Singular Value Thresholding for Matrix Completion

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Abstract

Matrix completion, is the task of filling in the unknown entries of a sparse matrix. This problem is central in recommender systems and collaborative filtering, which provide individualized product suggestions based on ratings gathered from a large set of users. Matrix completion algorithms are used effectively today to provide helpful recommendations for users of services such as Netflix and Amazon.com. In the present paper, the baseline algorithms for matrix completion, such as eigentaste, nearest K-neighbor, and column mean are compared against the more sophisticated matrix completion algorithms, such as Spectral Matrix Completion (SMC) and Singular Value Thresholding (SVT). The comparison metric used in the present paper include, accuracy, convergence time, and the nature of the input matrix. All of the presented methods have their advantages and disadvantages based on the problem. In the present paper, the detailed results are presented for synthetic data and the Jester data. While Eigentaste needs at least few fully filled columns to complete the sparse matrix, SMC and SVT does not have that limitation. Between SMC and SVT, SMC provides better accuracy but is much slower than the SVT. Detailed comparison of the methods is presented in the later sections.

1 Introduction

How can we predict what movies a person will enjoy? On what webpages will they click? Which emails will they read? *Collaborative Filtering* problems like these are tricky. Consider predicting a person's movie preferences using Netflix's database. We can predict how highly a user will rate the movie *Rambo* using a regression model trained on the ratings of users that have rated *Rambo* before. We quickly run into problems when we realize that most users have not rated *Rambo*, and most users who have, have not rated the same ones as the user for whom we are making the prediction. This means that our model would have to ignore a majority of the data avaialable, limiting performance. Furthermore a separate model would need to be trained for each movie in the system, leading to computational difficulties.

A better framework for these sorts of problem is *matrix completion*. We can view each user-item pair as an entry in a matrix that has only a few entries filled in. Our task is then to complete the matrix, assuming that it has some simple underlying structure. Low rank is a common structural assumption. Recovering a low rank matrix from a few entries is a problem with applications not only in collaborative filtering [14], but also in dimensionality reduction [10, 16] and multi-task learning [1, 11]. This problem turns out to be computationally hard. In fact, rank minimization is NP-hard, meaning that is at least as hard as all those that can be solved by a nondeterministic Turing machine in polynomial time, like the Hamiltonian Cycle and Traveling Salesman problems. In both

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theory and practice, all algorithms that perform this task have time complexity exponential in the size of the matrix. We must settle for an approximation.

Candes and Recht showed that most low-rank matrices could be recovered by instead minimizing the nuclear norm, defined as the sum of the singular values [3]. Theoretically, this is because the nuclear norm is the tightest convex lower bound on the rank function for singular values no greater than 1. Intuitively, while the rank function counts the number of nonzero singular values, the nuclear norm sums their amplitude, much like how the 1-norm is a useful substitute for counting the number of nonzero entries in a vector. Conveniently, the nuclear norm can be minimized efficiently subject to equality constraints via semidefinite programming.

Nuclear norm minimization has long been observed to produce fairly low-rank solutions [6, 7, 15]), but only recently was there any theoretical basis for when it produced the true minimum rank solution. The first paper to provide such foundations was [13], where Recht, Fazel, and Parrilo developed probabilistic techniques to study average case behavior and showed that the nuclear norm heuristic could solve most instances of the rank minimization problem when the number of linear constraints was sufficiently large. This inspired a groundswell of interest in theoretical guarantees for rank minimization, and these results lay the foundation for [3]. Candes and Rechts bounds were subsequently improved by Candes and Tao [4] and Keshavan, Montanari, and Oh [9] to show that one could, in special cases, reconstruct a low-rank matrix by observing a set of entries of size at most a polylogarithmic factor larger than the intrinsic dimension of matrix.

To test the validity of these theoretical guarantees in practice, we explore two competing algorithms for matrix completion via nuclear norm-minimization and compare their performance on collaborative filtering baselines, using both synthetic and real-world data. The Singular Value Thresholding (SVT) algorithm was introduced in "A Singular Value Thresholding Algorithm for Matrix Completion" [2]. Spectral Matrix Comletion (SMC) was introduced in "Matrix Completion from a Few Entries" [9] and compare their effectiveness.

2 Singular Value Thresholding

Formally, Singular Value Thresholding (SVT) addresses the optimization problem

$$\begin{array}{ll} \underset{X}{\text{minimize}} & \|X\|_* \\ \text{subject to} & \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M), \end{array}$$

where $\|\cdot\|_*$ is the *nuclear norm*, or the sum of the singular values and $\mathcal{P}_\Omega(\cdot)$ is the projection operator, which makes zero all entries $(i,j) \notin \Omega$.

