UCLA ANDERSON SCHOOL OF MANAGEMENT
Daniel Andrei, Derivative Markets 237D, Winter 2017

Assignment #2

Due date: Tuesday March 14, 2017 (Week 10)

1 Black-Scholes: Closed Form Solution vs. Monte-Carlo Simulation

The purpose of this exercise is to price a vanilla call option with Monte-Carlo simulation and compare the result with the closed-form formula. The simulation will also provide a confidence interval for the option price. We will understand how the length of this confidence interval changes as we increase the number of simulated paths.

The price of a stock today is $S_0 = 100$. Consider a European call option with maturity 3 months ($T = 1/4$) and strike price $K = 100$. We make the following parametric assumptions: $r = 0.05$, $\sigma = 0.2$, and $\delta = 0$.

For this exercise, reset the random number generator before each simulation run by `randn('seed',0)`.

a. Simulate and plot 5 paths for the stock price under the risk-neutral measure. Use 5 minutes time-increments (there are $8 \times 12 = 96$ increments each day and 90 days until maturity).

b. Find the Black-Scholes call option price.

c. Find the option price and its 95% confidence interval by Monte-Carlo simulation. *You do not have to simulate the entire paths here—since the option is European, you will have to simulate only the final price S_T.* Do this for 100; 1,000; 1,000,000; and 100,000,000 simulations. Discuss how the length of the confidence interval changes with the number of simulations. Compare the Monte-Carlo price with the Black-Scholes price.

2 Down-And-Out Put Option Price by Monte-Carlo Simulation

The purpose of this exercise is to price a down-and-out put option with Monte-Carlo simulation and to understand how the volatility of the underlying affects the proportion of crossed paths and consequently the price of the option.

The price of a stock today is $S_0 = 100$. Consider a down-and-out put option (an option that ceases to exist when the price of the underlying security hits a specific barrier price level) with maturity 3 months ($T = 1/4$), strike price $K = 95$, and a barrier level of $S_b = 75$. We make the following parametric assumptions: $r = 0.05$, $\sigma = 0.2$, and $\delta = 0$.

For this exercise, reset the random number generator before each simulation run by `randn('seed',0)`.

a. Simulate (under the risk-neutral measure) 1,000 paths for the stock price at 5 minute intervals (there are $8 \times 12 = 96$ increments each day and 90 days until maturity). *You have to simulate the entire paths here, since the option is now path-dependent.*

   How many of the paths are "out," i.e., when the underlying price crossed the barrier S_b? What is the price of the put option and its 95% confidence interval? How does this price compare with the Black-Scholes put option price? Explain the difference.

b. Assume the following levels of volatility for the underlying: 5%, 10%, 15%, 20%, 25%, 30%, 35%, 40%. Simulate 1,000 paths for the stock price for each level of volatility. Plot a graph that has the volatility on the X axis and the proportion of "out" paths on the Y axis. What do you observe? Explain/interpret your findings.

c. For the same levels of the volatility, plot a graph that has the same X axis as above and the price of the down-and-out put on the Y axis, together with its confidence interval. Also, plot on the same graph the Black-Scholes price for each level of volatility. What do you observe? Explain/interpret your findings.

3 Pricing Exotic Options in Complicated Market Structures

Consider a financial market where uncertainty is described by a 2-dimensional Brownian motion $B = (B_1\ B_2)$. There are two risky assets $(S_1\ S_2)$ and one locally riskless money market account S^0. We assume the following dynamics under the risk-neutral measure $\mathbb{Q}$ (NB: B is a $\mathbb{Q}$-Brownian motion)

$$dS_{1t} = r_t S_{1t} dt + \sigma_{11}\sqrt{S_{1t}} dB_{1t} + \sigma_{12} S_{1t} dB_{2t} \tag{1}$$

$$dS_{2t} = r_t S_{2t} dt + \sigma_{21}(S_{1t} - S_{2t}) dB_{1t} \tag{2}$$

$$dr_t = \alpha(\beta - r_t)dt + \delta\sqrt{r_t} dB_{1t} \tag{3}$$

This structure is too complicated to allow analytical solutions for derivative prices. We therefore use Monte-Carlo simulation. We make the following parametric assumptions: $r_0 = \beta = 0.05$, $\alpha = 0.6$, $\sigma_{11} = 0.1$, $\sigma_{12} = 0.2$, $\sigma_{21} = 0.3$, $S_{10} = S_{20} = 10$, $\delta = 0.1$.

