Statistical Modeling and Inference – Problem Set #6

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November 30^{th}, 2015

Solution to proposed exercises.

Exercise 1

Part (a)

Given $t_n \sim \text{Laplace}(\mu, \Lambda)$, the likelihood function is:

$$\mathcal{L}(\text{Laplace}(t_n|\mu,\Lambda)) = \prod_{n=1}^{N} f(t_n|\mu,\Lambda) = \left(\frac{\Lambda}{2}\right)^N \exp\left\{-\Lambda\sum_{n=1}^{N}|t_n-\mu|\right\}$$

Part (b)

If we take the MLE:

$$\max_{\mu,\Lambda} \log\mathcal{L} = \max_{\mu,\Lambda} N\log\Lambda - \Lambda\sum_{n=1}^{N}|t_n-\mu| + C$$

With C containing the terms not dependent on μ or Λ. Then:

$$\begin{aligned}\frac{\partial\log\mathcal{L}}{\partial\mu} = 0 \quad &\Leftrightarrow \quad -\Lambda\sum_{n=1}^{N}\frac{t_n-\mu}{|t_n-\mu|} = 0 \\ &\Leftrightarrow \quad \sum_{n=1}^{N}\text{sgn}(t_n-\mu_{MLE}) = 0 \qquad (1)\end{aligned}$$

Note that with odd N, for (1) to hold, $(N-1)/2$ observations need to have $t_n \geq \mu$ and $(N-1)/2$ have $t_n \leq \mu$. The unpaired observation will have $t_n = \mu$. Thus:

$$\mathbb{E}[t] = \mu_{MLE} = \text{median}(t_1, t_2, ..., t_n)$$

is the expected value and also the median.

Part (c)

We derive Λ_{MLE}:

$$\begin{aligned}\frac{\partial\log\mathcal{L}}{\partial\Lambda} = 0 \quad &\Leftrightarrow \quad N\frac{1}{\Lambda} - \sum_{n=1}^{N}|t_n-\mu| = 0 \\ &\Leftrightarrow \quad \Lambda_{MLE} = \left(\frac{1}{N}\sum_{n=1}^{N}|t_n-\mu|\right)^{-1}\end{aligned}$$

Part (d)

By the equivariance property[1] of the MLE:

$$\mathrm{var}[t] = \Sigma_{MLE} = \frac{2}{\Lambda_{MLE}^2} = 2\left(\frac{1}{N}\sum_{n=1}^{N}|t_n - \mu|\right)^2.$$

[1] Wasserman's Theorem 10.14.

Exercise 2

Part (a)

Given:

$$t_n \sim \mathcal{N}(t_n|\mathbf{x}_n, \mathbf{w}, \eta_n, q) = \frac{(\eta_n q)^{1/2}}{\sqrt{2\pi}} \exp\left\{-\frac{\eta_n q}{2}(t_n - \phi(\mathbf{x}_n)\mathbf{w})^T(t_n - \phi(\mathbf{x}_n)\mathbf{w})\right\}$$

and:

$$\eta_n \sim \text{Gam}\left(\eta_n|\frac{\nu}{2}, \frac{\nu}{2} - 1\right) = \frac{\left(\frac{\nu}{2}-1\right)^{\nu/2}}{\left(\frac{\nu}{2}-1\right)!}\eta_n^{\left(\frac{\nu}{2}-1\right)} \exp\left\{-\left(\frac{\nu}{2}-1\right)\eta_n\right\}$$

We know:

$$\begin{aligned}
p(\eta_n|t_n, \mathbf{x}_n, \mathbf{w}, q) &\propto p(t_n|\eta_n, \mathbf{x}_n, \mathbf{w}, q)p(\eta_n) \\
&= C\eta_n^{\left(\frac{\nu}{2}-1+\frac{1}{2}\right)} \exp\left\{-\left(\frac{\nu}{2}-1\right)\eta_n - \frac{\eta_n q}{2}\mathbf{e}_n^T\mathbf{e}_n\right\} \\
&= C\eta_n^{\left(\frac{\nu+1}{2}-1\right)} \exp\left\{\left(-\left(\frac{\nu}{2}-1\right) - \frac{q}{2}\mathbf{e}_n^T\mathbf{e}_n\right)\eta_n\right\} \\
&= C\eta_n^{\left(\frac{\nu+1}{2}-1\right)} \exp\left\{-\left(\frac{\nu + q\mathbf{e}_n^T\mathbf{e}_n}{2} - 1\right)\eta_n\right\}
\end{aligned}$$

Where C contains the multiplication of the terms of the density functions not dependent on η_n. Observe this is a Gamma distribution with parameters:

$$\begin{aligned}
\alpha_n &= \frac{\nu+1}{2} \\
\beta_n &= \frac{\nu + q\mathbf{e}_n^T\mathbf{e}_n}{2} - 1
\end{aligned}$$

This finishes the proof.