Singular Value Thresholding works in an iterative, alternating fashion reminiscent of the Alternating Direction Method of Multipliers (ADMM). The complete matrix X is contructed iteratively using a low-rank, low-singular value approximation to an auxiliary sparse matrix Y. Y is then adjusted to ensure the resulting approximation in the subsequent step has matching entries $X_{ij} = M_{ij}$. Each iteration consists of the inductive steps

$$\begin{cases} X^k = \operatorname{shrink}(Y^{k-1}, \tau) \\ Y^k = Y^{k-1} + \delta_k \mathcal{P}_\Omega(M - X^k), \end{cases}$$

where $\operatorname{shrink}(\cdot,\cdot)$ is the *singular value shrinkage operator*, which reduces each singular value by aGiven a singular value decomposition $X = U\Sigma V^T$, $\Sigma = \operatorname{diag}(\{\sigma_i\}_{1\le i\le r})$, we can write this as

$$\operatorname{shrink}(X,\tau) = U\Sigma_\tau V^T, \;\; \Sigma_\tau = \operatorname{diag}(\{(\sigma_i - \tau)_+\}).$$

These two operations, when repeated, approach a low-nuclear norm solution by repeatedly shrinking the singular values of X. This algorithm has shown success in recovering accurate low-rank solutions when the source of M is also low-rank, even though it does not optimize this objective directly. The original authors discuss its theoretical guarantees in detail, but we choose to omit them in this discussion.

SVT has hyperparameters that must be carefully tuned to guarantee convergence. The shrinkage value τ must be set high in order for the algorithm to converge, but not too high that it exceeds the true singular values. The stepsizes δ_k are similarly sensitive. The stepsize can be set dynamically as well, though we choose to maintain a fixed stepsize throughout. Furthermore, we compute the decomposition of Y^K in batches, which introduces a new batch size parameter l that effects the speed of convergence. Also important is the initialization of Y, for which the authors provide helpful strategies. Finally, we use the relative error $\|\mathcal{P}_\Omega(X^k - M)\|_F / \|P_\Omega(M)\|$ as a stopping criterion. We terminate when this drops below a small ϵ.

We find from experimentation that SVT is frustratingly sensitive to the settings of these parameters. The suggested settings only lead to convergence for a limited range of matrix sizes and ranks, and choosing the correct settings often requires unrealistic knowledge of the underlying matrix structure.

3 Spectral Matrix Completion

Spectral Matrix Completion (SMC), presented in Keshavan et al. [9] minimizes the reconstruction error of a low-rank matrix. Let M be the complete matrix and M^E be the $m \times n$ matrix with known entries $(i,j) \in E$ filled in and zero entries otherwise. That is,

$$M^E_{i,j} = \begin{cases} M_{i,j} & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

As presented in [9], the SMC algorithm has three steps.

1. Trim M^E, giving $\widetilde{M}^E$.
2. Project $\widetilde{M}^E$ to $T_r(\widetilde{M}^E)$.
3. Clean, minimizing the reconstruction error $F(X,Y)$.

3.1 Trimming

In the trimming step, we throw out information in order to make the underlying true-rank structure more apparent. First, we zero all columns in M^E with degree larger than $2|E|/n$ and set to zero all rows with degree larger than $2|E|/m$, where $|E|$ is the number of non-zero entries in M. This step is crucially important when the number of revealed entries per row/column follows a heavy tail distribution, as is the case for real data.

3.2 Projection

In the projection step, we build a low-rank reconstruction of M^E. We begin by computing its the singular value decomposition (SVD). Let the singular values be $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{\min(m,n)} \geq 0$. Therefore, the decomposition can be written

$$M^E = \sum_{i=1}^{\min(m,n)} \sigma_i x_i y_i^T. \tag{2}$$

The projection

$$T_r(M^E) = \frac{mn}{|E|} \sum_{i=1}^{r} \sigma_i x_i y_i^T \tag{3}$$

is obtained by zeroing all but the r largest singular values. Note that apart from the rescaling factor $(mn/|E|)$, $T_r(M^E)$ is the orthogonal projection of M^E onto the set of rank-r matrices. TODO: how do we choose r?

