On the Ergodic Sum-Capacity of Decentralized Multiple Access Channels

Nitish Mital*, Kamal Singh†, Sibi Raj B Pillai†, *Member, IEEE*

***Abstract*—We consider a fast fading AWGN multiple-access channel (MAC) with full receiver CSI and distributed CSI at the transmitters. The objective is to evaluate the ergodic sum-capacity of this decentralized model, under identical average powers and channel statistics across users. While an optimal water-filling solution can be found for centralized MACs with full CSI at all terminals, such an explicit solution is not considered feasible in distributed CSI models. Our main contribution is an upper-bound on the ergodic sum-capacity when each transmitter is aware only of its own fading coefficients. Interestingly, our techniques also suggest an appropriate lower bound. These bounds are shown to be very close to each other, suggesting the tight nature of the results.**

I. Introduction

The multiple access channel (MAC) is an important communication model comprising multiple transmitters and a common receiver. The uplink channel in a cellular network is an example. The availability of channel state information (CSI) at the receiver and the transmitters plays an important role in computing the capacity region of a fading MAC. We consider a decentralized system where each transmitter knows only its own fading coefficients [3], [5]. We assume identical fading statistics and the same average power constraint across users, this is referred to as *identical users*. The receiver has full CSI, and we assume a fast fading model where the channel varies IID (independent and identically distributed) across time.

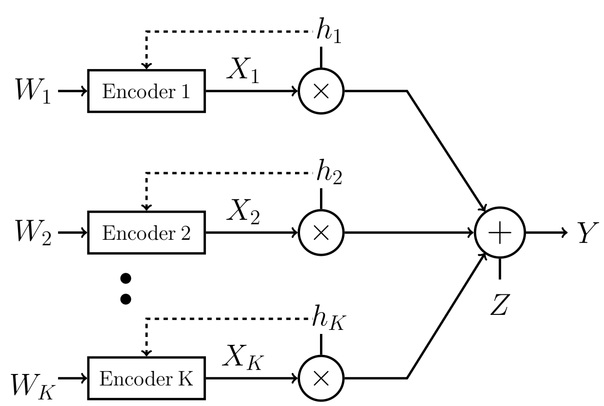

Fig. 1. Decentralized Gaussian fading MAC

*Nitish Mital was with the Department of Electrical Engineering at IIT Bombay, Mumbai, INDIA. Currently, he works in Cognito Tech Solutions LLP, Chandigarh, INDIA-160036 (e-mail: nitish.mital@cognitotec.com).
†Kamal Singh and †Sibi Raj B Pillai are with the Department of Electrical Engineering at IIT Bombay, Mumbai, INDIA-400076 (e-mail: {kamalsingh, bsraj}@ee.iitb.ac.in).

In a fast fading AWGN MAC, with full CSI at the receiver and partial CSI at transmitters (CSIT), it is well known that Gaussian codebooks with optimal power control can achieve the capacity region [3], [4]. The power control problem for the decentralized CSIT was first considered in [5], and a heuristic ON-OFF power control was proposed to circumvent the lack of analytical solutions for the optimum. For numerical comparisons, the full CSIT case of [7] is taken as an upper bound. In this paper, we provide an improved upper bound on the ergodic sum-capacity of the decentralized CSI MAC, which in turn, also suggests a better lower bound, improving the existing results for distributed power control [5], [6]. Numerical results suggest that the resulting bounds are very close for Rayleigh fading models. Notice that while the IID fading assumption makes the analytical treatment simpler, our results here are also applicable to ergodic fading models where coding can be performed over sufficiently long intervals. See [1], [2], [3] for further motivations and applications.

The rest of the paper is organized as follows. Section II introduces the distributed fading MAC model, along with some definitions and notations. The upper and lower bounds to the ergodic sum-capacity are derived in Sections III and IV respectively. Numerical results for Rayleigh distribution are shown in Section V. Section VI concludes the paper.

