Assignment 2

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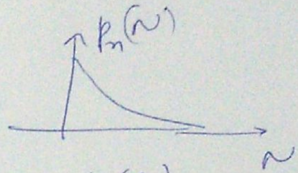
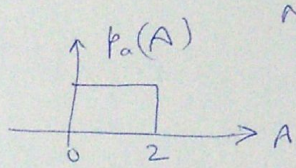
Q1 Find the MMSE, MAP & ML estimator of A





$$r = \ln a + n$$

$$p_n(N) = e^{-N}\, U(N)$$

$$p_a(A) = \frac{1}{2} A \left[U(A) - U(A-2)\right]$$

$$\Rightarrow \hat{a}_{MMSE} = \int_{-\infty}^{\infty} A\, p_{a|r=R}(A)\, dA$$

$$p_{r|a=A}(R) = p_n(r - \ln A)$$
$$= e^{-(R-\ln A)}\, U(R - \ln A)$$
$$= A e^{-R}\, U(R - \ln A)$$
$$= \underline{\underline{A e^{-R}}} \qquad R \geq \ln A \text{ or } A \leq e^{R}$$

$$p_r(R) = \int_{-\infty}^{\infty} p_{r|a=A}(R)\, p_a(A)\, dA$$

for $R < 0$ $$= \int_0^{R} A e^{-R} \cdot \frac{A}{2}\, dA = \frac{e^{2R}}{6}$$

for $R \geq 0$ $$= \int_0^{2} A e^{-R} \cdot \frac{A}{2}\, dA = \frac{4}{3} e^{-R}$$

$$\therefore\ p_{a|r=R}(A) = \frac{A e^{-R} \cdot A/2}{\frac{e^{2R}}{6}} = 3A^2 e^{-3R} \qquad R < 0$$

$$= \frac{A e^{-R} \cdot A/2}{\frac{4}{3} e^{-R}} = \frac{3}{8} A^2 \qquad R \geq 0$$

$$\therefore\ \hat{a}_{MMSE} = \int_0^{e^R} A \cdot 3A^2 \cdot e^{-3R}\, dA = \frac{3}{4} e^{R} \qquad R < 0$$

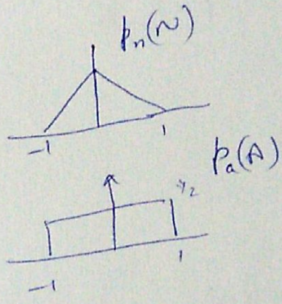
$$= \int_0^{2} A\, \frac{3A^2}{8}\, dA = \frac{3}{2} \qquad R \geq 0$$

$$\hat{a}_{MAP} = \max \{ p_{a/r=R}(A) \}$$
$$= \max_a \left(3A^2 e^{-3R}\right) = \left.\begin{matrix} 3e^{-R} & R<0 \\ \frac{3}{2} & R \geq 0 \end{matrix}\right\}$$

$$\hat{a}_{ML} = \max_a \left(p_{r/a=A}(R) \right)$$
$$= \max_a \left[A e^{-R} \, U(R - \ln A) \right]$$
$$= \boxed{1}$$ because max value is attained for $A=1$ & $R = \ln A$

Q2 Find ML & MAP estimator for 'A' given

$$r = 2a + n$$
$$p_n(N) = (1-|N|)\left[U(N+1) - U(N-1)\right]$$
$$p_a(A) = \frac{1}{2}\left[U(A+1) - U(A-1)\right]$$



$\Rightarrow$ $$\hat{a}_{ML} = \max_a \left(p_{r/a=A}(R) \right)$$

$$p_{r/a=A}(R) = p_n(R-2A)$$
$$= \left(1 - |R-2A|\right)\left[U(R-2A+1) - U(R-2A-1)\right]$$
$$= \left(1 - |R-2A|\right) \quad 2A-1 \leq R \leq 2A+1$$

$\therefore$ it is max at $R = 2A$

or $A = R/2$

$$\therefore \boxed{\hat{a}_{ML} = R/2}$$

$$\hat{a}_{MAP} = \max_a \{ p_{a/r=R}(A) \}$$

from Bayes Rule

$$p_{a/r=R}(A) = \frac{p_{r/a=A}(R) \; p_a(A)}{p_r(R)}$$

$p_r(R) =$ convolution of pdf of (n) & pdf $(2a)$

$$p_{2a}(2A) = U(2A+1) - U(2A-1)$$

$$\therefore\ p_r(R) = \int_{-1}^{1} \frac{1}{2}\left(1 - |R-y|\right) dy \qquad \text{, wh } y = 2A,\ \frac{dy}{2} = dA$$

