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Dynamics of a falling projectile under wind effect

Independent Project

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Problem Statement: We consider a plane moving with a constant speed at an elevated height above the Earth's surface. In the course of its flight, the plane drops a package from its luggage compartment. What will be the path of the package and where will it be with respect to the plane? And how can the motion of the package be described?

When the package leaves the moving airplane it will not fall in a straight line. From the moment the package leaves the plane its trajectory will vary in curvature as shown in fig.1.

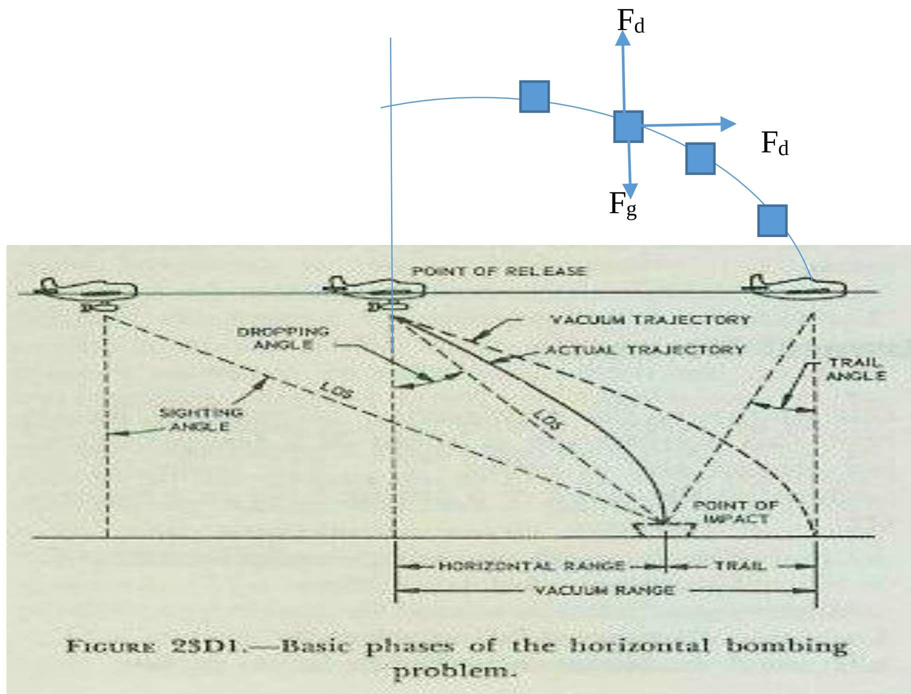

Figure 1: Idea on how path will be followed.

During the fall, the package follows a parabolic path and remains directly below the plane at all times. As the package falls, it undergoes a vertical acceleration; that is, there is a change in its vertical velocity. This vertical acceleration is attributed to the downward force of gravity which acts upon the package. If the package's motion could be approximated as projectile motion (that is, if the influence of air resistance could be assumed negligible), then there would be no horizontal acceleration. In the absence of horizontal forces, there would be a constant velocity in the horizontal direction. This explains why the package would be located directly under the plane from which it is dropped.

Many would insist that there is a horizontal force acting upon the package since it has a horizontal motion. This is simply not the case. The horizontal motion of the package is the result of its own inertia. When dropped from the plane, the package already possessed a horizontal motion. The package will maintain this state of horizontal motion unless acted upon by a horizontal force. An object in motion will continue in motion with the same speed and in the same direction (Newton's first law).

Point to remember: forces do not cause motion; rather, forces cause accelerations.

The following terms are important to understand:

1. *Sighting angle:* At any instant the angle between the LOS and the true vertical determined by the bomb sight gyro (spin axis).
2. *Dropping angle:* The angle between the LOS and the spin axis of the bomb sight at the instant of release. At this instant the sighting angle is equal to the dropping angle.
3. *Trail angle*: The angle between the spin axis of the bomb sight and the LOS at the time of impact, if the airplane maintains course and speed.
4. *Horizontal range in air*: The horizontal distance that the bomb travels between release and Impact trail (T). The horizontal distance between the points of impact of an actual trajectory and a vacuum trajectory.
5. *Horizontal range in vacuum*: Equal to horizontal range in air, plus trail.

Trail is primarily caused by the horizontal force (air resistance) acting on the bomb after release.

