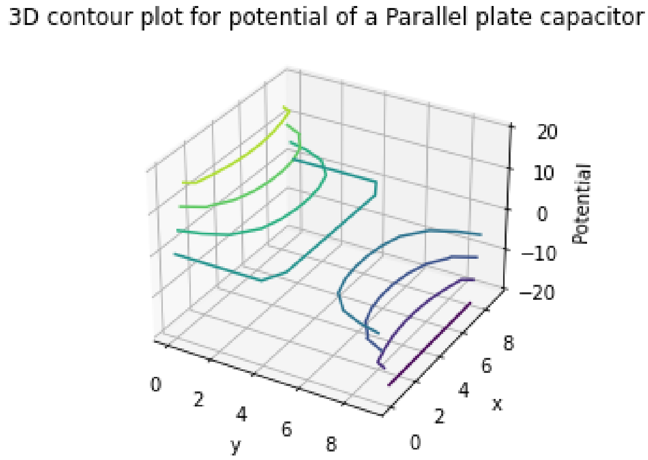

Figure 7: Equipotential surfaces in Capacitor

4 Results

- Dimensions used is 30×30 for Square boxes.
- Dimensions used is 10×10 for Capacitor plates.
- Numpy library used to create a mesh grid.
- MatPlot library used to plot the graphs.

The finite difference approximation method was used in this report to solve the Laplace equation in electrostatics numerically. Using this iterative method is quick and simple, but the issue arises in terms of tolerance level as the truncation error was ignored. The Colour gradient changes from red to blue in the plot as potential decreases, and the plots are obtained as expected. This method can also be used to solve other PDEs, and the Laplace equation also has other applications, like in fluid dynamics and thermal and gravitational fields. This method was not very precise, but it will be helpful in huge and fast calculations.

References

- finite-difference-pde
- lecture/finite-difference-Laplace
- article/FDMLaplace/statperson.com
- libretexts/LaplaceEq
- JupyterLab
- P346 diffeq Slides