$$= \frac{1}{2}\int_{R}^{1}\left(1-(y-R)\right)dy + \frac{1}{2}\int_{-1}^{R}\left(1-(R-y)\right)dy$$

$$= \frac{1}{2}\left[\left| y - \frac{y^2}{2} + Ry \right|_R^1 + \left| y - Ry + \frac{y^2}{2}\right|_{-1}^{R}\right]$$

$$= \frac{1}{2}(1-R^2)$$

$$\therefore\ p_{a/r=R}(A) = \frac{(1-|R-2A|)\,\frac{1}{2}}{\frac{1}{2}(1-R^2)}$$

$$\hat{a}_{map} = \max_a \left\{ \frac{(1-|R-2A|)}{(1-R^2)} \right\}$$

max at $A = \frac{R}{2}$

$$\boxed{\hat{a}_{MAP} = \frac{R}{2}}$$

Q3 Show that $\hat{a}_{ML} = e^R$, given $r = \ln a + n$, if $p_n(N)$ is max at $N=0$

$$\Rightarrow\ \hat{a}_{ML} = \max_a \left\{ p_{r/a=A}(R) \right\}$$

$$= \max_a \left\{ p_n(R - \ln A) \right\}$$

Since $p_n(N)$ is max at $N=0$

$\therefore\ \hat{a}_{ML}$ is at $R = \ln A$

or $\boxed{\hat{A}_{ML} = e^R}$

Hence proved

Q4 Find $\hat{a}_{ML}$ & $\hat{b}_{ML}$ for

$$s_i = a + b n_i \quad , i = 1, 2, \ldots N$$

n_i are iid with $N(0,1)$

$$\Rightarrow\ \hat{a}_{ML} = \max_a \left\{ p_{s/a=A}(S) \right\} \quad , \quad p_{n_i}(N_i) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{N^2}{2}\right)$$

$$p_{s/a=A}(S) = \prod_{i=1}^{N} p_{s_i/a=A}(S_i) = \prod_{i=1}^{N} p_n\frac{(S_i - a_i)}{b_i} = \frac{1}{(2\pi)^{N/2}} \exp\left(\frac{-1}{2b^2}\sum (S_i - A)^2\right)$$

Differentiate both sides w.r.t A & equate to 0

$$\frac{\partial}{\partial A}\left(p_{s/a=A}(S)\right)=0$$

or $$\frac{\partial}{\partial A}\left(\ln\left(p_{s/a=A}(S)\right)\right)=0$$

$$\sum_{i=1}^{N}(S_i-A)=0$$

$$\therefore NA=\sum_i S_i$$

or $$\boxed{\hat{a}_{ML}=\frac{1}{N}\sum_i S_i}$$

$$p_{n}(N)=\mathcal{N}(0,b^2)$$

$$\therefore \hat{b}_{ML}=\max_b\left(p_{s/b=B}(S)\right)$$

$$=\max_b\left(\prod_{i=1}^{N}p_{nb}(S_i-a)\right)$$

$$=\max_b\left(\frac{1}{(\sqrt{2\pi B^2})^N}\exp\left[-\frac{1}{2}\frac{\sum(S_i-a)^2}{B^2}\right]\right)$$

$$\therefore \frac{\partial}{\partial B}\left(\ln p_{s/b=B}(S)\right)=0$$

$$\frac{\partial}{\partial B}\left[-\frac{N}{2}\ln(2\pi B^2)-\frac{1}{2}\left[\frac{\sum(S_i-a)^2}{B^2}\right]\right]=0$$

$$-\frac{N}{B}+\frac{1}{B^3}\sum(S_i-a)^2=0$$

or $$\boxed{B^2=\frac{1}{N}\sum(S_i-a)^2}$$

Q-5 Find $\hat{a}_{MSE}$ if $r=\lambda\ln a+n$

$$p_\lambda(\lambda)=\frac{1}{2}\delta(\lambda)+\frac{1}{2}\delta(\lambda-1)$$

$$p_n(N)=\mathcal{N}(0,1);\ p_a(A)=\mathcal{N}(1,1)$$

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$$p_{r/a=A}(R) = \int_{-\infty}^{\infty} p_n(R-\lambda \ln A)\, p_\lambda(\lambda)\, d\lambda$$ Using convolution formula

$$= \frac{1}{2} p_n(R) + \frac{1}{2} p_n(R - \ln A)$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} e^{-R^2/2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(R-\ln A)^2}{2}}\right]$$

$$\hat{a}_{ML}(R) = \max_a \left(p_{r/a=A}(R)\right)$$

max at $R = \ln A$

or $\boxed{\hat{A}_{ML}(R) = e^R}$

Q6 $r = \sqrt{a} + n$ $\hat{a}_{MAP}$?

