Assignment 1

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Question 1

$$A_N = 1 + \frac{2}{N}\sum_{j=1}^{N} A_{j-1} for N > 0 \tag{1}$$

Multiplying both sides by N gives

$$NA_N = N + 2\sum_{j=1}^{N} A_{j-1} for N > 0 \tag{2}$$

This holds for N-1 as

$$(N-1)A_{N-1} = (N-1) + 2\sum_{j=1}^{N-1} A_{j-1} for N > 1 \tag{3}$$

Subtracting the last two

$$NA_N = 1 + (N+1)A_{N-1} \tag{4}$$

Dividing both sides by $N(N+1)$ gives

$$\frac{A_N}{N+1} = \frac{1}{N(N+1)} + \frac{A_{N-1}}{N} \tag{5}$$

Iterating

$$\frac{A_N}{N+1} = \sum_{k=2}^{N} \frac{1}{k(k+1)} + \frac{A_1}{2} \tag{6}$$

Assuming $A_0 = 0$ as mentioned in the errata and solving the equation gives

$$\boxed{A_N = N}$$

Question 2

Let's assume our pivot is the $x + 1^{th}$ smallest element in the array of N numbers after it is partitioned. Hence, there will be x elements to the left of the element once it gets partitioned.

Hence, we need to calculate the following -

$$\frac{1}{N}\sum_{x=0}^{N-1} E(\textbf{number of elements greater than the pivot in the first x elements}) \tag{7}$$

This is so because only the number of elements greater than the pivot in the first x elements will need to be exchanged.

The quantity in the summation can be calculated using linearity of expectation.Say, there are variables $X_1 \ldots X_x$ where $X_i = 1i$ if i^{th} element is greater than pivot else 0

Now, by linearity of expectation

$$E(\textbf{Number of elements greater than pivot in the first x elements}) \\ = E(X_1 + X_2 + \ldots + X_x) \tag{8}$$

But, notice that $E(X_i) = Pr(X_i)$ and

$$P(X_i = 1) = 1 - \frac{x}{N-1} \tag{9}$$

Hence, using the above observations and plugging it into (7), we get the required result.

$$\boxed{\text{Number of exchanges before partitioning } = \frac{(N-2)}{6}}$$

Question 3

We solve this with exactly the same method for solving the usual Quicksort recurrence. Notice that on the iteration step we can't go all the way to C_1 but have to stop at C_M and this gives us the following recurrence.

$$\frac{C_N}{N+1} = 2\sum_{k=M+2} N + 1\frac{1}{k} + \frac{C_M}{M+1} \tag{10}$$

Replacing the value of C_M gives us $\boxed{C_N = 2\sum_{k=M+2}^{N+1} \frac{1}{k} + \frac{M(M-1)}{4(M+1)}}$

Question 4

Using the approximation to H_N gives us the following value for $f(M)$

$$f(M) = \frac{M(M-1)}{4(M+1)} - 2ln(M+2) \tag{11}$$

Throwing away $\frac{M-1}{M+1}$ gives us the minimum to occur at $\boxed{M = 6}$
Pretty cool huh!