3.3 Cleaning

This is the step where all the magic happens in SMC algorithm. We "clean" the errors produced by the projection step by iteratively minimizing the reconstruction error. Given $X \in R^{m \times r}$, $Y \in R^{n \times r}$ with $X^T X = mI$ and $Y^T Y = nI$,

$$F(X,Y) = \min_{S \in R^{r \times r}} F(X,Y,S) \tag{4}$$

$$F(X,Y,S) = \frac{1}{2} \sum_{(i,j)\in E} (M_{ij} - (XSY^T)_{ij})^2 \tag{5}$$

We initiliaze $T_r(\widetilde{M}^E) = X_0 S_0 Y_0^T$ and minimize $F(X,Y)$ locally with initial condition $X = X_0$, $Y = Y_0$. Note that $F(X,Y)$ is easy to evaluate since it is defined by minimizing the quadratic function $S \mapsto F(X,Y,S)$ over the low-dimensional matrix S. Further it depends on X and Y only through their column spaces. In geometric terms, F is a function defined over the cartesian product of two Grassmann manifolds. Optimization over Grassmann manifolds is a well understood topic [5] and efficient algorithms (in particular Newton and conjugate gradient) can be applied. In our implementation, we minimize $F(X,Y)$, using gradient descent with line search.

4 Baseline Algorithms

For comparison, we introduce two simple baseline algorithms for matrix completion.

4.1 Eigentaste

Eigentaste is a collaborative filtering algorithm specifically designed for the Jester Dataset. It relies on the assumption that a subset of the columns will be dense, corresponding to a few "seed" items which every user is required to review. Eigentaste applies principal component analysis to this dense subset of columns, reducing dimensionality and allowing rapid clustering. Recommendation is done by averaging ratings over clusters.

Mean rating of the jth item in the gauge set is given by

$$\mu_{ij} = \frac{1}{n} \sum_{i \epsilon U_j} \widetilde{r}_{ij}$$

$$\sigma_j^2 = \frac{1}{n-1} \sum_{i \epsilon U_j} (\widetilde{r}_{ij} - \mu_j)^2$$

In A, the normalized rating r_{ij} is set to $(\widetilde{r}_{ij} - \mu_j)/\sigma_j$. The global correlation matrix is given by

$$C = \frac{1}{n-1} A^T A = E^T \Lambda E$$

The data is projected along the first v eigenvectors $x = R{E_v}^T$

TODO: make this next part more concise

Recursive Rectangular Clustering:

1. Find the minimal rectangular cell that encloses all the points in the eigenplane.
2. Bisect along x and y axis to form 4 rectangular sub-cells.
3. Bisect the cells in step 2 with origin as a vertex to form sub-cells at next hierarchial level.

Online Computation of Recommendations

1. Collect ratings of all items in gauge set.
2. Use PCA to project this vector to eigenplane.
3. Find the representative cluster.
4. Look up appropriate recommendations, present them to the new user, and collect ratings.

4.2 K-Nearest Neighbors

In the K-Nearest Neighbors algorithm, we predicts ratings based on the mean ratings of each person's nearest neighbors.

$$p_{ij} = \overline{r_i} + \kappa \sum_{k=1}^{n} w(i,p)(\overline{r_{pj}} - \overline{r_p})$$

where $\overline{r_i}$ is the average joke rating for user i, and κ is a normalizing factor ensuring that the absolute value of the weights sum to 1. We implemented the weighted nearest neighbor algorithm. We used a function of Euclidean distance from user i to user p as the weight w(i, p), and $\kappa = \sum_{k=1}^{n} w(i,p)$. Specifically, if we are interested in q nearest neighbors, $w(i;p) = d(i, q+1) - d(i,p)$. This ensures that is closest neighbor has the largest weight.

5 Experimental Results

5.1 Synthetic Data

The performance of the SVT and SMC algorithms were compared on synthetically generated matrices of varying ranks, sizes and sparsity. Specifically, the algorithms were tested on all combinations of matrix size(1000, 5000), rank (5, 10 and 20), and sparsity(30%, 50% and 70%). As a baseline to SVT and SMC, a column mean matrix completion method was implemented. This method predicts the average column value for all missing entries. In the Tables 5.1, 2 the mean absolute error and the run time is averaged over the categories of size, rank and sparsity. For example, out of the eighteen tests that were run, nine tests include the matrix of size 1000x1000. Therefore the mean absolute error value for all nine tests are averaged in order to represent the values in the size 1000 column of the table below. Mean absolute error is the absolute value of the average error on the unknown entries. Run time is measured in seconds and all simulations were run at University of Michigan's CAEN lab computers (8GB memory with one core i7 processors). The columns in Tables 5.1, 2 contain the averaged mean absolute error and the run time for all experiments.

