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Assignment 2

1)

$T(x_1, x_2) = (4x_1, -2x_2, 3|x_2|)$

Proof By ~~cont~~ counter example

$T(0,1) + T(0,-1) = (-2,3) + (2,3) = (0,6)$

But

$T((0,1) + (0,-1)) = T(0,0) = (0,0)$

Both not equal $\therefore$ T does not preserve vector addition $\therefore$ not linear transformation

2)

$T: R^3 \to R^2$

$(1,0,0)$, $(0,1,0)$ & $(0,0,1)$ are standard basis for R^3

& we want

$T([x_1, x_2, x_3]) = [x_1, x_2]$

i.e $T([1,0,0]) \to [1,0]$

$T([0,1,0]) \to [0,1]$

$T([0,0,1]) \to [0,0]$

i.e for some transformation T, defined as $T \times A = B$, where $A \in R^3$ & T is a transformation matrix to project A into a 2D plane as B (transformation)

So, let $T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

i.e

$a_1x_1 + a_2x_2 + a_3x_3 = x_1 \Rightarrow a_2 = a_3 = 0$ & $a_1 = 1$

& $b_1x_1 + b_2x_2 + b_3x_3 = x_2 \Rightarrow b_1 = b_3 = 0$ & $b_2 = 1$

$$\therefore\ T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

3) i)

let's condider a polynomial of degree 3 be

$p(x) = a + bx + cx^2 + dx^3$
where $d \neq 0$

then $D(p(x)) = \dfrac{d(a+bx+cx^2+dx^3)}{dx}$

$= b + 2cx + 3dx^2$

then

$Ker(D) = \{a + bx + cx^2 + dx^3 : D(a+bx+cx^2+dx^3) = 0\}$

$= \{a + bx + cx^2 + dx^3 : b + 2cx + 3dx^2 = 0\}$

Here

$b + 2cx + 3dx^2 = 0$ iff
$b = 2c = 3d = 0 \Rightarrow b = c = d = 0$

so, $Ker(D) = \{a + bx + cx^2 + dx^3 : b = c = d = 0\}$
$= \{a \mid a \in R\}$

$\Rightarrow$ Ker(D) = Set of constant polynomial

~~Range(D)~~

Range (D) = ~~Set of c~~

The range of D is all of P^2 since every polynomial in P^2 is the range under Derivation of some polynomial in P^3

So, if $a+bx+cx^2$ is in P^2 then

$$a+bx+cx^2 = D\left(ax+\left(\frac{b}{2}\right)x^2+\left(\frac{c}{3}\right)x^3\right)$$

3)(ii) $S: P' \to P$,

$$S(p(x)) = \int_0^1 p(x)\,dx$$

let $a+bx$ be a polynomial of degree 1

then
$$S(a+bx) = \int_0^1 (a+bx)\,dx$$
$$= \left[ax+\frac{bx^2}{2}\right]_0^1$$
$$= a+\frac{b}{2}$$

$$\text{Ker}(S) = \{a+bx : a+\frac{b}{2}=0\}$$
$$= \{a+bx : a=-b/2\}$$
$$= \{-\frac{b}{2}+bx\}$$

The range of S is R since every real # can be obtained as the image under S of some polynomial P'

if $a \in \mathbb{R}$

then $\int_0^1 a\,dx = [ax]_0^1 = a$

So $a = S(a)$

(iii) $T : M_{22} \to M_{22}$, $T(A) = A^T$

M_{22} is a 2×2 Matrix &

T is a transpose of Matrix

Kernal $(T) = \{ A \text{ in } M_{22} : T(A) = 0 \}$

$= \{ A \text{ in } M_{22} : A^T = 0 \}$

where 0 is NULL Matrix

If $A^T = 0$, then $A = (A^T)^T = 0^T = 0$

So, $\text{Ker}(T) = \{0\}$

Range (T):

Since for any matrix A in M_{22}, we have

$A = (A^T)^T = T(A^T)$ & A^T is in M_{22}. So range $(T) = M_{22}$

4)

Let $\{v_1, \ldots v_k\}$ is linearly independent in V & let

$$c_1[v_1]B + \ldots + c_k[v_k]B = 0 \quad \text{In } R_n$$

But then

$$[c_1 u_1 + \ldots c_k u_k]B = 0$$

[Since coordinate vector preserves linear combinations]

So, the coordinates of the vector

$$c_1 v_1 + c_2 v_2 + \ldots + c_k u_k \quad \text{wrt } B$$

are all 0.

or $c_1 u_1 + c_2 u_2 + \ldots + c_k u_k = 0v_1 + 0v_2 + \ldots + 0v_n = 0$

So, linear independence of $\{v_1, \ldots u_k\}$

makes $c_1 = c_2 = \ldots c_k = 0$

$\Rightarrow \{[v_1]B, \ldots, [v_k]B\}$ is linearly Independent.

