Statistical Modeling and Inference – Problem Set #5

Group 3: Niti Mishra · Miquel Torrens · Bálint Ván

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Solution to proposed exercises.

Exercise 1

Part (a)

We define:

$$\text{NdEF}(z|\theta, q) \equiv p(z|\theta, q) = \exp\{q[z\theta - c(\theta)] + h(z, q)\} \tag{1}$$

Let's start by the **normal** distribution:

$$\begin{aligned}
\mathcal{N}(t|\mu, q) &= \frac{q^{1/2}}{\sqrt{2\pi}} \exp\left\{-\frac{q}{2}(t-\mu)^2\right\} \\
&= \frac{q^{1/2}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}q(t^2 + \mu^2 - 2z\mu)\right\} \\
&= \frac{q^{1/2}}{\sqrt{2\pi}} \exp\left\{q\left[t\mu - \frac{1}{2}\left(z^2 + \mu^2\right)\right]\right\} \\
&= \exp\left\{q\left[t\mu - \frac{1}{2}\mu^2\right] + \log\left(\frac{q^{1/2}}{\sqrt{2\pi}}\right) - \frac{1}{2}qz^2\right\}
\end{aligned} \tag{2}$$

Now using (1) we can see the following in (2):

$$\mathcal{N}(t|\mu, q) = \exp\left\{\underbrace{q}_{=q}\left[\underbrace{t}_{=z}\underbrace{\mu}_{=\theta} - \underbrace{\frac{1}{2}\mu^2}_{=c(\theta)}\right] + \underbrace{\log\left(\frac{q^{1/2}}{\sqrt{2\pi}}\right) - \frac{1}{2}qz^2}_{=h(z,q)}\right\} \tag{3}$$

Thus with the normal distribution: $z = t$, $q = q$, $\theta = \mu$ and $c(\theta) = \frac{1}{2}\mu^2 = \frac{1}{2}\theta^2$.

For the **Bernouilli** distribution:

$$\begin{aligned}
\text{Bern}(t|p) &= p^t(1-p)^{1-t} \\
&= \exp\{t\log p + (1-t)\log(1-p)\} \\
&= \exp\left\{t\log\left(\frac{p}{1-p}\right) + \log(1-p)\right\}
\end{aligned}$$

Mimetizing the analysis in (3), we have that $z = t$, $q = 1$, $\theta = \log\left(\frac{p}{1-p}\right)$ and that $c(\theta) = -\log(1-p) = \log(1 + e^\theta)$, because:

$$\theta = \log\left(\frac{p}{1-p}\right) \Leftrightarrow p = \frac{e^\theta}{1 + e^\theta}. \tag{4}$$

For the **binomial** distribution:

$$\begin{aligned}
\mathrm{Bin}(t|n,p) &= \binom{n}{t} p^t (1-p)^{n-t} \\
&= \exp\left\{\log\binom{n}{t} + t\log p + (n-t)\log(1-p)\right\} \\
&= \exp\left\{t\log\left(\frac{p}{1-p}\right) + n\log(1-p) + \log\binom{n}{t}\right\} \\
&= \exp\left\{n\left[\frac{t}{n}\log\left(\frac{p}{1-p}\right) + \log(1-p)\right] + \log\binom{n}{t}\right\}
\end{aligned}$$

Then for the binomial: $z = t/n$, $q = n$, $\theta = \log\left(\frac{p}{1-p}\right)$ and $c(\theta) = -\log(1-p) = \log(1+e^\theta)$.

Finally, for the **Poisson** distribution:

$$\begin{aligned}
\mathrm{Pois}(t|\lambda) &= \frac{\lambda^t e^{-\lambda}}{t!} \\
&= \exp\{t\log\lambda - \lambda - \log t!\} \\
&= \exp\{[t\log\lambda - \lambda] - \log t!\}
\end{aligned}$$

And using (1) that means that for Poisson $z = t$, $q = 1$, $\theta = \log\lambda$ and $c(\theta) = \lambda = e^\theta$.