Part (b)

Given[2]:

$$Q(\theta, \theta') = \int \log\left(p(t, \eta|\theta)\right) p(\eta|t, \theta')d\eta = \mathbb{E}\left[\log\left(p(t, \eta|\theta)\right)\right]$$

[2]Equations (4.3) and (4.4) in the lecture notes on the appendix of the lecture.

We can use $\theta = (q, \mathbf{w})$ and through maximum likelihood compute:

$$\begin{aligned}
\mathbb{E}\left[\log p(t,\eta|\theta)\right] &= \mathbb{E}\left[\log\left(p(t|\theta)p(\eta|t,\theta')\right)\right] \\
&= \mathbb{E}\left[\log p(t|\theta) + \log p(\eta|t,\theta')\right] \\
&= \mathbb{E}\left[\log\left(\prod_{n=1}^{N}\left(\frac{q\eta_n}{2\pi}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}qe_n^2\eta_n\right\}\right) + \log p(\eta|t,\theta')\right] \\
&= \mathbb{E}\left[\log\left(\prod_{n=1}^{N}\left(\frac{q\eta_n}{2\pi}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}qe_n^2\eta_n\right\}\right)\right] + \mathbb{E}\left[\log p(\eta|t,\theta')\right] \\
&= \mathbb{E}\left[\log\left(\left(\frac{q\eta_n}{2\pi}\right)^{\frac{N}{2}}\exp\left\{-\frac{1}{2}q\sum_{n=1}^{N}e_n^2\eta_n\right\}\right)\right] + c \\
&= \mathbb{E}\left[\frac{N}{2}\log q - \frac{q}{2}\sum_{n=1}^{N}(t_n - \phi(\mathbf{x}_n)\mathbf{w})^2\eta_n\right] + c \\
&= \frac{N}{2}\log q - \frac{q}{2}(\mathbf{t} - \mathbf{\Phi w})^T\mathbb{E}\left[\mathbf{H}\right](\mathbf{t} - \mathbf{\Phi w}) + c
\end{aligned}$$

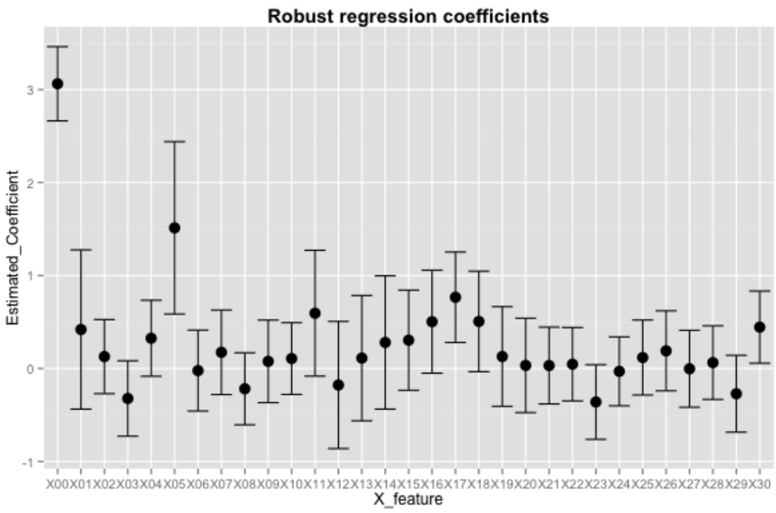
Where $\mathbf{H}$ is a diagonal matrix with $\mathbf{H}_{nn} = \eta_n$, and c contains all terms not dependent on θ. The distribution of η_n will not depend on θ but on the previous $\theta' = (q', \mathbf{w}')$, and given that for $t \sim \mathrm{Gam}(\alpha, \beta)$ we have $\mathbb{E}[t] = \alpha/\beta$, then:

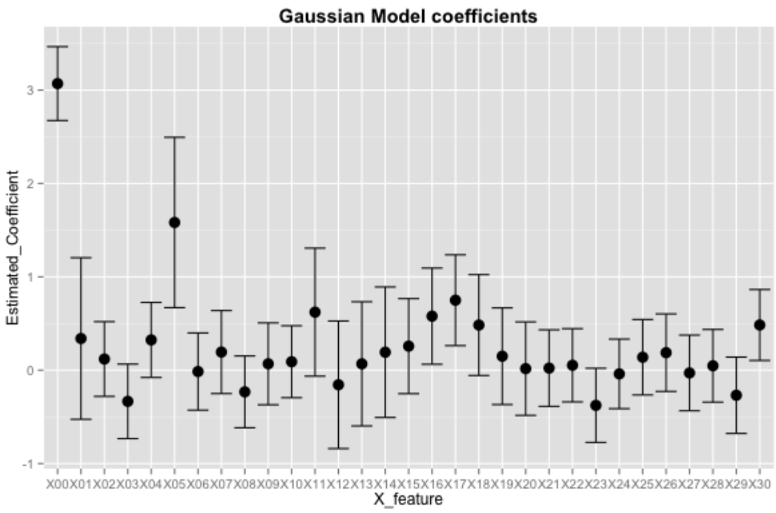
$$\mathbb{E}[\eta_n] = \frac{\frac{\nu+1}{2}}{\frac{\nu+q'(e_n')^2}{2} - 1} = \frac{\nu+1}{\nu + q'(e_n')^2 - 2} = \frac{\nu+1}{\nu + q'(t_n - \phi(\mathbf{x}_n)\mathbf{w}')^2 - 2}$$

Hence proved.

Exercise 3

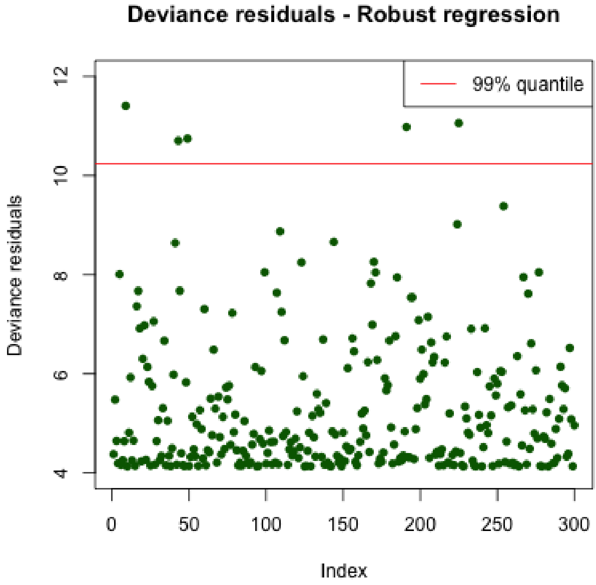
Part (a)

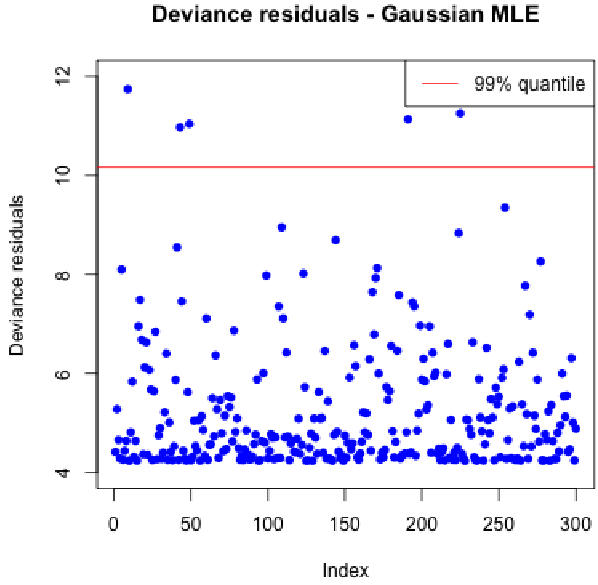




Part (b)

For completeness, we plot the values including the deviance constant terms. The residuals resemble in both models and the simulations produce very similar 99% quantiles, though not exact. This indicates that with high probability the input data is indeed Gaussian.