While low rank is a desirable property and a suitable target for optimization, we are primarily concerend with accurately predicting unknown entries in a matrix.

	Rank			Sparsity			Size ($n \times n$)	
Algorithm	5	10	20	30%	50%	70%	1000	5000
Mean	5.65	9.90	13.98	9.10	10.21	10.21	9.43	10.26
SVT	2.05	7.2E-04	1.4E-03	2.05	8.4E-04	9.6E-04	1.37	7.6E-04
SMC	2.1E-04	4.3E-07	4.5E-07	2.1E-04	4.2E-07	3.8E-07	1.4E-04	3.7E-07

Table 1: Mean Absolute Error of predictions of unknown entries. TODO: add more details about experiment parameters

Table 5.1 shows that both SVT and SMC outperform the baseline column-mean method, which simply fills in all unknown entries in a column with the mean of its known entries. In addition, SMC consistently outperforms SVT in terms of mean absolute error, often by several orders of magnitude.

We are also concerned with the speed at which we arrive at accurate predictions. Table 2 shows that, while SMC significantly outperforms SVT in terms of accuracy, SVT significantly outperforms SMC in terms of time. It appears that the computational cost of SMC grows exponentially with the rank. Even at rank 5, the smallest tested rank value, SMC runs 9.6 times slower than SVT.

5.2 Jester Dataset

The Jester Joke dataset contains 4.1 million ratings for 100 jokes from 73,421 users [8]. A set of 10 "seed" jokes were chosen to be presented to users before any others, and users that did not rate all of the seed jokes were discarded. This leaves us with 10 completely dense columns, allowing us to apply both supervised learning algorithms (Eigentaste) and matrix completion algorithms (SVT,

Algorithm	Rank			Sparsity			Size ($n \times n$)	
	5	10	20	30%	50%	70%	1000	5000
Mean	0.12	0.12	0.10	0.11	0.11	0.12	0.01	0.22
SVT	26	39	92	42	54	60	6	98
SMC	252	405	1623	1021	510	748	86	1433

Table 2: Run Time (in seconds) TODO: add more details about experiment parameters

SMC). We hypothesize that the matrix completion algorithms will outperform those that only take advantage of the dense columns because they are capable of utilizing all data during training.

For the purpose of evaluation we randomly select subsets of 100, 200, and 1000 users' ratings from the Jester Dataset. We choose two ratings at random from each user as test points and leave the remainder for training. We evaluate three algorithms, Eigentaste, SMC, and SVT, using the Normalized Mean Absolute Error (NMAE) of the reconstruction on the test points, similar to [12].

The NMAE, a commonly used performance metric in collaborative filtering, is defined in terms of The Mean Absolute Error (MAE):

$$MAE = \frac{1}{|T|} \sum_{(u,i) \in T} |M_{ui} - \widetilde{M}_{ui}| \tag{6}$$

where M_{ui} is the original ratings, $\widetilde{M}_{ui}$ is the predicted rating for user u, item i, and T is the test set. The NMAE can be defined as:

$$NMAE = \frac{MAE}{M_{max} - M_{min}} \tag{7}$$

where M_{max} and M_{min} are the upper and lower bounds for the ratings. In the Jester joke dataset, the rating are in the range $[-10, 10]$.

In the Table 3, we present numerical results on the Jester joke dataset with the SMC algorithm.

# users	# jokes	samp. ratio	NMAE	Time (s)
100	100	5353	0.1573	25.31
200	100	10921	0.1603	17.44
1000	100	57578	0.1647	44.95

Table 3: Numerical results on the Jester joke dataset with SMC algorithm. Times reported are from a University of Michigan's CAEN lab computer(8GB memory with one core i7 processor)