Since the bomb is released in the direction of the plane's motion, the trail is always astern of the aircraft.

The amount of trail is dependent upon two factors:

(a)The amount of air resistance,

(b)The length of time that air resistance acts on the bomb.

The air exerts a resistance upon the package. It decreases the vertical velocity at any instant and thereby increases the time of fall and it causes the horizontal velocity to diminish and thus causes the package to trail behind the vertical from the plane. The above effects vary with:

(i) Shape, weight, and skin friction of the package (ii) Altitude (iii) Air speed.

Drag Equation

As the projectile leaves the plane, the air that it moves through will provide this resistance, called *drag force* F_d which depends on the density of the air ρ, the relative velocity v, the air's viscosity and compressibility D, and the reference or Frontal area A by the relation

$$F_d = \frac{1}{2} D\rho A v_0\, v \qquad \text{----- (1A)}$$

"D" used in the drag equation, where a lower drag coefficient indicates the object will have less aerodynamic or hydrodynamic drag. The drag coefficient is always associated with a particular surface area. In case of stream lined objects, it is less as compared to others (e.g. 0.04).

Conditions:

- When the relative velocity is *small* (v_0 is taken as const.) and F_d varies as 'v':

  {NOTE: that equation 1(A) is dimensionally true eqn.}

  and $F_d = \frac{1}{2} D\rho A v^2$ ------ (1B)

- When the relative velocity is *large*.

Note: ***We have discussed Case I just for better understanding, otherwise we are working on case II, i.e. when relative velocity is high.***

Case-I: *When the relative velocity is small*

Drag force is proportional to the velocity, given by eq. 1(A).

The motion of the vertical fall of the package will now have a drag force since air resistance is present in all real applications. Since the projectile motion is two-dimensional motion, it can be resolved into motion along the x − axis and the other along the y-axis.

Let us consider the case for small relative velocity for drag force i.e. eq. 1(A)

- **Along vertical direction:**

$$F_{net} = F_g - F_d \qquad \text{-- (2)}$$

Here Fg is the force due to gravity and Fd is the force due to drag acting in opposite direction and after filling all the values, we get

or $$ma = mg - \frac{1}{2} D\rho A v_0\, v \qquad \text{-- (3)}$$

or $$a = g - \frac{1}{2m} D\rho A v_0\, v \qquad \text{-- (4)}$$

or $$a = g - (kv_0)\, v \qquad \text{-- (5)}$$

where $$k = \frac{F_d}{m} = \frac{D\rho A}{2m} \qquad \text{-- (6)}$$

Eq. (5) can be written as

$$\frac{dv}{dt} = g - (kv_0)v$$

$$\int \frac{dv}{g - (kv_0)v} = \int dt$$

Putting, $K = kv_0$, the velocity (v) will be given by the relation

$$v_y = \frac{g}{K}(1 - e^{-Kt}) \qquad \text{--- (7)}$$

The displacement (y) can be calculated from eq. (7) by writing

$$\frac{dy}{dt} = \frac{g}{K}(1 - e^{-Kt}) \qquad \text{Or}$$

$$\int dy = \frac{g}{K}\int(1 - e^{-Kt})dt \qquad \text{(Integrating both sides)}$$

Which will yield the vertical displacement (y) at any time (t) given as

$$y = y_0 + \frac{g}{K}t - \frac{g}{K^2}(1 - e^{-Kt})$$

Take y_0=0, and solve for y (as given in article on aerial bombing)

$$\text{i.e.} \quad y = \frac{g}{K}t + \frac{g}{K^2}(1 - e^{-Kt}) \qquad \text{-- (8)}$$

- **<u>Along horizontal direction</u>:**

$$F_{net} = F_g - F_d$$

Here Fg is zero in horizontal direction. And force due to drag is Fd.

$$ma = 0 - \frac{1}{2}D\rho A v_0\, v$$

$$a = \frac{dv}{dt} = -K\, v$$

$$\text{Or} \quad v_x = v_0\, e^{-Kt} \quad \text{(after integrating)} \qquad \text{-- (9)}$$

Where v_0 is the velocity of the airplane projectile at the point of drop i.e. time t=0