II. System Model

The K-user discrete-time Gaussian multiple-access channel is described by the received sample at instant n as,

$$Y(n) = \sum_{i=1}^{K} H_i(n)X_i(n) + Z(n),$$

where $X_i(n)$ and $H_i(n)$ are the transmitted signal and the multiplicative fading coefficient for user-i. The noise process $Z(n)$ is assumed to be normalized IID complex Gaussian, independent of both $X_i(n)$ and $H_i(n)$. The fading process $H_i(n)$ varies IID over time and across users.

In our model, the channel state information available at the i-th encoder is the respective fading state h_i at that instant. Using this, an instantaneous transmit power of $P_i(h_i)$ is chosen. Let P_i^{av} be the average transmit power constraint at user-i. Thus, $\mathbb{E}\, P_i(H_i) \leq P_i^{\text{av}}, 1 \leq i \leq K$. We further assume that the full state vector $(h_1, \cdots, h_K)$ is available at the receiver at all instants.

We will write the vector $(u_1, \cdots, u_K)$ as u^K, and $(u_j, \cdots, u_K)$ as u_j^K. Random variables are in capitals and their realizations are denoted by corresponding lower case letters. For example, $X^K \sim p(x^K)$ denotes a random vector X^K

according to the probability distribution $p(x^K)$, and $p(x^K|v^K)$ denotes the conditional distribution given v^K. We will denote the indicator function by $\mathbb{1}_{\{\cdot\}}$, and $\max\{0,x\}$ by $(x)^+$.

Definition 1. *The ergodic sum-capacity is given by [5]*

$$C_E^* = \max_{P_1(h_1),\ldots,P_K(h_K)} \mathbb{E}\log\left(1+\sum_{i=1}^{K}|H_i|^2 P_i(H_i)\right), \quad (1)$$
$$s.t. \qquad \mathbb{E}\,P_i(H_i) \le P_i^{\text{av}},\ 1\le i\le K.$$

Under full CSIR and decentralized CSI at the respective transmitters, the sum-capacity in (1) depends only on the fading magnitudes. Thus, we denote $|H_i|^2$ as V_i and write $P_i(H_i)$ as $P_i(V_i)$, $1\le i\le K$. For the *identical users* case, due to the convexity of the objective function (see [5]), we can take $P_i(v) = P_j(v) = P(v)$. Let $\Psi(v_j^K)$ be the cumulative distribution function (CDF) of V_j^K, and we take $P_i^{\text{av}} = P^{\text{av}}$, $1\le i\le K$.

Proposition 2. *For identical users, a necessary and sufficient condition for the power control $Q(v)$ to be optimal is*

$$c(v) := v\int_{v_2}\cdots\int_{v_K}\frac{d\Psi(v_2^K)}{1+vQ(v)+\sum_{i=2}^{K}v_iQ(v_i)} = constant$$

Proof: From (1), using KKT conditions, the optimal power $Q(v)$, whenever non-zero, satisfies

$$v\int_{v_2}\cdots\int_{v_K}\frac{d\Psi(v_2^K)}{1+vQ(v)+\sum_{i=2}^{K}v_iQ(v_i)} = \mu,$$

where μ is a constant, known as the Lagrange multiplier. The LHS is the function $c(v)$ itself. Thus, $c(v)=\mu$ (a constant) for any optimal $Q(v)$. ∎

It is difficult to analytically solve the KKT conditions and find the optimum. However, we can make the following structural observations.

Lemma 3. *There exists positive constants C_1 and C_2 such that the optimal power control $Q(v)$ obeys the following boundary conditions:*

1) $\lim_{v\to\lambda^+}\frac{dQ(v)}{dv} = \frac{C_1}{v^2}$.
2) $\lim_{v\to\infty}\frac{dQ(v)}{dv} = \frac{C_2}{v^2}$.

where λ is the channel threshold above which the power allocation is non-zero.

Proof: The proof is given in Appendix A. ∎

We now propose a set of upper and lower bounds to the achievable rates, in the coming sections.

