CHAPTER 4

ENERGY AND POTENTIAL

In the previous two chapters we became acquainted with Coulomb's law and its use in finding the electric field about several simple distributions of charge, and also with Gauss's law and its application in determining the field about some symmetrical charge arrangements. The use of Gauss's law was invariably easier for these highly symmetrical distributions, because the problem of integration always disappeared when the proper closed surface was chosen.

However, if we had attempted to find a slightly more complicated field, such as that of two unlike point charges separated by a small distance, we would have found it impossible to choose a suitable gaussian surface and obtain an answer. Coulomb's law, however, is more powerful and enables us to solve problems for which Gauss's law is not applicable. The application of Coulomb's law is laborious, detailed, and often quite complex, the reason for this being precisely the fact that the electric field intensity, a vector field, must be found directly from the charge distribution. Three different integrations are needed in general, one for each component, and the resolution of the vector into components usually adds to the complexity of the integrals.

Certainly it would be desirable if we could find some as yet undefined scalar function with a single integration and then determine the electric field from this scalar by some simple straightforward procedure, such as differentiation.

This scalar function does exist and is known as the *potential* or *potential field*. We shall find that it has a very real physical interpretation and is more familiar to most of us than is the electric field which it will be used to find.

We should expect, then, to be equipped soon with a third method of finding electric fields—a single scalar integration, although not always as simple as we might wish, followed by a pleasant differentiation.

The remaining difficult portion of the task, the integration, we intend to remove in Chap. 7.

4.1 ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD

The electric field intensity was defined as the force on a unit test charge at that point at which we wish to find the value of this vector field. If we attempt to move the test charge against the electric field, we have to exert a force equal and opposite to that exerted by the field, and this requires us to expend energy, or do work. If we wish to move the charge in the direction of the field, our energy expenditure turns out to be negative; we do not do the work, the field does.

Suppose we wish to move a charge Q a distance $d\mathbf{L}$ in an electric field $\mathbf{E}$. The force on Q due to the electric field is

$$\boxed{\mathbf{F}_E = Q\mathbf{E}} \tag{1}$$

where the subscript reminds us that this force is due to the field. The component of this force in the direction $d\mathbf{L}$ which we must overcome is

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

where $\mathbf{a}_L$ = a unit vector in the direction of $d\mathbf{L}$.

The force which we must apply is equal and opposite to the force due to the field,

$$F_{\text{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

and our expenditure of energy is the product of the force and distance. That is,

$$\text{Differential work done by external source moving } Q$$
$$= -Q\mathbf{E} \cdot \mathbf{a}_L dL = -Q\mathbf{E} \cdot d\mathbf{L}$$

or

$$\boxed{dW = -Q\mathbf{E} \cdot d\mathbf{L}} \tag{2}$$

where we have replaced $\mathbf{a}_L dL$ by the simpler expression $d\mathbf{L}$.

This differential amount of work required may be zero under several conditions determined easily from (2). There are the trivial conditions for which $\mathbf{E}$, Q, or $d\mathbf{L}$ is zero, and a much more important case in which $\mathbf{E}$ and $d\mathbf{L}$ are

perpendicular. Here the charge is moved always in a direction at right angles to the electric field. We can draw on a good analogy between the electric field and the gravitational field, where, again, energy must be expended to move against the field. Sliding a mass around with constant velocity on a frictionless surface is an effortless process if the mass is moved along a constant elevation contour; positive or negative work must be done in moving it to a higher or lower elevation, respectively.

Returning to the charge in the electric field, the work required to move the charge a finite distance must be determined by integrating,

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \tag{3}$$

where the path must be specified before the integral can be evaluated. The charge is assumed to be at rest at both its initial and final positions.

This definite integral is basic to field theory, and we shall devote the following section to its interpretation and evaluation.

✔ **D4.1.** Given the electric field $\mathbf{E} = \frac{1}{z^2}(8xyz\mathbf{a}_x + 4x^2z\mathbf{a}_y - 4x^2y\mathbf{a}_z)$ V/m, find the differential amount of work done in moving a 6-nC charge a distance of 2 μm, starting at $P(2, -2, 3)$ and proceeding in the direction $\mathbf{a}_L =$: (*a*) $-\frac{6}{7}\mathbf{a}_x + \frac{3}{7}\mathbf{a}_y + \frac{2}{7}\mathbf{a}_z$; (*b*) $\frac{6}{7}\mathbf{a}_x - \frac{3}{7}\mathbf{a}_y - \frac{2}{7}\mathbf{a}_z$; (*c*) $\frac{3}{7} + \frac{6}{7}\mathbf{a}_y$.

Ans. −149.3; 149.3; 0 fJ

4.2 THE LINE INTEGRAL

The integral expression for the work done in moving a point charge Q from one position to another, Eq. (3), is an example of a line integral, which in vector-analysis notation always takes the form of the integral along some prescribed path of the dot product of a vector field and a differential vector path length $d\mathbf{L}$. Without using vector analysis we should have to write

$$W = -Q \int_{\text{init}}^{\text{final}} E_L \, dL$$

where E_L = component of $\mathbf{E}$ along $d\mathbf{L}$.

A line integral is like many other integrals which appear in advanced analysis, including the surface integral appearing in Gauss's law, in that it is essentially descriptive. We like to look at it much more than we like to work it out. It tells us to choose a path, break it up into a large number of very small segments, multiply the component of the field along each segment by the length of the segment, and then add the results for all the segments. This is a summation, of course, and the integral is obtained exactly only when the number of segments becomes infinite.

This procedure is indicated in Fig. 4.1, where a path has been chosen from an initial position B to a final position[1] A and a *uniform electric field* selected for simplicity. The path is divided into six segments, $\Delta\mathbf{L}_1, \Delta\mathbf{L}_2, \ldots, \Delta\mathbf{L}_6$, and the components of $\mathbf{E}$ along each segment denoted by $E_{L1}, E_{L2}, \ldots, E_{L6}$. The work involved in moving a charge Q from B to A is then approximately

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \ldots + E_{L6}\Delta L_6)$$

or, using vector notation,

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \ldots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

and since we have assumed a uniform field,

$$\mathbf{E}_1 = \mathbf{E}_2 = \ldots = \mathbf{E}_6$$
$$W = -Q\mathbf{E} \cdot (\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \ldots + \Delta\mathbf{L}_6)$$

What is this sum of vector segments in the parentheses above? Vectors add by the parallelogram law, and the sum is just the vector directed from the initial point B to the final point A, $\mathbf{L}_{BA}$. Therefore

$$W = -Q\mathbf{E} \cdot \mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)} \tag{4}$$

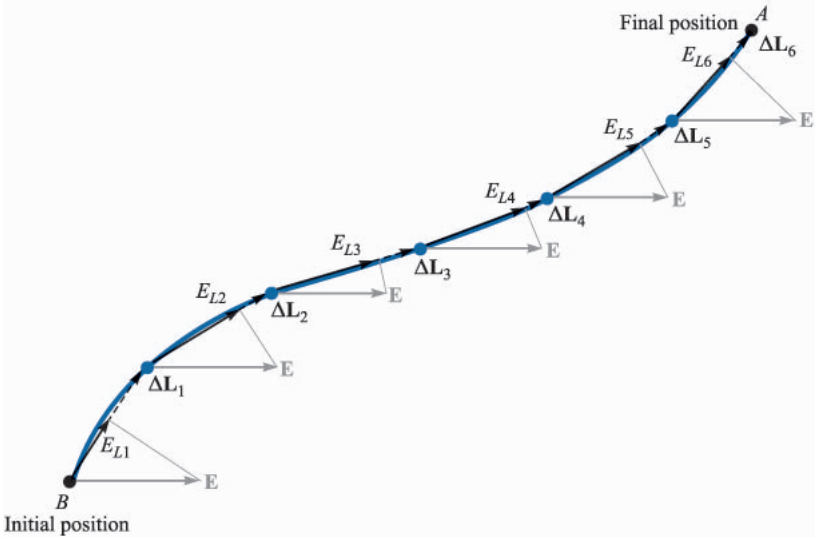

FIGURE 4.1
A graphical interpretation of a line integral in a uniform field. The line integral of $\mathbf{E}$ between points B and A is independent of the path selected, even in a nonuniform field; this result is not, in general, true for time-varying fields.

[1] The final position is given the designation A to correspond with the convention for potential difference, as discussed in the following section.

Remembering the summation interpretation of the line integral, this result for the uniform field can be obtained rapidly now from the integral expression

$$W = -Q \int_B^A \mathbf{E} \cdot d\mathbf{L} \tag{5}$$

as applied to a uniform field

$$W = -Q\mathbf{E} \cdot \int_B^A d\mathbf{L}$$

where the last integral becomes $\mathbf{L}_{BA}$ and

$$W = -Q\mathbf{E} \cdot \mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$

For this special case of a uniform electric field intensity, we should note that the work involved in moving the charge depends only on Q, $\mathbf{E}$, and $\mathbf{L}_{BA}$, a vector drawn from the initial to the final point of the path chosen. It does not depend on the particular path we have selected along which to carry the charge. We may proceed from B to A on a straight line or via the Old Chisholm Trail; the answer is the same. We shall show in Sec. 4.5 that an identical statement may be made for any nonuniform (static) $\mathbf{E}$ field.

Let us use several examples to illustrate the mechanics of setting up the line integral appearing in (5).