$p_a(A) = e^{-A} u(A)$, $p_n(N) = N(0,1)$

$$\Rightarrow \hat{a}_{MAP} = \max_a \left\{ p_{a/r=R}(A) \right\}$$

$$p_{r/a=A}(R) = p_n(R - \sqrt{A}) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(R-\sqrt{a})^2\right\}$$

$$p_r(R) = \int_{-\infty}^{\infty} p_{r/a=A}(R) \cdot p_a(A)\, dA$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(R-\sqrt{a})^2\right\} \cdot e^{-A}\, dA$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{R^2}{2} - \frac{3A}{2} + R\sqrt{a}\right\} dA$$

$$p_{a/r=R}(A) = \max_a \left\{ \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(R-\sqrt{A})^2\right) \exp(-A)}{p_r(R)} \right\}$$

max only if

$$(R-\sqrt{A})^2 + 2A = 0$$

$$\therefore \sqrt{A} = \frac{2R \pm \sqrt{4R^2 - 12R^2}}{6}$$

$$\hat{A}_{MAP} = R^2\left(\frac{1 \pm j\sqrt{2}}{3}\right)^2$$

Q7 $r = \lambda a + n$, find $\hat{a}_{MMSE}$, $\hat{a}_{MAP}$

$$p_\lambda(\lambda) = U(\lambda) - U(\lambda-1)$$

$$p_n(N) = e^{-N} U(N) \quad \& \quad p_a(A) = \frac{1}{2}\delta(A) + \frac{1}{2}\delta(A-1)$$

$$\Rightarrow \hat{a}_{MMSE}(R) = \int_{-\infty}^{\infty} A \, p_{a/r=R}(A)\, dA$$

$$p_{r/a=A}(R) = \int_{-\infty}^{\infty} p_n(R-\lambda A)\, p_\lambda(\lambda)\, d\lambda \quad \text{using convolution property}$$

$$= \int_0^1 p_n(R-\lambda A)\, d\lambda$$

$$= \int_0^1 e^{-(R-\lambda A)}\, d\lambda = \frac{e^{-(R-A)} - e^{-R}}{A}$$

$$p_r(R) = \int_{-\infty}^{\infty} p_{r/a=A}(R) \cdot p_a(A)\, dA$$

$$= e^{-R}\left(\frac{e^{A}-1}{A}\right)\cdot\frac{1}{2}\delta(A) + \frac{1}{2}e^{-R}\left(\frac{e^{A}-1}{A}\right)\delta(A-1)$$

$$= 0 + \frac{1}{2}e^{-R}(e^{1}-1)$$

$$\therefore p_{a/r=R}(A) = \frac{e^{-R}\left(\frac{e^{A}-1}{A}\right)\frac{1}{2}\delta(A) + e^{-R}\left(\frac{e^{A}-1}{A}\right)\frac{1}{2}\delta(A-1)}{\frac{1}{2}e^{-R}(e^{1}-1)}$$

$$\hat{a}_{MAP} = \max\left(p_{a/r=R}(A)\right)$$

$$\boxed{\hat{a}_{MAP} = 1}$$

$$\boxed{A=1}$$

$$\boxed{\cancel{\hat{a}_{MAP}}}$$

$$\hat{a}_{MMSE} = \int_{-\infty}^{\infty} A \cdot p_{a/r=R}(A)\, dA = \int_{-\infty}^{\infty} A\, dA = \underline{\text{not realizable}}$$