III. Upper Bound on Ergodic Sum-capacity

Theorem 4. *For any power allocation $Q(v)$ meeting the constraint $\int Q(v)\,d\Psi(v)\le P^{\text{av}}$, the ergodic sum-capacity C_E^* of the decentralized Gaussian MAC obeys*

$$C_E^* \le \mathbb{E}\log_2 Q_y + \max_{P(v)}\mathbb{E}\left(\frac{P_y}{Q_y}-1\right)\cdot\log_2 e, \quad (2)$$

where $Q_y = 1+\sum_{i=1}^{K}V_iQ(V_i)$ and $P_y = 1+\sum_{i=1}^{K}V_iP(V_i)$.

Proof: We know from the converse to MAC coding theorem [1] that the sum-rate is bounded by the mutual information $I(X^K;Y|V^K)$. Now

$$\begin{aligned} I(X^K;Y|V^K) &= h(Y|V^K) - h(Y|X^K,V^K)\\ &= \mathbb{E}\,h(Y|V^K) - h(Z)\\ &= \mathbb{E}\int f(y|V^K)\log_2\frac{1}{f(y|V^K)}\,dy - h(Z). \quad (3)\end{aligned}$$

Notice that $f(y|v^K)$ is the conditional distribution function of the channel output Y given the state vector $V^K=v^K$, and the expectation is over V^K. The conditional differential entropy $h(Y|v^K)$ is

$$\begin{aligned} &\int f(y|v^K)\log_2\frac{1}{f(y|v^K)}dy\\ &\quad= \int f(y|v^K)\log_2\frac{g(y|v^K)}{f(y|v^K)\cdot g(y|v^K)}dy\\ &\quad= \int f(y|v^K)\log_2\frac{1}{g(y|v^K)}dy - D(f\|g|v^K)\\ &\quad\le \int f(y|v^K)\log_2\frac{1}{g(y|v^K)}dy, \quad (4)\end{aligned}$$

where $g(y|v^K)$ is any valid conditional distribution and $D(f\|g|v^K)\ge 0$ is the divergence between $f(y|v^K)$ and $g(y|v^K)$. For some power control scheme $Q(v)$ satisfying the average power constraints, let us take

$$g(y|v^K) = \frac{1}{\pi(1+\sum_{i=1}^{K}v_iQ(v_i))}\exp\left(\frac{-|y|^2}{1+\sum_{i=1}^{K}v_iQ(v_i)}\right).$$

For convenience, we denote $Q_y = 1+\sum_{i=1}^{K}v_iQ(v_i)$ and $P_y = 1+\sum_{i=1}^{K}v_iP(v_i)$. Now

$$\begin{aligned} &\int f(y|v^K)\ \log_2\frac{1}{g(y|v^K)}\,dy\\ &\quad\le \log_2\pi\,Q_y + \frac{\log_2 e}{Q_y}\int|y|^2 f(y|v^K)\,dy\\ &\quad= \log_2\pi\,Q_y + \frac{P_y}{Q_y}\cdot\log_2 e. \quad (5)\end{aligned}$$

Combining (3), (4) and (5), we get

$$\begin{aligned} I(X^K;Y|V^K) &\le \mathbb{E}\log_2\pi\,Q_y + \mathbb{E}\frac{P_y}{Q_y}\cdot\log_2 e - h(Z)\\ &= \mathbb{E}\log_2\pi\,Q_y + \mathbb{E}\frac{P_y}{Q_y}\cdot\log_2 e - \log_2\pi e\\ &= \mathbb{E}\log_2 Q_y + \mathbb{E}\left(\frac{P_y}{Q_y}-1\right)\cdot\log_2 e.\end{aligned}$$

Thus, the ergodic sum-capacity is bounded by

$$C_E^* \le \mathbb{E}\log_2 Q_y + \max_{P(v)}\mathbb{E}\left(\frac{P_y}{Q_y}-1\right)\cdot\log_2 e,$$

for any power control scheme $Q(v)$. ∎

Remark 5. *The first expectation term in* (2) *is the ergodic sum-rate for the power control $Q(v)$ and hence can serve as a lower bound to the ergodic sum-capacity.*

The following lemma simplifies (2) further.