Example 4.1

We are given the nonuniform field

$$\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$$

and we are asked to determine the work expended in carrying 2 C from $B(1, 0, 1)$ to $A(0.8, 0.6, 1)$ along the shorter arc of the circle

$$x^2 + y^2 = 1 \quad z = 1$$

Solution. We use $W = -Q \int_B^A \mathbf{E} \cdot d\mathbf{L}$, where $\mathbf{E}$ is not necessarily constant. Working in cartesian coordinates, the differential path $d\mathbf{L}$ is $dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$, and the integral becomes

$$\begin{aligned} W &= -Q \int_B^A \mathbf{E} \cdot d\mathbf{L} \\ &= -2 \int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z) \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z) \\ &= -2 \int_1^{0.8} y\,dx - 2 \int_0^{0.6} x\,dy - 4 \int_1^1 dz \end{aligned}$$

where the limits on the integrals have been chosen to agree with the initial and final values of the appropriate variable of integration. Using the equation of the circular path (and selecting the sign of the radical which is correct for the quadrant involved), we have

$$W = -2\int_1^{0.8} \sqrt{1-x^2}\,dx - 2\int_0^{0.6} \sqrt{1-y^2}\,dy - 0$$
$$= -\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1} y\right]_0^{0.6}$$
$$= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0)$$
$$= -0.96\,\text{J}$$

Example 4.2

Again find the work required to carry 2 C from B to A in the same field, but this time use the straight-line path from B to A.

Solution. We start by determining the equations of the straight line. Any two of the following three equations for planes passing through the line are sufficient to define the line:

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$
$$z - z_B = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$
$$x - x_B = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

From the first equation above we have

$$y = -3(x - 1)$$

and from the second we obtain

$$z = 1$$

Thus,

$$W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$
$$= 6\int_1^{0.8} (x-1)dx - 2\int_0^{0.6} \left(1 - \frac{y}{3}\right) dy$$
$$= -0.96\,\text{J}$$

This is the same answer we found using the circular path between the same two points, and it again demonstrates the statement (unproved) that the work done is independent of the path taken in any electrostatic field.

It should be noted that the equations of the straight line show that $dy = -3\,dx$ and $dx = -3\,dy$. These substitutions may be made in the first two integrals above, along with a change in limits, and the answer may be obtained by evaluating the new integrals. This method is often simpler if the integrand is a function of only one variable.

Note that the expressions for $d\mathbf{L}$ in our three coordinate systems utilize the differential lengths obtained in the first chapter (cartesian in Sec. 1.3, cylindrical in Sec. 1.8, and spherical in Sec. 1.9):

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(cartesian)} \qquad (6)$$

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)} \qquad (7)$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\;d\phi\,\mathbf{a}_\phi \qquad \text{(spherical)} \qquad (8)$$

The interrelationships among the several variables in each expression are determined from the specific equations for the path.

As a final example illustrating the evaluation of the line integral, let us investigate several paths which we might take near an infinite line charge. The field has been obtained several times and is entirely in the radial direction,

$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho$$

Let us first find the work done in carrying the positive charge Q about a circular path of radius ρ_b centered at the line charge, as illustrated in Fig. 4.2*a*. Without lifting a pencil, we see that the work must be nil, for the path is always perpendicular to the electric field intensity, or the force on the charge is always exerted at right angles to the direction in which we are moving it. For practice, however, let us set up the integral and obtain the answer.

The differential element $d\mathbf{L}$ is chosen in cylindrical coordinates, and the circular path selected demands that $d\rho$ and dz be zero, so $d\mathbf{L} = \rho_1 d\phi\,\mathbf{a}_\phi$. The work is then

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1\,d\phi\,\mathbf{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0} d\phi\,\mathbf{a}_\rho \cdot \mathbf{a}_\phi = 0$$

Let us now carry the charge from $\rho = a$ to $\rho = b$ along a radial path (Fig. 4.2*b*). Here $d\mathbf{L} = d\rho\,\mathbf{a}_\rho$ and

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho \cdot d\rho\,\mathbf{a}_\rho = -Q\int_a^b \frac{\rho_L}{2\pi\epsilon_0}\frac{d\rho}{\rho}$$

or

$$W = -\frac{Q\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Since b is larger than a, $\ln(b/a)$ is positive, and we see that the work done is negative, indicating that the external source that is moving the charge receives energy.

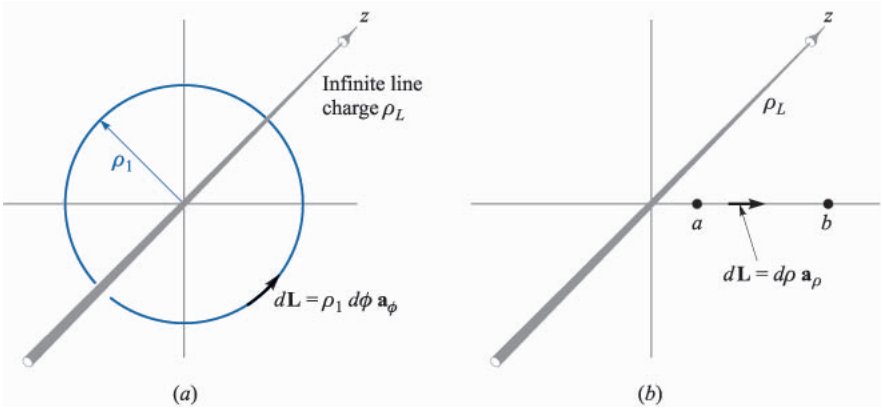

FIGURE 4.2
(*a*) A circular path and (*b*) a radial path along which a charge of Q is carried in the field of an infinite line charge. No work is expected in the former case.

One of the pitfalls in evaluating line integrals is a tendency to use too many minus signs when a charge is moved in the direction of a *decreasing* coordinate value. This is taken care of completely by the limits on the integral, and no misguided attempt should be made to change the sign of $d\mathbf{L}$. Suppose we carry Q from b to a (Fig. 4.2*b*). We still have $d\mathbf{L} = d\rho\, \mathbf{a}_\rho$ and show the different direction by recognizing $\rho = b$ as the initial point and $\rho = a$ as the final point,

$$W = -Q \int_b^a \frac{\rho_L}{2\pi\epsilon_0} \frac{d\rho}{\rho} = \frac{Q\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

This is the negative of the previous answer and is obviously correct.

✔ **D4.2.** Calculate the work done in moving a 4-C charge from $B(1, 0, 0)$ to $A(0, 2, 0)$ along the path $y = 2 - 2x$, $z = 0$ in the field $\mathbf{E} =$: (*a*) $5\mathbf{a}_x$ V/m; (*b*) $5x\mathbf{a}_x$ V/m; (*c*) $5x\mathbf{a}_x + 5y\mathbf{a}_y$ V/m.

Ans. 20 J; 10 J; −30 J

✔ **D4.3.** We shall see later that a time-varying **E** field need not be conservative. (If it is not conservative, the work expressed by Eq. (3) may be a function of the path used.) Let $\mathbf{E} = y\mathbf{a}_x$ V/m at a certain instant of time, and calculate the work required to move a 3-C charge from (1, 3, 5) to (2, 0, 3) along the straight line segments joining: (*a*) (1, 3, 5) to (2, 3, 5) to (2, 0, 5) to (2, 0, 3); (*b*) (1, 3, 5) to (1, 3, 3) to (1, 0, 3) to (2, 0, 3).

Ans. −9 J; 0

4.3 DEFINITION OF POTENTIAL DIFFERENCE AND POTENTIAL

We are now ready to define a new concept from the expression for the work done by an external source in moving a charge Q from one point to another in an electric field $\mathbf{E}$,

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

In much the same way as we defined the electric field intensity as the force on a *unit* test charge, we now define *potential difference* V as the work done (by an external source) in moving a *unit* positive charge from one point to another in an electric field,

$$\text{Potential difference} = V = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \tag{9}$$

We shall have to agree on the direction of movement, as implied by our language, and we do this by stating that V_{AB} signifies the potential difference between points A and B and is the work done in moving the unit charge from B (last named) to A (first named). Thus, in determining V_{AB}, B is the initial point and A is the final point. The reason for this somewhat peculiar definition will become clearer shortly, when it is seen that the initial point B is often taken at infinity, whereas the final point A represents the fixed position of the charge; point A is thus inherently more significant.

Potential difference is measured in joules per coulomb, for which the *volt* is defined as a more common unit, abbreviated as V. Hence the potential difference between points A and B is

$$V_{AB} = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L} \quad \text{V} \tag{10}$$

and V_{AB} is positive if work is done in carrying the positive charge from B to A.

From the line-charge example of the last section we found that the work done in taking a charge Q from $\rho = b$ to $\rho = a$ was

$$W = \frac{Q\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a}$$

Thus, the potential difference between points at $\rho = a$ and $\rho = b$ is

$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\epsilon_0} \ln \frac{b}{a} \tag{11}$$

We can try out this definition by finding the potential difference between points A and B at radial distances r_A and r_B from a point charge Q. Choosing an origin at Q,

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

and

$$d\mathbf{L} = dr\, \mathbf{a}_r$$

we have

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_A} - \frac{1}{r_B} \right) \tag{12}$$

If $r_B > r_A$, the potential difference V_{AB} is positive, indicating that energy is expended by the external source in bringing the positive charge from r_B to r_A. This agrees with the physical picture showing the two like charges repelling each other.

It is often convenient to speak of the *potential*, or *absolute potential*, of a point, rather than the potential difference between two points, but this means only that we agree to measure every potential difference with respect to a specified reference point which we consider to have zero potential. Common agreement must be reached on the zero reference before a statement of the potential has any significance. A person having one hand on the deflection plates of a cathode-ray tube which are "at a potential of 50 V" and the other hand on the cathode terminal would probably be too shaken up to understand that the cathode is not the zero reference, but that all potentials in that circuit are customarily measured with respect to the metallic shield about the tube. The cathode may be several thousands of volts negative with respect to the shield.

Perhaps the most universal zero reference point in experimental or physical potential measurements is "ground," by which we mean the potential of the surface region of the earth itself. Theoretically, we usually represent this surface by an infinite plane at zero potential, although some large-scale problems, such as those involving propagation across the Atlantic Ocean, require a spherical surface at zero potential.

Another widely used reference "point" is infinity. This usually appears in theoretical problems approximating a physical situation in which the earth is relatively far removed from the region in which we are interested, such as the static field near the wing tip of an airplane that has acquired a charge in flying through a thunderhead, or the field inside an atom. Working with the *gravitational* potential field on earth, the zero reference is normally taken at sea level; for an interplanetary mission, however, the zero reference is more conveniently selected at infinity.