Q8 $\Pr\{r \text{ events}/a\} = {}^nC_r\, a^r (1-a)^{n-r}$, $r = 0, 1, \ldots n$

(i) $\hat{a}_{ML}$ & $\mathrm{var}(\hat{a}_{ML})$

(ii) Is it efficient

$\Rightarrow$ (i) $\hat{a}_{ML}(R) = \max_{a} \{ p_{r|a=A}(R) \}$

$= \max_{a} \{ {}^{n}C_R A^R (1-A)^{n-R} \}$

Taking derivative w.r.t A & equating to 0

$\frac{\partial}{\partial A} \{ p_{r|a=A}(R) \} = 0$

$${}^{n}C_R \, R A^{R-1} (1-A)^{n-R} - {}^{n}C_R A^R (n-R)(1-A)^{n-R-1} = 0$$

$$\frac{R}{A} = \frac{n-R}{1-A}$$

or $\frac{1-A}{A} = \frac{n-R}{R}$

or $\frac{1}{A} - 1 = \frac{n}{R} - 1$

or $\boxed{A = R/n}$

$\therefore \boxed{\hat{a}_{ML}(R) = R/n}$

$\text{var}\{\hat{a}_{ML}(R)\} = E\{(\hat{a}_{ML}(R) - E[\hat{a}_{ML}(R)])^2\}$

$= E[\hat{a}_{ML}(R)^2] - E[\hat{a}_{ML}(R)]^2$

$= \frac{1}{N^2} \text{var}(R)$

Since R is a Binomial RV

$\therefore \text{var}(R) = n(A)(1-A)$

$\therefore \text{var}(\hat{a}_{ML}(R)) = n \frac{A(1-A)}{n^2} = \boxed{\frac{1}{n} A(1-A)}$

(ii) as $N \to \infty$

$\text{var}(\hat{a}_{ML}(R)) \to 0$

$\therefore$ It is an efficient estimator

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Q9 $r = ab+n$, $a = N(0, \sigma_a^2)$, $b = N(0, \sigma_b^2)$, $n = N(0, \sigma_n^2)$

(i) $\hat{a}_{MAP}$

(ii) is this equivalent to simultaneously finding $\hat{a}_{map}, \hat{b}_{map}$

(iii) $r = a + \sum_{i=1}^{K} b_i + n$, $b_i = N(0, \sigma_{b_i}^2)$

a) $\hat{a}_{MAP}$

b) Is this equivalent to find $\hat{a}_{map}$, $\hat{b}_{iMAP}$

$\Rightarrow$ $\hat{a}_{MAP} = \max_a \{ p_{a/r=R}(A) \}$

$$p_{r/a=A}(R) = N(0, A^2\sigma_b^2 + \sigma_n^2))$$

$$p_r(R) = \int_{-\infty}^{\infty} p_{r/a=A}(R)\, p_a(A)\, dA$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,(A^2\sigma_b^2+\sigma_n^2)} \exp\left\{ -\frac{R^2}{2(\sigma_b^2 A^2 + \sigma_n^2)} \right\} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_a^2} \exp\left(-\frac{A^2}{2\sigma_a^2} \right) dA$$

$$p_{a/r=R}(A) = \frac{p_{r/a=A}(R) \cdot p_a(A)}{p_r(R)}$$

$\hat{a}_{MAP} = \max \{ p_{a/r=R}(A) \}$, denominator can be neglected as it is not a function of A

$$= \max \{ p_{r/a=A}(R) \cdot p_a(A) \}$$

$$= \max_a \{ N(0, A^2\sigma_b^2 + \sigma_n^2) \cdot N(0, \sigma_a^2) \}$$

$$= \max_a \left\{ \frac{1}{2\pi\,(\sigma_a^2\,(A^2\sigma_b^2+\sigma_n^2))^{1/2}} \exp\left\{ -\frac{A^2}{2\sigma_A^2} - \frac{R^2}{2(A^2\sigma_b^2+\sigma_n^2)} \right\} \right]$$

maximum only if exponent is 0

$$\therefore \quad A^2(A^2\sigma_b^2 + \sigma_n^2) + R^2\sigma_A^2 = 0$$

$$\therefore \quad A^2 = \frac{-\sigma_n^2 \pm \sqrt{\sigma_n^4 - 4R^2\sigma_b^2\sigma_A^2}}{2\sigma_b^2}$$

$$\hat{a}_{MAP}(R) = (\ A^2\ )^{1/2}$$

b) let $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$

$$p_{r/\theta}(R) = N(ab, \sigma_n^2)$$

$$\hat{\theta}_{MAP} = \max_{\theta} \{ p_{\theta/r=R}(\theta) \}$$

$$= \max_{\theta} \{ N(ab, \sigma_n^2) . N(0, \sigma_a^2) . N(0, \sigma_b^2) \}$$

maximum value when exponent is min

$$\therefore \frac{(R-AB)^2}{2\sigma_n^2} + \frac{A^2}{2\sigma_A^2} + \frac{B^2}{2\sigma_B^2} = 0$$

derivative w.r.t. $A = 0$

$$\frac{(R-AB)B}{\sigma_n^2} - \frac{A}{\sigma_A^2} = 0$$

w.r.t. B $\quad \frac{(R-AB)A}{\sigma_n^2} - \frac{B}{\sigma_B^2} = 0$

$$R - AB = \frac{A}{B}\frac{\sigma_n^2}{\sigma_A^2} \quad \& \quad R - AB = \frac{B}{A}\frac{\sigma_n^2}{\sigma_B^2}$$

