Problem Solutions for E-text

1.3. The vector from the origin to the point A is given as $(6, -2, -4)$, and the unit vector directed from the origin toward point B is $(2, -2, 1)/3$. If points A and B are ten units apart, find the coordinates of point B.

With $\mathbf{A} = (6, -2, -4)$ and $\mathbf{B} = \frac{1}{3}B(2, -2, 1)$, we use the fact that $|\mathbf{B} - \mathbf{A}| = 10$, or
$|(6 - \frac{2}{3}B)\mathbf{a}_x - (2 - \frac{2}{3}B)\mathbf{a}_y - (4 + \frac{1}{3}B)\mathbf{a}_z| = 10$
Expanding, obtain
$36 - 8B + \frac{4}{9}B^2 + 4 - \frac{8}{3}B + \frac{4}{9}B^2 + 16 + \frac{8}{3}B + \frac{1}{9}B^2 = 100$
or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64-176}}{2} = 11.75$ (taking positive option) and so

$$\mathbf{B} = \frac{2}{3}(11.75)\mathbf{a}_x - \frac{2}{3}(11.75)\mathbf{a}_y + \frac{1}{3}(11.75)\mathbf{a}_z = \underline{7.83\mathbf{a}_x - 7.83\mathbf{a}_y + 3.92\mathbf{a}_z}$$

1.17. Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18, -10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle.

a) Find a unit vector perpendicular to the triangle: Use

$$\mathbf{a}_p = \frac{\mathbf{R}_{AM} \times \mathbf{R}_{AN}}{|\mathbf{R}_{AM} \times \mathbf{R}_{AN}|} = \frac{(350, -200, 340)}{527.35} = \underline{(0.664, -0.379, 0.645)}$$

The vector in the opposite direction to this one is also a valid answer.

b) Find a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$:

$$\mathbf{a}_{AN} = \frac{(-10, 8, 15)}{\sqrt{389}} = (-0.507, 0.406, 0.761)$$

Then

$$\mathbf{a}_{pAN} = \mathbf{a}_p \times \mathbf{a}_{AN} = (0.664, -0.379, 0.645) \times (-0.507, 0.406, 0.761) = \underline{(-0.550, -0.832, 0.077)}$$

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at A: A non-unit vector in the required direction is $(1/2)(\mathbf{a}_{AM} + \mathbf{a}_{AN})$, where

$$\mathbf{a}_{AM} = \frac{(20, 18, -10)}{|(20, 18, -10)|} = (0.697, 0.627, -0.348)$$

Now

$$\frac{1}{2}(\mathbf{a}_{AM} + \mathbf{a}_{AN}) = \frac{1}{2}[(0.697, 0.627, -0.348) + (-0.507, 0.406, 0.761)] = (0.095, 0.516, 0.207)$$

Finally,

$$\mathbf{a}_{bis} = \frac{(0.095, 0.516, 0.207)}{|(0.095, 0.516, 0.207)|} = \underline{(0.168, 0.915, 0.367)}$$

1.27. The surfaces $r = 2$ and 4, $\theta = 30°$ and $50°$, and $\phi = 20°$ and $60°$ identify a closed surface.

a) Find the enclosed volume: This will be

$$\text{Vol} = \int_{20°}^{60°}\int_{30°}^{50°}\int_{2}^{4} r^2 \sin\theta dr d\theta d\phi = \underline{2.91}$$

where degrees have been converted to radians.

b) Find the total area of the enclosing surface:

$$\text{Area} = \int_{20°}^{60°}\int_{30°}^{50°} (4^2 + 2^2)\sin\theta d\theta d\phi + \int_{2}^{4}\int_{20°}^{60°} r(\sin 30° + \sin 50°) dr d\phi$$
$$+ 2\int_{30°}^{50°}\int_{2}^{4} r dr d\theta = \underline{12.61}$$

c) Find the total length of the twelve edges of the surface:

$$\text{Length} = 4\int_{2}^{4} dr \; + \; 2\int_{30°}^{50°} (4+2) d\theta + \int_{20°}^{60°} (4\sin 50° + 4\sin 30° + 2\sin 50° + 2\sin 30°) d\phi$$
$$= \underline{17.49}$$

d) Find the length of the longest straight line that lies entirely within the surface: This will be from $A(r = 2, \theta = 50°, \phi = 20°)$ to $B(r = 4, \theta = 30°, \phi = 60°)$ or

$$A(x = 2\sin 50° \cos 20°, y = 2\sin 50° \sin 20°, z = 2\cos 50°)$$

to

$$B(x = 4\sin 30° \cos 60°, y = 4\sin 30° \sin 60°, z = 4\cos 30°)$$

or finally $A(1.44, 0.52, 1.29)$ to $B(1.00, 1.73, 3.46)$. Thus $\mathbf{B} - \mathbf{A} = (-0.44, 1.21, 2.18)$ and

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = \underline{2.53}$$

2.5. Let a point charge $Q_1 25$ nC be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3, 4, -2)$.

a) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1, 2, 3)$: This field will be