5)

Any two bases for a vector space have the same # of vectors

Hence, let $C = [a, b, c]$

& $P_{C \leftarrow B} = [\,[x]_C \;\; [1+x]_C \;\; [1-x+x^2]_C\,]$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \text{ (given)}$$

Equating:

$\Rightarrow x = 1(a) + 0(b) + (-1)c$

$\Rightarrow x = a - c \quad - ①$

$\Rightarrow 1 + x = 0(a) + 2(b) + 1(c)$

$\Rightarrow 1 + x = 2b + c \quad - ②$

$\Rightarrow 1 - x + x^2 = 0(a) + 1(b) + 1(c)$

$\Rightarrow 1 - x + x^2 = b + c \quad - ③$

$① + ② = 1 + 2x = a + 2b$

$- ① + ③ = -1 - x^2 = -a - b$

$= 2x - x^2 = b$

using this eqn in ③

$\Rightarrow 1 - x + x^2 = 2x - x^2 + c$

$\Rightarrow c = 1 - 3x + 2x^2$

From ①, $x = a - (1 - 3x + 2x^2)$

$\Rightarrow a = 1 - 2x + 2x^2$

$\therefore$ Basis $C = [2x^2 - 2x + 1, 2x - x^2, 2x^2 - 3x + 1]$

6 (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -4b \\ a + 5b \end{bmatrix}$$

Soln: First, find the matrix representation of T with respect to standard basis

$B = \{e_1, e_2\}$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then matrix $[T]_B = \Big[[T(e_1)]_B \ [T(e_2)]_B \Big]$

$$= \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}_B \begin{bmatrix} -4 \\ 5 \end{bmatrix}_B \right]$$

$$= \begin{bmatrix} 0 & -4 \\ 1 & 5 \end{bmatrix}$$

It is Matrix representation of T wrt standard Basis, $B = \{e_1, e_2\}$

Now, we will find a basis C for $\mathbb{R}^2$ such that the matrix $[T]_C$ is a diagonal matrix when you work in an eigen basis i.e basis made up of eigen vectors, so we need to find the eigen vectors of $[T]_B$ & that'll be the basis 'C' which we want

Let $A = [T]_B$

Find eigen values: $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} -\lambda & -4 \\ 1 & 5 - \lambda \end{bmatrix} \right| = 0$$

$\Rightarrow -\lambda(5-\lambda)+4=0$

$\Rightarrow \lambda^2-5\lambda+4=0 \quad \Rightarrow (\lambda-1)(\lambda-4)=0$

$\Rightarrow \lambda=1$ & $\lambda=4$ (2 eigen values, $\therefore$ A can be diagonalizable)

Find Eigen Vectors:

1) $\lambda=1$: $(A-\lambda I)x=0$

$\Rightarrow (A-I)x=0$

$$\Rightarrow \begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from above, $x+4y=0 \Rightarrow x=-4y$

Let $y=1$ then $x=-4$

Eigen vector for $\lambda=1$ is $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

2) $\lambda=4$: $(A-\lambda I)x=0$

$\Rightarrow (A-4I)x=0$

$$\Rightarrow \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow x+y=0 \Rightarrow x=-y$

Let $y=1$ then $x=-1$

$\therefore$ Eigen vector $\lambda=4$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore$ Basis $C = \left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ &

$[T_c] = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ which is diagonal.