Lemma 6. *For* $Q_y = 1 + \sum_{i=1}^{K} V_i Q(V_i)$ *and* $P_y = 1 + \sum_{i=1}^{K} V_i P(V_i)$,

$$\max_{P(v)} \mathbb{E} \frac{P_y}{Q_y} = K \cdot \max_{v_0} \gamma \int_{v_0}^{\infty} c(v)\, d\Psi(v) + \mathbb{E} \frac{1}{Q_y}, \quad (6)$$

where $c(v) = v \int_{v_2} \cdots \int_{v_K} \frac{d\Psi(v_2^K)}{Q_y}$.

Proof: Before proving, observe that RHS of (6) is a single parameter optimization, which is easily tractable using numerical techniques. The proof details are provided in Appendix B. ■

IV. Lower Bound on Ergodic Sum-Capacity

Using the structural properties stated in Lemma 3, we now propose efficient power control schemes. The following recipe seems to yield excellent performance, not only outperforming the existing results based on heuristic power control schemes employed in [5], [6] etc., but also achieving rates very close to the outer bound in (2). Let us take,

$$Q(v) = \left(\frac{1}{\lambda} - \frac{1}{v}\right)^{+} G(v - \lambda),$$

where the function $G(x) = 1 + \alpha \exp(-\beta/x)$ and λ is a parameter such that $\int Q(v)\, d\Psi(v) = P^{\text{av}}$. Thus, the scheme is parameterized in terms of α and β. The derivative for $v > \lambda$ is as follows:

$$\frac{dQ(v)}{dv} = \frac{G(v-\lambda)}{v^2} + G^{'}(v-\lambda)\left(\frac{1}{\lambda} - \frac{1}{v}\right).$$

The boundary conditions for this power control (see Lemma 3) are:

- $\lim_{v \to \lambda^+} \frac{dQ(v)}{dv} = \frac{G(0)}{v^2}$.
- $\lim_{v \to \infty} \frac{dQ(v)}{dv} = \frac{G(\infty)}{v^2}$.

By optimizing the parameters α and β, we can try to maximize the sum-rate. Then

$$C_E^* \geq \max_{\alpha, \beta} \mathbb{E} \log_2 Q_y. \quad (7)$$

V. Numerical Results

We show the utility of the proposed power control scheme by numerically comparing the upper bounds and lower bounds computed using (2) and (7) respectively. We assume that all channels encounter independent normalized Rayleigh fading, and each user has the same average transmit power constraint, i.e. $\mathbb{E}\, Q(V_i) \leq P^{\text{av}}$. Notice that the fading gain V_i is an exponential random variable with unit mean. The bounds achieved by this class of functions are found to be very tight as shown in Figure 2, suggesting that the performance of this power scheme is very close to optimal.

VI. Conclusions

We have obtained improved bounds (upper and lower) to ergodic sum-capacity of decentralized Gaussian MACs. The proposed power allocation was found to have near-optimal performance. While we have assumed identical fading statistics and average powers, the bounding techniques can be extended to more general models as well.

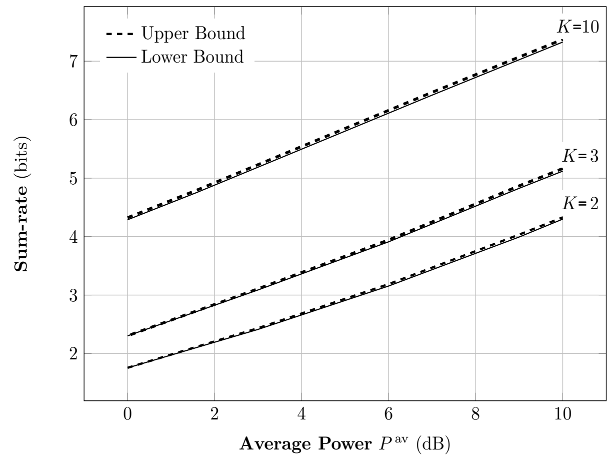

Fig. 2. Bounds for $K = 2, 3$ and 10 user decentralized MAC. Dashed and solid curves represent the upper bound and the lower bound to the achievable rates. Notice that the power is in log scale.