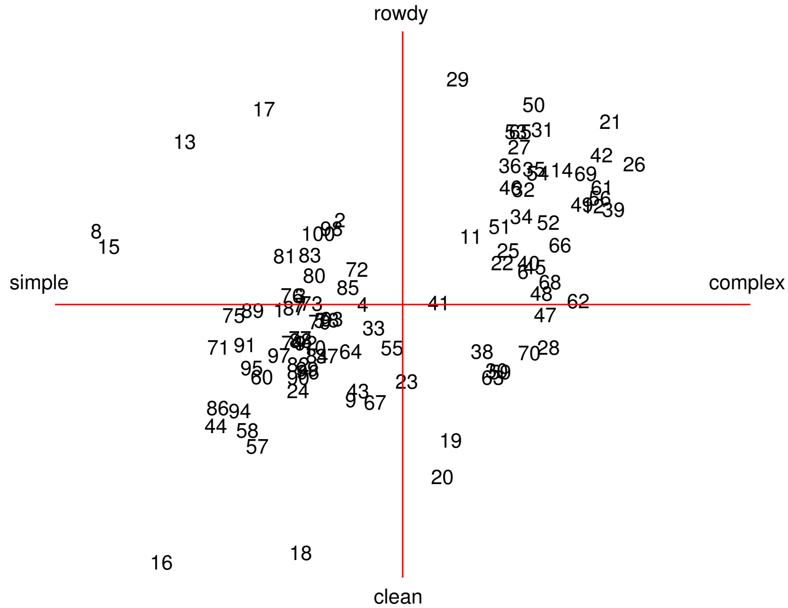
# user	# jokes	NMAE		
		SMC	SVT	Eigentaste
100	100	0.1573	0.1865	0.187
200	100	0.1603	0.1843	0.190
1000	100	0.1647	0.1714	0.237

Table 4: NMAE comparison on the Jester joke dataset for SMC, SVT, and Eigentaste algorithms.

We have eigentaste, SMC and SVT on subsets of the Jester dataset. Compare accuracy and time. Mention that we used LMSVD in SMC for this version.

5.2.1 Visualization

Each of the algorithms compared previously rely on the assumption that the Jester Dataset can approximated using a low-rank matrix. To test this claim, we plot each of the 100 jester jokes in a two-dimensional plane and attempt to explain the meaning of the directions. In axes according to the top two eigenvectors of the Singular Value Thresholding solution.



Joke #	Joke Text
8	Q. Did you hear about the dyslexic devil worshiper? A. He sold his soul to Santa.
26	A guy walks into a bar and sits down next to an extremely gorgeous woman. The first thing he notices about her though, are her pants. They were skin-tight, high-waisted and had no obvious mechanism (zipper, buttons or velcro) for opening them. After several minutes of puzzling over how she got the pants up over her hips, he finally worked up the nerve to ask her. ”Excuse me miss, but how do you get into your pants?” ”Well,” she replied, ”you can start by buying me a drink.”
29	An old Scotsmen is sitting with a younger Scottish gentleman and says the boy. ”Ah, lad look out that window. You see that stone wall there, I built it with me own bare hands, placed every stone meself. But do they call me MacGregor the wall builder? No! He Takes a few sips of his beer then says, ”Aye, and look out on that lake and eye that beautiful pier. I built it meself, laid every board and hammered each nail but do they call me MacGregor the pier builder? No! He continues...”And lad, you see that road? That too I build with me own bare hands. Laid every inch of pavement meself, but do they call MacGregor the road builder? No!” Again he returns to his beer for a few sips, then says, ”Agh, but you screw one sheep...”
18	A dog walks into Western Union and asks the clerk to send a telegram. He fills out a form on which he writes down the telegram he wishes to send: ”Bow wow wow, Bow wow wow.” The clerk says, ”You can add another ’Bow wow’ for the same price.” The dog responded, ”Now wouldn’t that sound a little silly?”

Figure 1: Two-dimensional projection of the Jester joke dataset. The horizontal and vertical axes correspond to the directions of the first and second eigenvectors, respectively, of the completed matrix found by the SVT algorithm on the full Jester dataset. Also given are the most extreme jokes in each direction, which characterize the axes.

In figure 1 we find that the two principal components in the space of jokes correspond to short, simple puns (left) versus longer, complex jokes (right), and rowdy jokes (top) versus clean jokes (bottom). Only a few jokes are given here, but at closer inspection these observations seem to hold true in general.

6 Conclusions

General Points: synthetic data conclusions: SMC and SVT are better than baseline mean method. SMC attains better accuracy than SVT but takes longer. One potential improvement to SMC is to use the LMSVD mentioned in the SVT paper. We used this upgrade in the Jester dataset tests and we found that

7 Group Member's Accomplishments

All team members participated in data processing early in the project. All team members contributed to the final report. Each member was responsible for the writeup regarding their individual experiments. Sriram spearheaded the implementation of Eigentaste. He discovered ways of modifying these methods to supercharge performance and also contributed a great deal towards the performance evaluation for different algorithms. Nitin and Jeeheh implemented the SMC algorithm. Jeeheh took charge in getting the performance comparison for the synthetic data. Nitin contributed towards the performance metric for different algorithms and also helped integrating the final report together. Jonathan was in command of implementing the SVT algorithm and tuning the parameters for experiments. He also created the nice-looking visualizations of the Jester joke in the report.

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