A cylindrical surface of some definite radius may occasionally be used as a zero reference when cylindrical symmetry is present and infinity proves inconvenient. In a coaxial cable the outer conductor is selected as the zero reference for potential. And, of course, there are numerous special problems, such as those for which a two-sheeted hyperboloid or an oblate spheroid must be selected as the zero-potential reference, but these need not concern us immediately.

If the potential at point A is V_A and that at B is V_B, then

$$V_{AB} = V_A - V_B \tag{13}$$

where we necessarily agree that V_A and V_B shall have the same zero reference point.

✔ **D4.4.** An electric field is expressed in cartesian coordinates by $\mathbf{E} = 6x^2\mathbf{a}_x + 6y\mathbf{a}_y + 4\mathbf{a}_z$ V/m. Find: (*a*) V_{MN} if points M and N are specified by $M(2, 6, -1)$ and $N(-3, -3, 2)$; (*b*) V_M if $V = 0$ at $Q(4, -2, -35)$; (*c*) V_N if $V = 2$ at $P(1, 2, -4)$.

Ans. −139.0 V; −120.0 V; 19.00 V

4.4 THE POTENTIAL FIELD OF A POINT CHARGE

In the previous section we found an expression (12) for the potential difference between two points located at $r = r_A$ and $r = r_B$ in the field of a point charge Q placed at the origin,

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B \tag{14}$$

It was assumed that the two points lay on the same radial line or had the same θ and ϕ coordinate values, allowing us to set up a simple path on this radial line along which to carry our positive charge. We now should ask whether different θ and ϕ coordinate values for the initial and final position will affect our answer and whether we could choose more complicated paths between the two points without changing the results. Let us answer both questions at once by choosing two general points A and B (Fig. 4.3) at radial distances of r_A and r_B, and any values for the other coordinates.

The differential path length $d\mathbf{L}$ has r, θ, and ϕ components, and the electric field has only a radial component. Taking the dot product then leaves us only

$$V_{AB} = -\int_{r_B}^{r_A} E_r dr = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

We obtain the same answer and see, therefore, that the potential difference between two points in the field of a point charge depends only on the distance of

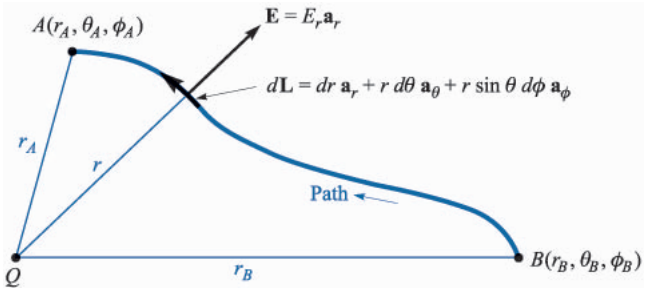

FIGURE 4.3
A general path between general points B and A in the field of a point charge Q at the origin. The potential difference V_{AB} is independent of the path selected.

each point from the charge and does not depend on the particular path used to carry our unit charge from one point to the other.

How might we conveniently define a zero reference for potential? The simplest possibility is to let $V = 0$ at infinity. If we let the point at $r = r_B$ recede to infinity the potential at r_A becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

or, since there is no reason to identify this point with the A subscript,

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}} \tag{15}$$

This expression defines the potential at any point distant r from a point charge Q at the origin, the potential at infinite radius being taken as the zero reference. Returning to a physical interpretation, we may say that $Q/4\pi\epsilon_0 r$ joules of work must be done in carrying a 1-C charge from infinity to any point r meters from the charge Q.

A convenient method to express the potential without selecting a specific zero reference entails identifying r_A as r once again and letting $Q/4\pi\epsilon_0 r_B$ be a constant. Then

$$V = \frac{Q}{4\pi\epsilon_0 r} + C_1 \tag{16}$$

and C_1 may be selected so that $V = 0$ at any desired value of r. We could also select the zero reference indirectly by electing to let V be V_0 at $r = r_0$.

It should be noted that the *potential difference* between two points is not a function of C_1.

Equation (15) or (16) represents the potential field of a point charge. The potential is a scalar field and does not involve any unit vectors.

Let us now define an *equipotential surface* as a surface composed of all those points having the same value of potential. No work is involved in moving a unit charge around on an equipotential surface, for, by definition, there is no potential difference between any two points on this surface.

The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.

An inspection of the form of the potential field of a point charge shows that it is an inverse-distance field, whereas the electric field intensity was found to be an inverse-square-law relationship. A similar result occurs for the gravitational force field of a point mass (inverse-square law) and the gravitational potential field (inverse distance). The gravitational force exerted by the earth on an object one million miles from it is four times that exerted on the same object two million miles away. The kinetic energy given to a freely falling object starting from the end of the universe with zero velocity, however, is only twice as much at one million miles as it is at two million miles.

✔ **D4.5.** A 15-nC point charge is at the origin in free space. Calculate V_1 if point P_1 is located at $P_1(-2, 3, -1)$ and: (*a*) $V = 0$ at $(6, 5, 4)$; (*b*) $V = 0$ at infinity; (*c*) $V = 5$ V at $(2, 0, 4)$

Ans. 20.7 V; 36.0 V; 10.89 V

4.5 THE POTENTIAL FIELD OF A SYSTEM OF CHARGES: CONSERVATIVE PROPERTY

The potential at a point has been defined as the work done in bringing a unit positive charge from the zero reference to the point, and we have suspected that this work, and hence the potential, is independent of the path taken. If it were not, potential would not be a very useful concept.

Let us now prove our assertion. We shall do so by beginning with the potential field of the single point charge for which we showed, in the last section, the independence with regard to the path, noting that the field is linear with respect to charge so that superposition is applicable. It will then follow that the potential of a system of charges has a value at any point which is independent of the path taken in carrying the test charge to that point.

Thus the potential field of a single point charge, which we shall identify as Q_1 and locate at $\mathbf{r}_1$, involves only the distance $|\mathbf{r} - \mathbf{r}_1|$ from Q_1 to the point at $\mathbf{r}$ where we are establishing the value of the potential. For a zero reference at infinity, we have

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|}$$

The potential due to two charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is a function only of $|\mathbf{r} - \mathbf{r}_1|$ and $|\mathbf{r} - \mathbf{r}_2|$, the distances from Q_1 and Q_2 to the field point, respectively.

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|}$$

Continuing to add charges, we find that the potential due to n point charges is

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|} + \ldots + \frac{Q_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|}$$

or

$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|} \tag{17}$$

If each point charge is now represented as a small element of a continuous volume charge distribution $\rho_v \Delta v$, then

$$V(\mathbf{r}) = \frac{\rho_v(\mathbf{r}_1)\Delta v_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{\rho_v(\mathbf{r}_2)\Delta v_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|} + \ldots + \frac{\rho_v(\mathbf{r}_n)\Delta v_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|}$$

As we allow the number of elements to become infinite, we obtain the integral expression

$$V(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{18}$$

We have come quite a distance from the potential field of the single point charge, and it might be helpful to examine (18) and refresh ourselves as to the meaning of each term. The potential $V(\mathbf{r})$ is determined with respect to a zero reference potential at infinity and is an exact measure of the work done in bringing a unit charge from infinity to the field point at $\mathbf{r}$ where we are finding the potential. The volume charge density $\rho_y(\mathbf{r}')$ and differential volume element dv' combine to represent a differential amount of charge $\rho_v(\mathbf{r}')\,dv'$ located at $\mathbf{r}'$. The distance $|\mathbf{r}-\mathbf{r}'|$ is that distance from the source point to the field point. The integral is a multiple (volume) integral.

If the charge distribution takes the form of a line charge or a surface charge, the integration is along the line or over the surface:

$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{19}$$

$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} \tag{20}$$

The most general expression for potential is obtained by combining (17), (18), (19), and (20).

These integral expressions for potential in terms of the charge distribution should be compared with similar expressions for the electric field intensity, such as (18) in Sec. 2.3:

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

The potential again is inverse distance, and the electric field intensity, inverse-square law. The latter, of course, is also a vector field.

To illustrate the use of one of these potential integrals, let us find V on the z axis for a uniform line charge ρ_L in the form of a ring, $\rho = a$, in the $z = 0$ plane, as shown in Fig. 4.4. Working with (19), we have $dL' = a\,d\phi'$, $\mathbf{r} = z\mathbf{a}_z$, $\mathbf{r}' = a\mathbf{a}_\rho$, $|\mathbf{r} - \mathbf{r}'| = \sqrt{a^2 + z^2}$, and

$$V = \int_0^{2\pi} \frac{\rho_L a\, d\phi'}{4\pi\epsilon_0\sqrt{a^2 + z^2}} = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2 + z^2}}$$

For a zero reference at infinity, then:

1. The potential due to a single point charge is the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, and the work is independent of the path chosen between those two points.
2. The potential field in the presence of a number of point charges is the sum of the individual potential fields arising from each charge.
3. The potential due to a number of point charges or any continuous charge distribution may therefore be found by carrying a unit charge from infinity to the point in question along any path we choose.

In other words, the expression for potential (zero reference at infinity),

$$V_A = -\int_\infty^A \mathbf{E} \cdot d\mathbf{L}$$

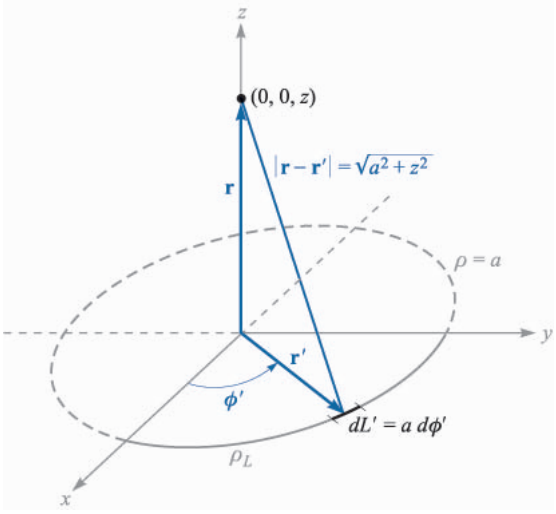

FIGURE 4.4
The potential field of a ring of uniform line charge density is easily obtained from $V = \int \rho_L(\mathbf{r}')\, dL' / (4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|)$.

or potential difference,

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

is not dependent on the path chosen for the line integral, regardless of the source of the **E** field.