Part (b)

To obtain the canonical links we apply the following formula[1]:

$$g(\mathbb{E}[z]) = \int \frac{1}{c''(\theta)} d\mathbb{E}[z].$$

For the **normal** distribution, where $\theta = \mu$:

$$c(\theta) = \frac{1}{2}\theta^2.$$

This means that $c'(\theta) = \theta$, $c''(\theta) = 1$ and $1/c''(\theta) = 1$. Then:

$$g(\mu) = \int d\mu = \mu.$$

For the **Bernouilli** and **binomial** distributions, which share $c(\theta)$, recall (4):

$$c(\theta) = -\log(1-p) = -\log\left(1 - \frac{e^\theta}{1+e^\theta}\right) = -\log\left(\frac{1}{1+e^\theta}\right) = \log(1+e^\theta)$$

From this we obtain:

$$\begin{aligned}
c'(\theta) &= \frac{e^\theta}{1+e^\theta} \\
c''(\theta) &= \frac{e^\theta}{(1+e^\theta)^2} \\
\frac{1}{c''(\theta)} &= \frac{(1+e^\theta)^2}{e^\theta} = \frac{\left(1+\frac{p}{1-p}\right)^2}{\frac{p}{1-p}}
\end{aligned}$$

[1] GLMs lecture, slide number 11.

And finally recover:

$$g(p) = \int \frac{(1 + \frac{p}{1-p})^2}{\frac{p}{1-p}} dp = \log p - \log(1-p) = \log\left(\frac{p}{1-p}\right).$$

In the case of the **Poisson** distribution, we have that $\theta = \log \lambda$ and $c(\theta) = \lambda = e^{\theta}$, thus:

$$\begin{aligned} c'(\theta) &= e^{\theta} \\ c''(\theta) &= e^{\theta} = \lambda \\ \frac{1}{c''(\theta)} &= \lambda^{-1} \end{aligned}$$

This implies:

$$g(\lambda) = \int \lambda^{-1} d\lambda = \log \lambda.$$

<u>Part (c)</u>

From the log-likelihood function:

$$\log \mathcal{L}\left(\mathrm{NdEF}(t_n|\theta(\mathbf{x}_n, \mathbf{w}), q\gamma_n)\right) = \sum_{n=1}^{N} \gamma_n q[t_n \theta(\phi(\mathbf{x}_n)^T \mathbf{w}) - c(\theta(\phi(\mathbf{x}_n)^T \mathbf{w}))] + \sum_{n=1}^{N} h(t_n, q\gamma_n)$$

From the GLM lecture, slide 12:

$$\begin{aligned} \frac{\partial \log \mathcal{L}}{\partial \mathbf{w}} &= \sum_{n=1}^{N} \frac{\gamma_n}{V(y(\mathbf{x}_n, \mathbf{w}))} \frac{1}{g'(y(\mathbf{x}_n, \mathbf{w}))} q(t_n - y(\mathbf{x}_n, \mathbf{w}))\phi(\mathbf{x}_n) \\ &= \sum_{n=1}^{N} \gamma_n q(t_n - y(\mathbf{x}_n, \mathbf{w}))\phi(\mathbf{x}_n) \\ &= q\sum_{n=1}^{N} \gamma_n t_n \phi(\mathbf{x}_n) - q\sum_{n=1}^{N} \gamma_n y(\mathbf{x}_n, \mathbf{w})\phi(\mathbf{x}_n) \end{aligned}$$

We can make the simplification because under the canonical form: $g'(y) = 1/V(y)$. Also under canonical form $\mathbb{E}[t_n] = \theta = g(y(\mathbf{x}_n, \mathbf{w})) = y(\mathbf{x}_n, \mathbf{w}) = \phi(\mathbf{x}_n)^T \mathbf{w}$. Then:

$$\begin{aligned} \frac{\partial \log \mathcal{L}}{\partial \mathbf{w}} &= q\sum_{n=1}^{N} \gamma_n t_n \phi(\mathbf{x}_n) - q\sum_{n=1}^{N} \gamma_n \phi(\mathbf{x}_n)^T \mathbf{w} \phi(\mathbf{x}_n) \\ \frac{\partial^2 \log \mathcal{L}}{\partial \mathbf{w}^2} &= -q\sum_{n=1}^{N} \gamma_n \phi(\mathbf{x}_n)^T \phi(\mathbf{x}_n) \qquad (5) \end{aligned}$$

In matrix form write (5) as:

$$-q\mathbf{\Phi}^T \mathbf{\Gamma} \mathbf{\Phi},$$

where $\mathbf{\Gamma} = \mathrm{diag}(\gamma_n)$. We know that $\mathbf{\Phi}^T\mathbf{\Phi}$ would be positive semi-definite (proved in Exercise 1 of Problem Set 1), thus log-convex. All elements of $\mathbf{\Gamma}$ are non-negative and q is as well, so this multiplication remains log-convex. Then the negative sign makes the log-likelihood log-concave.