Thus it is diffent than estimating only $\hat{a}_{map}$

c) $\quad r = a + \sum_{i=1}^{K} b_i + n$

$$p_{r/a=A}(R) = N(A, K\sigma_b^2 + \sigma_n^2)$$

$$p_{a/r=R}(A) = \frac{p_{r/a=A}(R)\, p_a(A)}{p_r(R)}$$

$$\hat{a}_{MAP} = \max \{ p_{r/a=A}(R) . p_a(A) \}$$

$$= \frac{1}{2\pi} \frac{1}{(K\sigma_b^2 + \sigma_n^2)^{1/2} (\sigma_a^2)^{1/2}} \exp\left\{ -\frac{1}{2}\frac{(R-A)^2}{K\sigma_b^2 + \sigma_n^2} - \frac{A^2}{2\sigma_a^2} \right\}$$

Taking derivative & equating to 0

$$\frac{\partial}{\partial A}\left\{ -\frac{1}{2}\frac{(R-A)^2}{(K\sigma_b^2 + \sigma_n^2)} - \frac{A^2}{2\sigma_a^2} \right\} = 0$$

$$\frac{R}{A} = \frac{K\sigma_b^2 + \sigma_n^2}{\sigma_a^2} + 1$$

$$\therefore \boxed{\hat{A}_{MAP} = R\left(\frac{\sigma_a^2}{K\sigma_b^2 + \sigma_n^2 + \sigma_a^2} \right)}$$

b) $p_{r/a,b}(R) = \mathcal{N}(a + \sum b_i, \sigma_n^2)$

let $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\hat{\theta}_{MAP} = \max_{A,B} \left(p_{r/A,B}(R) \cdot p_a(A) \cdot p_b(B) \right)$$

$$= \max \left\{ \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left\{ -\frac{(R - A - \sum B_i)^2}{2\sigma_n^2} \right. \cdot \frac{1}{\sqrt{2\pi\sigma_a^2}} \exp\left(-\frac{A^2}{2\sigma_a^2}\right) \cdot \frac{1}{(2\pi)^{K/2} (\sigma_b^2)^{K/2}} \exp\left\{ -\frac{\sum B_i^2}{2\sigma_B^2} \right\} \right\}$$

Taking derivative with respect to A & B_i & equating to 0

$$\frac{R - A - \sum B_i}{\sigma_n^2} = \frac{A}{\sigma_a^2}$$

$$\frac{R - A - \sum B_i}{\sigma_n^2} = \frac{B}{\sigma_B^2}$$

for K = 2

$$\begin{bmatrix} \frac{1}{\sigma_n^2} + \frac{1}{\sigma_a^2} & \frac{1}{\sigma_n^2} & \frac{1}{\sigma_n^2} \\ \frac{1}{\sigma_n^2} & \frac{1}{\sigma_n^2} + \frac{1}{\sigma_b^2} & \frac{1}{\sigma_n^2} \\ \frac{1}{\sigma_n^2} & \frac{1}{\sigma_n^2} & \frac{1}{\sigma_n^2} + \frac{1}{\sigma_b^2} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ B_2 \end{bmatrix} = \frac{R}{\sigma_n^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for K = 1

$$\underbrace{\begin{pmatrix} \frac{1}{\sigma_n^2} + \frac{1}{\sigma_a^2} & 1/\sigma_n^2 \\ \frac{1}{\sigma_n^2} & \frac{1}{\sigma_n^2} + \frac{1}{\sigma_b^2} \end{pmatrix}}_{P} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \frac{R}{\sigma_n^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A_1 = P^{-1} \frac{R}{\sigma_n^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{R}{\sigma_n^2} P^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma_b^2}\right)\left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma_a^2}\right) - \frac{1}{\sigma_n^4}} \begin{bmatrix} \frac{1}{\sigma_n^2} + \frac{1}{\sigma_a^2} & -\frac{1}{\sigma_n^2} \\ -\frac{1}{\sigma_n^2} & \frac{1}{\sigma_n^2} + \frac{1}{\sigma_b^2} \end{bmatrix}$$

$$\therefore A_1 = \frac{R}{\sigma_n^2} \frac{1}{\left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma_a^2}\right)^2 - \frac{1}{\sigma_n^4}} \left[\frac{1}{\sigma_n^2} + \frac{1}{\sigma_b^2} \quad -\frac{1}{\sigma_n^2} \right] = R \left(\frac{\sigma_a^2}{\sigma_a^2 + \sigma_b^2 + \sigma_n^2} \right)$$

$\therefore$ It is same

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Q10 $p_{r/a_1,a_2}(R/A_1, A_2) = (2\pi A_2)^{-1/2} \exp\left\{-\frac{(R-A_1)^2}{2A_2}\right\}$

(i) Find joint ML estimate of A_1 & A_2 using n obs^n

(ii) Are they biased

(iii) Are they coupled

(iv) Find error cov^n matrix?