This result is often stated concisely by recognizing that no work is done in carrying the unit charge around any *closed path*, or

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \tag{21}$$

A small circle is placed on the integral sign to indicate the closed nature of the path. This symbol also appeared in the formulation of Gauss's law, where a closed *surface* integral was used.

Equation (21) is true for *static* fields, but we shall see in Chap. 10 that Faraday demonstrated it was incomplete when time-varying magnetic fields were present. One of Maxwell's greatest contributions to electromagnetic theory was in showing that a time-varying electric field produces a magnetic field, and therefore we should expect to find later that (21) is not correct when either **E** or the magnetic field varies with time.

Restricting our attention to the static case where **E** does not change with time, consider the dc circuit shown in Fig. 4.5. Two points, A and B, are marked, and (21) states that no work is involved in carrying a unit charge from A through R_2 and R_3 to B and back to A through R_1, or that the sum of the potential differences around any closed path is zero.

Equation (21) is therefore just a more general form of Kirchhoff's circuital law for voltages, more general in that we can apply it to any region where an electric field exists and we are not restricted to a conventional circuit composed of wires, resistances, and batteries. Equation (21) must be amended before we can apply it to time-varying fields. We shall take care of this in Chap. 10, and in Chap. 13 we will then be able to establish the general form of Kirchhoff's voltage law for circuits in which currents and voltages vary with time.

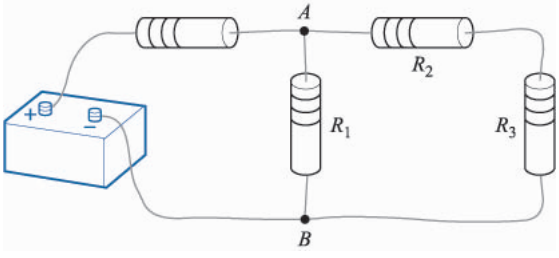

FIGURE 4.5
A simple dc-circuit problem which must be solved by applying $\oint \mathbf{E} \cdot d\mathbf{L} = 0$ in the form of Kirchhoff's voltage law.

Any field that satisfies an equation of the form of (21), (i.e., where the closed line integral of the field is zero) is said to be a *conservative field*. The name arises from the fact that no work is done (or that energy is *conserved*) around a closed path. The gravitational field is also conservative, for any energy expended in moving (raising) an object against the field is recovered exactly when the object is returned (lowered) to its original position. A nonconservative gravitational field could solve our energy problems forever.

Given a *nonconservative* field, it is of course possible that the line integral may be zero for certain closed paths. For example, consider the force field, $\mathbf{F} = \sin \pi\rho\, \mathbf{a}_\phi$. Around a circular path of radius $\rho = \rho_1$, we have $d\mathbf{L} = \rho\, d\phi\, \mathbf{a}_\phi$, and

$$\oint \mathbf{F} \cdot d\mathbf{L} = \int_0^{2\pi} \sin \pi\rho_1 \mathbf{a}_\phi \cdot \rho_1 d\phi\, \mathbf{a}_\phi = \int_0^{2\pi} \rho_1 \sin \pi\rho_1\, d\phi$$

$$= 2\pi\rho_1 \sin \pi\rho_1$$

The integral is zero if $\rho_1 = 1, 2, 3, \ldots$, etc., but it is not zero for other values of ρ_1, or for most other closed paths, and the given field is not conservative. A conservative field must yield a zero value for the line integral around every possible closed path.

✔ **D4.6.** If we take the zero reference for potential at infinity, find the potential at $(0, 0, 2)$ caused by this charge configuration in free space: (*a*) 12 nC/m on the line $\rho = 2.5$ m, $z = 0$; (*b*) point charge of 18 nC at $(1, 2, -1)$; (*c*) 12 nC/m on the line $y = 2.5$, $z = 0$.

Ans. 529 V; 43.2 V; 67.4 V

4.6 POTENTIAL GRADIENT

We now have two methods of determining potential, one directly from the electric field intensity by means of a line integral, and another from the basic charge distribution itself by a volume integral. Neither method is very helpful in determining the fields in most practical problems, however, for as we shall see later, neither the electric field intensity nor the charge distribution is very often known. Preliminary information is much more apt to consist of a description of two equipotential surfaces, such as the statement that we have two parallel conductors of circular cross section at potentials of 100 and −100 V. Perhaps we wish to find the capacitance between the conductors, or the charge and current distribution on the conductors from which losses may be calculated.

These quantities may be easily obtained from the potential field, and our immediate goal will be a simple method of finding the electric field intensity from the potential.

We already have the general line-integral relationship between these quantities,

$$V = -\int \mathbf{E} \cdot d\mathbf{L} \tag{22}$$

but this is much easier to use in the reverse direction: given $\mathbf{E}$, find V.

However, (22) may be applied to a very short element of length $\Delta\mathbf{L}$ along which $\mathbf{E}$ is essentially constant, leading to an incremental potential difference ΔV,

$$\Delta V \doteq -\mathbf{E} \cdot \Delta\mathbf{L} \tag{23}$$

Let us see first if we can determine any new information about the relation of V to $\mathbf{E}$ from this equation. Consider a general region of space, as shown in Fig. 4.6, in which $\mathbf{E}$ and V both change as we move from point to point. Equation (23) tells us to choose an incremental vector element of length $\Delta\mathbf{L} = \Delta L\, \mathbf{a}_L$ and multiply its magnitude by the component of $\mathbf{E}$ in the direction of $\mathbf{a}_L$ (one interpretation of the dot product) to obtain the small potential difference between the final and initial points of $V\mathbf{L}$.

If we designate the angle between $\Delta\mathbf{L}$ and $\mathbf{E}$ as θ, then

$$\Delta V \doteq -E\Delta L \cos\theta$$

We now wish to pass to the limit and consider the derivative dV/dL. To do this, we need to show that V may be interpreted as a *function* $V(x, y, z)$. So far, V is merely the result of the line integral (22). If we assume a specified starting point or zero reference and then let our end point be (x, y, z), we know that the result of the integration is a unique function of the end point (x, y, z) because $\mathbf{E}$ is a conservative field. Therefore V is a single-valued function $V(x, y, z)$. We may then pass to the limit and obtain

$$\frac{dV}{dL} = -E\cos\theta$$

In which direction should $\Delta\mathbf{L}$ be placed to obtain a maximum value of ΔV? Remember that $\mathbf{E}$ is a definite value at the point at which we are working and is independent of the direction of $\Delta\mathbf{L}$. The magnitude ΔL is also constant, and our

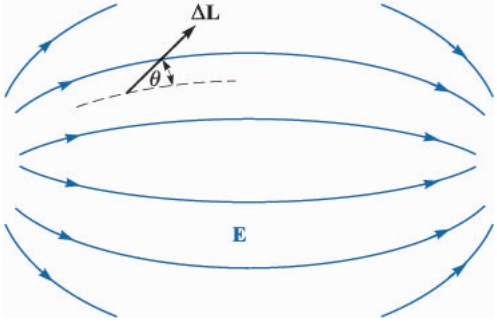

FIGURE 4.6
A vector incremental element of length $\Delta\mathbf{L}$ is shown making an angle of θ with an $\mathbf{E}$ field, indicated by its streamlines. The sources of the field are not shown.

variable is $\mathbf{a}_L$, the unit vector showing the direction of $\Delta\mathbf{L}$. It is obvious that the maximum positive increment of potential, $\Delta V_{\max}$, will occur when $\cos\theta$ is -1, or $\Delta\mathbf{L}$ points in the direction *opposite* to $\mathbf{E}$. For this condition,

$$\left.\frac{dV}{dL}\right|_{\max} = E$$

This little exercise shows us two characteristics of the relationship between $\mathbf{E}$ and V at any point:

1. The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.
2. This maximum value is obtained when the direction of the distance increment is opposite to $\mathbf{E}$ or, in other words, the direction of $\mathbf{E}$ is *opposite* to the direction in which the potential is *increasing* the most rapidly.

Let us now illustrate these relationships in terms of potential. Fig. 4.7 is intended to show the information we have been given about some potential field. It does this by showing the equipotential surfaces (shown as lines in the two-dimensional sketch). We desire information about the electric field intensity at point P. Starting at P, we lay off a small incremental distance $\Delta\mathbf{L}$ in various directions, hunting for that direction in which the potential is changing (increasing) the most rapidly. From the sketch, this direction appears to be left and slightly upward. From our second characteristic above, the electric field intensity is therefore oppositely directed, or to the right and slightly downward at P. Its magnitude is given by dividing the small increase in potential by the small element of length.

It seems likely that the direction in which the potential is increasing the most rapidly is perpendicular to the equipotentials (in the direction of *increasing* potential), and this is correct, for if $\Delta\mathbf{L}$ is directed along an equipotential, $\Delta V = 0$ by our definition of an equipotential surface. But then

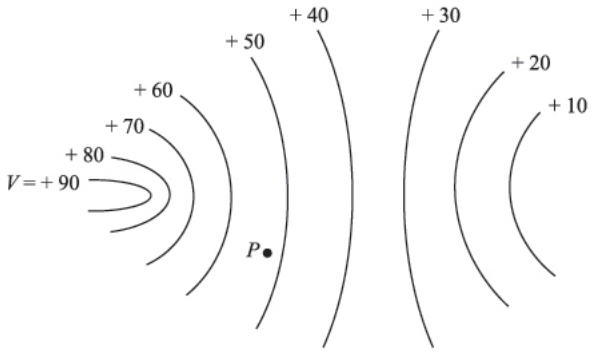

FIGURE 4.7
A potential field is shown by its equipotential surfaces. At any point the $\mathbf{E}$ field is normal to the equipotential surface passing through that point and is directed toward the more negative surfaces.

$$\Delta V = -\mathbf{E} \cdot \Delta \mathbf{L} = 0$$

and since neither $\mathbf{E}$ nor $\Delta \mathbf{L}$ is zero, $\mathbf{E}$ must be perpendicular to this $\Delta \mathbf{L}$ or perpendicular to the equipotentials.

Since the potential field information is more likely to be determined first, let us describe the direction of $\Delta \mathbf{L}$ which leads to a maximum increase in potential mathematically in terms of the potential field rather than the electric field intensity. We do this by letting $\mathbf{a}_N$ be a unit vector normal to the equipotential surface and directed toward the higher potentials. The electric field intensity is then expressed in terms of the potential,

$$\mathbf{E} = -\left.\frac{dV}{dL}\right|_{\max} \mathbf{a}_N \tag{24}$$

which shows that the magnitude of $\mathbf{E}$ is given by the maximum space rate of change of V and the direction of $\mathbf{E}$ is *normal* to the equipotential surface (in the direction of *decreasing* potential).

Since $dV/dL|_{\max}$ occurs when $\Delta \mathbf{L}$ is in the direction of $\mathbf{a}_N$, we may remind ourselves of this fact by letting

$$\left.\frac{dV}{dL}\right|_{\max} = \frac{dV}{dN}$$

and

$$\mathbf{E} = -\frac{dV}{dN}\mathbf{a}_N \tag{25}$$

Equation (24) or (25) serves to provide a physical interpretation of the process of finding the electric field intensity from the potential. Both are descriptive of a general procedure, and we do not intend to use them directly to obtain quantitative information. This procedure leading from V to $\mathbf{E}$ is not unique to this pair of quantities, however, but has appeared as the relationship between a scalar and a vector field in hydraulics, thermodynamics, and magnetics, and indeed in almost every field to which vector analysis has been applied.

The operation on V by which $-\mathbf{E}$ is obtained is known as the *gradient*, and the gradient of a scalar field T is defined as

$$\boxed{\text{Gradient of } T = \text{ grad } T = \frac{dT}{dN}\mathbf{a}_N} \tag{26}$$

where $\mathbf{a}_N$ is a unit vector normal to the equipotential surfaces, and that normal is chosen which points in the direction of increasing values of T.

Using this new term, we now may write the relationship between V and $\mathbf{E}$ as

$$\boxed{\mathbf{E} = -\text{ grad } V} \tag{27}$$

Since we have shown that V is a unique function of x, y, and z, we may take its total differential

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

But we also have

$$dV = -\mathbf{E}\cdot d\mathbf{L} = -E_x\,dx - E_y\,dy - E_z\,dz$$

Since both expressions are true for any dx, dy, and dz, then

$$E_x = -\frac{\partial V}{\partial x}$$
$$E_y = -\frac{\partial V}{\partial y}$$
$$E_z = -\frac{\partial V}{\partial z}$$

These results may be combined vectorially to yield

$$\boxed{\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)} \tag{28}$$

and comparison of (27) and (28) provides us with an expression which may be used to evaluate the gradient in cartesian coordinates,

$$\boxed{\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z} \tag{29}$$

The gradient of a scalar is a vector, and old quizzes show that the unit vectors which are often incorrectly added to the divergence expression appear to be those which were incorrectly removed from the gradient. Once the physical interpretation of the gradient, expressed by (26), is grasped as showing the maximum space rate of change of a scalar quantity and *the direction in which this maximum occurs*, the vector nature of the gradient should be self-evident.

The vector operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

may be used formally as an operator on a scalar, T, ∇T, producing

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{a}_x + \frac{\partial T}{\partial y}\mathbf{a}_y + \frac{\partial T}{\partial z}\mathbf{a}_z$$

from which we see that

$$\boxed{\nabla T = \text{grad}\, T}$$

This allows us to use a very compact expression to relate $\mathbf{E}$ and V,

$$\boxed{\mathbf{E} = -\nabla V} \tag{30}$$

The gradient may be expressed in terms of partial derivatives in other coordinate systems through application of its definition (26). These expressions are derived in Appendix A and repeated below for convenience when dealing with problems having cylindrical or spherical symmetry. They also appear inside the back cover.

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cartesian)}} \tag{31}$$

$$\boxed{\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}} \tag{32}$$

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}} \tag{33}$$

Note that the denominator of each term has the form of one of the components of $d\mathbf{L}$ in that coordinate system, except that partial differentials replace ordinary differentials; for example, $r\sin\theta\, d\phi$ becomes $r\sin\theta\, \partial\phi$.

Let us now hasten to illustrate the gradient concept with an example.

Example 4.3

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Solution. The potential at $P(-4, 5, 6)$ is

$$V_P = 2(-4)^2(3) - 5(6) = 66 \quad \text{V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

The value of **E** at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \quad \text{V/m}$$

The direction of **E** at P is given by the unit vector

$$\begin{aligned}\mathbf{a}_{E,P} &= (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9 \\ &= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z\end{aligned}$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\mathbf{a}_x - 17.71x^2\mathbf{a}_y + 44.3\mathbf{a}_z \quad \text{pC/m}^3$$

Finally, we may use the divergence relationship to find the volume charge density that is the source of the given potential field,

$$\rho_v = \nabla \cdot \mathbf{D} = -35.4y \quad \text{pC/m}^3$$

At P, $\rho_v = -106.2$ pC/m^3.

✔ **D4.7.** A portion of a two-dimensional ($E_z = 0$) potential field is shown in Fig. 4.8. The grid lines are 1 mm apart in the actual field. Determine approximate values for **E** in cartesian coordinates at: (*a*) *a*; (*b*) *b*; (*c*) *c*.

Ans. $-1075\mathbf{a}_y$ V/m; $-600\mathbf{a}_x - 700\mathbf{a}_y$ V/m; $-500\mathbf{a}_x - 650\mathbf{a}_y$ V/m

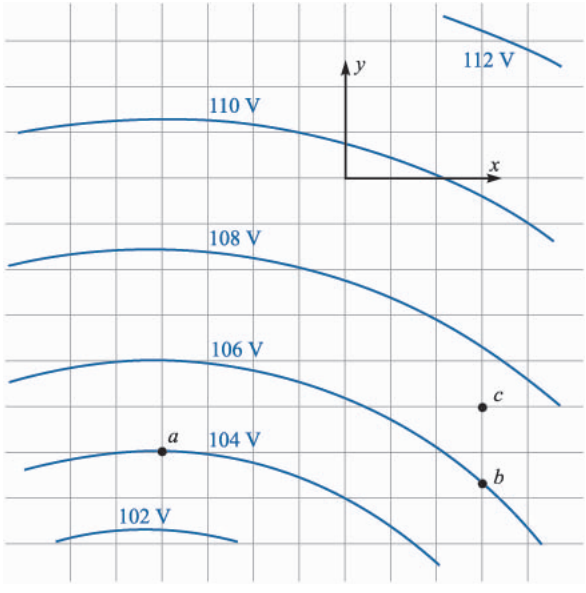

FIGURE 4.8
See Prob. D4.7.

✔ **D4.8.** Given the potential field in cylindrical coordinates, $V = \frac{100}{z^2+1}\rho\cos\phi$ V, and point P at $\rho = 3$ m, $\phi = 60°$, $z = 2$ m, find values at P for: (*a*) V; (*b*) $\mathbf{E}$; (*c*) E; (*d*) dV/dN; (*e*) $\mathbf{a}_N$; (*f*) ρ_v in free space.

Ans. 30.0 V; $-10.00\mathbf{a}_\rho + 8.66\mathbf{a}_\phi + 24.0\mathbf{a}_z$ V/m; 27.4 V/m; $-0.365\mathbf{a}_\rho - 0.316\mathbf{a}_\phi - 0.876\mathbf{a}_z$; 27.4 V/m; -234 pC/m^3.

4.7 THE DIPOLE

The dipole fields which we shall develop in this section are quite important because they form the basis for the behavior of dielectric materials in electric fields, as discussed in part of the following chapter, as well as justifying the use of images, as described in Sec. 5.5 of the next chapter. Moreover, this development will serve to illustrate the importance of the potential concept presented in this chapter.

An *electric dipole*, or simply a *dipole*, is the name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P at which we want to know the electric and potential fields. The dipole is shown in Fig. 4.9*a*. The distant point P is described by the spherical coordinates r, θ, and $\phi - 90°$, in view of the azimuthal symmetry. The positive and negative point charges have separation d and cartesian coordinates $(0, 0, \frac{1}{2}d)$ and $(0, 0, -\frac{1}{2}d)$, respectively.

So much for the geometry. What would we do next? Should we find the total electric field intensity by adding the known fields of each point charge? Would it be easier to find the total potential field first? In either case, having found one, we shall find the other from it before calling the problem solved.

If we choose to find $\mathbf{E}$ first, we shall have two components to keep track of in spherical coordinates (symmetry shows E_ϕ is zero), and then the only way to find V from $\mathbf{E}$ is by use of the line integral. This last step includes establishing a suitable zero reference for potential, since the line integral gives us only the potential difference between the two points at the ends of the integral path.

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

Note that the plane $z = 0$, midway between the two point charges, is the locus of points for which $R_1 = R_2$, and is therefore at zero potential, as are all points at infinity.