Exercise 2

Part (a)

From the slides on MLE regression[2], we know that:

$$-2\log p(\mathbf{t}|\mathbf{X},\mathbf{w},q) = -N\log q + q(\mathbf{t}-\mathbf{\Phi w})^T(\mathbf{t}-\mathbf{\Phi w}) + c,$$

where c gathers all constant terms not dependent on $\mathbf{X}$. This yielded:

$$q_{MLE} = \left(\frac{1}{N}\mathbf{e}^T\mathbf{e}\right)^{-1}.$$

Then:

$$\begin{aligned} -2\log p(\mathbf{t}|\mathbf{X},\mathbf{w}_{MLE},q_{MLE}) &= -N\log\left(\frac{1}{N}\mathbf{e}^T\mathbf{e}\right)^{-1} + \left(\frac{1}{N}\mathbf{e}^T\mathbf{e}\right)^{-1}(\mathbf{t}-\mathbf{\Phi w}_{MLE})^T(\mathbf{t}-\mathbf{\Phi w}_{MLE}) + c \\ &= N\log\mathbf{e}^T\mathbf{e} + N\left(\mathbf{e}^T\mathbf{e}\right)^{-1}\left(\mathbf{e}^T\mathbf{e}\right) + c \\ &= N\log\mathbf{e}^T\mathbf{e} + c. \end{aligned}$$

Thus proved.

Part (b)

The problem in Part (a) also yielded:

$$\mathbf{w}_{MLE} = \left(\mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^T\mathbf{t}.$$

In the null model, $\mathbf{\Phi}$ is a column vector with N components all of which are equal to 1. Consequently $\mathbf{\Phi}^T$ is a row vector also of length N. Then:

$$\mathbf{\Phi}^T\mathbf{\Phi} = \sum_{i=1}^{N} 1\times 1 = N.$$

This implies:

$$\left(\mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1} = \frac{1}{N}.$$

Also:

$$\mathbf{\Phi}^T\mathbf{t} = \sum_{i=1}^{N} 1\times t_i = \sum_{i=1}^{N} t_i.$$

Finally:

$$w_{0,MLE} = \left(\mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^T\mathbf{t} = \frac{1}{N}\sum_{i=1}^{N} t_i = \bar{t}.$$

Hence proved.

[2] Slide 8

Part (c)

Using the results in Exercise 2.1:

$$\begin{aligned}
D_0 &= -2\log p(\mathbf{t}|\mathbf{X}_{m_0}\mathbf{w}_{m_0,MLE}, q_{m_0,MLE}) \\
&= N\log(\mathbf{t}-\bar{t})^T(\mathbf{t}-\bar{t}) + c \\
&= N\log q^{-1} + c \\
&= -N\log q + c
\end{aligned}$$

$$\begin{aligned}
D_1 &= -2\log p(\mathbf{t}|\mathbf{X}_{m_1}\mathbf{w}_{m_1,MLE}, q_{m_1,MLE}) \\
&= N\log(\mathbf{e}^T\mathbf{e}) + c
\end{aligned}$$

Also note that:

$$R^2 = 1 - \frac{\mathrm{var}(\mathbf{t}|\mathbf{x})}{\mathrm{var}(\mathbf{t})} = 1 - \frac{(\mathbf{t}-\hat{\mathbf{t}})^T(\mathbf{t}-\hat{\mathbf{t}})}{(\mathbf{t}-\bar{t})^T(\mathbf{t}-\bar{t})} = 1 - \frac{\mathbf{e}^T\mathbf{e}}{q^{-1}} = 1 - q\mathbf{e}^T\mathbf{e}$$

Now we calculate:

$$\begin{aligned}
D_0 - D_1 &= (-N\log q + c) - (N\log \mathbf{e}^T\mathbf{e} + c) \\
&= -N\log q - N\log \mathbf{e}^T\mathbf{e} \\
&= -N\log q\mathbf{e}^T\mathbf{e} \\
&= -N\log\left(1 - (1 - q\mathbf{e}^T\mathbf{e})\right) \\
&= -N\log\left(1 - R^2\right).
\end{aligned}$$

This finishes the proof.