6)b) $T: P_2 \to P_2$ defined by $T(p(x)) = p(3x+2)$

Soln: Trying to find representation of T wrt standard basis $B = \{1, x, x^2\}$

Then $T(1) = 1$;

$$T(x) = 3x+2$$

$$T(x^2) = (3x+2)^2 = 9x^2 + 12x + 4$$

then matrix $[T]_B = [[T(1)]_B \, [T(x)]_B \, [T(x^2)]_B]$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 12 \\ 0 & 0 & 9 \end{bmatrix}$$

It is Matrix representation of T wrt standard basis. $B = \{1, x, x^2\}$

Finding a basis C s.t $[T]_C$ is diagonal

Finding Eigen values: $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 4 \\ 0 & 3-\lambda & 12 \\ 0 & 0 & 9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(3-\lambda)(9-\lambda) - 0] = 0$$

$$(1-\lambda)(3-\lambda)(9-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 3, 9$$

As there are 3 eigen values, A can be diagonalizable

Find Eigen vector

1) $\lambda = 1$; $(A - \lambda I)X = 0$

$\Rightarrow (A - I)X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 12 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 8z = 0 \Rightarrow z = 0$

$2y + 12z = 0 \Rightarrow y = 0$

x can be any value.

so for eigen vector for $\lambda = 1$ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2) $\lambda = 3$; $(A - \lambda I)X = 0$

$\Rightarrow (A - 3I)X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 2 & 4 \\ 0 & 0 & 12 \\ 0 & 6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$6z = 0 \Rightarrow z = 0$, & $-2x + 2y + 4z = 0 \Rightarrow x = y$

Let $x = 1$ then $y = 1$

$\therefore$ Eigen vector $= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

3) $\lambda = 9$; $(A - \lambda I)X = 0$

$\Rightarrow (A - 9I)X = 0$

$$\Rightarrow \begin{bmatrix} -8 & 2 & 4 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$6 \; 6y + 12z = 0 \Rightarrow y = 2z$

$-8x + 2y + 4z = 0 \Rightarrow -8x + 4z + 4z = 0$

$\Rightarrow x = z$

Let $z = 1$ then $x = 1$ & $y = 2$

$\therefore$ Eigen vector for $\lambda = 9$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ The Basis $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

$= \{ 1, 1 + x, 1 + 2x + x^2 \}$

& $[T]_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ which is diagonal.

7)

$$B = \{x^2, x, 1\}$$

$$T(x^2) = x + m$$

$$T(x) = (m-1)x$$

$$T(1) = x^2 + m$$

a) Consider standard basis, $\{1, x, x^2\}$ of P_2, using P_2, using this basis, we can write the elements using co-ordinate vectors as

$$[x^2] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

By writing an element as a linear combination of the basis elements

Now since $\dim(P_2) = 3$, this set is basis iff these 3 vectors are linearly Independent.

$$\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = -1 \neq 0$$

So, this matrix is non-singular & so the columns are linearly Independent

Thus, the set $\{x^2, x, 1\} = B$ forms Basis

7b) Showing that T is linear transformation

Given $T(x^2) = x + m$; $T: P^2 \rightarrow P^2$
$(V) \quad (W)$

$T(x) = (m-1)x$

$T(1) = x^2 + m$

& Basis $B = \{x^2, x, 1\}$

Assuming B is basis for both V & W

Theorem: let V, W be vector spaces over F

Let $B = \{v_1, v_2, \dots v_n\}$ be a basis for V

& $A = \{w_1, w_2, \dots w_n\}$ be any subset of W

Then a transformation:

$T(\alpha_1 v_1 + \dots + \alpha_n v_n) = \alpha_1 w_1 + \dots + \alpha_n w_n$

is linear.

So we will use above theorem.

Let $p(x) = a_1x^2 + b_1x + c_1 \in V$

$q(x) = a_2x^2 + b_2x + c_2 \in V$

then $T(p(x) + q(x))$

$= T(a_1x^2 + b_1x + c_1 + a_2x^2 + b_2x + c_2)$

$= T((a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2))$

$= (a_1 + a_2)(x + m) + (b_1 + b_2)(m-1)x + (c_1 + c_2)(x^2 + m)$

using above theorem.