For a distant point, $R_1 \doteq R_2$, and the $R_1 R_2$ product in the denominator may be replaced by r^2. The approximation may not be made in the numerator, however, without obtaining the trivial answer that the potential field approaches zero as we go very far away from the dipole. Coming back a little closer to the dipole, we see from Fig. 4.9*b* that $R_2 - R_1$ may be approximated very easily if R_1 and R_2 are assumed to be parallel,

$$R_2 - R_1 \doteq d\cos\theta$$

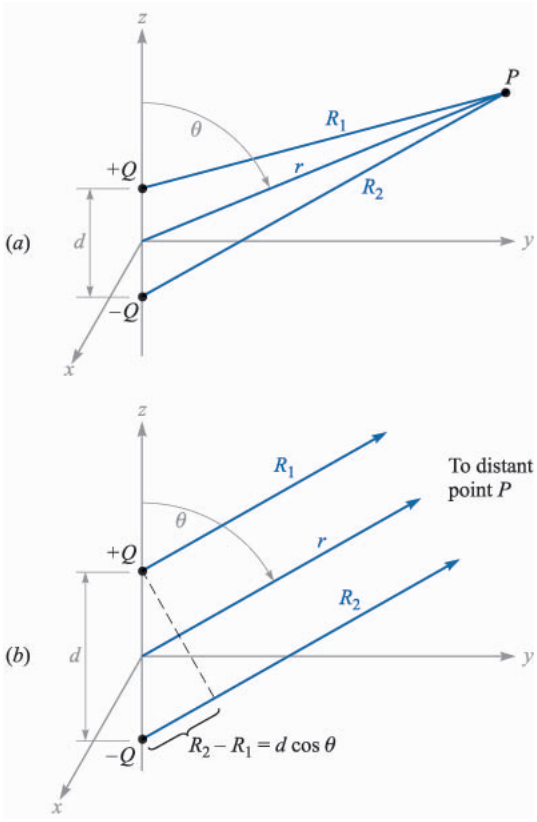

FIGURE 4.9
(*a*) The geometry of the problem of an electric dipole. The dipole moment $\mathbf{p} = Q\mathbf{d}$ is in the $\mathbf{a}_z$ direction. (*b*) For a distant point P, R_1 is essentially parallel to R_2, and we find that $R_2 - R_1 = d\cos\theta$.

The final result is then

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} \tag{34}$$

Again we note that the plane $z = 0$ ($\theta = 90^\circ$) is at zero potential.

Using the gradient relationship in spherical coordinates,

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right)$$

we obtain

$$\mathbf{E} = -\left(-\frac{Qd\cos\theta}{4\pi\epsilon_0 r^3}\mathbf{a}_r - \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\mathbf{a}_\theta\right) \tag{35}$$

or

$$\boxed{\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)} \tag{36}$$

These are the desired distant fields of the dipole, obtained with a very small amount of work. Any student who has several hours to spend may try to work the problem in the reverse direction—the authors consider the process too long and detailed to include, even for effect.

To obtain a plot of the potential field, we may choose a dipole such that $Qd/(4\pi\epsilon_0) = 1$, and then $\cos\theta = Vr^2$. The colored lines in Fig. 4.10 indicate equipotentials for which $V = 0, +0.2, +0.4, +0.6, +0.8$, and $+1$, as indicated. The dipole axis is vertical, with the positive charge on the top. The streamlines for the electric field are obtained by applying the methods of Sec. 2.6 in spherical coordinates,

$$\frac{E_\theta}{E_r} = \frac{r\,d\theta}{dr} = \frac{\sin\theta}{2\cos\theta}$$

or

$$\frac{dr}{r} = 2\cot\theta\,d\theta$$

from which we obtain

$$r = C_1 \sin^2\theta$$

The black streamlines shown in Fig. 4.10 are for $C_1 = 1, 1.5, 2$, and 2.5.

The potential field of the dipole, Eq. (34), may be simplified by making use of the dipole moment. Let us first identify the vector length directed from $-Q$ to $+Q$ as $\mathbf{d}$ and then define the *dipole moment* as $Q\mathbf{d}$ and assign it the symbol $\mathbf{p}$. Thus

$$\boxed{\mathbf{p} = Q\mathbf{d}} \tag{37}$$

The units of $\mathbf{p}$ are C · m.

Since $\mathbf{d} \cdot \mathbf{a}_r = d\cos\theta$, we then have

$$V = \frac{\mathbf{p} \cdot \mathbf{a}_r}{4\pi\epsilon_0 r^2} \tag{38}$$

This result may be generalized as

$$V = \frac{1}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2}\mathbf{p} \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{39}$$

where $\mathbf{r}$ locates the field point P, and $\mathbf{r}'$ determines the dipole center. Equation (39) is independent of any coordinate system.

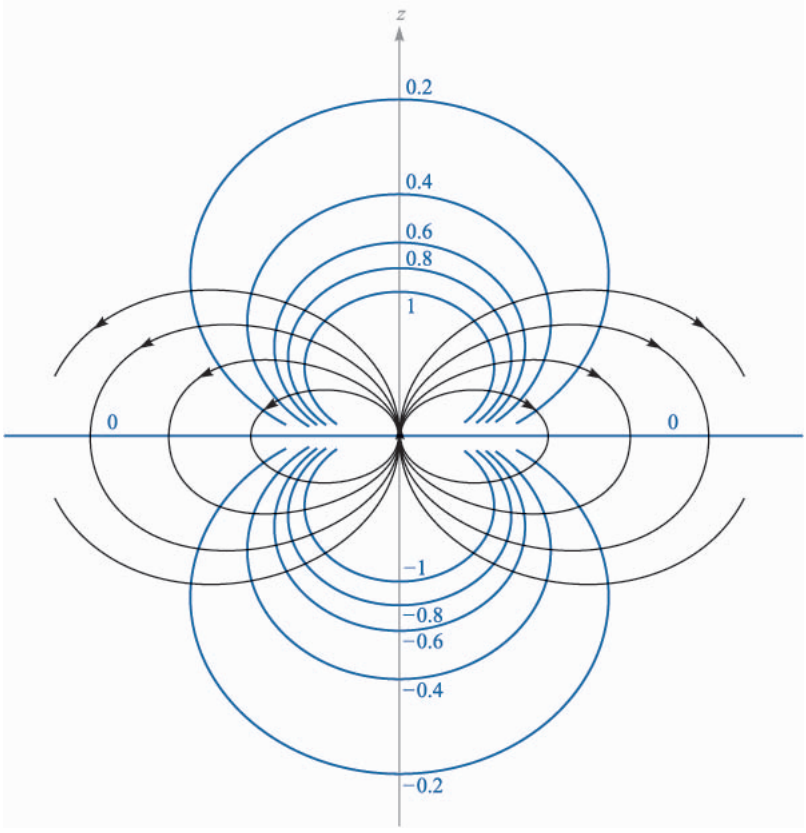

FIGURE 4.10
The electrostatic field of a point dipole with its moment in the $\mathbf{a}_z$ direction. Six equipotential surfaces are labeled with relative values of V.

The dipole moment **p** will appear again when we discuss dielectric materials. Since it is equal to the product of the charge and the separation, neither the dipole moment nor the potential will change as Q increases and **d** decreases, provided the product remains constant. The limiting case of a *point dipole* is achieved when we let **d** approach zero and Q approach infinity such that the product **p** is finite.

Turning our attention to the resultant fields, it is interesting to note that the potential field is now proportional to the inverse *square* of the distance, and the electric field intensity is proportional to the inverse *cube* of the distance from the dipole. Each field falls off faster than the corresponding field for the point charge, but this is no more than we should expect because the opposite charges appear to be closer together at greater distances and to act more like a single point charge of 0 C.

Symmetrical arrangements of larger numbers of point charges produce fields proportional to the inverse of higher and higher powers of r. These charge distributions are called *multipoles*, and they are used in infinite series to approximate more unwieldy charge configurations.

✔ **D4.9.** An electric dipole located at the origin in free space has a moment $\mathbf{p} = 3\mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$ nC · m. (*a*) Find V at $P_A(2, 3, 4)$. (*b*) Find V at $r = 2.5$, $\theta = 30°$, $\phi = 40°$.

Ans. 0.230 V; 1.973 V

✔ **D4.10.** A dipole of moment $\mathbf{p} = 6\mathbf{a}_z$ nC · m is located at the origin in free space. (*a*) Find V at $P(r = 4, \theta = 20°, \phi = 0°)$. (*b*) Find $\mathbf{E}$ at P.

Ans. 3.17 V; $1.584\mathbf{a}_r + 0.288\mathbf{a}_\theta$ V/m

4.8 ENERGY DENSITY IN THE ELECTROSTATIC FIELD

We have introduced the potential concept by considering the work done, or energy expended, in moving a point charge around in an electric field, and now we must tie up the loose ends of that discussion by tracing the energy flow one step further.

Bringing a positive charge from infinity into the field of another positive charge requires work, the work being done by the external source moving the charge. Let us imagine that the external source carries the charge up to a point near the fixed charge and then holds it there. Energy must be conserved, and the energy expended in bringing this charge into position now represents potential energy, for if the external source released its hold on the charge, it would accelerate away from the fixed charge, acquiring kinetic energy of its own and the capability of doing work.

In order to find the potential energy present in a system of charges, we must find the work done by an external source in positioning the charges.

We may start by visualizing an empty universe. Bringing a charge Q_1 from infinity to any position requires no work, for there is no field present.[2] The positioning of Q_2 at a point in the field of Q_1 requires an amount of work given by the product of the charge Q_2 and the potential at that point due to Q_1. We represent this potential as $V_{2,1}$, where the first subscript indicates the location and the second subscript the source. That is, $V_{2,1}$ is the potential at the location of Q_2 due to Q_1. Then

$$\text{Work to position } Q_2 = Q_2 V_{2,1}$$

[2] However, somebody in the workshop at infinity had to do an infinite amount of work to create the point charge in the first place! How much energy is required to bring two half-charges into coincidence to make a unit charge?