a. Discretize the above SDEs using the Euler discretization scheme.[1] Using 1,000 simulations and a step size of 1/250 approximate the distribution (draw the histogram) of $(r_T \mid r_0 = 0.05)$, for $T = 1$.

b. Simulate one trajectory of the interest rate from $t = 0$ to $T = 100$, with a step size of 1/52. Draw a plot of the trajectory.

c. Using the Euler discretization and the initial parameters, price a Call option on asset 1 with strike price equal to $K = 10$ and maturity $T = 0.5$ using Monte-Carlo simulation. Use 10,000 simulations and a time step of 1/250.

d. Using Monte-Carlo simulation price an option with payoff

$$f(T) = \max\left[\max\left(\max_{0\le t\le T} S_{1t}, \max_{0\le t\le T} S_{2t}\right) - K, 0\right], \tag{5}$$

maturity $T = 0.5$ and $K = 10$. Again use 10,000 simulations.

[1] The Euler discretization scheme for the dynamics $dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t$ is

$$S_{t+dt} = S_t + \mu(S_t, t)dt + \sigma(S_t, t)\sqrt{dt}Z \tag{4}$$

where Z is a $N(0, 1)$ random variable.

4 Hedging, Large Price Movements, and Transaction Costs

The purpose of this exercise is to hedge a call option at five minutes time intervals, during 30 days (for simplicity, assume that there are 365 trading days during a year; each day is supposed to have 8 trading hours, so there are $8 \times 12 = 96$ five minutes increments per day).

We will simulate one price trajectory at 5 minute level, for an horizon of 30 days. Once this trajectory is constructed, we compute the hedge ratio (the delta of the option) at each point in time. We then compare the evolution of the Black-Scholes price and of the hedging portfolio. To assess the effectiveness of the hedge, we will allow for a 10% jump (positive or negative) in the middle of the sample.

Consider a call option that expires in 60 days from today, i.e., $T = 60/365$. The price of the underlying today is $S_0 = 100$ and the strike of the option is $K = 100$. Other parameters are: $r = 0.05$, $\sigma = 0.3$, the expected return of the underlying $\mu = 0.2$, and the dividend yield $\delta = 0$.

a. Simulate a price path for 30 days at five minutes intervals *(we are <u>NOT</u> trying to price the option yet $\Rightarrow$ simulate under the physical probability measure and not under the risk neutral measure, i.e., use μ instead of r).*

b. For every simulated price, compute the Black-Scholes price of the option. Take into account the fact that the maturity of the option changes (for instance, at the latest simulated point the maturity of the option should be 30/365).

c. At time 0, the value of the hedging portfolio is exactly the value of the portfolio replicating the option, i.e., $C_0 = \Delta S_0 + B_0$. Then, at each new data point, perform the following changes in the hedging portfolio:

$$V_t = \Delta_{t-dt} S_t + B_{t-dt} e^{r \times dt} \tag{6}$$

$$B_t = V_t - \Delta_t S_t \tag{7}$$

***Interpretation**: The value of the hedge portfolio at time t, V_t, changes due to changes in the underlying (from S_{t-dt} to S_t) and in the risk-free part (from B_{t-dt} to $B_{t-dt}e^{r \times dt}$); because now we have a new hedge ratio Δ_t, adjust the position in the risk-free asset in such a way to have exactly $\Delta_t S_t$ in the new portfolio. If the intervals dt (which in our case are 5 minutes) are infinitesimally small, then these changes in the portfolio would be self-financed. Otherwise the value of the call and the value of the hedge portfolio will be slightly different.*

Plot the following three graphs: (1) The evolution of the underlying price, (2) The evolution of the call option and of the replicating portfolio, and (3) The evolution of the hedge ratio.

d. Assume now that exactly in the middle of the simulation there is an unexpected downward jump of 10% in the value of the asset. Plot the same graphs as before. What do you observe?

e. Perform the same exercise but assume an upward jump of 10%. Interpret your findings.

f. Get back to point (c) and assume that there are transaction fees of 20 basis points at each period. That is, if the number of risky assets in the hedging portfolio changes by $\Delta_t - \Delta_{t-dt}$, then the total transaction cost to be paid is:

$$|\Delta_t - \Delta_{t-dt}| \times S_t \times 0.002 \tag{8}$$

Plot the same graphs as requested at point (c). What do you observe?