References

[1] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, 2011.
[2] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge University Press, 2005.
[3] A. Das and P. Narayan, "Capacities of time-varying multiple-access channels with side information," *IEEE Transactions on Information Theory*, vol. 48, no. 1, pp. 4–25, Jan. 2002.
[4] D. N. C. Tse and S. V. Hanly, "Multiaccess fading channels. i. polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Transactions on Information Theory*, vol. 44, no. 7, pp. 2796–2815, Nov. 1998.
[5] S. Shamai and E. Telatar, "Some information theoretic aspects of decentralized power control in multiple access fading channels," in *Information Theory and Networking Workshop, IEEE*, June 1999.
[6] K. Singh and S. R. B. Pillai, "Decentralized power adaptation in ergodic fading multiple access channels," in *Communications (NCC), 2015 Twenty First National Conference on*, pp. 1–6, Feb. 2015.
[7] R. Knopp and P. Humblet, "Information capacity and power control in single-cell multiuser communications," in *ICC '95 Seattle*, vol. 1, pp. 331–335, June 1995.
[8] D. Bertsimas and J. N. Tsitsiklis, *Introduction to Linear Optimization*. Athena Scientific, 1997.

Appendix A
Optimal Power Boundary conditions

The $c(v)$ function expands to

$$c(v) = v \frac{\Psi^{K-1}(\lambda)}{1 + vQ(v)} + v \sum_{n=2}^{K} \Psi^{K-n-1}(\lambda) \times \int_{v_2^n} \frac{d\Psi(v_2^n)}{1 + vQ(v) + \sum_{j=2}^{n} v_j Q(v_j)}.$$

Differentiating $c(v)$ for optimality (see Proposition 2), we get

$$\frac{dc(v)}{dv} = \frac{\Psi^{K-1}(\lambda)}{(1+vQ(v))^2}\left(1 - v^2 \frac{dQ(v)}{dv}\right) + \sum_{n=2}^{K} \Psi^{K-n-1}(\lambda) \times \left(\int_{v_2^n} \frac{d\Psi(v_2^n)}{1 + vQ(v) + \sum_{j=2}^{n} v_j Q(v_j)} - \left(vQ(v) + v^2 \frac{dQ(v)}{dv}\right) \times \int_{v_2^n} \frac{d\Psi(v_2^n)}{(1 + vQ(v) + \sum_{j=2}^{n} v_j Q(v_j))^2}\right) = 0. \quad (8)$$

Case 1: Near the cut-off λ, i.e. $v \to \lambda^+$, $1 + vQ(v) + \sum_{j=2}^{n} v_j Q(v_j) \approx 1 + \sum_{j=2}^{n} v_j Q(v_j)$. Thus, (8) simplifies to

$$\Psi^{K-1}(\lambda)\left(1 - v^2\frac{dQ(v)}{dv}\right) + \sum_{n=2}^{K}\Psi^{K-n-1}(\lambda) \times$$
$$\left(\int_{v_2^K}\frac{d\Psi(v_2^n)}{1+\sum_{j=2}^{n} v_j Q(v_j)} - \int_{v_2^n}\frac{v^2\frac{dQ(v)}{dv}\, d\Psi(v_2^n)}{(1+\sum_{j=2}^{n} v_j Q(v_j))^2}\right) = 0$$
$$\Rightarrow \qquad \lim_{v\to v_0^+}\frac{dQ(v)}{dv} = \frac{C_1}{v^2} \text{ with } C_1 > 0.$$

Case 2: When $v \to \infty$, $1 + vQ(v) + \sum_{j=2}^{n} v_j Q(v_j) \approx vQ(v)$. Thus,

$$\frac{\Psi^{K-1}(\lambda)}{(vQ(v))^2}\left(1 - v^2\frac{dQ(v)}{dv}\right) + \sum_{n=2}^{K}\Psi^{K-n-1}(\lambda) \times$$
$$\left(\frac{1}{vQ(v)}\int_{v_2^n} d\Psi(v_2^n) - \frac{vQ(v) + v^2\frac{dQ(v)}{dv}}{(vQ(v))^2}\int_{v_2^n} d\Psi(v_2^n)\right) = 0$$
$$\Rightarrow \qquad \lim_{v\to\infty}\frac{dQ(v)}{dv} = \frac{C_2}{v^2} \text{ with } C_2 > 0.$$

This completes the proof of Lemma 3.