Similarly, we may express the work required to position each additional charge in the field of all those already present:

$$\text{Work to position } Q_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$$
$$\text{Work to position } Q_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$

and so forth. The total work is obtained by adding each contribution:

$$\begin{aligned}\text{Total positioning work} &= \text{potential energy of field}\\ &= W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1}\\ &\quad + Q_4 V_{4,2} + Q_4 V_{4,3} + \ldots \end{aligned} \tag{40}$$

Noting the form of a representative term in the above equation,

$$Q_3 V_{3,1} = Q_3 \frac{Q_1}{4\pi\epsilon_0 R_{13}} = Q_1 \frac{Q_3}{4\pi\epsilon_0 R_{31}}$$

where R_{13} and R_{31} each represent the scalar distance between Q_1 and Q_3, we see that it might equally well have been written as $Q_1 V_{1,3}$. If each term of the total energy expression is replaced by its equal, we have

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \ldots \tag{41}$$

Adding the two energy expressions (40) and (41) gives us a chance to simplify the result a little:

$$\begin{aligned} 2W_E = {} & Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \ldots)\\ & + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \ldots)\\ & + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \ldots)\\ & + \ldots \end{aligned}$$

Each sum of potentials in parentheses is the combined potential due to all the charges except for the charge at the point where this combined potential is being found. In other words,

$$V_{1,2} + V_{1,3} + V_{1,4} + \ldots = V_1$$

the potential at the location of Q_1 due to the presence of $Q_2, Q_3, \ldots$. We therefore have

$$W_E = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \ldots) = \tfrac{1}{2} \sum_{m=1}^{m=N} Q_m V_m \tag{42}$$

In order to obtain an expression for the energy stored in a region of continuous charge distribution, each charge is replaced by $\rho_v dv$, and the summation becomes an integral,

$$W_E = \tfrac{1}{2} \int_{\text{vol}} \rho_v V \, dv \tag{43}$$

Equations (42) and (43) allow us to find the total potential energy present in a system of point charges or distributed volume charge density. Similar expressions may be easily written in terms of line or surface charge density. Usually we prefer to use (43) and let it represent all the various types of charge which may have to be considered. This may always be done by considering point charges, line charge density, or surface charge density as continuous distributions of volume charge density over very small regions. We shall illustrate such a procedure with an example shortly.

Before we undertake any interpretation of this result, we should consider a few lines of more difficult vector analysis and obtain an expression equivalent to (43) but written in terms of **E** and **D**.

We begin by making the expression a little bit longer. Using Maxwell's first equation, replace ρ_v by its equal $\nabla \cdot \mathbf{D}$ and make use of a vector identity which is true for any scalar function V and any vector function **D**,

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V) \tag{44}$$

This may be proved readily by expansion in cartesian coordinates. We then have, successively,

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \rho_v V dv = \tfrac{1}{2}\int_{\text{vol}} (\nabla \cdot \mathbf{D}) V \, dv$$

$$= \tfrac{1}{2}\int_{\text{vol}} [\nabla \cdot (V\mathbf{D}) - \mathbf{D} \cdot (\nabla V)] \, dv$$

Using the divergence theorem from the last chapter, the first volume integral of the last equation is changed into a closed surface integral, where the closed surface surrounds the volume considered. This volume, first appearing in (43), must contain *every* charge, and there can then be no charges outside of the volume. We may therefore consider the volume as *infinite* in extent if we wish. We have

$$W_E = \tfrac{1}{2}\oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \tfrac{1}{2}\int_{\text{vol}} \mathbf{D} \cdot (\nabla V) \, dv$$

The surface integral is equal to zero, for over this closed surface surrounding the universe we see that V is approaching zero at least as rapidly as $1/r$ (the charges look like a point charge from there), **D** is approaching zero at least as rapidly as $1/r^2$, while the differential element of surface, looking more and more like a portion of a sphere, is increasing only as r^2. The integrand therefore approaches zero at least as rapidly as $1/r$. In the limit the integrand and the integral are zero. Substituting $\mathbf{E} = -\nabla V$ in the remaining volume integral, we have our answer,

$$\boxed{W_E = \tfrac{1}{2}\int_{\text{vol}} \mathbf{D} \cdot \mathbf{E} \, dv = \tfrac{1}{2}\int_{\text{vol}} \epsilon_0 E^2 \, dv} \tag{45}$$

Let us now use this last expression to calculate the energy stored in the electrostatic field of a section of a coaxial cable or capacitor of length L. We found in Sec. 3.3 of the previous chapter that

$$D_\rho = \frac{a\rho_S}{\rho}$$

Hence,

$$\mathbf{E} = \frac{a\rho_S}{\epsilon_0 \rho}\mathbf{a}_\rho$$

where ρ_S is the surface charge density on the inner conductor, whose radius is a. Thus,

$$W_E = \tfrac{1}{2}\int_0^L \int_0^{2\pi} \int_a^b \epsilon_0 \frac{a^2\rho_S^2}{\epsilon_0^2\rho^2}\,\rho\, d\rho\, d\phi\, dz = \frac{\pi L a^2 \rho_S^2}{\epsilon_0}\ln\frac{b}{a}$$

This same result may be obtained from (43). We choose the outer conductor as our zero-potential reference, and the potential of the inner cylinder is then

$$V_a = -\int_b^a E_\rho\, d\rho = -\int_b^a \frac{a\rho_S}{\epsilon_0\rho} d\rho = \frac{a\rho_S}{\epsilon_0}\ln\frac{b}{a}$$

The surface charge density ρ_S at $\rho = a$ can be interpreted as a volume charge density $\rho_v = \rho_S/t$, extending from $\rho = a - \frac{1}{2}t$ to $\rho = a + \frac{1}{2}t$, where $t \ll a$. The integrand in (43) is therefore zero everywhere between the cylinders (where the volume charge density is zero), as well as at the outer cylinder (where the potential is zero). The integration is therefore performed only within the thin cylindrical shell at $\rho = a$,

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \rho_v V\, dV = \tfrac{1}{2}\int_0^L \int_0^{2\pi} \int_{a-t/2}^{a+t/2} \frac{\rho_S}{t} a \frac{\rho_S}{\epsilon_0}\ln\frac{b}{a}\rho\, d\rho\, d\phi\, dz$$

from which

$$W_E = \frac{a^2\rho_S^2 \ln(b/a)}{\epsilon_0}\pi L$$

once again.

This expression takes on a more familiar form if we recognize the total charge on the inner conductor as $Q = 2\pi a L \rho_S$. Combining this with the potential difference between the cylinders, V_a, we see that

$$W_E = \tfrac{1}{2} Q V_a$$

which should be familiar as the energy stored in a capacitor.

The question of where the energy is stored in an electric field has not yet been answered. Potential energy can never be pinned down precisely in terms of physical location. Someone lifts a pencil, and the pencil acquires potential energy. Is the energy stored in the molecules of the pencil, in the gravitational

field between the pencil and the earth, or in some obscure place? Is the energy in a capacitor stored in the charges themselves, in the field, or where? No one can offer any proof for his or her own private opinion, and the matter of deciding may be left to the philosophers.

Electromagnetic field theory makes it easy to believe that the energy of an electric field or a charge distribution is stored in the field itself, for if we take (45), an exact and rigorously correct expression,

$$W_E = \tfrac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\, dv$$

and write it on a differential basis,

$$dW_E = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E}\, dv$$

or

$$\frac{dW_E}{dv} = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} \tag{46}$$

we obtain a quantity $\frac{1}{2}\mathbf{D}\cdot\mathbf{E}$, which has the dimensions of an energy density, or joules per cubic meter. We know that if we integrate this energy density over the entire field-containing volume, the result is truly the total energy present, but we have no more justification for saying that the energy stored in each differential volume element dv is $\mathbf{D}\cdot\mathbf{E}\, dv$ than we have for looking at (43) and saying that the stored energy is $\frac{1}{2}\rho_v V dv$. The interpretation afforded by (46), however, is a convenient one, and we shall use it until proved wrong.

✔ **D4.11.** Find the energy stored in free space for the region $2\,\text{mm} < r < 3\,\text{mm}$, $0 < \theta < 90°$, $0 < \phi < 90°$, given the potential field $V =$: (*a*) $\dfrac{200}{r}$ V; (*b*) $\dfrac{300\cos\theta}{r^2}$ V.

Ans. 1.391 pJ; 36.7 J

SUGGESTED REFERENCES

1. Attwood, S. S.: "Electric and Magnetic Fields," 3d ed., John Wiley & Sons, Inc., New York, 1949. There are a large number of well-drawn field maps of various charge distributions, including the dipole field. Vector analysis is not used.
2. Skilling, H. H.: (see Suggested References for Chap. 3). Gradient is described on pp. 19–21.
3. Thomas, G. B., Jr., and R. L. Finney: (see Suggested References for Chap. 1). The directional derivative and the gradient are presented on pp. 823–830.

PROBLEMS

4.1 The value of $\mathbf{E}$ at $P(\rho = 2,\ \phi = 40^\circ,\ z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20-μC charge a distance of 6 μm in the direction of: (*a*) $\mathbf{a}_\rho$; (*b*) $\mathbf{a}_\phi$; (*c*) $\mathbf{a}_z$; (*d*) $\mathbf{E}$; (*e*) $\mathbf{G} = 2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z$.

4.2 Let $\mathbf{E} = 400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z$ V/m in the neighborhood of point $P(6, 2, -3)$. Find the incremental work done in moving a 4-C charge a distance of 1 mm in the direction specified by: (*a*) $\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$; (*b*) $-2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z$.

4.3 If $\mathbf{E} = 120\mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a 50-μC charge a distance of 2 mm from: (*a*) $P(1, 2, 3)$ toward $Q(2, 1, 4)$; (*b*) $Q(2, 1, 4)$ toward $P(1, 2, 3)$.

4.4 Find the amount of energy required to move a 6-C charge from the origin to $P(3, 1, -1)$ in the field $\mathbf{E} = 2x\mathbf{a}_x - 3y^2\mathbf{a}_y + 4\mathbf{a}_z$ V/m along the straight-line path $x = -3z$, $y = x + 2z$.

4.5 Compute the value of $\int_A^P \mathbf{G} \cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1, -1, 2)$ and $P(2, 1, 2)$ using the path: (*a*) straight-line segments $A(1, -1, 2)$ to $B(1, 1, 2)$ to $P(2, 1, 2)$; (*b*) straight-line segments $A(1, -1, 2)$ to $C(2, -1, 2)$ to $P(2, 1, 2)$.