Now

$$T(p(x)) = T(a_1x^2 + b_1x + c_1)$$

$$= a_1(x+m) + b_1(m-1)x + c_1(x^2+m)$$

$$= a_1x + a_1m + b_1(m-1)x + c_1x^2 + c_1m$$

$$= c_1x^2 + (a_1 + b_1m - b_1)x + (a_1m + c_1m)$$

Similarly

$$T(q(x)) = c_2x^2 + (a_2 + b_2m - b_2)x + (a_2m + c_2m)$$

Then $T(p(x)) + T(q(x))$

$$= (c_1 + c_2)x^2 + (a_1 + a_2 + b_1m - b_1 + b_2m - b_2)x$$

$$+ (a_1m + a_2m + c_1m + c_2m)$$

$$\therefore T(p(x) + q(x)) = T(p(x)) + T(q(x))$$

Now, let $\alpha \in \mathbb{R}$ then

$$T(\alpha p(x)) = T(a_1\alpha x^2 + b_1\alpha x + c_1\alpha)$$

$$= a_1\alpha(x+m) + b_1\alpha(m-1)x + c_1\alpha(x^2+m)$$

$$= c_1\alpha x^2 + (\alpha a_1 + b_1\alpha m - b_1\alpha)x + (a_1\alpha m + c_1\alpha m)$$

& $\alpha T(p(x)) = \alpha[c_1x^2 + (a_1 + b_1m - b_1)x + (a_1m + c_1m)]$

$$= c_1\alpha x^2 + (a_1\alpha + b_1\alpha m - b_1\alpha)x + (a_1\alpha m + c_1\alpha m)$$

$$\therefore T(\alpha p(x)) = \alpha T(p(x))$$

Hence, T is linear transformation.

7) c)

$T(x^2) = x + m = 0.x^2 + x.1 + 1.m$

$T(x) = (m-1)x = 0.x^2 + (m-1).x + 0.1$

$T(1) = x^2 + m = 1.x^2 + 0.x + m.1$

So the transformation matrix will be

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} = \text{Let it be "A"}$$

7) d) To find kernal (A) we need to find ($X \in$ Vector space) such that $AX = 0$,

Two case possible → det(A) = 0 & det (A) ≠ 0

det (A) = 0 or $X \neq 0$

So for det (A) = 0

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{vmatrix} = 0$$

$\Rightarrow 1(-m(m-1)) = 0$

$\Rightarrow m = 0$ or 1

for $m = 0$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So,

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $v(p, q, r) \in \ker(A)$

$\Rightarrow r = 0$

& $p - q = 0 \Rightarrow p = q \in R$

So, $\ker(A) = \{ p(x^2 + px \mid p \in R \}$

for $m = 1$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow r = 0, \; p = 0, \; p + r = 0$

& $q \in R$

So, $\ker(A) = \{ qx \mid q \text{ in } R \}$

for second case when $\det(A) \neq 0$

for $\ker(A)$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{vmatrix} \neq 0 \Rightarrow m(1-m) \neq 0$$

$\Rightarrow m \neq 0$ or $m \neq 1$

so, the only solⁿ of

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = 0$$

is $p = q = \gamma = 0$

so, $\ker(A) = \{0\}$

e) let $v(p, q, \gamma) \in$ image (T)

$\Rightarrow$ There exist a vector $w(a, b, c)$

s.t $T(w) = v$

so $Aw = v$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ \gamma \end{bmatrix}$$

Again two cases possible

$\det(A) \neq 0$ & $\det(A) = 0$

first case: when $\det(A) \neq 0$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{vmatrix} \neq 0 \Rightarrow m(1-m) = 0$$

$\Rightarrow m \neq 0$ & $m \neq 1$.

In this case there is for each $V(p, q, r)$ there exists a suitable $W(a, b, c)$ with value $W = \frac{V}{(\det(A))}$

So, Image of T is V

Case II) when $\det(A) = 0$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{vmatrix} = 0$$

$\Rightarrow m(1-m) = 0 \Rightarrow m = 0$ or $m = 1$

when $m = 0$

let $V(p, q, r) \in$ image (t)

there is a vector $W(a, b, c)$ such that

$t(W) = V$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$\Rightarrow$ $c = p$

$a - b = q$

$0 = r$

so the Image of t consist of all vectors $V(p, q, 0)$

$\Rightarrow$ image $(t) = \{ px^2 + qx \mid p \& q \in R \}$

when $m = 1$

Again $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

$$\Rightarrow \quad c = p$$
$$a = q$$
$$a + c = r$$

So $\text{image}(t) = \{ p x^2 + q x + (p+q) \mid p, q \in \mathbb{R} \}$

8)

$B = [E_{22}, E_{21}, E_{12}, E_{11}]$

Here E_{ij} are standard basis vectors

then $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

So,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 4E_{22} + 3E_{21} + 2E_{12} + 1E_{11}$$

so, coordinate vector of A

$$= \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$