Assigment 1
Mgmt 237E: Empirical Methods
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Question 1

a)

We know the jump at time t is $J_t = B_t(\mu_j + \sigma_j\delta_t)$

where δ_t = N(0,1)

and the log returns $r_t = \mu + \sigma\epsilon_t + J_t$

Mean
$\mathbb{E}(J_t) = \mathbb{E}[B_t(\mu_j + \sigma_j\delta_t)] = \mathbb{E}(B_t)\mathbb{E}(\mu_j + \sigma_j\delta_t) = \mathbf{p}\mu_\mathbf{j}$

$\mathbb{E}(r_t) = \mathbb{E}(\mu) + \mathbb{E}(\sigma\epsilon_t) + \mathbb{E}(J_t) = \underline{\mu + \mathbf{p}\mu_\mathbf{j}}$

Variance
Var(J_t) = $\mathbb{E}(J_t^2)$ - $(\mathbb{E}(J_t))^2$
$= \mathbb{E}({B_t}^2(\mu_j + \sigma_j\delta_t)^2) - {\mu_j}^2p^2$
$= \underline{\mathbb{E}({B_t}^2)\mathbb{E}((\mu_j + \sigma_j\delta_t)^2)} - {\mu_j}^2p^2$

$\underline{\mathbb{E}({B_t}^2)} = Var(B_t) + (E(B_t))^2 = p(1-p) + p^2 = p$
$\underline{\mathbb{E}((\mu_j + \sigma_j\delta_t)^2)} = Var(\mu_j + \sigma_j\delta_t) + (\mathbb{E}(\mu_j + \sigma_j\delta_t))^2 = {\sigma_j}^2 + {\mu_j}^2$

So, Var(J_t) = $\mathbf{p}{\sigma_\mathbf{j}}^\mathbf{2} + \mathbf{p}{\mu_\mathbf{j}}^\mathbf{2} - {\mu_\mathbf{j}}^\mathbf{2}\mathbf{p^2}$

Var(r_t) = Var$(\mu + \sigma\epsilon_t)$ + Var(J_t) = $\underline{\sigma^\mathbf{2} + \mathbf{p}{\sigma_\mathbf{j}}^\mathbf{2} + \mathbf{p}{\mu_\mathbf{j}}^\mathbf{2} - {\mu_\mathbf{j}}^\mathbf{2}\mathbf{p^2}}$

Skewness
If Skew in terms of mean and variance = $\frac{\mathbb{E}({J_t}^3) - 3\mathbb{E}(J_t)Var(J_t) - (\mathbb{E}(J_t))^3}{(Var(J_t))^{3/2}}$
Then 3rd moment $\mu_3(J_t) = \mathbb{E}({J_t}^3) - 3\mathbb{E}(J_t)Var(J_t) - (\mathbb{E}(J_t))^3$

$\mathbb{E}({J_t}^3) = \mathbb{E}({B_t}^3(\mu_j + \sigma_j\delta_t)^3) = \underline{\mathbb{E}({B_t}^3)\mathbb{E}((\mu_j + \sigma_j\delta_t)^3)}$

We can find $\underline{\mathbb{E}({B_t}^3)}$ by using bernoulli distribution mean, variance and skew($\frac{1-2p}{\sqrt{p(1-p)}}$). Substitute in skew formula in terms of mean and variance
$\mathbb{E}({B_t}^3) = \frac{(1-2p)(p(1-p))^{3/2}}{\sqrt{p(1-p)}} + 3p^2(1-p) - p^3$
$= (1-2p)(p-p^2) + 3p^2 - 3p^3 + p^3 = \mathbf{p}$

We can find $\underline{\mathbb{E}((\mu_j + \sigma_j\delta_t)^3)}$ by using normal distribution mean, variance and skew (0). Substitute in skew formula in terms of mean and variance

$\mathbb{E}((\mu_j + \sigma_j\delta_t)^3) = \mathbf{3}\mu_\mathbf{j}{\sigma_\mathbf{j}}^\mathbf{2} + {\mu_\mathbf{j}}^\mathbf{3}$

so $\mathbb{E}({J_t}^3) = \mathbf{p}(\mathbf{3}\mu_\mathbf{j}{\sigma_\mathbf{j}}^\mathbf{2} + {\mu_\mathbf{j}}^\mathbf{3})$

so $\mu_3(J_t) = \mathbf{3}\mu_\mathbf{j}{\sigma_\mathbf{j}}^\mathbf{2}\mathbf{p} + {\mu_\mathbf{j}}^\mathbf{3}\mathbf{p} - \mathbf{3p^2}\mu_\mathbf{j}{\sigma_\mathbf{j}}^\mathbf{2} - \mathbf{3p^2}{\mu_\mathbf{j}}^\mathbf{3} + \mathbf{2p^3}{\mu_\mathbf{j}}^\mathbf{3}$

Skew(r_t) = Skew($\mu + \sigma\epsilon_t + J_t$)

We know, Skew(A + B) = $\frac{\mu_3(A)+\mu_3(B)}{(Var(A)+Var(B))^{3/2}}$,

Substitute A = $\mu + \sigma\epsilon_t$ and B = J_t

Skew(r_t) = $\frac{0+\mu_3(B)}{\sigma^2+p{\sigma_j}^2+p{\mu_j}^2-{\mu_j}^2p^2)^{3/2}}$

$= \underline{\boldsymbol{\frac{3\mu_j{\sigma_j}^2p+{\mu_j}^3p-3p^2\mu_j{\sigma_j}^2-3p^2{\mu_j}^3+2p^3{\mu_j}^3}{(\sigma^2+p{\sigma_j}^2+p{\mu_j}^2-{\mu_j}^2p^2)^{3/2}}}}$