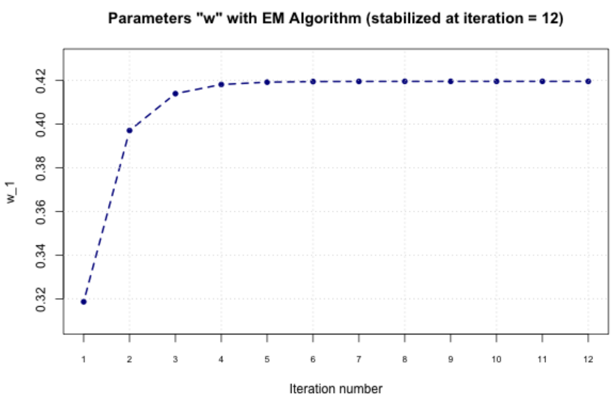


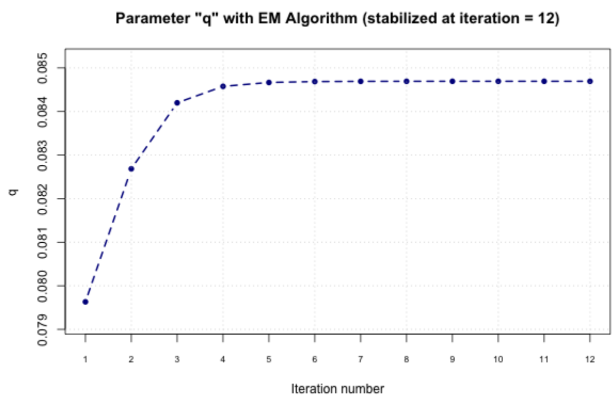


Part (c)

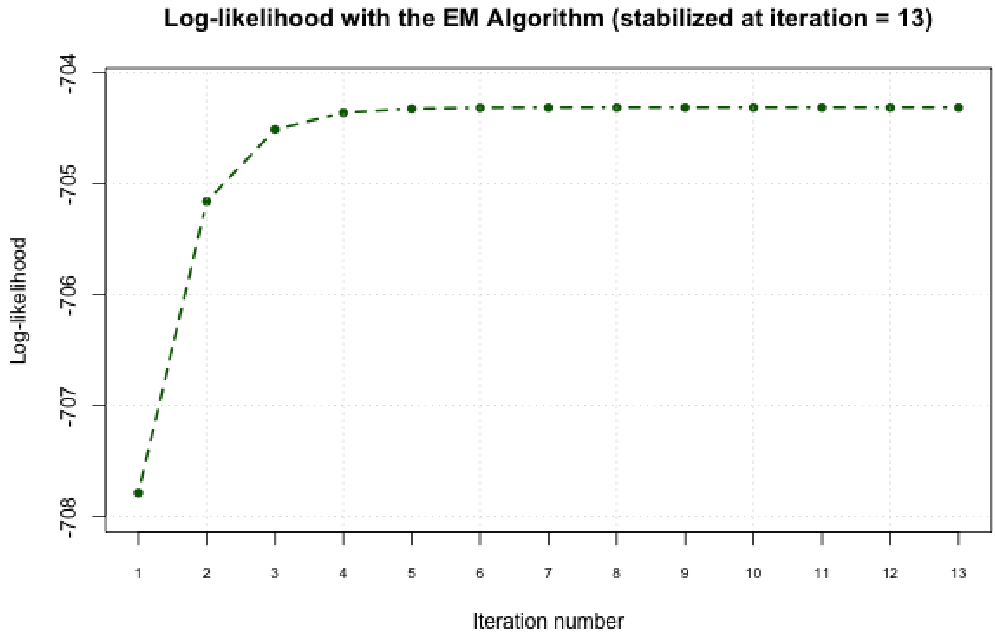
We can decide to stop the EM algorithm when the values for the estimated parameters θ estabilize, which is very similar to saying when the log-likelihood stops increasing (the sequential iterations increase its value by a very small number, say 10^{-6}).

Example (parameter stabilization: q and $\mathbf{w}$, here only w_1 for illustration):





Example (log-likelihood stabilization):



Part (d)

We can choose ν by running the EM algorithm for different values of ν and then choosing the smallest ν that maximizes the log-likelihood, i.e. the one for which increasing ν raises log-likelihood by a reasonably small amount, here we use 0.1.

Graphical representation: