CHAPTER
5

CONDUCTORS, DIELECTRICS, AND CAPACITANCE

In this chapter we intend to apply the laws and methods of the previous chapters to some of the materials with which an engineer must work. After defining current and current density and developing the fundamental continuity equation, we shall consider a conducting material and present Ohm's law in both its microscopic and macroscopic forms. With these results we may calculate resistance values for a few of the simpler geometrical forms that resistors may assume. Conditions which must be met at conductor boundaries are next obtained, and this knowledge enables us to introduce the use of images.

After a brief consideration of a general semiconductor, we shall investigate the polarization of dielectric materials and define relative permittivity, or the dielectric constant, an important engineering parameter. Having both conductors and dielectrics, we may then put them together to form capacitors. Most of the work of the previous chapters will be required to determine the capacitance of the several capacitors which we shall construct.

The fundamental electromagnetic principles on which resistors and capacitors depend are really the subject of this chapter; the inductor will not be introduced until Chap. 9.

5.1 CURRENT AND CURRENT DENSITY

Electric charges in motion constitute a *current*. The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I, and therefore

$$I = \frac{dQ}{dt} \tag{1}$$

Current is thus defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons, as we shall see shortly.

In field theory we are usually interested in events occurring at a point rather than within some large region, and we shall find the concept of *current density*, measured in amperes per square meter (A/m²), more useful. Current density is a vector[1] represented by **J**.

The increment of current ΔI crossing an incremental surface ΔS normal to the current density is

$$\Delta I = J_N \Delta S$$

and in the case where the current density is not perpendicular to the surface,

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

Total current is obtained by integrating,

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{S}} \tag{2}$$

Current density may be related to the velocity of volume charge density at a point. Consider the element of charge $\Delta Q = \rho_v \Delta v = \rho_v \, \Delta S \, \Delta L$, as shown in Fig. 5.1a. To simplify the explanation, let us assume that the charge element is oriented with its edges parallel to the coordinate axes, and that it possesses only an x component of velocity. In the time interval Δt, the element of charge has moved a distance Δx, as indicated in Fig. 5.1b. We have therefore moved a charge $\Delta Q = \rho_v \, \Delta S \, \Delta x$ through a reference plane perpendicular to the direction of motion in a time increment Δt, and the resultant current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta x}{\Delta t}$$

As we take the limit with respect to time, we have

$$\Delta I = \rho_v \, \Delta S \, v_x$$

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[1] Current is not a vector, for it is easy to visualize a problem in which a total current I in a conductor of nonuniform cross section (such as a sphere) may have a different direction at each point of a given cross section. Current in an exceedingly fine wire, or a *filamentary current*, is occasionally defined as a vector, but we usually prefer to be consistent and give the direction to the filament, or path, and not to the current.
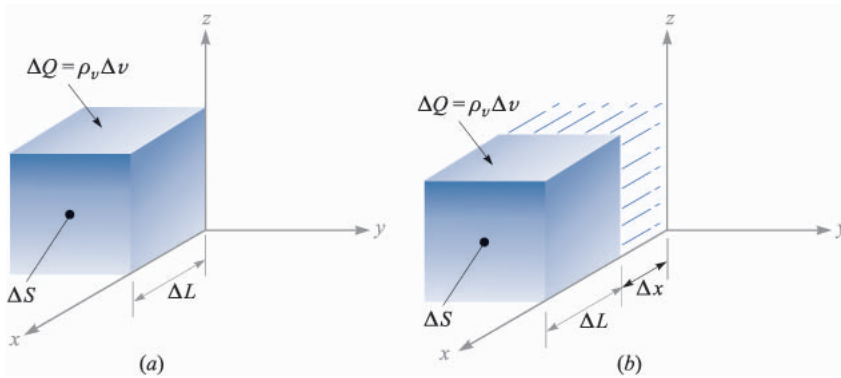
FIGURE 5.1
An increment of charge, $\Delta Q = \rho_v \Delta S\, \Delta L$, which moves a distance Δx in a time Δt, produces a component of current density in the limit of $J_x = \rho_v v_x$.

where v_x represents the x component of the velocity $\mathbf{v}$.[2] In terms of current density, we find

$$J_x = \rho_v\, v_x$$

and in general

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}} \qquad (3)$$

This last result shows very clearly that charge in motion constitutes a current. We call this type of current a *convention current*, and $\mathbf{J}$ or $\rho_v\mathbf{v}$ is the *convection current density*. Note that the convection current density is related linearly to charge density as well as to velocity. The mass rate of flow of cars (cars per square foot per second) in the Holland Tunnel could be increased either by raising the density of cars per cubic foot, or by going to higher speeds, if the drivers were capable of doing so.

✔ **D5.1.** Given the vector current density $\mathbf{J} = 10\rho^2 z\mathbf{a}_\rho - 4\rho\cos^2\phi\,\mathbf{a}_\phi$ A/m²: (*a*) find the current density at $P(\rho = 3,\ \phi = 30°,\ z = 2)$; (*b*) determine the total current flowing outward through the circular band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

Ans. $180\mathbf{a}_\rho - 9\mathbf{a}_\phi$ A/m²; 518 A

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[2] The lowercase v is used both for volume and velocity. Note, however, that velocity always appears as a vector $\mathbf{v}$, a component v_x, or a magnitude $|\mathbf{v}|$, while volume appears only in differential form as dv or Δv.

5.2 CONTINUITY OF CURRENT

Although we are supposed to be studying static fields at this time, the introduction of the concept of current is logically followed by a discussion of the conservation of charge and the continuity equation. The principle of conservation of charge states simply that charges can be neither created nor destroyed, although equal amounts of positive and negative charge may be *simultaneously* created, obtained by separation, destroyed, or lost by recombination.

The continuity equation follows from this principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

and this *outward flow* of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i, then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} \tag{4}$$

It might be well to answer here an often-asked question. "Isn't there a sign error? I thought $I = dQ/dt$." The presence or absence of a negative sign depends on what current and charge we consider. In circuit theory we usually associate the current flow *into* one terminal of a capacitor with the time rate of increase of charge on that plate. The current of (4), however, is an *outward-flowing* current.

Equation (4) is the integral form of the continuity equation, and the differential, or point, form is obtained by using the divergence theorem to change the surface integral into a volume integral:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J}) \, dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J}) \, dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v \, dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J}) \, dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t} \, dv$$

Since the expression is true for any volume, however small, it is true for an incremental volume,

$$(\nabla \cdot \mathbf{J}) \, \Delta v = -\frac{\partial \rho_v}{\partial t} \Delta v$$

from which we have our point form of the continuity equation,

$$(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t} \qquad (5)$$

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

As a numerical example illustrating some of the concepts from the last two sections, let us consider a current density that is directed radially outward and decreases exponentially with time,

$$\mathbf{J} = \frac{1}{r}e^{-t}\mathbf{a}_r \quad \text{A/m}^2$$

Selecting an instant of time $t = 1\,\text{s}$, we may calculate the total outward current at $r = 5\,\text{m}$:

$$I = J_r S = \left(\tfrac{1}{5}e^{-1}\right)(4\pi 5^2) = 23.1 \quad \text{A}$$

At the same instant, but for a slightly larger radius, $r = 6\,\text{m}$, we have

$$I = J_r S = \left(\tfrac{1}{6}e^{-1}\right)(4\pi 6^2) = 27.7 \quad \text{A}$$

Thus, the total current is larger at $r = 6$ than it is at $r = 5$.

To see why this happens, we need to look at the volume charge density and the velocity. We use the continuity equation first:

$$-\frac{\partial \rho_v}{\partial t} = \nabla \cdot \mathbf{J} = \nabla \cdot \left(\frac{1}{r}e^{-t}\mathbf{a}_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r}e^{-t}\right) = \frac{1}{r^2}e^{-t}$$

We next seek the volume charge density by integrating with respect to t. Since ρ_v is given by a partial derivative with respect to time, the "constant" of integration may be a function of r:

$$\rho_v = -\int \frac{1}{r^2}e^{-t}\,dt + K(r) = \frac{1}{r^2}e^{-t} + K(r)$$

If we assume that $\rho_v \to 0$ as $t \to \infty$, then $K(r) = 0$, and

$$\rho_v = \frac{1}{r^2}e^{-t} \quad \text{C/m}^3$$

We may now use $\mathbf{J} = \rho_v \mathbf{v}$ to find the velocity,

$$v_r = \frac{J_r}{\rho_v} = \frac{\dfrac{1}{r}e^{-t}}{\dfrac{1}{r^2}e^{-t}} = r \quad \text{m/s}$$

The velocity is greater at $r = 6$ than it is at $r = 5$, and we see that some (unspecified) force is accelerating the charge density in an outward direction.

In summary, we have a current density that is inversely proportional to r, a charge density that is inversely proportional to r^2, and a velocity and total current that are proportional to r. All quantities vary as e^{-t}.

✔ **D5.2.** Current density is given in cylindrical coordinates as $\mathbf{J} = -10^6 z^{1.5} \mathbf{a}_z$ A/m^2 in the region $0 \le \rho \le 20\,\mu$m; for $\rho \ge 20\,\mu$m, $\mathbf{J} = 0$. (*a*) Find the total current crossing the surface $z = 0.1$ m in the $\mathbf{a}_z$ direction. (*b*) If the charge velocity is 2×10^6 m/s at $z = 0.1$ m, find ρ_v there. (*c*) If the volume charge density at $z = 0.15$ m is -2000 C/m^3, find the charge velocity there.

Ans. -39.7 mA; -15.81 kC/m^3; -2900 m/s

5.3 METALLIC CONDUCTORS

Physicists today describe the behavior of the electrons surrounding the positive atomic nucleus in terms of the total energy of the electron with respect to a zero reference level for an electron at an infinite distance from the nucleus. The total energy is the sum of the kinetic and potential energies, and since energy must be given to an electron to pull it away from the nucleus, the energy of every electron in the atom is a negative quantity. Even though the picture has some limitations, it is convenient to associate these energy values with orbits surrounding the nucleus, the more negative energies corresponding to orbits of smaller radius. According to the quantum theory, only certain discrete energy levels, or energy states, are permissible in a given atom, and an electron must therefore absorb or emit discrete amounts of energy, or quanta, in passing from one level to another. A normal atom at absolute zero temperature has an electron occupying every one of the lower energy shells, starting outward from the nucleus and continuing until the supply of electrons is exhausted.

In a crystalline solid, such as a metal or a diamond, atoms are packed closely together, many more electrons are present, and many more permissible energy levels are available because of the interaction forces between adjacent atoms. We find that the energies which may be possessed by electrons are grouped into broad ranges, or "bands," each band consisting of very numerous, closely spaced, discrete levels. At a temperature of absolute zero, the normal solid also has every level occupied, starting with the lowest and proceeding in order until all the electrons are located. The electrons with the highest (least negative) energy levels, the valence electrons, are located in the *valence band.* If there are permissible higher-energy levels in the valence band, or if the valence band merges smoothly into a *conduction band*, then additional kinetic energy may be given to the valence electrons by an external field, resulting in an electron flow. The solid is called a *metallic conductor*. The filled valence band and the unfilled conduction band for a conductor at 0 K are suggested by the sketch in Fig. 5.2*a*.
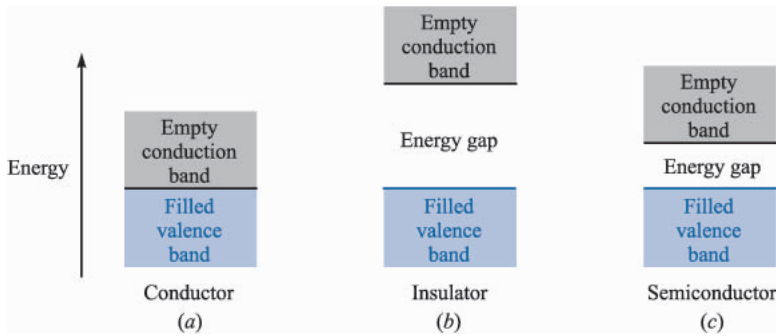
FIGURE 5.2
The energy-band structure in three different types of materials at 0 K. (*a*) The conductor exhibits no energy gap between the valence and conduction bands. (*b*) The insulator shows a large energy gap. (*c*) The semiconductor has only a small energy gap.

If, however, the electron with the greatest energy occupies the top level in the valence band and a gap exists between the valence band and the conduction band, then the electron cannot accept additional energy in small amounts, and the material is an insulator. This band structure is indicated in Fig. 5.2*b*. Note that if a relatively large amount of energy can be transferred to the electron, it may be sufficiently excited to jump the gap into the next band where conduction can occur easily. Here the insulator breaks down.

An intermediate condition occurs when only a small "forbidden region" separates the two bands, as illustrated by Fig. 5.2*c*. Small amounts of energy in the form of heat, light, or an electric field may raise the energy of the electrons at the top of the filled band and provide the basis for conduction. These materials are insulators which display many of the properties of conductors and are called *semiconductors*.

Let us first consider the conductor. Here the valence electrons, or *conduction*, or *free*, electrons, move under the influence of an electric field. With a field **E**, an electron having a charge $Q = -e$ will experience a force

$$\mathbf{F} = -e\mathbf{E}$$

In free space the electron would accelerate and continuously increase its velocity (and energy); in the crystalline material the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity $\mathbf{v}_d$ is termed the *drift velocity*, and it is linearly related to the electric field intensity by the *mobility* of the electron in the given material. We designate mobility by the symbol μ (mu), so that

$$\mathbf{v}_d = -\mu_e \mathbf{E} \tag{6}$$

where μ_ϵ is the mobility of an electron and is positive by definition. Note that the electron velocity is in a direction opposite to the direction of **E**. Equation (6) also

shows that mobility is measured in the units of square meters per volt-second; typical values[3] are 0.0012 for aluminum, 0.0032 for copper, and 0.0056 for silver.

For these good conductors a drift velocity of a few inches per second is sufficient to produce a noticeable temperature rise and can cause the wire to melt if the heat cannot be quickly removed by thermal conduction or radiation.

Substituting (6) into Eq. (3) of Sec. 5.1, we obtain

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E} \tag{7}$$

where ρ_e is the free-electron charge density, a negative value. The total charge density ρ_v is zero, since equal positive and negative charge is present in the neutral material. The negative value of ρ_e and the minus sign lead to a current density $\mathbf{J}$ that is in the same direction as the electric field intensity $\mathbf{E}$.

The relationship between $\mathbf{J}$ and $\mathbf{E}$ for a metallic conductor, however, is also specified by the conductivity σ (sigma),

$$\mathbf{J} = \sigma \mathbf{E} \tag{8}$$

where σ is measured is siemens[4] per meter (S/m). One siemens (1 S) is the basic unit of conductance in the SI system, and is defined as one ampere per volt. Formerly, the unit of conductance was called the mho and symbolized by an *inverted* Ω. Just as the siemens honors the Siemens brothers, the reciprocal unit of resistance which we call the ohm (1 Ω is one volt per ampere) honors Georg Simon Ohm, a German physicist who first described the current-voltage relationship implied by (8). We call this equation the *point form of Ohm's law*; we shall look at the more common form of Ohm's law shortly.

First, however, it is informative to note the conductivity of several metallic conductors; typical values (in siemens per meter) are 3.82×10^7 for aluminum, 5.80×10^7 for copper, and 6.17×10^7 for silver. Data for other conductors may be found in Appendix C. On seeing data such as these, it is only natural to assume that we are being presented with *constant* values; this is essentially true. Metallic conductors obey Ohm's law quite faithfully, and it is a *linear* relationship; the conductivity is constant over wide ranges of current density and electric field intensity. Ohm's law and the metallic conductors are also described as *isotropic*, or having the same properties in every direction. A material which is not isotropic is called *anisotropic*, and we shall mention such a material a few pages from now.

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[3] Wert and Thomson, p. 238, listed in the Suggested References at the end of this chapter.

[4] This is the family name of two German-born brothers, Karl Wilhelm and Werner von Siemens, who were famous engineer-inventors in the nineteenth century. Karl became a British subject and was knighted, becoming Sir William Siemens.

The conductivity is a function of temperature, however. The resistivity, which is the reciprocal of the conductivity, varies almost linearly with temperature in the region of room temperature, and for aluminum, copper, and silver it increases about 0.4 percent for a 1 K rise in temperature.[5] For several metals the resistivity drops abruptly to zero at a temperature of a few kelvin; this property is termed *superconductivity*. Copper and silver are not superconductors, although aluminum is (for temperatures below 1.14 K).

If we now combine (7) and (8), the conductivity may be expressed in terms of the charge density and the electron mobility,

$$\boxed{\sigma = -\rho_e \mu_e} \tag{9}$$

From the definition of mobility (6), it is now satisfying to note that a higher temperature infers a greater crystalline lattice vibration, more impeded electron progress for a given electric field strength, lower drift velocity, lower mobility, lower conductivity from (9), and higher resistivity as stated.
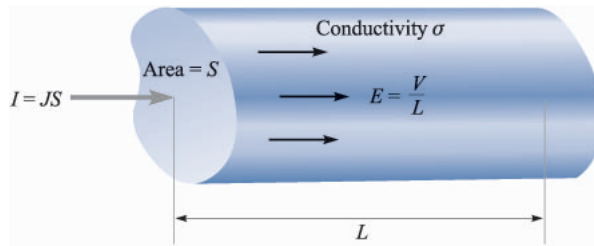
The application of Ohm's law in point form to a macroscopic (visible to the naked eye) region leads to a more familiar form. Initially, let us assume that **J** and **E** are *uniform*, as they are in the cylindrical region shown in Fig. 5.3. Since they are uniform,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = J\,S \tag{10}$$

and

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L} = -\mathbf{E} \cdot \int_b^a d\mathbf{L} = -\mathbf{E} \cdot \mathbf{L}_{ba}$$

$$= \mathbf{E} \cdot \mathbf{L}_{ab} \tag{11}$$

or

$$V = EL$$

Conductivity σ

Area = S

$I = JS$

$E = \dfrac{V}{L}$

L

FIGURE 5.3
Uniform current density **J** and electric field intensity **E** in a cylindrical region of length L and cross-sectional area S. Here $V = IR$, where $R = L/\sigma S$.

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[5] Copious temperature data for conducting materials are available in the "Standard Handbook for Electrical Engineers," listed among the Suggested References at the end of this chapter.

Thus

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$$

or

$$V = \frac{L}{\sigma S} I$$

The ratio of the potential difference between the two ends of the cylinder to the current entering the more positive end, however, is recognized from elementary circuit theory as the *resistance* of the cylinder, and therefore

$$\boxed{V = IR} \tag{12}$$

where

$$\boxed{R = \frac{L}{\sigma S}} \tag{13}$$

Equation (12) is, of course, known as *Ohm's law*, and (13) enables us to compute the resistance R, measured in ohms (abbreviated as Ω), of conducting objects which possess uniform fields. If the fields are not uniform, the resistance may still be defined as the ratio of V to I, where V is the potential difference between two specified equipotential surfaces in the material and I is the total current crossing the more positive surface into the material. From the general integral relationships in (10) and (11), and from Ohm's law (8), we may write this general expression for resistance when the fields are nonuniform,

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}} \tag{14}$$

The line integral is taken between two equipotential surfaces in the conductor, and the surface integral is evaluated over the more positive of these two equipotentials. We cannot solve these nonuniform problems at this time, but we should be able to solve several of them after perusing Chaps. 6 and 7.

▌▌▌▶Example 5.1

As an example of the determination of the resistance of a cylinder, let us find the resistance of a 1-mile length of #16 copper wire, which has a diameter of 0.0508 in.

Solution. The diameter of the wire is $0.0508 \times 0.0254 = 1.291 \times 10^{-3}$ m, the area of the cross section is $\pi(1.291 \times 10^{-3}/2)^2 = 1.308 \times 10^{-6}$ m^2, and the length is 1609 m. Using a conductivity of 5.80×10^7 S/m, the resistance of the wire is therefore

$$R = \frac{1609}{(5.80 \times 10^7)(1.308 \times 10^{-6})} = 21.2 \quad \Omega$$

This wire can safely carry about 10 A dc, corresponding to a current density of $10/(1.308 \times 10^{-6}) = 7.65 \times 10^6$ A/m^2, or 7.65 A/mm^2. With this current the potential difference between the two ends of the wire is 212 V, the electric field intensity is 0.312 V/m, the drift velocity is 0.000 422 m/s, or a little more than one furlong a week, and the free-electron charge density is -1.81×10^{10} C/m^3, or about one electron in a cube two angstroms on a side.

✔ **D5.3.** Find the magnitude of the current density in a sample of silver for which $\sigma = 6.17 \times 10^7$ S/m and $\mu_e = 0.0056$ m^2/V · s if: (*a*) the drift velocity is 1.5 μm/*s*; (*b*) the electric field intensity is 1 mV/m; (*c*) the sample is a cube 2.5 mm on a side having a voltage of 0.4 mV between opposite faces; (*d*) the sample is a cube 2.5 mm on a side carrying a total current of 0.5 A.

Ans. 16.53 kA/m^2; 61.7 kA/m^2; 9.87 MA/m^2; 80.0 kA/m^2

✔ **D5.4.** A copper conductor has a diameter of 0.6 in and it is 1200 ft long. Assume that it carries a total dc current of 50 A. (*a*) Find the total resistance of the conductor. (*b*) What current density exists in it? (*c*) What is the dc voltage between the conductor ends? (*d*) How much power is dissipated in the wire?

Ans. 0.0346 Ω; 2.74 × 10^5 A/m^2; 1.729 V; 86.4 W

5.4 CONDUCTOR PROPERTIES AND BOUNDARY CONDITIONS

Once again we must temporarily depart from our assumed static conditions and let time vary for a few microseconds to see what happens when the charge distribution is suddenly unbalanced within a conducting material. Let us suppose, for the sake of the argument, that there suddenly appear a number of electrons in the interior of a conductor. The electric fields set up by these electrons are not counteracted by any positive charges, and the electrons therefore begin to accelerate away from each other. This continues until the electrons reach the surface of the conductor or until a number of electrons equal to the number injected have reached the surface.

Here the outward progress of the electrons is stopped, for the material surrounding the conductor is an insulator not possessing a convenient conduction band. No charge may remain within the conductor. If it did, the resulting electric field would force the charges to the surface.

Hence the final result within a conductor is zero charge density, and a surface charge density resides on the exterior surface. This is one of the two characteristics of a good conductor.

The other characteristic, stated for static conditions in which no current may flow, follows directly from Ohm's law: the electric field intensity within the conductor is zero. Physically, we see that if an electric field were present, the

conduction electrons would move and produce a current, thus leading to a non-static condition.

Summarizing for electrostatics, no charge and no electric field may exist at any point *within* a conducting material. Charge may, however, appear on the surface as a surface charge density, and our next investigation concerns the fields *external* to the conductor.

We wish to relate these external fields to the charge on the surface of the conductor. The problem is a simple one, and we may first talk our way to the solution with little mathematics.

If the external electric field intensity is decomposed into two components, one tangential and one normal to the conductor surface, the tangential component is seen to be zero. If it were not zero, a tangential force would be applied to the elements of the surface charge, resulting in their motion and nonstatic conditions. Since static conditions are assumed, the tangential electric field intensity and electric flux density are zero.

Gauss's law answers our questions concerning the normal component. The electric flux leaving a small increment of surface must be equal to the charge residing on that incremental surface. The flux cannot penetrate into the conductor, for the total field there is zero. It must then leave the surface normally. Quantitatively, we may say that the electric flux density in coulombs per square meter leaving the surface normally is equal to the surface charge density in coulombs per square meter, or $D_N = \rho_S$.

If we use some of our previously derived results in making a more careful analysis (and incidentally introducing a general method which we must use later), we should set up a conductor-free space boundary (Fig. 5.4) showing tangential and normal components of **D** and **E** on the free-space side of the boundary. Both fields are zero in the conductor. The tangential field may be determined by applying Sec. 4.5, Eq. (21),

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$
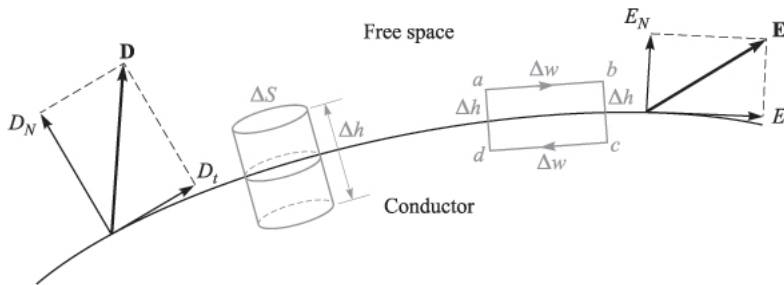
FIGURE 5.4
An appropriate closed path and gaussian surface are used to determine boundary conditions at a conductor-free space boundary; $E_t = 0$ and $D_N = \rho_S$.

around the small closed path *abcda*. The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Remembering that $\mathbf{E} = 0$ within the conductor, we let the length from *a* to *b* or *c* to *d* be Δw and from *b* to *c* or *d* to *a* be Δh, and obtain

$$E_t \Delta w - E_{N,\text{at } b} \tfrac{1}{2} \Delta h + E_{N,\text{at } a} \tfrac{1}{2} \Delta h = 0$$

As we allow Δh to approach zero, keeping Δw small but finite, it makes no difference whether or not the normal fields are equal at *a* and *b*, for Δh causes these products to become negligibly small. Hence

$$E_t \Delta w = 0$$

and therefore

$$E_t = 0$$

The condition on the normal field is found most readily by considering D_N rather than E_N and choosing a small cylinder as the gaussian surface. Let the height be Δh and the area of the top and bottom faces be ΔS. Again we shall let Δh approach zero. Using Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

we integrate over the three distinct surfaces

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

and find that the last two are zero (for different reasons). Then

$$D_N \Delta S = Q = \rho_S \Delta S$$

or

$$D_N = \rho_S$$

These are the desired *boundary conditions* for the conductor-free space boundary in electrostatics,

$$\boxed{D_t = E_t = 0} \tag{15}$$

$$\boxed{D_N = \epsilon_0 E_N = \rho_S} \tag{16}$$

The electric flux leaves the conductor in a direction normal to the surface, and the value of the electric flux density is numerically equal to the surface charge density.

An immediate and important consequence of a zero tangential electric field intensity is the fact that a conductor surface is an equipotential surface. The evaluation of the potential difference between any two points on the surface by the line integral leads to a zero result, because the path may be chosen on the surface itself where $\mathbf{E} \cdot d\mathbf{L} = 0$.

To summarize the principles which apply to conductors in electrostatic fields, we may state that

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface.

Using these three principles, there are a number of quantities that may be calculated at a conductor boundary, given a knowledge of the potential field.

Example 5.2

Given the potential,

$$V = 100(x^2 - y^2)$$

and a point $P(2, -1, 3)$ that is stipulated to lie on a conductor-free space boundary, let us find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_S at P, and also the equation of the conductor surface.

Solution. The potential at point P is

$$V_P = 100[2^2 - (-1)^2] = 300 \quad \text{V}$$

Since the conductor is an equipotential surface, the potential at the entire surface must be 300 V. Moreover, if the conductor is a solid object, then the potential everywhere in and on the conductor is 300 V, for $\mathbf{E} = 0$ within the conductor.

The equation representing the locus of all points having a potential of 300 V is

$$300 = 100(x^2 - y^2)$$

or

$$x^2 - y^2 = 3$$

This is therefore the equation of the conductor surface; it happens to be a hyperbolic cylinder, as shown in Fig. 5.5. Let us assume arbitrarily that the solid conductor lies above and to the right of the equipotential surface at point P, while free space is down and to the left.
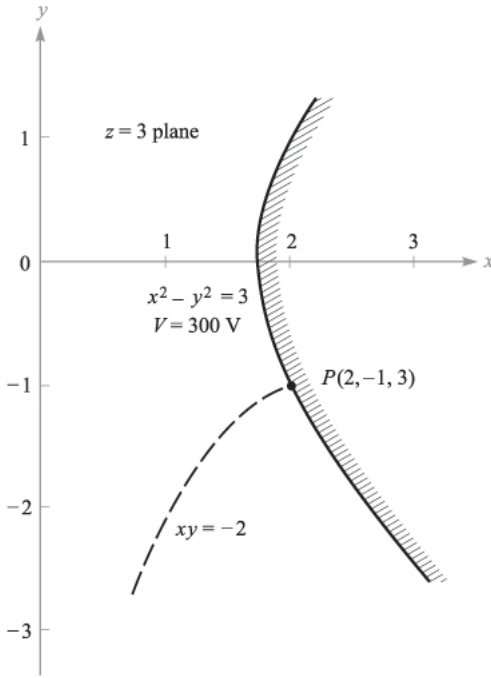
FIGURE 5.5
Given point $P(2, -1, 3)$ and the potential field, $V = 100(x^2 - y^2)$, we find the equipotential surface through P is $x^2 - y^2 = 3$, and the streamline through P is $xy = -2$.

Next, we find **E** by the gradient operation,

$$\mathbf{E} = -100\nabla(x^2 - y^2) = -200x\mathbf{a}_x + 200y\mathbf{a}_y$$

At point P,

$$\mathbf{E}_p = -400\mathbf{a}_x - 200\mathbf{a}_y \quad \text{V/m}$$

Since $\mathbf{D} = \epsilon_0\mathbf{E}$, we have

$$\mathbf{D}_P = 8.854 \times 10^{-12}\mathbf{E}_P = -3.54\mathbf{a}_x - 1.771\mathbf{a}_y \quad \text{nC/m}^2$$

The field is directed downward and to the left at P; it is normal to the equipotential surface. Therefore,

$$D_N = |\mathbf{D}_P| = 3.96 \quad \text{nC/m}^2$$

Thus, the surface charge density at P is

$$\rho_{S,P} = D_N = 3.96 \quad \text{nC/m}^2$$

Note that if we had taken the region to the left of the equipotential surface as the conductor, the **E** field would *terminate* on the surface charge and we would let $\rho_S = -3.96 \, \text{nC/m}^2$.

⫼▶Example 5.3

Finally, let us determine the equation of the streamline passing through *P*.

Solution. We see that

$$\frac{E_y}{E_x} = \frac{200y}{-200x} = -\frac{y}{x} = \frac{dy}{dx}$$

Thus,

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

and

$$\ln y + \ln x = C_1$$

Therefore,

$$xy = C_2$$

The line (or surface) through *P* is obtained when $C_2 = (2)(-1) = -2$. Thus, the streamline is the trace of another hyperbolic cylinder,

$$xy = -2$$

This is also shown on Fig. 5.5.

✔ **D5.5.** Given the potential field in free space, $V = 100 \sinh 5x \sin 5y$ V, and a point $P(0.1, 0.2, 0.3)$, find at *P*: (*a*) V; (*b*) **E**; (*c*) $|\mathbf{E}|$; (*d*) $|\rho_S|$ if it is known that *P* lies on a conductor surface.

Ans. 43.8 V; $-474\mathbf{a}_x - 140.8\mathbf{a}_y$ V/m; 495 V/m; 4.38 nC/m²

5.5 THE METHOD OF IMAGES

One important characteristic of the dipole field that we developed in the last chapter is the infinite plane at zero potential that exists midway between the two charges. Such a plane may be represented by a vanishingly thin conducting plane that is infinite in extent. The conductor is an equipotential surface at a potential $V = 0$, and the electric field intensity is therefore normal to the surface. Thus, if we replace the dipole configuration shown in Fig. 5.6*a* with the single charge and conducting plane shown in Fig. 5.6*b*, the fields in the upper half of each figure are the same. Below the conducting plane, all fields are zero since we have not provided any charges in that region. Of course, we might also substitute a single negative charge below a conducting plane for the dipole arrangement and obtain equivalence for the fields in the lower half of each region.

If we approach this equivalence from the opposite point of view, we begin with a single charge above a perfectly conducting plane and then see that we may maintain the same fields above the plane by removing the plane and locating a negative charge at a symmetrical location below the plane. This charge is called the *image* of the original charge, and it is the negative of that value.

If we can do this once, linearity allows us to do it again and again, and thus *any* charge configuration above an infinite ground plane may be replaced by an
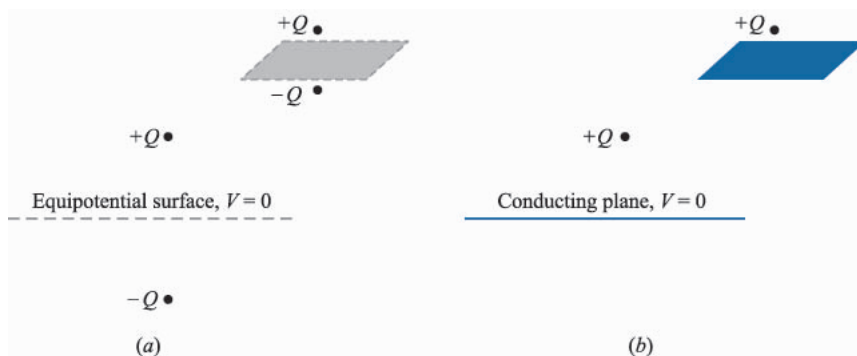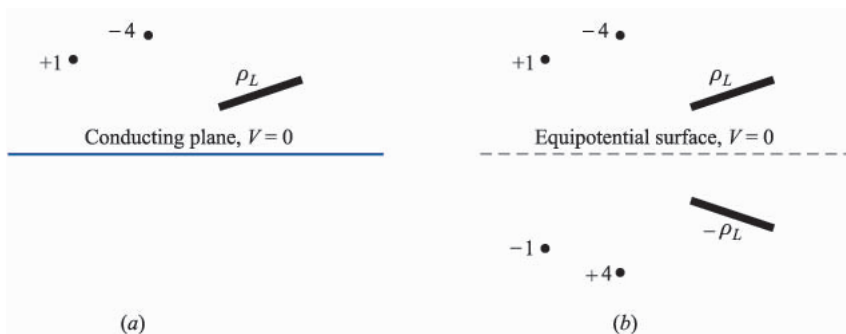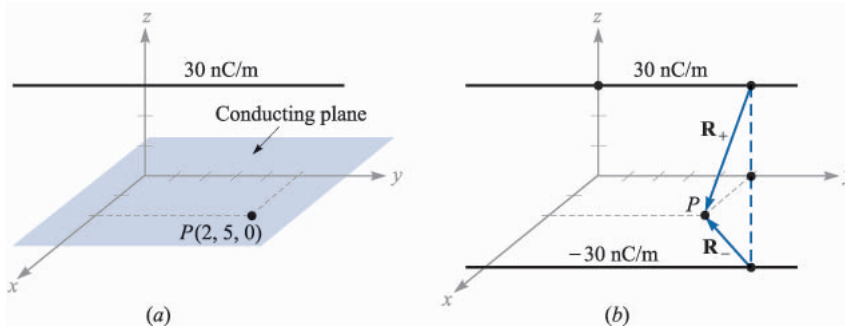
FIGURE 5.6
(a) Two equal but opposite charges may be replaced by (b) a single charge and a conducting plane without affecting the fields above the $V = 0$ surface.

arrangement composed of the given charge configuration, its image, and no conducting plane. This is suggested by the two illustrations of Fig. 5.7. In many cases, the potential field of the new system is much easier to find since it does not contain the conducting plane with its unknown surface charge distribution.

As an example of the use of images, let us find the surface charge density at $P(2, 5, 0)$ on the conducting plane $z = 0$ if there is a line charge of 30 nC/m located at $x = 0$, $z = 3$, as shown in Fig. 5.8a. We remove the plane and install an image line charge of -30 nC/m at $x = 0$, $z = -3$, as illustrated in Fig. 5.8b. The field at P may now be obtained by superposition of the known fields of the line charges. The radial vector from the positive line charge to P is $\mathbf{R}_+ = 2\mathbf{a}_x - 3\mathbf{a}_z$, while $\mathbf{R}_- = 2\mathbf{a}_x + 3\mathbf{a}_z$. Thus, the individual fields are

FIGURE 5.7
(a) A given charge configuration above an infinite conducting plane may be replaced by (b) the given charge configuration plus the image configuration, without the conducting plane.

FIGURE 5.8
(*a*) A line charge above a conducting plane. (*b*) The conductor is removed, and the image of the line charge is added.

$$\mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+}\mathbf{a}_{R+} = \frac{30 \times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}$$

and
$$\mathbf{E}_- = \frac{30 \times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}$$

Adding these results, we have
$$\mathbf{E} = \frac{-180 \times 10^{-9}\mathbf{a}_z}{2\pi\epsilon_0(13)} = -249\mathbf{a}_z \quad \text{V/m}$$

This then is the field at (or just above) P in both the configurations of Fig. 5.8, and it is certainly satisfying to note that the field is normal to the conducting plane, as it must be. Thus, $\mathbf{D} = \epsilon_0\mathbf{E} = -2.20\mathbf{a}_z$ nC/m^2, and since this is directed *toward* the conducting plane, ρ_S is negative and has a value of -2.20 nC/m^2 at P.

✔ **D5.6.** A perfectly conducting plane is located in free space at $x = 4$, and a uniform infinite line charge of 40 nC/m lies along the line $x = 6$, $y = 3$. Let $V = 0$ at the conducting plane. At $P(7, -1, 5)$ find: (*a*) V; (*b*) $\mathbf{E}$.

Ans. -316 V; $-45.4\mathbf{a}_x$ V/m.

5.6 SEMICONDUCTORS

If we now turn our attention to an intrinsic semiconductor material, such as pure germanium or silicon, two types of current carriers are present, electrons and *holes*. The electrons are those from the top of the filled valence band which have received sufficient energy (usually thermal) to cross the relatively small forbidden band into the conduction band. The forbidden-band energy gap in typical semiconductors is of the order of one electronvolt. The vacancies left by these electrons represent unfilled energy states in the valence band which may also move

from atom to atom in the crystal. The vacancy is called a *hole*, and many semiconductor properties may be described by treating the hole as if it had a positive charge of e, a mobility, μ_h, and an effective mass comparable to that of the electron. Both carriers move in an electric field, and they move in opposite directions; hence each contributes a component of the total current which is in the same direction as that provided by the other. The conductivity is therefore a function of both hole and electron concentrations and mobilities,

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h \tag{17}$$

For pure, or *intrinsic*, silicon the electron and hole mobilities are 0.12 and 0.025, respectively, while for germanium, the mobilities are, respectively, 0.36 and 0.17. These values are given in square meters per volt-second and range from 10 to 100 times as large as those for aluminum, copper, silver, and other metallic conductors.[6] The mobilities listed above are given for a temperature of 300 K.

The electron and hole concentrations depend strongly on temperature. At 300 K the electron and hole volume charge densities are both 0.0024 C/m^3 in magnitude in intrinsic silicon and 3.0 C/m^3 in intrinsic germanium. These values lead to conductivities of 0.000 35 S/m in silicon and 1.6 S/m in germanium. As temperature increases, the mobilities decrease, but the charge densities increase very rapidly. As a result, the conductivity of silicon increases by a factor of 10 as the temperature increases from 300 to about 330 K and decreases by a factor of 10 as the temperature drops from 300 to about 275 K. Note that the conductivity of the intrinsic semiconductor increases with temperature, while that of a metallic conductor decreases with temperature; this is one of the characteristic differences between the metallic conductors and the intrinsic semiconductors.

Intrinsic semiconductors also satisfy the point form of Ohm's law; that is, the conductivity is reasonably constant with current density and with the direction of the current density.

The number of charge carriers and the conductivity may both be increased dramatically by adding very small amounts of impurities. *Donor* materials provide additional electrons and form *n-type* semiconductors, while *acceptors* furnish extra holes and form *p-type* materials. The process is known as *doping*, and a donor concentration in silicon as low as one part in 10^7 causes an increase in conductivity by a factor of 10^5.

The range of value of the conductivity is extreme as we go from the best insulating materials to semiconductors and the finest conductors. In siemens per meter, σ ranges from 10^{-17} for fused quartz, 10^{-7} for poor plastic insulators, and roughly unity for semiconductors to almost 10^8 for metallic conductors at room temperature. These values cover the remarkably large range of some 25 orders of magnitude.

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[6] Mobility values for semiconductors are given in Refs. 2, 4, and 7 listed at the end of this chapter.

✔ **D5.7.** Using the values given in this section for the electron and hole mobilities in silicon at 300 K, and assuming hole and electron charge densities are 0.0029 C/m^3 and −0.0029 C/m^3, respectively, find: (*a*) the component of the conductivity due to holes; (*b*) the component of the conductivity due to electrons; (*c*) the conductivity.

Ans. 0.0725 S/m; 0.348 S/m; 0.421 S/m

5.7   THE NATURE OF DIELECTRIC MATERIALS

Although we have mentioned insulators and dielectric materials, we do not as yet have any quantitative relationships in which they are involved. We shall soon see, however, that a dielectric in an electric field can be viewed as a free-space arrangement of microscopic electric dipoles which are composed of positive and negative charges whose centers do not quite coincide.

These are not free charges, and they cannot contribute to the conduction process. Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called *bound* charges, in contrast to the free charges that determine conductivity. The bound charges can be treated as any other sources of the electrostatic field. If we did not wish to, therefore, we would not need to introduce the dielectric constant as a new parameter or to deal with permittivities different from the permittivity of free space; however, the alternative would be to consider *every charge within a piece of dielectric material*. This is too great a price to pay for using all our previous equations in an unmodified form, and we shall therefore spend some time theorizing about dielectrics in a qualitative way; introducing polarization **P**, permittivity ϵ, and relative permittivity ϵ_R; and developing some quantitative relationships involving these new quantities.

The characteristic which all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their ability to store electric energy. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces.

This displacement against a restraining force is analogous to lifting a weight or stretching a spring and represents potential energy. The source of the energy is the external field, the motion of the shifting charges resulting perhaps in a transient current through a battery which is producing the field.

The actual mechanism of the charge displacement differs in the various dielectric materials. Some molecules, termed *polar* molecules, have a permanent displacement existing between the centers of "gravity" of the positive and negative charges, and each pair of charges acts as a dipole. Normally the dipoles are oriented in a random way throughout the interior of the material, and the action of the external field is to align these molecules, to some extent, in the same direction. A sufficiently strong field may even produce an additional displacement between the positive and negative charges.

A *nonpolar* molecule does not have this dipole arrangement until after a field is applied. The negative and positive charges shift in opposite directions against their mutual attraction and produce a dipole which is aligned with the electric field.

Either type of dipole may be described by its dipole moment $\mathbf{p}$, as developed in Sec. 4.7, Eq. (37),

$$\mathbf{p} = Q\mathbf{d} \tag{18}$$

where Q is the positive one of the two bound charges composing the dipole, and $\mathbf{d}$ is the vector from the negative to the positive charge. We note again that the units of $\mathbf{p}$ are coulomb-meters.

If there are n dipoles per unit volume and we deal with a volume Δv, then there are $n \Delta v$ dipoles, and the total dipole moment is obtained by the vector sum,

$$\mathbf{p}_{\text{total}} = \sum_{i=1}^{n \Delta v} \mathbf{p}_i$$

If the dipoles are aligned in the same general direction, $\mathbf{p}_{\text{total}}$ may have a significant value. However, a random orientation may cause $\mathbf{p}_{\text{total}}$ to be essentially zero.

We now define the polarization $\mathbf{P}$ as the *dipole moment per unit volume,*

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n \Delta v} \mathbf{p}_i \tag{19}$$

with units of coulombs per square meter. We shall treat $\mathbf{P}$ as a typical continuous field, even though it is obvious that it is essentially undefined at points within an atom or molecule. Instead, we should think of its value at any point as an average value taken over a sample volume Δv—large enough to contain many molecules ($n \Delta v$ in number), but yet sufficiently small to be considered incremental in concept.

Our immediate goal is to show that the bound volume charge density acts like the free volume charge density in producing an external field; we shall obtain a result similar to Gauss's law.

To be specific, let us assume that we have a dielectric containing nonpolar molecules. No molecule has a dipole moment, and $\mathbf{P} = 0$ throughout the material. Somewhere in the interior of the dielectric we select an incremental surface element $\Delta \mathbf{S}$, as shown in Fig. 5.9a, and apply an electric field $\mathbf{E}$. The electric field produces a moment $\mathbf{p} = Q\mathbf{d}$ in each molecule, such that $\mathbf{p}$ and $\mathbf{d}$ make an angle θ with $\Delta \mathbf{S}$, as indicated in Fig. 5.9b.

Now let us inspect the movement of bound charges across $\Delta \mathbf{S}$. Each of the charges associated with the creation of a dipole must have moved a distance $\frac{1}{2} d \cos \theta$ in the direction perpendicular to $\Delta \mathbf{S}$. Thus, any positive charges initially lying below the surface $\Delta \mathbf{S}$ and within the distance $\frac{1}{2} d \cos \theta$ of the surface must have crossed $\Delta \mathbf{S}$ going upward. Also, any negative charges initially lying above
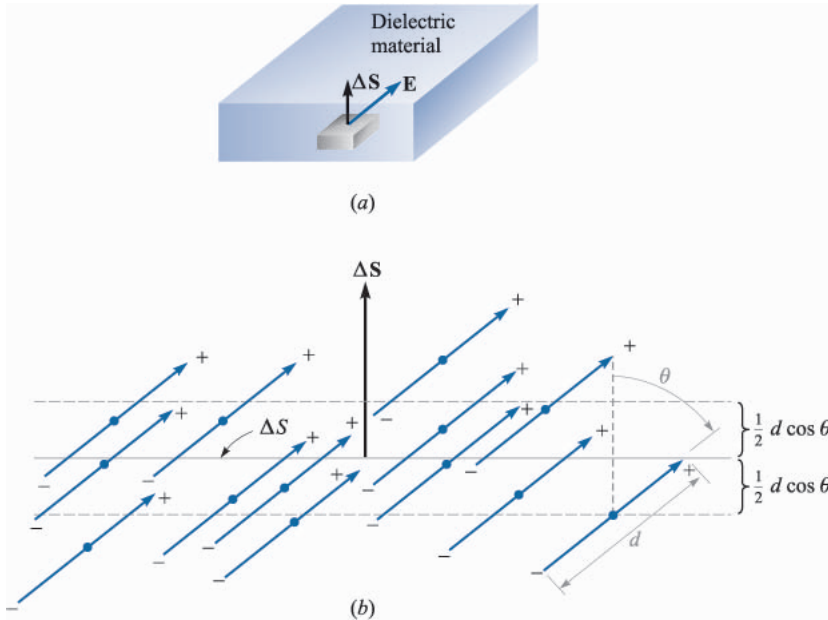
FIGURE 5.9
(*a*) An incremental surface element ΔS is shown in the interior of a dielectric in which an electric field $\mathbf{E}$ is present. (*b*) The nonpolar molecules form dipole moments $\mathbf{p}$ and a polarization $\mathbf{P}$. There is a net transfer of bound charge across $\Delta \mathbf{S}$.

the surface and within that distance $(\frac{1}{2}d\cos\theta)$ from $\Delta\mathbf{S}$ must have crossed $\Delta\mathbf{S}$ going downward. Therefore, since there are n molecules/m^3, the net total charge which crosses the elemental surface in an upward direction is equal to $nQd\cos\theta\Delta\mathbf{S}$, or

$$\Delta Q_b = nQ\mathbf{d}\cdot\Delta\mathbf{S}$$

where the subscript on Q_b reminds us that we are dealing with a bound charge and not a free charge. In terms of the polarization, we have

$$\Delta Q_b = \mathbf{P}\cdot\Delta\mathbf{S}$$

If we interpret $\Delta\mathbf{S}$ as an element of a *closed* surface inside the dielectric material, then the direction of $\Delta\mathbf{S}$ is outward, and the net increase in the bound charge *within* the closed surface is obtained through the integral

$$Q_b = -\oint_S \mathbf{P}\cdot d\mathbf{S} \tag{20}$$

This last relationship has some resemblance to Gauss's law, and we may now generalize our definition of electric flux density so that it applies to media other

than free space. We first write Gauss's law in terms of $\epsilon_0\mathbf{E}$ and Q_T, the *total* enclosed charge, bound plus free:

$$Q_T = \oint_S \epsilon_0\mathbf{E} \cdot d\mathbf{S} \tag{21}$$

where

$$Q_T = Q_b + Q$$

and Q is the total *free* charge enclosed by the surface S. Note that the free charge appears without subscript since it is the most important type of charge and will appear in Maxwell's equations.

Combining these last three equations, we obtain an expression for the free charge enclosed,

$$Q = Q_T - Q_b = \oint_S (\epsilon_0\mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} \tag{22}$$

We may now define $\mathbf{D}$ in more general terms than we did in Chap. 3,

$$\boxed{\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}} \tag{23}$$

There is thus an added term to D which appears when polarizable material is present. Thus,

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} \tag{24}$$

where Q is the free charge enclosed.

Utilizing the several volume charge densities, we have

$$Q_b = \int_v \rho_b \, dv$$

$$Q = \int_v \rho_v \, dv$$

$$Q_T = \int_v \rho_T \, dv$$

With the help of the divergence theorem, we may therefore transform (20), (21), and (24) into the equivalent divergence relationships,

$$\nabla \cdot \mathbf{P} = -\rho_b$$

$$\nabla \cdot \epsilon_0\mathbf{E} = \rho_T$$

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \tag{25}$$

We shall emphasize only (24) and (25), the two expressions involving the free charge, in the work that follows.

In order to make any real use of these new concepts, it is necessary to know the relationship between the electric field intensity $\mathbf{E}$ and the polarization $\mathbf{P}$ which results. This relationship will, of course, be a function of the type of material, and we shall essentially limit our discussion to those isotropic materials for which $\mathbf{E}$ and $\mathbf{P}$ are linearly related. In an isotropic material the vectors $\mathbf{E}$ and $\mathbf{P}$ are always parallel, regardless of the orientation of the field. Although most engineering dielectrics are linear for moderate-to-large field strengths and are also isotropic, single crystals may be anisotropic. The periodic nature of crystalline materials causes dipole moments to be formed most easily along the crystal axes, and not necessarily in the direction of the applied field.

In *ferroelectric* materials the relationship between $\mathbf{P}$ and $\mathbf{E}$ is not only nonlinear, but also shows hysteresis effects; that is, the polarization produced by a given electric field intensity depends on the past history of the sample. Important examples of this type of dielectric are barium titanate, often used in ceramic capacitors, and Rochelle salt.

The linear relationship between $\mathbf{P}$ and $\mathbf{E}$ is

$$\boxed{\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}} \qquad (26)$$

where χ_e (chi) is a dimensionless quantity called the *electric susceptibility* of the material.

Using this relationship in (23), we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$$

The expression within the parentheses is now defined as

$$\epsilon_R = \chi_e + 1 \qquad (27)$$

This is another dimensionless quantity and it is known as the *relative permittivity*, or *dielectric constant* of the material. Thus,

$$\mathbf{D} = \epsilon_0 \epsilon_R \mathbf{E} = \epsilon \mathbf{E} \qquad (28)$$

where

$$\boxed{\epsilon = \epsilon_0 \epsilon_R} \qquad (29)$$

and ϵ is the *permittivity*. The dielectric constants are given for some representative materials in Appendix C.

Anisotropic dielectric materials cannot be described in terms of a simple susceptibility or permittivity parameter. Instead, we find that each component of $\mathbf{D}$ may be a function of every component of $\mathbf{E}$, and $\mathbf{D} = \epsilon \mathbf{E}$ becomes a matrix equation where $\mathbf{D}$ and $\mathbf{E}$ are each 3×1 column matrices and ϵ is a 3×3 square matrix. Expanding the matrix equation gives

$$D_x = \epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z$$
$$D_y = \epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z$$
$$D_z = \epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z$$

Note that the elements of the matrix depend on the selection of the coordinate axes in the anisotropic material. Certain choices of axis directions lead to simpler matrices.[7]

$\quad$ **D** and **E** (and **P**) are no longer parallel, and although $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ remains a valid equation for anisotropic materials, we may continue to use $\mathbf{D} = \epsilon\mathbf{E}$ only by interpreting it as a matrix equation. We shall concentrate our attention on linear isotropic materials and reserve the general case for a more advanced text.

$\quad$ In summary, then, we now have a relationship between **D** and **E** which depends on the dielectric material present,

$$\boxed{\mathbf{D} = \epsilon\mathbf{E}} \tag{28}$$

where

$$\boxed{\epsilon = \epsilon_0\epsilon_R} \tag{29}$$

This electric flux density is still related to the free charge by either the point or integral form of Gauss's law:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \tag{25}$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q} \tag{24}$$

$\quad$ The use of the relative permittivity, as indicated by (29) above, makes consideration of the polarization, dipole moments, and bound charge unnecessary. However, when anisotropic or nonlinear materials must be considered, the relative permittivity, in the simple scalar form that we have discussed, is no longer applicable.

$\quad$ Let us now illustrate these new concepts with a numerical example.

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[7] A more complete discussion of this matrix may be found in the Ramo, Whinnery, and Van Duzer reference listed at the end of this chapter.

▶Example 5.4

We locate a slab of Teflon in the region $0 \le x \le a$, and assume free space where $x < 0$ and $x > a$. Outside the Teflon there is a uniform field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$ V/m. We seek values for $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$ everywhere.

Solution. The dielectric constant of Teflon is 2.1, and thus the electric susceptibility is 1.1.

Outside the slab, we have immediately $\mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$. Also since there is no dielectric material there, $\mathbf{P}_{\text{out}} = 0$. Now, any of the last four or five equations will enable us to relate the several fields inside the material to each other. Thus

$$\mathbf{D}_{\text{in}} = 2.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \le x \le a)$$
$$\mathbf{P}_{\text{in}} = 1.1\epsilon_0\mathbf{E}_{\text{in}} \qquad (0 \le x \le a)$$

As soon as we establish a value for any of these three fields within the dielectric, the other two can be found immediately. The difficulty lies in crossing over the boundary from the known fields external to the dielectric to the unknown ones within it. To do this we need a boundary condition, and this is the subject of the next exciting section. We shall complete this example then.

In the remainder of this text we shall describe polarizable materials in terms of $\mathbf{D}$ and ϵ rather than $\mathbf{P}$ and χ_e. We shall limit our discussion to isotropic materials.

✔ **D5.8.** A slab of dielectric material has a relative dielectric constant of 3.8 and contains a uniform electric flux density of 8 nC/m². If the material is lossless; find: (a) E; (b) P; (c) the average number of dipoles per cubic meter if the average dipole moment is 10^{-29} C · m

Ans. 238 V/m; 5.89 nC/m²; 5.89×10^{20} m^{-3}

5.8 BOUNDARY CONDITIONS FOR PERFECT DIELECTRIC MATERIALS

How do we attack a problem in which there are two different dielectrics, or a dielectric and a conductor? This is another example of a *boundary condition*, such as the condition at the surface of a conductor whereby the tangential fields are zero and the normal electric flux density is equal to the surface charge density on the conductor. Now we take the first step in solving a two-dielectric problem, or a dielectric-conductor problem, by determining the behavior of the fields at the dielectric interface.

Let us first consider the interface between two dielectrics having permittivities ϵ_1 and ϵ_2 and occupying regions 1 and 2, as shown in Fig. 5.10. We first examine the tangential components by using

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$
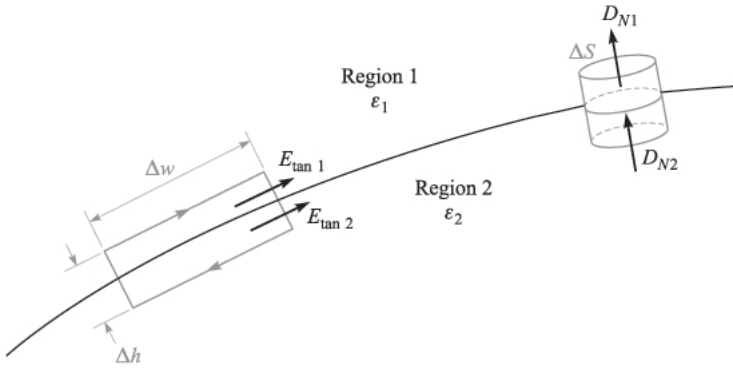
FIGURE 5.10
The boundary between perfect dielectrics of permittivities ϵ_1 and ϵ_2. The continuity of D_N is shown by the gaussian surface on the right, and the continuity of E_{tan} by the line integral about the closed path at the left.

around the small closed path on the left, obtaining

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

The small contribution to the line integral by the normal component of **E** along the sections of length Δh becomes negligible as Δh decreases and the closed path crowds the surface. Immediately, then,

$$\boxed{E_{\tan 1} = E_{\tan 2}} \tag{30}$$

and we might feel that Kirchhoff's voltage law is still applicable to this case. Certainly we have shown that the potential difference between any two points on the boundary that are separated by a distance Δw is the same immediately above or below the boundary.

If the tangential electric field intensity is continuous across the boundary, then tangential **D** is discontinuous, for

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

or

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \tag{31}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in Fig. 5.10. The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

from which

$$D_{N1} - D_{N2} = \rho_S \qquad (32)$$

What is this surface charge density? It cannot be a *bound* surface charge density, because we are taking the polarization of the dielectric into effect by using a dielectric constant different from unity; that is, instead of considering bound charges in free space, we are using an increased permittivity. Also, it is extremely unlikely that any *free* charge is on the interface, for no free charge is available in the perfect dielectrics we are considering. This charge must then have been placed there deliberately, thus unbalancing the total charge in and on this dielectric body. Except for this special case, then, we may assume ρ_S is zero on the interface and

$$D_{N1} = D_{N2} \qquad (33)$$

or the normal component of D is continuous. It follows that

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2} \qquad (34)$$

and normal **E** is discontinuous.

These conditions may be combined to show the change in the vectors **D** and **E** at the surface. Let $\mathbf{D}_1$ (and $\mathbf{E}_1$) make an angle θ_1 with a normal to the surface (Fig. 5.11). Since the normal components of **D** are continuous,
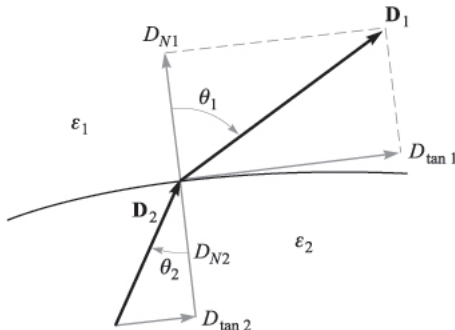
$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2} \qquad (35)$$

The ratio of the tangential components is given by (31) as

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

or

$$\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2 \qquad (36)$$

FIGURE 5.11
The refraction of **D** at a dielectric interface. For the case shown, $\epsilon_1 > \epsilon_2$; $\mathbf{E}_1$ and $\mathbf{E}_2$ are directed along $\mathbf{D}_1$ and $\mathbf{D}_2$, with $D_1 > D_2$ and $E_1 < E_2$.

and the division of this equation by (35) gives

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2} \tag{37}$$

In Fig. 5.11 we have assumed that $\epsilon_1 > \epsilon_2$, and therefore $\theta_1 > \theta_2$.

The direction of **E** on each side of the boundary is identical with the direction of **D**, because $\mathbf{D} = e\mathbf{E}$.

The magnitude of **D** in region 2 may be found from (35) and (36),

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1} \tag{38}$$

and the magnitude of $\mathbf{E}_2$ is

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1} \tag{39}$$

An inspection of these equations shows that D is larger in the region of larger permittivity (unless $\theta_1 = \theta_2 = 0°$ where the magnitude is unchanged) and that E is larger in the region of smaller permittivity (unless $\theta_1 = \theta_2 = 90°$, where its magnitude is unchanged).

These boundary conditions, (30), (31), (33), and (34), or the magnitude and direction relations derived from them, (37) to (39), allow us to find quickly the field on one side of a boundary *if we know the field on the other side*. In the example we began at the end of the previous section, this was the case. Now let's finish up that problem.

▒▶Example 5.5

Complete Example 5.4 by finding the fields within the Teflon ($\epsilon_R = 2.1$), given the uniform external field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$ in free space.

Solution. We recall that we had a slab of Teflon extending from $x = 0$ to $x = a$, as shown in Fig. 5.12, with free space on both sides of it and an external field $\mathbf{E}_{\text{out}} = E_0\mathbf{a}_x$. We also have $\mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$ and $\mathbf{P}_{\text{out}} = 0$.

Inside, the continuity of D_N at the boundary allows us to find that $\mathbf{D}_{\text{in}} = \mathbf{D}_{\text{out}} = \epsilon_0 E_0\mathbf{a}_x$. This gives us $\mathbf{E}_{\text{in}} = \mathbf{D}_{\text{in}}/\epsilon = \epsilon_0 E_0\mathbf{a}_x/(\epsilon_R\epsilon_0) = 0.476E_0\mathbf{a}_x$. To get the polarization field in the dielectric, we use $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and obtain

$$\mathbf{P}_{\text{in}} = \mathbf{D}_{\text{in}} - \epsilon_0\mathbf{E}_{\text{in}} = \epsilon_0 E_0\mathbf{a}_x - 0.476\epsilon_0 E_0\mathbf{a}_x = 0.524\epsilon_0 E_0\mathbf{a}_x$$

Summarizing then gives

$$\begin{aligned}
\mathbf{D}_{\text{in}} &= \epsilon_0 E_0\mathbf{a}_x & (0 \le x \le a) \\
\mathbf{E}_{\text{in}} &= 0.476E_0\mathbf{a}_x & (0 \le x \le a) \\
\mathbf{P}_{\text{in}} &= 0.524\epsilon_0 E_0\mathbf{a}_x & (0 \le x \le a)
\end{aligned}$$
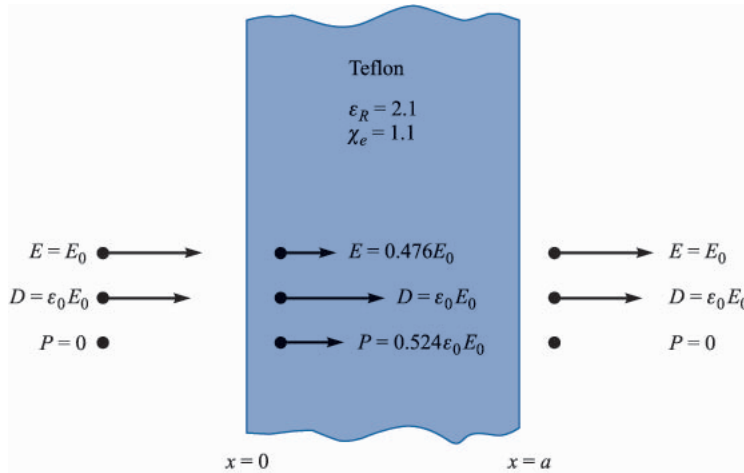
FIGURE 5.12
A knowledge of the electric field external to the dielectric enables us to find the remaining external fields first, and then to use the continuity of normal **D** to begin finding the internal fields.

A practical problem most often does not provide us with a direct knowledge of the field on either side of the boundary. The boundary conditions must be used to help us determine the fields on both sides of the boundary from the other information which is given. A simple problem of this type will be considered in Sec. 5.10.

The boundary conditions existing at the interface between a conductor and a dielectric are much simpler than those above. First, we know that **D** and **E** are both zero inside the conductor. Second, the tangential **E** and **D** field components must both be zero to satisfy

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

and

$$\mathbf{D} = \epsilon \mathbf{E}$$

Finally, the application of Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

shows once more that both **D** and **E** are normal to the conductor surface and that $D_N = \rho_S$ and $E_N = \rho_S/\epsilon$. We see, then, that the boundary conditions we developed previously for the conductor-free space boundary are valid for the conductor-dielectric boundary if we replace ϵ_0 by ϵ. Thus

$$\boxed{D_t = E_t = 0} \tag{40}$$

$$\boxed{D_N = \epsilon E_N = \rho_S} \tag{41}$$

It is interesting to spend a moment discovering how any change that is introduced *internally* within a conducting material arrives at the surface as a surface charge. We should understand that this is not a common occurrence, but it does give us some additional insight into the characteristics of a conductor,
    Given Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E}$$

and the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

in which $\mathbf{J}$ and ρ_v both involve only free charges, we have

$$\nabla \cdot \sigma \mathbf{E} = -\frac{\partial \rho_v}{\partial t}$$

or

$$\nabla \cdot \frac{\sigma}{\epsilon} \mathbf{D} = -\frac{\partial \rho_v}{\partial t}$$

If we assume that the medium is homogeneous, so that σ and ϵ are not functions of position,

$$\nabla \cdot \mathbf{D} = -\frac{\epsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$$

Now we may use Maxwell's first equation to obtain

$$\rho_v = -\frac{\epsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$$

Let us now make the simplifying assumption that σ is not a function of ρ_v. This is probably not a very good assumption, for we found in Sec. 5.3, Eq. (9), that σ depended on both ρ_v and the mobility, but it leads to an easy solution that at least permits us to compare different conductors. We simply rearrange and integrate directly, obtaining

$$\rho_v = \rho_0 e^{-(\sigma/\epsilon)t}$$

where $\rho_0 = $ charge density at $t = 0$. This shows an exponential decay of charge density at every point with a time constant of ϵ/σ. This time constant, often called the *relaxation time*, may be calculated for a relatively poor conductor, such as distilled water, from the data in Appendix C, giving

$$\frac{\epsilon}{\sigma} = \frac{80 \times 8.854 \times 10^{-12}}{2 \times 10^{-4}} = 3.54 \quad \mu s$$

In 3.54 μs any charge we place in the interior of a body of distilled water has dropped to about 37 percent of its initial value. This rapid decay is characteristic of good conductors and we see that, except for an extremely short transient period, we may safely consider the charge density to be zero within a good conductor.

With the physical materials with which we must work, no dielectric material is without some few free electrons; all have conductivities different from zero, and charge introduced internally in any of them will eventually reach the surface.

With the knowledge we now have of conducting materials, dielectric materials, and the necessary boundary conditions, we are ready to define and discuss capacitance.

✔ **D5.9.** Let the region $z < 0$ be composed of a uniform dielectric material for which $\epsilon_R = 3.2$, while the region $z > 0$ is characterized by $\epsilon_R = 2$. Let $\mathbf{D}_1 = -30\mathbf{a}_x + 50\mathbf{a}_y + 70\mathbf{a}_z$ nC/m² and find: (a) D_{N1}; (b) $\mathbf{D}_{t1}$; (c) D_{t1}; (d) D_1; (e) θ_1; (f) $\mathbf{P}_1$.

**Ans.** 70 nC/m²; $-30\mathbf{a}_x + 50\mathbf{a}_y$ nC/m²; 58.3 nC/m²; 91.1 nC/m²; 39.8°; $-20.6\mathbf{a}_x + 34.4\mathbf{a}_y + 48.1\mathbf{a}_z$ nC/m²

✔ **D5.10.** Continue Prob. D5.9 by finding: (a) $\mathbf{D}_{N2}$; (b) $\mathbf{D}_{t2}$; (c) $\mathbf{D}_2$; (d) $\mathbf{P}_2$; (e) θ_2

Ans. 70$\mathbf{a}_z$ nC/m²; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y$ nC/m²; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y + 70\mathbf{a}_z$ nC/m²; $-9.38\mathbf{a}_x + 15.63\mathbf{a}_y + 35\mathbf{a}_z$ nC/m²; 27.5°

5.9 CAPACITANCE

Now let us consider two conductors embedded in a homogeneous dielectric (Fig. 5.13). Conductor M_2 carries a total positive charge Q, and M_1 carries an equal
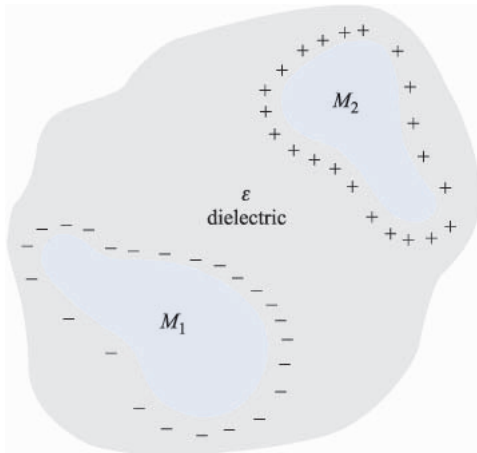
FIGURE 5.13
Two oppositely charged conductors M_1 and M_2 surrounded by a uniform dielectric. The ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them is the capacitance C.

negative charge. There are no other charges present, and the *total* charge of the system is zero.

We now know that the charge is carried on the surface as a surface charge density and also that the electric field is normal to the conductor surface. Each conductor is, moreover, an equipotential surface. Since M_2 carries the positive charge, the electric flux is directed from M_2 to M_1, and M_2 is at the more positive potential. In other words, work must be done to carry a positive charge from M_1 to M_2.

Let us designate the potential difference between M_2 and M_1 as V_0. We may now define the *capacitance* of this two-conductor system as the ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between conductors,

$$C = \frac{Q}{V_0} \qquad (42)$$

In general terms, we determine Q by a surface integral over the positive conductors, and we find V_0 by carrying a unit positive charge from the negative to the positive surface,

$$C = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot d\mathbf{L}} \qquad (43)$$

The capacitance is independent of the potential and total charge, for their ratio is constant. If the charge density is increased by a factor of N, Gauss's law indicates that the electric flux density or electric field intensity also increases by N, as does the potential difference. The capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the homogeneous dielectric.
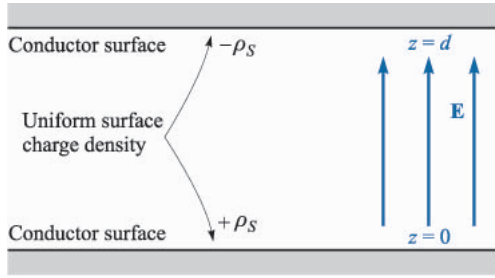
Capacitance is measured in *farads* (F), where a farad is defined as one coulomb per volt. Common values of capacitance are apt to be very small fractions of a farad, and consequently more practical units are the microfarad (μF), the nanofarad (nF), and the picofarad (pF).

We can apply the definition of capacitance to a simple two-conductor system in which the conductors are identical, infinite parallel planes with separation d (Fig. 5.14). Choosing the lower conducting plane at $z = 0$ and the upper one at $z = d$, a uniform sheet of surface charge $\pm \rho_S$ on each conductor leads to the uniform field [Sec. 2.5, Eq. (22)]

$$\mathbf{E} = \frac{\rho_S}{\epsilon} \mathbf{a}_z$$

where the permittivity of the homogeneous dielectric is ϵ, and

$$\mathbf{D} = \rho_S \mathbf{a}_z$$

Conductor surface $-\rho_S$ $z = d$

Uniform surface   E
charge density

                                         $z = 0$
Conductor surface $+\rho_S$

FIGURE 5.14
The problem of the parallel-plate capacitor. The
capacitance per square meter of surface area is
ϵ/d.

The charge on the lower plane must then be positive, since **D** is directed
upward, and the normal value of **D**,

$$D_N = D_z = \rho_S$$

is equal to the surface charge density there. On the upper plane,

$$D_N = -D_z$$

and the surface charge there is the negative of that on the lower plane.
The potential difference between lower and upper planes is

$$V_0 = -\int_{upper}^{lower} \mathbf{E} \cdot d\mathbf{L} = -\int_d^0 \frac{\rho_S}{\epsilon}\, dz = \frac{\rho_S}{\epsilon} d$$

Since the total charge on either plane is infinite, the capacitance is infinite. A
more practical answer is obtained by considering planes, each of area S, whose
linear dimensions are much greater than their separation d. The electric field and
charge distribution are then almost uniform at all points not adjacent to the
edges, and this latter region contributes only a small percentage of the total
capacitance, allowing us to write the familiar result

$$Q = \rho_S S$$
$$V_0 = \frac{\rho_S}{\epsilon} d$$

$$\boxed{C = \frac{Q}{V_0} = \frac{\epsilon S}{d}} \tag{44}$$

More rigorously, we might consider (44) as the capacitance of a portion of
the infinite-plane arrangement having a surface area S. Methods of calculating
the effect of the unknown and nonuniform distribution near the edges must wait
until we are able to solve more complicated potential problems.

▐▐▐▶Example 5.6

Calculate the capacitance of a parallel-plate capacitor having a mica dielectric, $\epsilon_R = 6$, a plate area of $10\,\text{in}^2$, and a separation of $0.01\,\text{in}$.

Solution. We may find that

$$S = 10 \times 0.0254^2 = 6.45 \times 10^{-3} \quad \text{m}^2$$
$$d = 0.01 \times 0.0254 = 2.54 \times 10^{-4} \quad \text{m}$$

and therefore

$$C = \frac{6 \times 8.854 \times 10^{-12} \times 6.45 \times 10^{-3}}{2.54 \times 10^{-4}} = 1.349 \quad \text{nF}$$

A large plate area is obtained in capacitors of small physical dimensions by stacking smaller plates in 50- or 100-decker sandwiches, or by rolling up foil plates separated by a flexible dielectric.

Table $C.1$ in Appendix C also indicates that materials are available having dielectric constants greater than 1000.

If more than two conductors are involved, *partial capacitances* between each pair of conductors must be defined. This is interestingly discussed in Maxwell's works.[8]

Finally, the total energy stored in the capacitor is

$$W_E = \frac{1}{2} \int_{\text{vol}} \epsilon E^2 \, dv = \frac{1}{2} \int_0^S \int_0^d \frac{\epsilon \rho_S^2}{\epsilon^2} \, dz \, dS = \frac{1}{2} \frac{\rho_S^2}{\epsilon} Sd = \frac{1}{2} \frac{\epsilon S}{d} \frac{\rho_S^2 d^2}{\epsilon^2}$$

or

$$\boxed{W_E = \tfrac{1}{2} C V_0^2 = \tfrac{1}{2} Q V_0 = \tfrac{1}{2} \frac{Q^2}{C}} \qquad (45)$$

which are all familiar expressions. Equation (45) also indicates that the energy stored in a capacitor with a fixed potential difference across it increases as the dielectric constant of the medium increases.

✔ **D5.11.** Find the relative permittivity of the dielectric material present in a parallel-plate capacitor if: (*a*) $S = 0.12\,\text{m}^2$, $d = 80\,\mu\text{m}$, $V_0 = 12\,\text{V}$, and the capacitor contains $1\,\mu\text{J}$ of energy; (*b*) the stored energy density is $100\,\text{J/m}^3$, $V_0 = 200\,\text{V}$, and $d = 45\,\mu\text{m}$; (*c*) $E = 200\,\text{kV/m}$, $\rho_S = 20\,\mu\text{C/m}^2$, and $d = 100\,\mu\text{m}$.

Ans. 6.27; 1.144; 11.29

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[8] See the Suggested References at the end of the chapter.

5.10   SEVERAL CAPACITANCE EXAMPLES

As a first brief example we choose a coaxial cable or coaxial capacitor of inner radius a, outer radius b, and length L. No great derivational struggle is required, because the potential difference is given as Eq. (11) in Sec. 4.3, and we find the capacitance very simply by dividing this by the total charge $\rho_L L$ in the length L. Thus,

$$C = \frac{2\pi\epsilon L}{\ln(b/a)} \qquad (46)$$

Next we consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b, $b > a$. The expression for the electric field was obtained previously by Gauss's law,

$$E_r = \frac{Q}{4\pi\epsilon r^2}$$

where the region between the spheres is a dielectric with permittivity ϵ. The expression for potential difference was found from this by the line integral [Sec. 4.3, Eq. (12)]. Thus,

$$V_{ab} = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Here Q represents the total charge on the inner sphere, and the capacitance becomes

$$C = \frac{Q}{V_{ab}} = \frac{4\pi\epsilon}{\dfrac{1}{a} - \dfrac{1}{b}} \qquad (47)$$

If we allow the outer sphere to become infinitely large, we obtain the capacitance of an isolated spherical conductor,

$$C = 4\pi\epsilon a \qquad (48)$$

For a diameter of 1 cm, or a sphere about the size of a marble,

$$C = 0.556 \quad \text{pF}$$

in free space.

Coating this sphere with a different dielectric layer, for which $\epsilon = \epsilon_1$, extending from $r = a$ to $r = r_1$,

$$D_r = \frac{Q}{4\pi r^2}$$

$$E_r = \frac{Q}{4\pi\epsilon_1 r^2} \qquad (a < r < r_1)$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r_1 < r)$$

and the potential difference is

$$V_a - V_\infty = -\int_{r_1}^{a} \frac{Q\,dr}{4\pi\epsilon_1 r^2} - \int_{\infty}^{r_1} \frac{Q\,dr}{4\pi\epsilon_0 r^2}$$

$$= \frac{Q}{4\pi}\left[\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}\right]$$

Therefore,

$$C = \frac{4\pi}{\dfrac{1}{\epsilon_1}\left(\dfrac{1}{a} - \dfrac{1}{r_1}\right) + \dfrac{1}{\epsilon_0 r_1}} \tag{49}$$

In order to look at the problem of multiple dielectrics a little more thoroughly, let us consider a parallel-plate capacitor of area S and spacing d, with the usual assumption that d is small compared to the linear dimensions of the plates. The capacitance is $\epsilon_1 S/d$, using a dielectric of permittivity ϵ_1. Now let us replace a part of this dielectric by another of permittivity ϵ_2, placing the boundary between the two dielectrics parallel to the plates (Fig. 5.15).

Some of us may immediately suspect that this combination is effectively two capacitors in series, yielding a total capacitance of

$$C = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$$

$$C = \frac{1}{\dfrac{d_1}{\varepsilon_1 S} + \dfrac{d_2}{\varepsilon_2 S}}$$
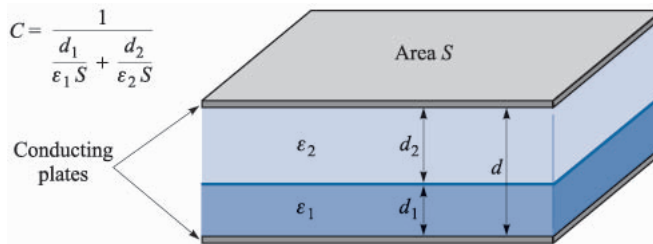
FIGURE 5.15
A parallel-plate capacitor containing two dielectrics with the dielectric interface parallel to the conducting plates.

where $C_1 = \epsilon_1 S/d_1$ and $C_2 = \epsilon_2 S/d_2$. This is the correct result, but we can obtain it using less intuition and a more basic approach.

Since our capacitance definition, $C = Q/V$, involves a charge and a voltage, we may assume either and then find the other in terms of it. The capacitance is not a function of either, but only of the dielectrics and the geometry. Suppose we assume a potential difference V_0 between the plates. The electric field intensities in the two regions, E_2 and E_1, are both uniform, and $V_0 = E_1 d_1 + E_2 d_2$. At the dielectric interface, E is normal and $D_{N1} = D_{N2}$, or $\epsilon_1 E_1 = \epsilon_2 E_2$. Eliminating E_2 in our V_0 relation, we have

$$E_1 = \frac{V_0}{d_1 + d_2(\epsilon_1/\epsilon_2)}$$

and the surface charge density therefore has the magnitude

$$\rho_{S1} = D_1 = \epsilon_1 E_1 = \frac{V_0}{\dfrac{d_1}{\epsilon_1} + \dfrac{d_2}{\epsilon_2}}$$

Since $D_1 = D_2$, the magnitude of the surface charge is the same on each plate. The capacitance is then

$$C = \frac{Q}{V_0} = \frac{\rho_S S}{V_0} = \frac{1}{\dfrac{d_1}{\epsilon_1 S} + \dfrac{d_2}{\epsilon_2 S}} = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$$

As an alternate (and slightly simpler) solution, we might assume a charge Q on one plate, leading to a charge density Q/S and a value of D that is also Q/S. This is true in both regions, as $D_{N1} = D_{N2}$ and D is normal. Then $E_1 = D/\epsilon_1 = Q/(\epsilon_1 S)$, $E_2 = D/\epsilon_2 = Q/(\epsilon_2 S)$, and the potential differences across the regions are $V_1 = E_1 d_1 = Q d_1/(\epsilon_1 S)$, and $V_2 = E_2 d_2 = Q d_2/(\epsilon_2 S)$. The capacitance is

$$C = \frac{Q}{V} = \frac{Q}{V_1 + V_2} = \frac{1}{\dfrac{d_1}{\epsilon_1 S} + \dfrac{d_2}{\epsilon_2 S}} \tag{50}$$

How would the method of solution or the answer change if there were a third conducting plane along the interface? We would now expect to find surface charge on each side of this conductor, and the magnitudes of these charges should be equal. In other words, we think of the electric lines not as passing directly from one outer plate to the other, but as terminating on one side of this interior plane and then continuing on the other side. The capacitance is unchanged, provided, of course, that the added conductor is of negligible thickness. The addition of a thick conducting plate will increase the capacitance if the separation of the outer plates is kept constant, and this is an example of a more general theorem which states that the replacement of any portion of the dielectric by a conducting body will cause an increase in the capacitance.

If the dielectric boundary were placed *normal* to the two conducting plates and the dielectrics occupied areas of S_1 and S_2, then an assumed potential difference V_0 would produce field strengths $E_1 = E_2 = V_0/d$. These are tangential fields at the interface, and they must be equal. Then we may find in succession $D_1, D_2, \rho_{S1}, \rho_{S2}$, and Q, obtaining a capacitance

$$C = \frac{\epsilon_1 S_1 + \epsilon_2 S_2}{d} = C_1 + C_2 \tag{51}$$

as we should expect.

At this time we can do very little with a capacitor in which two dielectrics are used in such a way that the interface is not everywhere normal or parallel to the fields. Certainly we know the boundary conditions at each conductor and at the dielectric interface; however, we do not know the fields to which to apply the boundary conditions. Such a problem must be put aside until our knowledge of field theory has increased and we are willing and able to use more advanced mathematical techniques.

✔ **D5.12.** Determine the capacitance of: (*a*) a 1-ft length of 35B/U coaxial cable, which has an inner conductor 0.1045 in in diameter, a polyethylene dielectric ($\epsilon_R = 2.26$ from Table C.1), and an outer conductor which has an inner diameter of 0.680 in; (*b*) a conducting sphere of radius 2.5 mm, covered with a polyethylene layer 2 mm thick, surrounded by a conducting sphere of radius 4.5 mm; (*c*) two rectangular conducting plates, 1 cm by 4 cm, with negligible thickness, between which are three sheets of dielectric, each 1 cm by 4 cm, and 0.1 mm thick, having dielectric constants of 1.5, 2.5, and 6.

Ans. 20.5 pF; 1.414 pF; 28.0 pF

5.11   CAPACITANCE OF A TWO-WIRE LINE

We conclude this chapter with the problem of the two-wire line. The final configuration will consist of two parallel conducting cylinders, each of circular cross section, and we shall be able to find complete information about the electric field intensity, the potential field, the surface-charge-density distribution, and the capacitance. This arrangement is an important type of transmission line, as is the coaxial cable we have discussed several times before.

We begin by investigating the potential field of two infinite line charges. Fig 5.16 shows a positive line charge in the xz plane at $x = a$ and a negative line charge at $x = -a$. The potential of a single line charge with zero reference at a radius of R_0 is

$$V = \frac{\rho_L}{2\pi\epsilon} \ln \frac{R_0}{R}$$

We now write the expression for the combined potential field in terms of the radial distances from the positive and negative lines, R_1 and R_2, respectively,
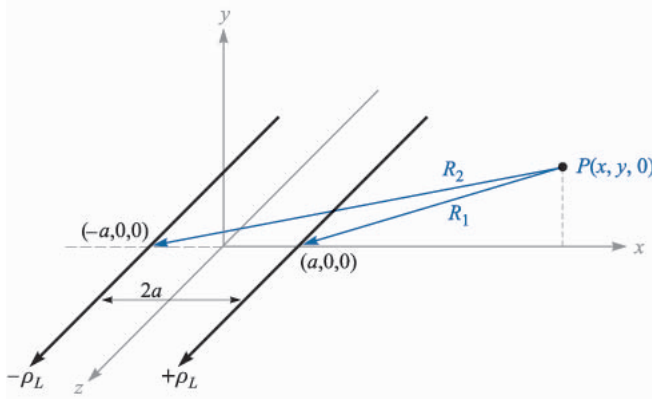
FIGURE 5.16
Two parallel infinite line charges carrying opposite charge. The positive line is at $x = a$, $y = 0$, and the negative line is at $x = -a$, $y = 0$. A general point $P(x, y, 0)$ in the xy plane is radially distant R_1 and R_2 from the positive and negative lines, respectively. The equipotential surfaces are circular cylinders.

$$V = \frac{\rho_L}{2\pi\epsilon}\left(\ln\frac{R_{10}}{R_1} - \ln\frac{R_{20}}{R_2}\right) = \frac{\rho_L}{2\pi\epsilon}\ln\frac{R_{10}R_2}{R_{20}R_1}$$

We choose $R_{10} = R_{20}$, thus placing the zero reference at equal distances from each line. This surface is the $x = 0$ plane. Expressing R_1 and R_2 in terms of x and y,

$$V = \frac{\rho_L}{2\pi\epsilon}\ln\sqrt{\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}} = \frac{\rho_L}{4\pi\epsilon}\ln\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \tag{52}$$

In order to recognize the equipotential surfaces and adequately understand the problem we are going to solve, some algebraic manipulations are necessary. Choosing an equipotential surface $V = V_1$, we define K_1 as a dimensionless parameter that is a function of the potential V_1,

$$K_1 = e^{4\pi\epsilon V_1/\rho_L} \tag{53}$$

so that

$$K_1 = \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

After multiplying and collecting like powers, we obtain

$$x^2 - 2ax\frac{K_1 + 1}{K_1 - 1} + y^2 + a^2 = 0$$

We next work through a couple of lines of algebra and complete the square,

$$\left(x - a\frac{K_1 + 1}{K_1 - 1}\right)^2 + y^2 = \left(\frac{2a\sqrt{K_1}}{K_1 - 1}\right)^2$$

This shows that the $V = V_1$ equipotential surface is independent of z (or is a cylinder) and intersects the xy plane in a circle of radius b,

$$b = \frac{2a\sqrt{K_1}}{K_1 - 1}$$

which is centered at $x = h$, $y = 0$, where

$$h = a\frac{K_1 + 1}{K_1 - 1}$$

Now let us attack a physical problem by considering a zero-potential conducting plane located at $x = 0$, and a conducting cylinder of radius b and potential V_0 with its axis located a distance h from the plane. We solve the last two equations for a and K_1 in terms of the dimensions b and h,

$$a = \sqrt{h^2 - b^2} \tag{54}$$

and

$$\sqrt{K_1} = \frac{h + \sqrt{h^2 - b^2}}{b} \tag{55}$$

But the potential of the cylinder is V_0, so (53) leads to

$$\sqrt{K_1} = e^{2\pi\epsilon V_0/\rho_L}$$

Therefore,

$$\rho_L = \frac{4\pi\epsilon V_0}{\ln K_1} \tag{56}$$

Thus, given h, b, and V_0, we may determine a, ρ_L, and the parameter K_1. The capacitance between the cylinder and plane is now available. For a length L in the z direction, we have

$$C = \frac{\rho_L L}{V_0} = \frac{4\pi\epsilon L}{\ln K_1} = \frac{2\pi\epsilon L}{\ln \sqrt{K_1}}$$

or

$$C = \frac{2\pi\epsilon L}{\ln[h + \sqrt{h^2 - b^2}/b]} = \frac{2\pi\epsilon L}{\cosh^{-1}(h/b)} \tag{57}$$

The heavy black circle in Fig. 5.17 shows the cross section of a cylinder of 5-m radius at a potential of 100 V in free space, with its axis 13 m distant from a plane at zero potential. Thus, $b = 5$, $h = 13$, $V_0 = 100$, and we rapidly find the location of the equivalent line charge from (54),
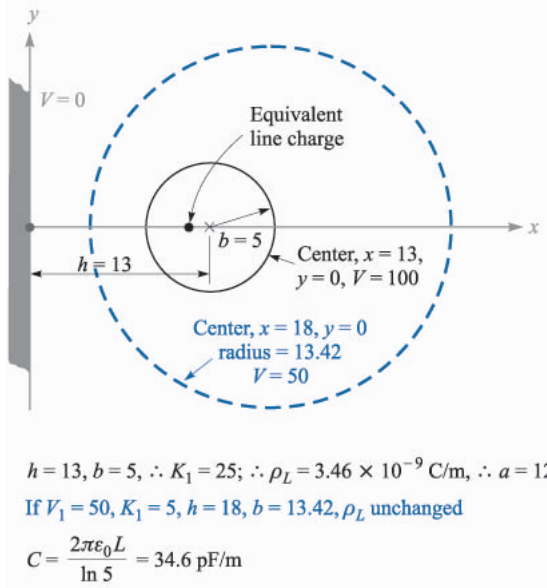
$h = 13, b = 5, \therefore K_1 = 25; \therefore \rho_L = 3.46 \times 10^{-9}$ C/m, $\therefore a = 12$

If $V_1 = 50, K_1 = 5, h = 18, b = 13.42, \rho_L$ unchanged

$C = \dfrac{2\pi\varepsilon_0 L}{\ln 5} = 34.6$ pF/m

FIGURE 5.17
A numerical example of the capacitance, linear charge density, position of an equivalent line charge, and characteristics of the mid-equipotential surface for a cylindrical conductor of 5-m radius at a potential of 100 V, parallel to and 13 m from a conducting plane at zero potential.

$$a = \sqrt{h^2 - b^2} = \sqrt{13^2 - 5^2} = 12\,\text{m}$$

the value of the potential parameter K_1 from (55),

$$\sqrt{K_1} = \frac{h + \sqrt{h^2 - b^2}}{b} = \frac{13 + 12}{5} = 5 \quad K_1 = 25$$

the strength of the equivalent line charge from (56)

$$\rho_L = \frac{4\pi\varepsilon V_0}{\ln K_1} = \frac{4\pi \times 8.854 \times 10^{-12} \times 100}{\ln 25} = 3.46 \quad \text{nC/m}$$

and the capacitance between cylinder and plane from (57),

$$C = \frac{2\pi\varepsilon}{\cosh^{-1}(h/b)} = \frac{2\pi \times 8.854 \times 10^{-12}}{\cosh^{-1}(13/5)} = 34.6 \quad \text{pF/m}$$

We may also identify the cylinder representing the 50-V equipotential surface by finding new values for K_1, h, and b. We first use (53) to obtain

$$K_1 = e^{4\pi\varepsilon V_1/\rho_L} = e^{4\pi \times 8.854 \times 10^{-12} \times 50/3.46 \times 10^{-9}} = 5.00$$

Then the new radius is

$$b = \frac{2a\sqrt{K_1}}{K_1 - 1} = \frac{2 \times 12\sqrt{5}}{5 - 1} = 13.42 \quad \text{m}$$

and the corresponding value of h becomes

$$h = a\frac{K_1 + 1}{K_1 - 1} = 12\frac{5 + 1}{5 - 1} = 18 \quad \text{m}$$

This cylinder is shown in color in Fig. 5.17.

The electric field intensity can be found by taking the gradient of the potential field, as given by (52),

$$\mathbf{E} = -\nabla\left[\frac{\rho_L}{4\pi\epsilon} \ln\frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}\right]$$

Thus,

$$\mathbf{E} = -\frac{\rho_L}{4\pi\epsilon}\left[\frac{2(x + a)\mathbf{a}_x + 2y\mathbf{a}_y}{(x + a)^2 + y^2} - \frac{2(x - a)\mathbf{a}_x + 2y\mathbf{a}_y}{(x - a)^2 + y^2}\right)$$

and

$$\mathbf{D} = e\mathbf{E} = -\frac{\rho_L}{2\pi}\left[\frac{(x + a)\mathbf{a}_x + y\mathbf{a}_y}{(x + a)^2 + y^2} - \frac{(x - a)\mathbf{a}_x + y\mathbf{a}_y}{(x - a)^2 + y^2}\right]$$

If we evaluate D_x at $x = h - b$, $y = 0$, we may obtain $\rho_{S,max}$

$$\rho_{S,max} = -D_{x,x=h-b,y=0} = \frac{\rho_L}{2\pi}\left[\frac{h - b + a}{(h - b + a)^2} - \frac{h - b - a}{(h - b - a)^2}\right]$$

For our example,

$$\rho_{S,max} = \frac{3.46 \times 10^{-9}}{2\pi}\left[\frac{13 - 5 + 12}{(13 - 5 + 12)^2} - \frac{13 - 5 - 12}{(13 - 5 - 12)^2}\right] = 0.1650 \quad \text{nC/m}^2$$

Similarly, $\rho_{S,min} = D_{x,x=h+b,y=0}$, and

$$\rho_{S,min} = \frac{3.46 \times 10^{-9}}{2\pi}\left[\frac{13 + 5 + 12}{30^2} - \frac{13 + 5 - 12}{6^2}\right] = 0.0734 \quad \text{nC/m}^2$$

Thus,

$$\rho_{S,max} = 2.25\rho_{S,min}$$

If we apply (57) to the case of a conductor for which $b \ll h$, then

$$\ln[(h + \sqrt{h^2 - b^2}/b] \doteq \ln([h + h)/b] \doteq \ln(2h/b)$$

and

$$C = \frac{2\pi\epsilon L}{\ln(2h/b)} \quad (b \ll h) \tag{58}$$

The capacitance between two circular conductors separated by a distance $2h$ is one-half the capacitance given by (57) or (58). This last answer is of interest

because it gives us an expression for the capacitance of a section of two-wire transmission line, one of the types of transmission lines studied later in Chap. 12.

✔ **D5.13.** A conducting cylinder with a radius of 1 cm and at a potential of 20 V is parallel to a conducting plane which is at zero potential. The plane is 5 cm distant from the cylinder axis. If the conductors are embedded in a perfect dielectric for which $\epsilon_R = 4.5$, find: (a) the capacitance per unit length between cylinder and plane; (b) $\rho_{S,\text{max}}$ on the cylinder.

Ans. 109.2 pF/m; 2.21 nC/m

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PROBLEMS

5.1 Given the current density $\mathbf{J} = -10^4(\sin 2x\, e^{-2y}\mathbf{a}_x + \cos 2x\, e^{-2y}\mathbf{a}_y)\,\text{kA/m}^2$: (*a*) find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$. Find the total current leaving the region $0 < x, y < 1$, $2 < z < 3$ by: (*b*) integrating $\mathbf{J} \cdot d\mathbf{S}$ over the surface of the cube; (*c*) employing the divergence theorem.

5.2 Let the current density be $\mathbf{J} = 2\rho\cos^2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\,\mathbf{a}_\phi\,\text{A/m}^2$ within the region $2.1 < \rho < 2.5$, $0 < \phi < 0.1$ rad, $6 < z < 6.1$. Find the total current I crossing the surface: (*a*) $\rho = 2.2$, $0 < \phi < 0.1$, $6 < z < 6.1$ in the $\mathbf{a}_\rho$ direction; (*b*) $\phi = 0.05$, $2.2 < \rho < 2.5$, $6 < z < 6.1$, in the $\mathbf{a}_\phi$ direction. (*c*) Evaluate $\nabla \cdot \mathbf{J}$ at $P(\rho = 2.4, \phi = 0.08, z = 6.05)$.

5.3 Let $\mathbf{J} = \dfrac{400\sin\theta}{r^2 + 4}\,\text{A/m}^2$. (*a*) Find the total current flowing through that portion of the spherical surface $r = 0.8$ bounded by $\theta = 0.1\pi$, $\theta = 0.3\pi$, $0 < \phi < 2\pi$. (*b*) Find the average value of J over the defined area.

5.4 The cathode of a planar vacuum tube is at $z = 0$. Let $\mathbf{E} = -4 \times 10^6\mathbf{a}_z$ V/m for $z > 0$. An electron ($e = 1.602 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg) is emitted from the cathode with zero initial velocity at $t = 0$. (*a*) Find $v(t)$. (*b*) Find $z(t)$, the electron location as a function of time. (*c*) Determine $v(z)$. (*d*) Make the assumption that electrons are emitted continuously as a beam with a 0.25-mm radius and a total current of 60 µA. Find $\mathbf{J}(z)$ and $\rho_v(z)$.

5.5 Let $\mathbf{J} = \dfrac{25}{\rho}\mathbf{a}_\rho - \dfrac{20}{\rho^2 + 0.01}\mathbf{a}_z\,\text{A/m}^2$, and: (*a*) find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$. (*b*) Calculate $\dfrac{\partial\rho_v}{\partial t}$. (*c*) Find the total outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$. (*d*) Show that the divergence theorem is satisified for $\mathbf{J}$ and the surface specified.

5.6 Let $\epsilon = \epsilon_0$ and $V = 90z^{4/3}$ in the region $z = 0$. (*a*) Obtain expressions for $\mathbf{E}$, $\mathbf{D}$, and ρ_v as functions of z. (*b*) If the velocity of the charge density is given as $v_x = 5 \times 10^6 z^{2/3}$ m/s, find J_z at $z = 0$ and $z = 0.1$ m.

5.7 Assuming that there is no transformation of mass to energy or vice versa, it is possible to write a continuity equation for mass. (*a*) If we use the continuity equation for charge as our model, what quantities correspond to $\mathbf{J}$ and ρ_v? (*b*) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, −9.85, 1.75, −2.00, −4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center.

5.8 The continuity equation for mass equates the divergence of the mass rate of flow (mass per second per square meter) to the negative of the density (mass per cubic meter). After setting up a cartesian coordinate system inside a star, Captain Kirk and his intrepid crew make measurements over the faces of a cube centered at the origin with edges 40 km long and

parallel to the coordinate axes. They find the mass rate of flow of mate-
rial outward across the six faces to be -1112, 1183, 201, -196, 1989, and
$-1920 \, \text{kg/km}^2 \cdot \text{s}$. (a) Estimate the divergence of the mass rate of flow at
the origin. (b) Estimate the rate of change of the density at the origin.

5.9 (a) Using data tabulated in Appendix C, calculate the required diameter
for a 2-m long nichrome wire that will dissipate an average power of
450 W when 120-V rms at 60 Hz is applied to it. (b) Calculate the rms
current density in the wire.

5.10 A steel wire has a radius of 2 mm and a conductivity of $6 \times 10^6 \, \text{S/m}$. The
steel wire has an aluminum ($\sigma = 3.8 \times 10^7 \, \text{S/m}$) coating of 2-mm thick-
ness. Let the total current carried by this hybrid conductor be 80 A dc.
Find: (a) J_{st}; (b) J_{Al}; (c) E_{st}; (d) E_{Al}; (e) the voltage between the ends of
the conductor if it is 1 mi long.

5.11 Two perfectly conducting cylindrical surfaces are located at $\rho = 3$ and
$\rho = 5 \, \text{cm}$. The total current passing radially outward through the med-
ium between the cylinders is 3 A dc. (a) Find the voltage and resistance
between the cylinders, and **E** in the region between the cylinders, if a
conducting material having $\sigma = 0.05 \, \text{S/m}$ is present for $3 < \rho < 5 \, \text{cm}$. (b)
Show that integrating the power dissipated per unit volume over the
volume gives the total dissipated power.