To calculate the horizontal distance (x) of fall, we can write eq. (9) as

$$\frac{dx}{dt} = v_0\, e^{-Kt}$$

Integrating and applying the limits, we get

$$x = \frac{v_0}{K}\,(1 - e^{-Kt}) \qquad \text{-- (10)}$$

<u>Further conditions (with some more observations):</u>

- **<u>For no Drag force:</u>** i.e. K = very small or zero, using $e^{-Kt} \approx (1 - Kt)$

Equations, (7), (8), (9) and (10) reduce to

$$v_y = g\,t\ ; \qquad v_x = v_0 \qquad \text{-- (11)}$$

$$y = \frac{1}{2}\,g\,t^2; \qquad x\,(Range) = v_0\,t = \sqrt{\frac{2y}{g}} = \qquad \text{-- (12)}$$

In this case, the package would remain vertically below the plane as it fell and the plane would be directly over the point of impact when the package hit. If the aircraft did not alter its course or speed. Here range (or vacuum range to be precise) refers to the distance from the point of release (on the ground) to the point of impact. Here 't' refers to time of fall.

Terminal Velocity (v_T) of the projectile is calculated:

when the weight and Drag of an object are equal and opposite. There is no net force acting on the projectile and the vertical acceleration is zero.

i.e. $F_d = F_g$ or $mg = \frac{1}{2} D\rho A v_0\, v_T$ or $v_T = \frac{g}{kv_0} = \frac{g}{K}$

i.e. when t → ∞ Thus from eq. (7):

$$v = v_T\,(1 - e^{-Kt}) \qquad \text{-- (13)}$$

Equation of path:

$$t = \sqrt{\frac{2\,y}{g}} \quad ; \quad \longrightarrow y = \frac{1}{2} g\,\frac{x^2}{u^2}$$

(This eqn. has been taken by considering the positive y-axis in the downward direction i.e. x=0, y=0 ; at time t=0, y=0)

Case-II: ***when the relative velocity is large:***

- **Along horizontal direction**, since the only net force acting on the ball is the drag:

$$F_{net} = \text{ma} = -F_d$$

$$or \qquad a = -\frac{1}{2m} D\rho A v^2 = -k\,v^2$$

(This is the equation we considered above also, but here for large relative velocity we have v^2)

Where "v" is the horizontal velocity. We can use the terminal velocity to simplify this equation:

$$a = \frac{dv}{dt} = -kv^2 \quad \Rrightarrow \quad \int \frac{dv}{v^2} = -k \int dt$$

$$Since\ for \qquad t = 0, v = v_0$$

$$Therefore \qquad v = \frac{v_0}{1+v_0\,kt} \qquad \text{---- (14)}$$

$$Now \quad v = \frac{dx}{dt} = \frac{v_0}{1+v_0\,kt}$$

$$\Rightarrow \quad \frac{1}{v_0}dx = \frac{1}{1+v_0\,kt}\,dt$$

$$\text{Or} \qquad \frac{\ln(1+v_0\,kt)}{v_0\,k} = \frac{1}{v_0}x + C$$

For t=0, x=0 gives C=0;

$$\text{Therefore,} \qquad x = \frac{ln(1+v_0\,kt)}{k} \qquad \text{--- (15)}$$

- **<u>In the vertical direction</u>**

Putting $F_d = \frac{1}{2}D\rho A v^2$ in eqn. (3)

We get, $ma = mg - \frac{1}{2}D\rho A v^2$ (again considering net forces in vertical direction)

$$\text{or} \qquad a = g - \frac{1}{2m}D\rho A v^2$$

$$\text{or} \qquad a = g - k\,v^2$$

$$\text{where} \qquad \boldsymbol{k = \frac{F_d}{m} = \frac{D\rho A}{2m}}$$

Equation for acceleration (a=dv/dt)

$$\frac{dv}{dt} = g - kv^2$$

$$\int \frac{dv}{(g/k) - v^2} = \int k\,dt$$

Integrating the left hand side yields .

