ROLL NO. 102103377

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Predictive Analysis Assignment
Parameter Estimation

Q.1 Let (X_1, X_2 ---) be a random sample of size n taken from a normal population with parameters : mean = θ1 & variance = θ2. Find the maximum likelihood Estimates of these two parameters.

Sol: $PMF(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\mu_i)^2/2\sigma^2}$ for normal dist.

here $\mu = \theta 1$
$\sigma^2 = \theta 2$

$$f(x_i|\theta_1,\theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_i-\theta_1)^2/2\theta_2}$$

Now likelihood function

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n} f(x_i|\theta_1,\theta_2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_i-\theta_1)^2/2\theta_2}$$

$$L(\theta_1,\theta_2) = \prod_{i=1}^{n} (\theta)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{-(x_i-\theta_i)^2/2\theta_2}.$$

$$L(\theta_1, \theta_2) = \left(\frac{\theta_2}{2}\right)^{-n/2} (2\pi)^{-n/2} e^{-\left(\sum_{i=1}^{n} (x_i - \theta_1)^2/2\theta_2\right)}$$

$$L(\theta_1, \theta_2) = \frac{\theta_2}{2}^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n} (x_i - \theta_1)^2}$$

taking log on both sides.

$$\ln L(\theta_1, \theta_2) = \ln \left[\frac{\theta_2}{2}^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n} (x_i - \theta_1)^2} \right]$$

$$= \ln \left(\frac{\theta_2}{2}\right)^{-n/2} + \ln (2\pi)^{-n/2} + \ln \left(e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n} (x_i - \theta_1)^2}\right)$$

$$\ln L(\theta_1, \theta_2) = -\frac{n}{2} \ln \theta_2 - \frac{n}{2} \ln 2\pi - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2 \quad -①$$

Now, differentiating both sides w.r.t θ_1

$$\frac{\partial L(\theta_1, \theta_2)}{\partial \theta_1} = \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)$$

now, $\frac{\partial L(\theta_1, \theta_2)}{\partial \theta_1} = 0$

hence, $\frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1) = 0$

$$(1/\theta_2) \left(\sum_{i=1}^{n} x_i - n\theta_1 \right) = 0$$

$$\sum_{i=1}^{n} x_i - n\theta_1 = 0.$$

$$\theta_1 = \sum_{i=1}^{n} \frac{x_i}{n}$$

$$\theta_1 = \overline{x}_n$$

$$\boxed{\theta_{1_{MLE}} = \overline{x}_n} \quad - (2)$$

Differentiating (1) w.r.t (2)

$$\frac{\partial L(\theta_1, \theta_2)}{\partial \theta_2} = \frac{-n}{2\theta_2} - \frac{1}{2(\theta_2)^2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

Since $\frac{\partial L(\theta_1, \theta_2)}{\partial \theta_2} = 0$

$$-\frac{n}{2\theta_2} - \frac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = 0$$

$$\frac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = \frac{-n}{2\theta_2}$$

$$\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

from (2):

$$\boxed{\theta_{2_{MLE}} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x}_n)^2}$$

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Q.2 Let $X_1, X_2 -- X_n$ be a random sample from $B(m,\theta)$ distribution, where $\theta \in (0,1)$ is unknown and 'm' is a known positive integer. Compute value of θ using MLE.

Sol. $PMF(x_i) = {}^nC_{x_i}\, p^{x_i} (1-p)^{n-x_i}$

here $n = m$, $p = \theta$

$$P(x_i \mid m, \theta) = {}^mC_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}$$

for likelihood function,

$$L(P) = \prod_{i=1}^{n} P(x_i \mid m, \theta)$$

$$= \prod_{i=1}^{n} \left({}^mC_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i} \right)$$

$$= \prod_{i=1}^{n} {}^mC_{x_i} \prod_{i=1}^{n} \theta^{x_i} \prod_{i=1}^{n} (1-\theta)^{m-x_i}$$

$$= \prod_{i=1}^{n} {}^mC_{x_i}\, \theta^{\sum_{i=1}^{n} \cancel{\theta} x_i} (1-\theta)^{\sum_{i=1}^{n}(m-x_i)}$$

$$= \prod_{i=1}^{n} {}^mC_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nm - \sum_{i=1}^{n} x_i}$$

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Taking ln on both sides,

$$\ln L(P) = \ln \left(\prod_{i=1}^{n} {}^{m}C_{x_i} \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{nm - \sum_{i=1}^{n} x_i} \right)$$

$$= \ln \left(\prod_{i=1}^{n} {}^{m}C_{x_i} \right) + \ln \left(\theta^{\sum_{i=1}^{n} x_i} \right) + \ln \left((1-\theta)^{nm - \sum_{i=1}^{n} x_i} \right),$$

$$= \ln \left(\prod_{i=1}^{n} {}^{n}C_{x_i} \right) + \ln(\theta) \cdot \sum_{i=1}^{n} x_i + \ln(1-\theta) \cdot \left(nm - \sum_{i=1}^{n} x_i \right)$$

Differentiating w.r.t. θ

$$\frac{\partial L(P)}{\partial \theta} = \frac{1}{\theta} \sum_{i=1}^{n} x_i + \left(\frac{-1}{1-\theta} \right) \left(nm - \sum_{i=1}^{n} x_i \right)$$

$$= \frac{1}{\theta} \sum_{i=1}^{n} x_i - \frac{1}{1-\theta} \left(nm - \sum_{i=1}^{n} x_i \right)$$

Now $\frac{\partial L(P)}{\partial \theta} = 0.$

$$\frac{1}{\theta} \sum_{i=1}^{n} x_i - \frac{1}{1-\theta} \left(nm - \sum_{i=1}^{n} x_i \right) = 0$$

$$\frac{1}{\theta} \sum_{i=1}^{n} x_i = \frac{1}{1-\theta} \left(nm - \sum_{i=1}^{n} x_i \right)$$

$$\frac{1-\theta}{\theta} = \frac{nm - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i}$$

$$\frac{nm}{\sum_{i=1}^{n} x_i} - 1 = \frac{1}{\theta} - 1$$

$$\theta = \sum_{i=1}^{n} \frac{x_i}{nm}$$

$$\theta = \frac{\bar{x}_n}{m}$$

$$\boxed{\theta_{MLE} \in (0,1) = \frac{\bar{x}_n}{m}}$$