

# Derivation of belief propagation equations for community / core-periphery structure

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$\psi_r^{u \rightarrow v}$  is the marginal probability that node  $u$  is type  $r$  in the absence of  $v$ , given the marginal probabilities for all other nodes besides  $v$  and knowledge about the adjacency matrix. Let  $I$  be the information:  $(\forall w \neq u, v, \forall t : \psi_t^{w \rightarrow u}) \wedge A \wedge v$  is absent. Then

$$\psi_r^{u \rightarrow v} = P(t_u = r \mid I) = \sum_{\{w'\}} P(t_u = r \wedge \{w'\} \mid I)$$

We sum over all possible type configurations  $(\{w'\})$  for nodes which don't include  $u$  and  $v$ . This is OK because each  $\{w'\}$  is mutually exclusive.

$$= \sum_{\{w'\}} P(t_u = r \mid \{w'\} \wedge I) P(\{w'\} \mid I)$$

We first expand the first probability as a product of independent probabilities (by our tree assumption)., where  $f$  is the function we described last Friday.

$$= \sum_{\{w'\}} \left[ \prod_{w \in \{w'\}} f(t_u = r \mid t_w \wedge A_{uw}) \right] P(\{w'\} \mid I)$$

We next expand the right term as a product of  $\psi'$ s. Each  $\psi_t^{w \rightarrow u}$  is unaffected by  $I$ : the only difference between  $I$  and the information  $\psi_t^{w \rightarrow u}$  is conditional on,  $I'$ , is that  $I$  includes the absence of  $v$  and  $I'$  includes the absence of  $u$ . However,  $\psi_t^{w \rightarrow u}$  is independent of  $u$  by assumption and independent of  $v$  by our treelike factor graph assumption (note the community detection factor graph is definitely not treelike: the influence of non-edges causes the factor graph to be a clique. However, the BP result is only for treelike factor graphs. I believe this is where our derivation fell through. Since our factor graph isn't treelike, the derivation isn't exact (but is pretty close, because the influence of  $v$  should be minimal).

$$= \sum_{\{w'\}} \left[ \prod_{w \in \{w'\}} f(t_u = r \mid t_w \wedge A_{uw}) \psi_{t_w}^{w \rightarrow u} \right]$$

The rest of the derivation should just be rearranging.