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{60\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$

where $\mathbf{R}_{13} = -3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{R}_{23} = 4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{41}$ and $|\mathbf{R}_{23}| = \sqrt{45}$. So

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25 \times (-3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z)}{(41)^{1.5}} + \frac{60 \times (4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z)}{(45)^{1.5}}\right]$$
$$= \underline{4.58\mathbf{a}_x - 0.15\mathbf{a}_y + 5.51\mathbf{a}_z}$$

b) At what point on the y axis is $E_x = 0$? P_3 is now at $(0, y, 0)$, so $\mathbf{R}_{13} = -4\mathbf{a}_x + (y+2)\mathbf{a}_y - 7\mathbf{a}_z$ and $\mathbf{R}_{23} = 3\mathbf{a}_x + (y-4)\mathbf{a}_y + 2\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{65 + (y+2)^2}$ and $|\mathbf{R}_{23}| = \sqrt{13 + (y-4)^2}$. Now the x component of $\mathbf{E}$ at the new P_3 will be:

$$E_x = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25 \times (-4)}{[65 + (y+2)^2]^{1.5}} + \frac{60 \times 3}{[13 + (y-4)^2]^{1.5}}\right]$$

To obtain $E_x = 0$, we require the expression in the large brackets to be zero. This expression simplifies to the following quadratic:

$$0.48y^2 + 13.92y + 73.10 = 0$$

which yields the two values: $y = \underline{-6.89, -22.11}$

2.19. A uniform line charge of 2 μC/m is located on the z axis. Find **E** in cartesian coordinates at $P(1,2,3)$ if the charge extends from

a) $-\infty < z < \infty$: With the infinite line, we know that the field will have only a radial component in cylindrical coordinates (or x and y components in cartesian). The field from an infinite line on the z axis is generally $\mathbf{E} = [\rho_l/(2\pi\epsilon_0\rho)]\mathbf{a}_\rho$. Therefore, at point P:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2} = \frac{(2\times10^{-6})}{2\pi\epsilon_0}\frac{\mathbf{a}_x + 2\mathbf{a}_y}{5} = \underline{7.2\mathbf{a}_x + 14.4\mathbf{a}_y \ \text{kV/m}}$$

where $\mathbf{R}_{zP}$ is the vector that extends from the line charge to point P, and is perpendicular to the z axis; i.e., $\mathbf{R}_{zP} = (1,2,3) - (0,0,3) = (1,2,0)$.

b) $-4 \le z \le 4$: Here we use the general relation

$$\mathbf{E}_P = \int \frac{\rho_l dz}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

where $\mathbf{r} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{r}' = z\mathbf{a}_z$. So the integral becomes

$$\mathbf{E}_P = \frac{(2\times10^{-6})}{4\pi\epsilon_0}\int_{-4}^{4}\frac{\mathbf{a}_x + 2\mathbf{a}_y + (3-z)\mathbf{a}_z}{[5+(3-z)^2]^{1.5}}\,dz$$

Using integral tables, we obtain:

$$\mathbf{E}_P = 3597\left[\frac{(\mathbf{a}_x + 2\mathbf{a}_y)(z-3) + 5\mathbf{a}_z}{(z^2 - 6z + 14)}\right]_{-4}^{4} \text{V/m} = \underline{4.9\mathbf{a}_x + 9.8\mathbf{a}_y + 4.9\mathbf{a}_z \ \text{kV/m}}$$

The student is invited to verify that when evaluating the above expression over the limits $-\infty < z < \infty$, the z component vanishes and the x and y components become those found in part a.

2.27. Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find:

a) the equation of the streamline that passes through the point $P(2,3,-4)$: We write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-(2x+4y)}{(4x-2y)}$$

Thus

$$2(x\,dy + y\,dx) = y\,dy - x\,dx$$

or

$$2\,d(xy) = \frac{1}{2}\,d(y^2) - \frac{1}{2}\,d(x^2)$$

So

$$C_1 + 2xy = \frac{1}{2}y^2 - \frac{1}{2}x^2$$

or

$$y^2 - x^2 = 4xy + C_2$$

Evaluating at $P(2, 3, -4)$, obtain:

$$9 - 4 = 24 + C_2, \text{ or } C_2 = -19$$

Finally, at P, the requested equation is

$$\underline{y^2 - x^2 = 4xy - 19}$$

b) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$: Have $\mathbf{E}_Q = [4(3) + 2(2)]\mathbf{a}_x - [2(3) - 4(2)]\mathbf{a}_y = 16\mathbf{a}_x + 2\mathbf{a}_y$. Then $|\mathbf{E}| = \sqrt{16^2 + 4} = 16.12$ So

$$\mathbf{a}_Q = \frac{16\mathbf{a}_x + 2\mathbf{a}_y}{16.12} = \underline{0.99\mathbf{a}_x + 0.12\mathbf{a}_y}$$