$\Rightarrow$ $\overline{a_1, a_2}_M = \max_{a_1, a_2} \{p_{r/a_1,a_2}\}$

Differentiate w.r.t to A_1 & A_2 & equate to 0

$$\frac{\partial}{\partial A_1}\{\ln(p_{r/a_1,a_2})\} = \frac{\partial}{\partial A_1}\left[-\frac{N}{2}\ln(2\pi A_2) - \frac{\Sigma(R-A_1)^2}{2A_2}\right] = 0$$

$$\Sigma(R-A_1) = 0$$

$$\boxed{A_1 = \frac{1}{N}\Sigma R_i}$$

$$\frac{\partial}{\partial A_2}\left[-\frac{N}{2}\ln(2\pi A_2) - \frac{\Sigma(R_i-A_1)^2}{2A_2}\right] = 0$$

$$-\frac{N}{A_2} + \frac{1}{A_2^2}\Sigma(R_i-A_1)^2 = 0$$

$$\therefore A_2 = \frac{1}{N}\Sigma(R_i-A_1)^2$$

(ii) $E[\hat{A}_{1ML}] = \frac{1}{N}\Sigma E[R_i] = E[R_i] = A$ Thus unbiased

$E[\hat{A}_{2ML}] = \frac{1}{N}\Sigma E(R_i-A)^2 = E(R_i-A)^2 = A_2$ Thus unbiased

(iii) Yes they are coupled

$\hat{A}_{2ML}$ requires $\hat{A}_{1ML}$

(iv) $\mathrm{Cov}[\hat{A}_{1ML}] = \frac{1}{N^2}\Sigma \mathrm{Cov}(R_i) = \boxed{\frac{A_2}{N}}$

$\mathrm{Cov}[\hat{A}_{2ML}] = \frac{1}{N^2}\left[\Sigma \mathrm{Cov}(R_i^2) + N\,\mathrm{Cov}(A_1^2) - 2A_1\Sigma \mathrm{Cov}(R_i)\right]$

$= \frac{1}{N^2}\left[\frac{A_2 + A_1^2}{N} + \ldots - 2A_1(NA_2)\right]$

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Q 11 Concider the biased estimate $\hat{a}(R)$ of non random parameter A

$$E[\hat{a}(R)] = A + B(A)$$

Prove $\operatorname{var}[\hat{a}(R)] \geq \dfrac{(1 + dB(A)/dA)^2}{E\left[\left[\dfrac{\partial \ln p_{r/a}(R|A)}{\partial A}\right]^2\right]}$

$$\Rightarrow \quad E[\hat{a}(R) - A] = B(A)$$

$$\int (\hat{a}(R) - A)\, p_{r/a=A}(R)\, dR = B(A)$$

Take derivative

$$-\int p_{r/a=A}(R)\,dR + \int (\hat{a}(R) - A) \frac{\partial}{\partial A} p_{r/a=A}(R)\,dR = B'(A)$$

$$1$$

But $\dfrac{\partial}{\partial A} p_{r/a=A}(R) = p_{r/a=A}(R) \dfrac{\partial}{\partial A} \ln\left(p_{r/a=A}(R)\right)$

$$\therefore \int (\hat{a}(R) - A) \sqrt{p_{r/a=A}(R)} \cdot \sqrt{p_{r/a=A}(R)}\, \frac{\partial}{\partial A} \ln\left(p_{r/a=A}(R)\right) dR = B'(A) + 1$$

Using Cauchy Schwarz inequality

$$\int f\bar{g} \leq \int f \cdot \int g$$

$$\therefore \; 1 + B'(A) \leq \int (\hat{a}(R) - A) \sqrt{p_{r/a=A}(R)}\, dR \cdot \int \sqrt{p_{r/a=A}(R)}\, \frac{\partial}{\partial A} \ln p_{r/a}(R)\, dR$$

$$(1 + B'(A))^2 \leq \operatorname{var}(\hat{a}(R)) \cdot E\left[\frac{\partial}{\partial A} \ln\left(p_{r/a=A}(R)\right)\right]$$

$$\therefore \operatorname{var}(\hat{a}(R)) \geq \frac{1 + B'(A)}{E\left\{\left[\frac{\partial}{\partial A} \ln\left(p_{r/A}(R)\right)\right]^2\right\}}$$