Appendix B
Lemma 6 proof details

Consider the following optimization:

$$\begin{aligned}\max_{P(v)} \ \mathbb{E}\ \frac{P_y}{Q_y} &= \max_{P(v)}\ \mathbb{E}\ \frac{1+\sum_{i=1}^{K} V_i P(V_i)}{Q_y}\\ &= \max_{P(v)}\ \mathbb{E}\ \frac{\sum_{i=1}^{K} V_i P(V_i)}{Q_y} + \mathbb{E}\ \frac{1}{Q_y}\\ &= K\cdot\max_{P(v)}\ \mathbb{E}\ \frac{VP(V)}{Q_y} + \mathbb{E}\ \frac{1}{Q_y}, \qquad (9)\end{aligned}$$

where the last equality follows because the expectation of $V_i P(V_i)$ is same for all choices of i. Consider the first term in (9),

$$\mathbb{E}\ \frac{VP(V)}{Q_y} = \int_v P(v)\, d\Psi(v)\, v \int_{v_2}\cdots\int_{v_K}\frac{d\Psi(v_2^K)}{Q_y}.$$

With the substitution $d\Psi(v) = f(v)\,dv$ and $c(v) = v\int_{v_2}\cdots\int_{v_K}\frac{d\Psi(v_2^K)}{Q_y}$, we can rewrite the optimization problem as

$$\begin{aligned}&\max_{\widehat{P}(v)} \quad \int_v \widehat{P}(v)c(v)\,dv,\\ &s.t. \quad \int_v \widehat{P}(v)\,dv \le P^{\text{av}}, \qquad (10)\end{aligned}$$

where $\widehat{P}(v) = P(v)f(v)$. Notice that (10) is a Linear program in $\widehat{P}(v)$. Recently, it was shown in [6] that any optimal distributed power control for the fading MAC model is monotonically non-decreasing in the fading magnitude. Thus, we can add the monotonicity constraints to (10) as follows.

$$P(\hat{v}) \ge P(\tilde{v}),\ \forall\ \hat{v},\tilde{v} \in \mathbb{R}^+ \text{ such that } \hat{v} > \tilde{v}. \qquad (11)$$

The corner points of this modified linear programming problem are $P(v) = \gamma\cdot\mathbb{1}_{\{v>v_0\}}$ with v_0 being a free variable (non-negative) and the parameter γ determined from the average power constraint (see Appendix C). Thus,

$$\max_{P(v)}\ \mathbb{E}\ \frac{P_y}{Q_y} = K\cdot\max_{v_0}\ \gamma\int_{v_0}^{\infty} c(v)\ d\Psi(v) + \mathbb{E}\ \frac{1}{Q_y}. \qquad (12)$$

Appendix C
Corner Points of Modified Linear Program

The corner points of a linear Program are characterized by the relaxation of any one of the constraints and the rest of the constraints are met with strict equality (see [8]). The constraints in our linear program are listed in (10) and (11). Constraint in (10) is the average power condition and (11) are the monotonicity constraints. Relaxing any one of (11) and keeping (10) with strict equality will give the corner points of the form $P(v) = \gamma\cdot\mathbb{1}_{\{v>v_0\}}$ where v_0 is the channel coefficient at which the constraint is relaxed.

Thus, we restrict our search to functions $P(v) = \gamma\cdot\mathbb{1}_{\{v>v_0\}}$ with v_0 as the only free variable and parameter γ gets fixed according to the average power constraint as shown below:

$$\int_{v_0}^{\infty} P(v)\,d\Psi(v) = P^{\text{av}} \Rightarrow \gamma = \frac{P^{\text{av}}}{1-\Psi(v_0)}.$$