(NOTE: $\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$; $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$

$\sinh x = -i\sin(ix)$; $\cosh x = \cos(ix)$; $\tanh x = -i\tan(ix)$)

$$\text{Put} \qquad v = \sqrt{g/_k}\ \tanh(\theta)$$

Squaring both sides, we get

$$k\, v^2 = g\, tanh^2(\theta)$$

$$g - k\, v^2 = g(1 - \ tanh^2(\theta) \ ) \quad \text{---(a)}$$

$$\frac{dv}{g - kv^2} = \frac{dv}{g(1 - \ tanh^2(\theta)} \qquad \text{---(b)}$$

$$\frac{dv}{d\theta} = \sqrt{\frac{g}{k}}\, (1 - \ tanh^2(\theta)$$

or $$dv = \sqrt{\frac{g}{k}}\, (1 - \ tanh^2(\theta)\,)\, d\theta$$

Therefore $$\frac{dv}{g - kv^2} = \frac{dv}{g(1 - tanh^2(\theta)} = \frac{\sqrt{\frac{g}{k}}\,(1 - tanh^2(\theta)\,)\, d\theta}{g(1 - tanh^2(\theta)} = \sqrt{\frac{1}{gk}}\ d\theta$$ --from (a) and (b)

$$\int \sqrt{\frac{1}{gk}}\ d\theta = \int dt$$

or $$\sqrt{\frac{1}{gk}}\ \theta = t$$

or $$\theta = t\sqrt{g\, k}$$

$$\sqrt{\frac{g}{k}} \tanh(\theta) = \sqrt{\frac{g}{k}} \tanh\ t\sqrt{g\, k}$$

$$\Rightarrow \quad v = \sqrt{\frac{g}{k}}\, tanh\ (t\sqrt{g\, k}) \qquad \text{----- (16)}$$

or $$v = v_T\ \ tanh\ (t\sqrt{g\, k})$$ (where $v_T = \sqrt{\frac{g}{k}}$, Terminal Velocity)

Now $$v_y = \frac{dy}{dt} = \sqrt{\frac{g}{k}}\, tanh\ (t\sqrt{g\, k})$$

Or $$dy = \sqrt{\frac{g}{k}}\, tanh\ (t\sqrt{g\, k})\ \ dt$$

$$For\ integrating \quad \int \tanh(c\ t) dt = \int \frac{e^{ct} - e^{-ct}}{e^{ct} + \ e^{-ct}}\ dt \qquad where\ c = \sqrt{gk}$$

$$Let\ u = \ e^{ct} + e^{-ct}$$

$$Then, \qquad du = c[\ e^{ct} - e^{-ct}]dt$$

$$\int \tanh ct \; dt = \frac{1}{c}\int \frac{du}{u} = \frac{1}{c}\ln(u) = \frac{1}{c}\ln\{\cosh(ct)\}$$

Therefore $\quad y = \int dy = \sqrt{\frac{g}{k}} \int \tanh(t\sqrt{g\,k})\, dt$

$$= \frac{1}{\sqrt{gk}} \cdot \sqrt{\frac{g}{k}} \ln\{\cosh\sqrt{gk}\, t\}$$

Or $\quad y = \frac{1}{k}\ln\{\cosh\sqrt{gk}\, t\}$ ----- (17)

Equations of the Trajectory and Range

Case-I: Without Drag

$$y = \frac{1}{2}\frac{gx^2}{u^2}$$

Range (x) = $\sqrt{\frac{2u^2y}{g}}$

Case-II: When Drag is small

- When Drag force is small and is proportional to velocity (v)

$$x = \frac{v_0}{K}(1 - e^{-Kt});$$

$y = y_0 - \frac{g}{k}t + \frac{g}{K^2}(1 - e^{-Kt})$ Where, $K = k.v_0$

Eliminate'‘t’ from these equations of ‘x’ and ‘y’ to get the formula of trajectory

$y = -\frac{g}{K^2}\left[\left\{ln\left(1 - \frac{xK}{v_0}\right)\right\} - \frac{xK}{v_0}\right]$ --- final equation

One can find Range by switching the above formula to find ‘x’ in terms of ‘y’

In Approximation, $\quad x\ (Range) \sim \frac{v_0}{K}\left[1 - e^{-\left(\frac{yK^2}{g}\right)}\right]$ -- final equation

Case-III: When Drag is large

- When Drag force is relatively is proportional to square of velocity (v^2)

$$\text{For Trajectory:} \quad y = \frac{1}{k}\, ln\left[\cosh \frac{e^{kx}-1}{v_0 k}\sqrt{gk}\right] \quad \text{--- (18)}$$

$$\text{For Range,}\ x = \frac{1}{k}\ln\left[1 + \frac{v_0\sqrt{k}}{\sqrt{g}}cosh^{-1}(e^{ky})\right] \quad \text{--- (19)}$$

Till now we have considered only 2D motion. If we see effect of wind which is also acting at some angle then that will be the 3D motion. Here wind has its components in considered to be in horizontal with a changes in x-z axis. The x-axis will get added to our earlier x-axis component and the vertical motion (in horizontal plane) will be as if that is the z-axis component. And in resultant to this velocity components there will be a drift in path of the object by an angle and change in directory will be there.