Hence proved

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Q3 verify

(i) $\frac{d A^T B}{dx} = \left(\frac{dA^T}{dx}\right)B + \left(\frac{dB^T}{dx}\right)A$, A + B are $m \times 1$ vectors

$\Rightarrow$ $A^T B = \sum_{i=1}^{m} a_i b_i$

vector derivative is $\frac{d}{d\underline{x}} = \left[\frac{d}{dx_1} \; \frac{d}{dx_2} \; \text{---} \; \frac{d}{dx_n}\right]^T$

$\therefore \frac{d}{d\underline{x}}(A^T B) = \left[\frac{d}{dx_1}\sum a_i b_i \;,\; \frac{d}{dx_2}\sum a_i b_i \; \text{---} \; \frac{d}{dx_n}\sum a_i b_i\right]^T$

$= \left[\sum \frac{da_i}{dx_1} b_i \quad \sum \frac{da_i}{dx_2} b_i \; \text{---} \; \sum \frac{da_i}{dx_n} b_i\right]^T$

$+ \left[\sum a_i \frac{db_i}{dx_1} \quad \sum a_i \frac{db_i}{dx_2} \; \text{---} \; \sum a_i \frac{db_i}{dx_n}\right]^T$

$= \left(\frac{d}{dx} A^T\right) B + \left(\frac{dB^T}{d\underline{x}}\right) A$

(ii) $\frac{d B^T x}{dx} = B$

$B^T x = \sum b_i x_i = b_1 x_1 + b_2 x_2 + \text{---} + b_n x_n$

$\therefore \frac{d}{dx} B^T x = \left[\frac{d}{dx_1}\sum b_i x_i \quad \frac{d}{dx_2}\sum b_i x_i \; \text{---} \; \frac{d}{dx_n}\sum b_i x_i\right]^T$

$= [b_1 \quad b_2 \; \text{---} \; b_n]^T$

$= B$

Hence proved

(iii) $\frac{d}{dx}(x^T c) = c$

$x^T c = \sum x_i c_i = x_1 c_1 + x_2 c_2 + \text{---} + x_n c_n$

$\frac{d}{dx}(x^T c) = \left[\frac{d}{dx_1}\sum x_i c_i \quad \frac{d}{dx_2}\sum x_i c_i \; \text{---} \; \frac{d}{dx}\sum x_i c_i\right]^T$

$= [c_1 \quad c_2 \; \text{---} \; c_n]^T$

$= \boxed{C}$ Hence proved

iv) $\frac{dx^T}{dx} = I$

$x^T = [x_1 \quad x_2 \quad \cdots \quad x_n]$

$$\frac{d}{dx}x^T = \begin{bmatrix} \frac{dx_1}{dx_1} & \frac{dx_2}{dx_1} & \frac{dx_3}{dx_1} & \cdots & \frac{dx_n}{dx_1} \\ \vdots & & & & \vdots \\ \frac{dx_1}{dx_n} & \cdots & & & \frac{dx_n}{dx_n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & & & & 1 \\ 0 & & & & \end{bmatrix} = \boxed{I_{n\times n}}$$

(v) $Q = A^T(x) \Lambda A(x)$ Λ is symetric positive definite matrix then show

$A(x)$ is $m\times 1$,

a) $\frac{dQ}{dx} = 2\left(\frac{d}{dx}A^T(x)\right)\Lambda A(x)$

$\Rightarrow \frac{dQ}{dx} = \frac{d}{dx}\left(A^T(x)\Lambda A(x)\right)$

$= \frac{dA^T(x)}{dx}\Lambda A(x) + A^T(x)\Lambda\frac{d}{dx}A(x)$

But $\frac{d}{dx}C^Tx = \frac{d}{dx}x^TC$

$\therefore \frac{dQ}{dx} = 2\left(\frac{dA^T(x)}{dx}\Lambda A(x)\right)$ where $C = \Lambda A(x)$ & $\Lambda^T = \Lambda$

b) if $A(x) = Bx$ then $\frac{dQ}{dx} = 2B^T\Lambda Bx$

$\Rightarrow \frac{dA^T(x)}{dx} = \frac{d}{dx}(x^TB^T) = B^T$

$\therefore \frac{dQ}{dx} = 2B^T\Lambda Bx$

c) if $Q = x^T\Lambda x$, $\frac{dQ}{dx} = 2\Lambda x$

$\Rightarrow \frac{dQ}{dx} = \Lambda x + (x^T\Lambda)^T = \boxed{2\Lambda x}$

Q14 For general gaussian problem.