5 Heston Model

The purpose of this exercise is to implement the formula of Heston's call, to plot the difference between the Black-Scholes price and Heston's option price for different underlying prices, and also to plot the Black-Scholes implied volatility in the Heston Model.

The Heston model is based on the following stock price and variance dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t} dZ_{1,t} \tag{9}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_{2,t} \tag{10}$$

where $\kappa, \theta, \sigma > 0$ are constant parameters. The two Brownian motions, $Z_{1,t}$ and $Z_{2,t}$, are correlated, i.e., $\text{Corr}[dZ_{1,t}, dZ_{2,t}] = \rho dt$.

The dynamics of the stock price in (9) is a geometric Brownian motion with time varying volatility. The variance v_t in (10) follows a square root process (also known as the Feller process or the Cox-Ingersoll-Ross process). The parameter θ corresponds to the long-run average of v_t, and κ controls the speed by which v_t returns to its long-run mean. Set $x_t = \log(S_t)$ and $\tau = T - t$. The call price is

$$C(S_t, v_t, K, T, t) = S_t\mathbb{P}_1 - Ke^{-r\tau}\mathbb{P}_2 \tag{11}$$

$$\mathbb{P}_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-iu\log(K)}\phi_{j;x_t,v_t,t}(u)}{iu}\right] du \tag{12}$$

$$\phi_{j;x_t,v_t,t}(u) = e^{A_j(\tau,u)+B_j(\tau,u)v_t+iux_t} \tag{13}$$

$$A_j(\tau, u) = rui\tau + \frac{\kappa\theta}{\sigma^2}\left[(b_j - \rho\sigma ui + d_j)\tau - 2\log\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right] \tag{14}$$

$$B_j(\tau, u) = \frac{b_j - \rho\sigma ui + d_j}{\sigma^2} \times \frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} \tag{15}$$

with

$$g_j = \frac{b_j - \rho\sigma ui + d_j}{b_j - \rho\sigma ui - d_j} \tag{16}$$

$$d_j = \sqrt{(\rho\sigma ui - b_j)^2 - \sigma^2(2u_j ui - u^2)} \tag{17}$$

$$u_1 = 1/2, \text{ and } u_2 = -1/2 \tag{18}$$

$$b_1 = \kappa + \lambda - \rho\sigma, \text{ and } b_2 = \kappa + \lambda \tag{19}$$

Even though these formulae look complicated, it is relatively easy to implement them numerically. The parameter λv_t can be interpreted as the volatility risk premium (Heston assumes that the volatility risk premium is a linear function of v_t). For our particular example we will assume $\lambda = 0$.

Consider the following parameters: $K = 100$, $r = 0.04$, $T = 0.5$, $t = 0$, $\sigma = 0.3$, $\rho = -0.5$, $v_t = 0.01$, $\kappa = 6$, $\theta = 0.02$, $S_t = 100$, and $\lambda = 0$.

a. What is the Black-Scholes prices of the option (assuming that the volatility is constant at its' long-run mean $\sqrt{\theta}$)? What is the Heston price of the option?

b. Plot the difference $C_{\text{Heston}} - C_{\text{Black-Scholes}}$ for different underlying prices, ranging from $S_t = 70$ to $S_t = 130$.

c. For each Heston price obtained at point (b), compute the Black-Scholes implied volatility. Plot the implied volatility as a function of the underlying, from $S_t = 70$ to $S_t = 130$.

d. Perform the same exercise as in (c), but now with $\rho = 0.5$. What do you observe?