Effect of Wind:

If the wind is blowing with velocity 'w' making an angle 'θ' with the direction of descent, then its velocity in the direction of descent will be

$$v_w = w\cos\theta$$

Since 'w' is constant, the integration will not be effected except v_0 to $(v_0 + v_w)$ as du = dv

Now change v_0 to $(v_0 + v_w)$ in various equations:

$$\text{Equation (14) :} \quad v_x = \frac{(v_0 + v_w)}{1+(v_0 + v_w)\,kt} \quad \text{---- (14A)}$$

$$\text{Equation (17) :} \quad y = \frac{1}{k}\ \ln\{\cosh(\sqrt{gk}\ t)\} \quad \text{----- (17A)}$$

$$\text{Equation (18) :} \quad y = \frac{1}{k}\ ln\{cosh\ (\sqrt{gk} * \frac{1}{k(v_0 + v_w)}(e^{kx} - 1))\} \quad \text{---- (18A)}$$

$$\text{Equation (19):} \quad x = \frac{1}{k}\left[ln\left\{1 + (v_0 + v_w)\sqrt{\frac{k}{g}}\ cosh^{-1}(e^{ky})\right\}\right] \quad \text{---- (19A)}$$

Now look at the picture in the x-z plane,

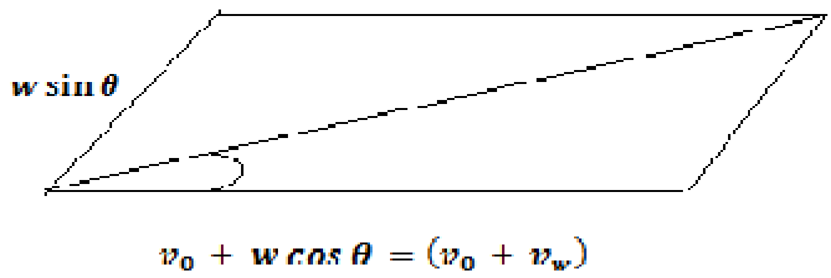

Figure2. Showing components of wind on object.

Along x-axis,

$$\text{Velocity} = v_0+v_w = v_0 + w\cos\theta$$

Along z-axis,

$$\text{Velocity is 'w sin θ'}$$

<u>Resultant Velocity </u>of v_0 and w in the x-z plane will be

$$\sqrt{{v_0}^2 + w^2 + 2v_0\, w\cos\theta}$$

If the angle made with x-axis (direction of original descent) is 'α' in the x-z plane,

$$\text{then} \quad \alpha = tan^{-1}\frac{w\sin\theta}{v_0+w\cos\theta} \qquad \text{---(20)}$$

Equation (14A) will now become after taking resultant velocity in this equation:

$$v_x = \frac{\sqrt{{v_0}^2+w^2+2v_0\, w\cos\theta}}{1+\sqrt{{v_0}^2+w^2+2v_0\, w\cos\theta}\; kt} \qquad \text{----} \quad (14B)$$

While (17A) and (18A) will remain the same and (19A) will transform to

$$\boldsymbol{x}'\, sec\,\alpha = \frac{1}{k}\left[ln\left\{1 + (v_0 + v_w)\sqrt{\frac{k}{g}}\; cosh^{-1}(e^{ky})\right\}\right]$$

Where "**x'**" is the distance covered corresponding to the resultant velocity in the x-z plane.

Thence we can write

$$\boldsymbol{x}' = \frac{1}{k\cos\alpha}\left[ln\left\{1 + (v_0 + v_w)\sqrt{\frac{k}{g}}\; cosh^{-1}(e^{ky})\right\}\right] \qquad \text{----} \ (19B)$$

Along y-axis we still have same equations as we discussed above.