

Assignment 1

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Question 1

$$A_N = 1 + \frac{2}{N} \sum_{j=1}^N A_{j-1} \text{ for } N > 0 \quad (1)$$

Multiplying both sides by N gives

$$NA_N = N + 2 \sum_{j=1}^N A_{j-1} \text{ for } N > 0 \quad (2)$$

This holds for N-1 as

$$(N-1)A_{N-1} = (N-1) + 2 \sum_{j=1}^{N-1} A_{j-1} \text{ for } N > 1 \quad (3)$$

Subtracting the last two

$$NA_N = 1 + (N+1)A_{N-1} \quad (4)$$

Dividing both sides by $N(N+1)$ gives

$$\frac{A_N}{N+1} = \frac{1}{N(N+1)} + \frac{A_{N-1}}{N} \quad (5)$$

Iterating

$$\frac{A_N}{N+1} = \sum_{k=2}^N \frac{1}{k(k+1)} + \frac{A_1}{2} \quad (6)$$

Assuming $A_0 = 0$ as mentioned in the errata and solving the equation gives

$$\boxed{A_N = N}$$

Question 2

Let's assume our pivot is the $x + 1^{th}$ smallest element in the array of N numbers after it is partitioned. Hence, there will be x elements to the left of the element once it gets partitioned.

Hence, we need to calculate the following -

$$\frac{1}{N} \sum_{x=0}^{N-1} E(\text{number of elements greater than the pivot in the first } x \text{ elements}) \quad (7)$$

This is so because only the number of elements greater than the pivot in the first x elements will need to be exchanged.

The quantity in the summation can be calculated using linearity of expectation. Say, there are variables $X_1 \dots X_x$ where $X_i = 1$ if i^{th} element is greater than pivot else 0

Now, by linearity of expectation

$$\begin{aligned} E(\text{Number of elements greater than pivot in the first } x \text{ elements}) \\ = E(X_1 + X_2 + \dots + X_x) \end{aligned} \quad (8)$$

But, notice that $E(X_i) = Pr(X_i)$ and

$$P(X_i = 1) = 1 - \frac{x}{N-1} \quad (9)$$

Hence, using the above observations and plugging it into (7), we get the required result.

Number of exchanges before partitioning $= \frac{(N-2)}{6}$

Question 3

We solve this with exactly the same method for solving the usual Quicksort recurrence. Notice that on the iteration step we can't go all the way to C_1 but have to stop at C_M and this gives us the following recurrence.

$$\frac{C_N}{N+1} = 2 \sum_{k=M+2}^{N+1} \frac{1}{k} + \frac{C_M}{M+1} \quad (10)$$

Replacing the value of C_M gives us $C_N = 2 \sum_{k=M+2}^{N+1} \frac{1}{k} + \frac{M(M-1)}{4(M+1)}$

Question 4

Using the approximation to H_N gives us the following value for $f(M)$

$$f(M) = \frac{M(M-1)}{4(M+1)} - 2\ln(M+2) \quad (11)$$

Throwing away $\frac{M-1}{M+1}$ gives us the minimum to occur at $M = 6$
Pretty cool huh!