5.12 The spherical surfaces $r = 3$ and $r = 5 \, \text{cm}$ are perfectly conducting, and
the total current passing radially outward through the medium between
the surfaces is 3 A dc. (a) Find the voltage and resistance between the
spheres, and **E** in the region between them, if a conducting material
having $\sigma = 0.05 \, \text{S/m}$ is present for $3 < r < 5 \, \text{cm}$. (b) Repeat if $\sigma =$
$0.0005/r$ for $3 < r < 5 \, \text{cm}$. (c) Show that integrating the power dissipated
per unit volume in part b over the volume gives the total dissipated
power.

5.13 A hollow cylindrical tube with a rectangular cross section has external
dimensions of 0.5 in by 1 in and a wall thickness of 0.05 in. Assume that
the material is brass for which $\sigma = 1.5 \times 10^7 \, \text{S/m}$. A current of 200 A dc
is flowing down the tube. (a) What voltage drop is present across a 1 m
length of the tube? (b) Find the voltage drop if the interior of the tube is
filled with a conducting material for which $\sigma = 1.5 \times 10^5 \, \text{S/m}$.

5.14 Find the magnitude of the electric field intensity in a conductor if: (a) the
current density is $5 \, \text{MA/m}^2$, the electron mobility is $3 \times 10^{-3} \, \text{m}^2/\text{V} \cdot \text{s}$,
and the volume charge density is $-2.4 \times 10^{10} \, \text{C/m}^3$; (b) $J = 3 \, \text{MA/m}^2$
and the resistivity is $3 \times 10^{-8} \, \Omega \cdot \text{m}$.

5.15 Let $V = 10(\rho + 1)z^2 \cos \phi \, \text{V}$ in free space. (a) Let the equipotential sur-
face $V = 20 \, \text{V}$ define a conductor surface. Find the equation of the con-
ductor surface. (b) Find ρ and **E** at that point on the conductor surface
where $\phi = 0.2\pi$ and $z = 1.5$. (c) Find $|\rho_S|$ at that point.

5.16 A potential field in free space is given as $V = (80 \cos \theta \sin \phi)/r^3 \, \text{V}$. Point
$P(r = 2, \theta = \pi/3, \phi = \pi/2)$ lies on a conducting surface. (a) Write the
equation of the conducting surface. (b) Find a unit normal directed out-

ward to the surface, assuming the origin is inside the surface. (*c*) Find $\mathbf{E}$ at *P*.

5.17 Given the potential field $V = \dfrac{100xz}{x^2 + 4}$ V in free space: (*a*) find $\mathbf{D}$ at the surface $z = 0$. (*b*) Show that the $z = 0$ surface is an equipotential surface. (*c*) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$.

5.18 Let us assume a field $\mathbf{E} = 3y^2z^3\mathbf{a}_x + 6xyz^3\mathbf{a}_y + 9xy^2z^2$ V/m in free space, and also assume that point $P(2, 1, 0)$ lies on a conducting surface. (*a*) Find ρ_v just adjacent to the surface at *P*. (*b*) Find ρ_S at *P*. (*c*) Show that $V = -3xy^2z^3$ V. (*d*) Determine V_{PQ}, given $Q(1, 1, 1)$.

5.19 Let $V = 20x^2yz - 10z^2$ V in free space. (*a*) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V. (*b*) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$-V surface where $x = 2$ and $z = 1$. It is known that $0 \leq V \leq 60$ V is the field-containing region. (*c*) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface.

5.20 A conducting plane is located at $z = 0$ in free space, and a 20-nC point charge is present at $Q(2, 4, 6)$. (*a*) If $V = 0$ at $z = 0$, find V at $P(5, 3, 1)$. (*b*) Find $\mathbf{E}$ at *P*. (*c*) Find ρ_S at $A(5, 3, 0)$.

5.21 Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$. (*a*) Let $V = 0$ at the plane $y = 0$, and find V at $P(1, 2, 0)$. (*b*) Find $\mathbf{E}$ at *P*.

5.22 Let the plane $x = 0$ be a perfect conductor in free space. Locate a point charge of 4 nC at $P_1(7, 1, -2)$, and a point charge of -3 nC at $P_2(4, 2, 1)$. (*a*) Find $\mathbf{E}$ at $A(5, 0, 0)$. (*b*) Find $|\rho_S|$ at $B(3, 0, 0)$.

5.23 A dipole with $\mathbf{p} = 0.1\mathbf{a}_z$ µC · m is located at $A(1, 0, 0)$ in free space, and the $x = 0$ plane is perfectly conducting. (*a*) Find V at $P(2, 0, 1)$. (*b*) Find the equation of the 200-V equipotential surface in cartesian coordinates.

5.24 The mobilities for intrinsic silicon at a certain temperature are $\mu_e = 0.14$ m^2/V · s and $\mu_h = 0.035$ m^2/V · s. The concentration of both holes and electrons is 2.2×10^{16} m^{-3}. Determine both the conductivity and resistivity of this silicon sample.

5.25 Electron and hole concentrations increase with temperature. For pure silicon suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}$ C/m^3. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}$ m^2/V · s and $\mu_e = 2.1 \times 10^5 T^{-2.5}$ m^2/V · s. Find σ at: (*a*) $0°$ C; (*b*) $40°$ C; (*c*) $80°$ C.

5.26 A little donor impurity, such as arsenic, is added to pure silicon so that the electron concentration is 2×10^{17} conduction electrons per cubic meter while the number of holes per cubic meter is only 1.1×10^{15}. If $\mu_e = 0.15$ m^2/V · s for this sample, and $\mu_h = 0.045$ m^2/V· s, determine the conductivity and resistivity.

5.27 Atomic hydrogen contains 5.5×10^{25} atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and the positive nucleus has an effective length of 7.1×10^{-19} m. Find: (a) P; (b) ϵ_R.

5.28 In a certain region where the relative permittivity is 2.4, $\mathbf{D} = 2\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z$ nC/m^2. Find: (a) $\mathbf{E}$; (b) $\mathbf{P}$; (c) $|\nabla V|$.

5.29 A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_R = 2.56$. If $\mathbf{P} = \dfrac{2}{\rho}\mathbf{a}_\rho$ nC/m^2 in the dielectric, find: (a) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ; (b) V_{ab} and χ_e. (c) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$.

5.30 Given the potential field $V = 200 - 50x + 20y$ V in a dielectric material for which $\epsilon_R = 2.1$, find: (a) $\mathbf{E}$; (b) $\mathbf{D}$; (c) $\mathbf{P}$; (d) ρ_v; (e) ρ_b; (f) ρ_T.

5.31 The surface $x = 0$ separates two perfect dielectrics. For $x > 0$ let $\epsilon_R = \epsilon_{R1} = 3$, while $\epsilon_{R2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find: (a) E_{N1}; (b) $\mathbf{E}_{t1}$; (c) E_{t1}; (d) E_1; (e) the angle θ_1 between $\mathbf{E}_1$ and a normal to the surface; (f) D_{N2}; (g) D_{t2}; (h) $\mathbf{D}_2$; (i) $\mathbf{P}_2$ (j) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface.

5.32 In Fig. 5.18 let $\mathbf{D}_1 = 3\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z$ nC/m^2 and find: (a) $\mathbf{D}_2$; (b) D_{N2}; (c) D_{t2}; (d) the energy density in each region; (e) the angle that $\mathbf{D}_2$ makes with $\mathbf{a}_z$; (f) D_2/D_1; (g) P_2/P_1.

5.33 Two perfect dielectrics have relative permittivities $\epsilon_{R1} = 2$ and $\epsilon_{R2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$.

5.34 Let the spherical surfaces $r = 4$ cm and $r = 9$ cm be separated by two perfect dielectric shells, $\epsilon_{R1} = 2$ for $4 < r < 6$ cm, and $\epsilon_{R2} = 5$ for $6 < r < 9$ cm. If $\mathbf{E}_1 = \dfrac{2000}{r^2}\mathbf{a}_r$ V/m, find: (a) $\mathbf{E}_2$; (b) the total electrostatic energy stored in each region.
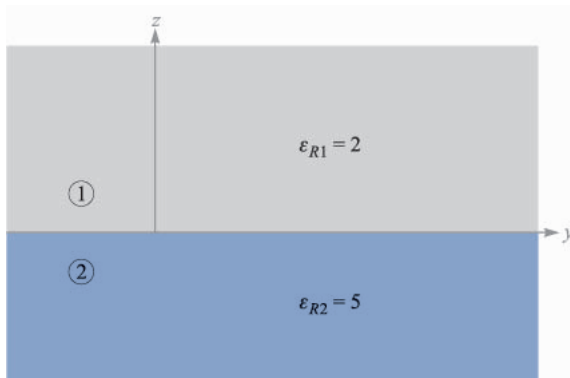
FIGURE 5.18
See Prob. 32.

5.35 Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{R1} = 2$ for $0 < \phi < \pi/2$, and $\epsilon_{R2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = \dfrac{2000}{\rho}\mathbf{a}_\rho$ V/m, find: (a) $\mathbf{E}_2$; (b) the total electrostatic energy stored in a 1-m length of each region.

5.36 Let $S = 120\,\text{cm}^2$, $d = 4$ mm, and $\epsilon_R = 12$ for a parallel-plate capacitor. (a) Calculate the capacitance. (b) After connecting a 40-V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy. (c) The source is now removed and the dielectric carefully withdrawn from between the plates. Again calculate E, D, Q, and the energy. (d) What is the voltage between the plates?

5.37 Capacitors tend to be more expensive as their capacitance and maximum voltage $V_{\max}$ increase. The voltage $V_{\max}$ is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest $CV_{\max}$ product for equal plate areas: (a) air: $\epsilon_R = 1$, $E_{BD} = 3$ MV/m; (b) barium titanate: $\epsilon_R = 1200$, $E_{BD} = 3$ MV/m; (c) silicon dioxide: $\epsilon_R = 3.78$, $E_{BD} = 16$ MV/m; (d) polyethylene: $\epsilon_R = 2.26$, $E_{BD} = 4.7$ MV/m.

5.38 A dielectric circular cylinder used between the plates of a capacitor has a thickness of 0.2 mm and a radius of 1.4 cm. The dielectric properties are $\epsilon_R = 400$ and $\sigma = 10^{-5}$ S/m. (a) Calculate C. (b) Find the quality factor $Q_{QF}(Q_{QF} = \omega RC)$ of the capacitor at $f = 10$ kHz. (c) If the maximum field strength permitted in the dielectric is 2 MV/m, what is the maximum permissible voltage across the capacitor? (d) What energy is stored when this voltage is applied?

5.39 A parallel-plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_R = 2 + 2 \times 10^6 x^2$, where x is the distance from one plate. If $S = 0.02\,\text{m}^2$ and $d = 1$ mm, find C.

5.40 (a) The width of the region containing ϵ_{R1} in Fig. 5.19 is 1.2 m. Find ϵ_{R1} if $\epsilon_{R2} = 2.5$ and the total capacitance is 60 nF. (b) Find the width of each region (containing ϵ_{R1} and ϵ_{R2}) if $C_{\text{total}} = 80$ nF, $\epsilon_{R2} = 3\epsilon_{R1}$, and $C_1 = 2C_2$.
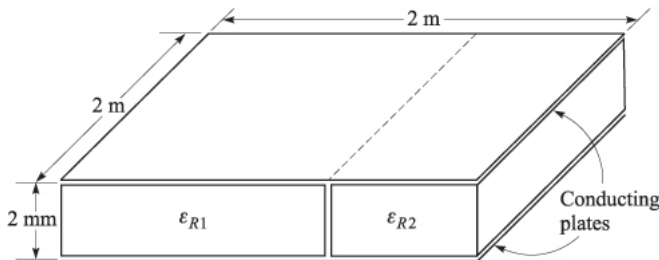

FIGURE 5.19
See Prob. 40.

5.41 Let $\epsilon_{R1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{R2} = 4$ for $1 < y < 3$ mm, and ϵ_{R3} for $3 < y < 5$ mm. Conducting surfaces are present at $y = 0$ and $y = 5$ mm. Calculate the capacitance per square meter of surface area if: (a) ϵ_{R3} is air; (b) $\epsilon_{R3} = \epsilon_{R1}$; (c) $\epsilon_{R3} = \epsilon_{R2}$; (d) ϵ_{R3} is silver.

5.42 Cylindrical conducting surfaces are located at $\rho = 0.8$ cm and 3.6 cm. The region 0.8 cm $< \rho < a$ contains a dielectric for which $\epsilon_R = 4$, while $\epsilon_R = 2$ for $a < \rho < 3.6$ cm. (a) Find a so that the voltage across each dielectric layer is the same. (b) Find the total capacitance per meter.

5.43 Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1 m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_R = 4$. Find the capacitance if: (a) $c = 2$ cm, $d = 3$ cm; (b) $d = 4$ cm, and the volume of dielectric is the same as in part a.

5.44 Conducting cylinders lie at $\rho = 3$ and 12 mm; both extend from $z = 0$ to $z = 1$ m. Perfect dielectrics occupy the interior region: $\epsilon_R = 1$ for $3 < \rho < 6$ mm, $\epsilon_R = 4$ for $6 < \rho < 9$ mm, and $\epsilon_R = 8$ for $9 < \rho < 12$ mm. (a) Calculate C. (b) If the voltage between the cylinders is 100 V, plot $|E_\rho|$ versus ρ.

5.45 Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_R = 8$. (a) Find C. (b) A portion of the dielectric is now removed so that $\epsilon_R = 1$, $0 < \phi < \pi/2$, and $\epsilon_R = 8$, $\pi/2 < \phi < 2\pi$. Again find C.

5.46 Conducting cylinders lie at $\rho = 3$ and 12 mm; both extend from $z = 0$ to $z = 1$ m. Perfect dielectrics occupy the interior region: $\epsilon_R = 1$ for $3 < \rho < 6$ mm, $\epsilon_R = 4$ for $6 < \rho < 9$ mm, and $\epsilon_R = 8$ for $9 < \rho < 12$ mm. (a) Calculate C. (b) If the voltage between the cylinders is 100 V, plot $|E_\rho|$ versus ρ.

5.47 With reference to Fig. 5.17, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C.

5.48 A potential function in free space is given by $V = -20 + 10 \ln \dfrac{(5+y)^2 + x^2}{(5-y)^2 + x^2}$ V. Describe: (a) the 0-V equipotential surface; (b) the 10-V equipotential surface.

5.49 A 2-cm-diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. Find the surface charge density on the: (a) cylinder at a point nearest the plane; (b) plane at a point nearest the cylinder.