$$r_1 = H_1 a + n_1$$
$$r_2 = H_2 a + n_2 \quad , n_1 \,\&\, n_2 \text{ are independent}$$

show $\hat{a}$ base on $[r_1\ r_2]^T$ can be written as

$$\hat{a} = A\hat{a}_1 + B\hat{a}_2$$

when $\hat{a}_1 = E(a/R_1)$, $\hat{a}_2 = E(a/R_2)$

$$A+B=I \ \& \ Va^{-1}=0$$

$$\Rightarrow \quad p_{R/a=A}(R) = p_{r_1/a=A}(R_1) \cdot p_{r_2/a=A}(R_2)$$

when $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}^T$

$$\therefore \hat{a}_{ML} = \max_a \left\{ p_{R/a=A}(R) \right\}$$

$$= \max_a \left\{ \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{ -\frac{(R_1 - H_1 A)^2}{2\sigma_1^2} \right\} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{ -\frac{(R_2 - H_2 A)^2}{2\sigma_2^2} \right\} \right\}$$

Taking derivative both sides & equating to 0

$$\frac{(R_1 - H_1 A)H_1}{\sigma_1^2} + \frac{(R_2 - H_2 A)H_2}{\sigma_2^2} = 0$$

$$\text{or} \quad A\left(\frac{H_1^2}{\sigma_1^2} + \frac{H_2^2}{\sigma_2^2} \right) = \frac{R_1 H_1}{\sigma_1^2} + \frac{R_2 H_2}{\sigma_2^2}$$

$$\text{or } A = \frac{R_1}{H_1}\left(\frac{H_1^2 \sigma_2^2}{H_1^2\sigma_1^2 + H_2^2\sigma_2^2} \right) + \frac{R_2}{H_2}\left(\frac{H_2^2\sigma_2^2}{H_1^2\sigma_1^2 + H_2^2\sigma_2^2} \right)$$

Also $E\{A/r_1\} = \frac{R_1}{H_1} = \hat{a}_1$

& $E\{A/r_2\} = R/H_2 = \hat{a}_2$

$$\therefore \boxed{\begin{gathered} \hat{A}_{ML} = A\hat{a}_1 + B\hat{a}_2 \\ \text{with } A+B = I \end{gathered}}$$

Hence proved

Q15 A find ML estimate of $A = [C_1\ C_2]^T$ for observations

$$r_i = \sqrt{C_1}\, n_i + C_2 \quad ; i = 1, 2 \ldots N$$

$$n_i = N(0,1)$$

a) $\hat{A}_{ML}$

$\Rightarrow$ ML estimate of A is

$$\max_{A_1 A_2} \{ p_{R/A_1A_2}(R) \} = \prod_{i=1}^{N} p_{r_i/A_1A_2}(R_i)$$

$$= \frac{1}{(2\pi C_1)^{N/2}} \exp\left\{ -\frac{1}{2C_1} \Sigma (R_i - C_2)^2 \right\}$$

Taking log & derivative w.r.t A & equating to 0

$$\frac{\partial}{\partial A}\left[-\frac{N}{2} \ln(2\pi C_1) - \frac{1}{2C_1} \Sigma(R_i - C_2)^2 \right] = 0$$

$$\Rightarrow \begin{bmatrix} -\frac{N}{2C_1} + \frac{1}{2C_1^2} \Sigma(R_i - C_2)^2 \\ \frac{1}{C_1} \Sigma(R_i - C_2) \end{bmatrix} = 0$$

$$\therefore \boxed{\begin{aligned} \hat{C}_{1_{ML}} &= \frac{1}{N} \Sigma(R_i - \hat{C}_{2_{ML}})^2 \\ \hat{C}_{2_{ML}} &= \frac{1}{N} \Sigma R_i \end{aligned}}$$

b) ML estimate has a sequential form as

$$\hat{C}_{2ML} = \frac{1}{N} \Sigma R_i$$

at any instant $N+1$ $\qquad \hat{C}_{2,N+1} = \frac{(1+N)\hat{X}_{2,N} + r_{N+1}}{N+1}$

$$\therefore \boxed{C_{2,i} = \frac{i-1}{i} C_{2,i-1} + \frac{1}{i} r_i}$$ Because $N = i-1$

Similarly $\hat{C}_{1,i} = \frac{(i-1)}{i} \hat{C}_{1,i-1} + \frac{1}{i}(r_i - \hat{C}_{2,i})^2$