4.6 Let $\mathbf{G} = 4x\mathbf{a}_x + 2z\mathbf{a}_y + 2y\mathbf{a}_z$. Given an initial point $P(2, 1, 1)$ and a final point $Q(4, 3, 1)$, find $\int \mathbf{G} \cdot d\mathbf{L}$ using the path: (*a*) straight line: $y = x - 1$, $z = 1$; (*b*) parabola: $6y = x^2 + 2$, $z = 1$.

4.7 Repeat Prob. 6 for $\mathbf{G} = 3xy^2\mathbf{a}_x + 2z\mathbf{a}_y$.

4.8 A point charge Q_1 is located at the origin in free space. Find the work done in carrying a charge Q_2 from: (*a*) $B(r_B, \theta_B, \phi_B)$ to $C(r_A, \theta_B, \phi_B)$ with θ and ϕ held constant; (*b*) $C(r_A, \theta_B, \phi_B)$ to $D(r_A, \theta_A, \phi_B)$ with r and ϕ held constant; (*c*) $D(r_A, \theta_A, \phi_B)$ to $A(r_A, \theta_A, \phi_A)$ with r and θ held constant.

4.9 A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6$ cm in free space. (*a*) Find the absolute potential at $P(r = 1\text{ cm},\ \theta = 25^\circ,\ \phi = 50^\circ)$. (*b*) Find V_{AB}, given points $A(2\text{ cm}, \theta = 30^\circ, \phi = 60^\circ)$ and $B(3\text{ cm}, 45^\circ, 90^\circ)$.

4.10 Given a surface charge density of 8 nC/m^2 on the plane $x = 2$, a line charge density of 30 nC/m on the line $x = 1$, $y = 2$, and a 1-μC point charge at $P(-1, -1, 2)$, find V_{AB} for points $A(3, 4, 0)$ and $B(4, 0, 1)$.

4.11 Let a uniform surface charge density of 5 nC/m^2 be present at the $z = 0$ plane, a uniform line charge density of 8 nC/m be located at $x = 0$, $z = 4$, and a point charge of 2 μC be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$.

4.12 Three point charges, 0.4 μC each, are located at $(0, 0, -1)$, $(0, 0, 0)$, and $(0, 0, 1)$, in free space. (*a*) Find an expression for the absolute potential as a function of z along the line $x = 0$, $y = 1$. (*b*) Sketch $V(z)$.

4.13 Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work

must be done to move one charge to a point equidistant from the other two and on the line joining them?

4.14 Two 6-nC point charges are located at $(1, 0, 0)$ and $(-1, 0, 0)$ in free space. (*a*) Find V at $P(0, 0, z)$. (*b*) Find $V_{\max}$. (*c*) Calculate $|dV/dz|$ on the z axis. (*d*) Find $|dV/dz|_{\max}$.

4.15 Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$, in free space. If the potential at the origin is 100 V, find V at $P(4, 1, 3)$.

4.16 Uniform surface charge distributions of 6, 4, and 2 nC/m^2 are present at $r = 2, 4$, and 6 cm, respectively, in free space. (*a*) Assume $V = 0$ at infinity, and find $V(r)$. (*b*) Calculate V at $r = 1, 3, 5$, and 7 cm. (*c*) Sketch V versus r for $1 < r < 10$ cm.

4.17 Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm, respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at $\rho =$ (*a*) 5 cm; (*b*) 7 cm.

4.18 The nonuniform linear charge density, $\rho_L = 8/(z^2 + 1)$ nC/m, lies along the z axis. Find the potential at $P(\rho = 1, 0, 0)$ in free space if $V = 0$ at $\rho = \infty$.

4.19 The annular surface, 1 cm $< \rho <$ 3 cm, $z = 0$, carries the nonuniform surface charge density $\rho_S = 5\rho$ nC/m^2. Find V at $P(0, 0, 2\text{ cm})$ if $V = 0$ at infinity.

4.20 Fig. 4.11 shows three separate charge distributions in the $z = 0$ plane in free space. (*a*) Find the total charge for each distribution. (*b*) Find the potential at $P(0, 0, 6)$ caused by each of the three charge distributions acting alone. (*c*) Find V_P.

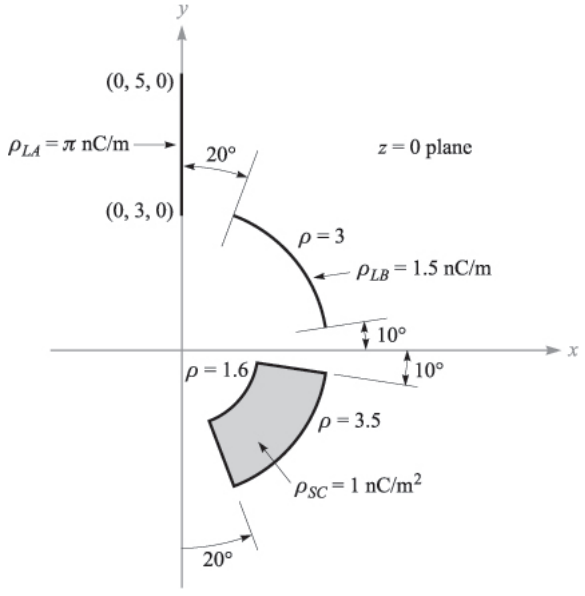

FIGURE 4.11
See Prob. 20.

4.21 Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$ V in free space. Evaluate each of the following quantities at $P(3, 2, -1)$: (*a*) V; (*b*) $|V|$; (*c*) $\mathbf{E}$; (*d*) $|\mathbf{E}|$; (*e*) $\mathbf{a}_N$; (*f*) $\mathbf{D}$.

4.22 It is known that the potential is given as $V = 80r^{0.6}$ V. Assuming free-space conditions, find: (*a*) $\mathbf{E}$; (*b*) the volume charge density at $r = 0.5$ m; (*c*) the total charge lying within the surface $r = 0.6$.

4.23 It is known that the potential is given as $V = 80\rho^{0.6}$ V. Assuming free-space conditions, find: (*a*) $\mathbf{E}$; (*b*) the volume charge density at $\rho = 0.5$ m; (*c*) the total charge lying within the closed surface $\rho = 0.6, 0 < z < 1$.

4.24 Given the potential field $V = 80r^2 \cos\theta$ and a point $P(2.5, \theta = 30°, \phi = 60°)$ in free space, find at P: (*a*) V; (*b*) $\mathbf{E}$; (*c*) $\mathbf{D}$; (*d*) ρ_v; (*e*) dV/dN; (*f*) $\mathbf{a}_N$

4.25 Within the cylinder $\rho = 2, 0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho \sin\phi$ V. (*a*) Find $V, \mathbf{E}, \mathbf{D}$, and ρ_v at $P(1, 60°, 0.5)$ in free space. (*b*) How much charge lies within the cylinder?

4.26 A dipole having $Qd/(4\pi\epsilon_0) = 100\text{V} \cdot \text{m}^2$ is located at the origin in free space and aligned so that its moment is in the $\mathbf{a}_z$ direction. (*a*) Sketch $|V(r = 1, \theta, \phi = 0)|$ versus θ on polar graph paper (homemade if you wish). (*b*) Sketch $|\mathbf{E}(r = 1, \theta, \phi = 0)|$ versus θ on polar paper.

4.27 Two point charges, 1 nC at $(0, 0, 0.1)$ and -1 nC at $(0, 0, -0.1)$, are in free space. (*a*) Calculate V at $P(0.3, 0, 0.4)$, (*b*) Calculate $|\mathbf{E}|$ at P. (*c*) Now treat the two charges as a dipole at the origin and find V at P.

4.28 A dipole located at the origin in free space has a moment $\mathbf{p} = 2 \times 10^{-9}\mathbf{a}_z$ C · m. At what points on the line $y = z, x = 0$ is: (*a*) $|E_\theta| = 1$ mV/m? (*b*) $|E_r| = 1$ mV/m?

4.29 A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC · m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$.

4.30 A dipole, having a moment of $\mathbf{p} = 2\mathbf{a}_z$ nC · m, is located at the origin in free space. Give the magnitude of $\mathbf{E}$ and its direction $\mathbf{a}_E$ in cartesian components at $r = 100$ m, $\phi = 90°$, and $\theta =$: (*a*) 0°; (*b*) 30°; (*c*) 90°.

4.31 A potential field in free space is expressed as $V = 20/(xyz)$ V. (*a*) Find the total energy stored within the cube $1 < x, y, z < 2$. (*b*) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube?

4.32 In the region of free space where $2 < r < 3, 0.4\pi < \theta < 0.6\pi$, $0 < \phi < \pi/2$, let $\mathbf{E} = \dfrac{k}{r^2}\mathbf{a}_r$. (*a*) Find a positive value for k so that the total energy stored is exactly 1 J. (*b*) Show that the surface $\theta = 0.6\pi$ is an equipotential surface. (*c*) Find V_{AB} given points $A(2, \theta = \pi/2, \phi = \pi/3)$ and $B(3, \pi/2, \pi/4)$.

4.33 A copper sphere of radius 4 cm carries a uniformly distributed total charge of 5 μC on its surface in free space. (*a*) Use Gauss's law to find $\mathbf{D}$ external to the sphere. (*b*) Calculate the total energy stored in the electrostatic field. (*c*) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere.

4.34 Given the potential field in free space, $V = 80\phi \mathbf{a}_\phi$ V (cyl. coord.), find: (*a*) the energy stored in the region $2 < \rho < 4$ cm, $0 < \phi < 0.2\pi$, $0 < z < 1$ m; (*b*) the potential difference V_{AB} for $A(3\text{ cm}, \phi = 0, z = 0)$ and $B(3\text{ cm}, 0.2\pi, 1\text{ m})$; (*c*) the maximum value of the energy density in the specified region.

4.35 Four 0.8-nC point charges are located in free space at the corners of a square 4 cm on a side. (*a*) Find the total potential energy stored. (*b*) A fifth 0.8-μC charge is installed at the center of the square. Again find the total stored energy.