3.5. Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2 + z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m: Of the 6 surfaces to consider, only 2 will contribute to the net outward flux. Why? First consider the planes at $y = 0$ and 3. The y component of $\mathbf{D}$ will penetrate those surfaces, but will be inward at $y = 0$ and outward at $y = 3$, while having the same magnitude in both cases. These fluxes will thus cancel. At the $x = 0$ plane, $D_x = 0$ and at the $z = 0$ plane, $D_z = 0$, so there will be no flux contributions from these surfaces. This leaves the 2 remaining surfaces at $x = 2$ and $z = 5$. The net outward flux becomes:

$$\begin{aligned}\Phi &= \int_0^5 \int_0^3 \mathbf{D}\big|_{x=2} \cdot \mathbf{a}_x \, dy \, dz + \int_0^3 \int_0^2 \mathbf{D}\big|_{z=5} \cdot \mathbf{a}_z \, dx \, dy \\ &= 5 \int_0^3 4(2)y \, dy \;+\; 2\int_0^3 4(5)y \, dy = \underline{360 \text{ C}}\end{aligned}$$

3.21. Calculate the divergence of $\mathbf{D}$ at the point specified if

a) $\mathbf{D} = (1/z^2)\left[10xyz\,\mathbf{a}_x + 5x^2 z\,\mathbf{a}_y + (2z^3 - 5x^2 y)\,\mathbf{a}_z\right]$ at $P(-2, 3, 5)$: We find

$$\nabla \cdot \mathbf{D} = \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2 y}{z^3}\right]_{(-2,3,5)} = \underline{8.96}$$

b) $\mathbf{D} = 5z^2\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45°, 5)$: In cylindrical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45°,5)} = \underline{71.67}$$

c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45°, -45°)$: In spherical coordinates, we have

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \\ &= \left[6\sin\theta\sin\phi + \frac{\cos 2\theta \sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45°,-45°)} = \underline{-2}\end{aligned}$$

3.27. Let $\mathbf{D} = 5r^2\mathbf{a}_r$ mC/m^2 for $r < 0.08$ m and $\mathbf{D} = 0.1\,\mathbf{a}_r/r^2$ mC/m^2 for $r > 0.08$ m.

a) Find ρ_v for $r = 0.06$ m: This radius lies within the first region, and so

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{1}{r^2}\frac{d}{dr}(5r^4) = 20r \text{ C/m}^3$$

which when evaluated at $r = 0.06$ yields $\rho_v(r = .06) = \underline{1.20 \text{ mC/m}^3}$.

b) Find ρ_v for $r = 0.1$ m: This is in the region where the second field expression is valid. The $1/r^2$ dependence of this field yields a zero divergence (shown in Problem 3.23), and so the volume charge density is zero at 0.1 m.

c) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m? The total surface charge should be equal and opposite to the total volume charge. The latter is

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{.08} 20r(\text{mC/m}^3)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 2.57 \times 10^{-3} \text{ mC} = 2.57\,\mu\text{C}$$

So now

$$\rho_s = -\left[\frac{2.57}{4\pi(.08)^2}\right] = \underline{-32\,\mu\text{C/m}^2}$$

4.5. Compute the value of $\int_A^P \mathbf{G}\cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1, -1, 2)$ and $P(2, 1, 2)$ using the path:

a) straight-line segments $A(1, -1, 2)$ to $B(1, 1, 2)$ to $P(2, 1, 2)$: In general we would have

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_A^P 2y\, dx$$

The change in x occurs when moving between B and P, during which $y = 1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_B^P 2y\, dx = \int_1^2 2(1)dx = \underline{2}$$

b) straight-line segments $A(1, -1, 2)$ to $C(2, -1, 2)$ to $P(2, 1, 2)$: In this case the change in x occurs when moving from A to C, during which $y = -1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_A^C 2y\, dx = \int_1^2 2(-1)dx = \underline{-2}$$

4.17. Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at:

a) $\rho = 5$ cm: Since $V = 0$ at 4 cm, the potential at 5 cm will be the potential difference between points 5 and 4:

$$V_5 = -\int_4^5 \mathbf{E}\cdot d\mathbf{L} = -\int_4^5 \frac{a\rho_{sa}}{\epsilon_0 \rho}\, d\rho = -\frac{(.02)(6\times 10^{-9})}{\epsilon_0}\ln\left(\frac{5}{4}\right) = \underline{-3.026\text{ V}}$$

b) $\rho = 7$ cm: Here we integrate piecewise from $\rho = 4$ to $\rho = 7$:

$$V_7 = -\int_4^6 \frac{a\rho_{sa}}{\epsilon_0 \rho}\,d\rho - \int_6^7 \frac{(a\rho_{sa} + b\rho_{sb})}{\epsilon_0 \rho}\,d\rho$$

With the given values, this becomes

$$V_7 = -\left[\frac{(.02)(6\times 10^{-9})}{\epsilon_0}\right]\ln\left(\frac{6}{4}\right) - \left[\frac{(.02)(6\times 10^{-9}) + (.06)(2\times 10^{-9})}{\epsilon_0}\right]\ln\left(\frac{7}{6}\right)$$
$$= \underline{-9.678\,\mathrm{V}}$$

4.31. (continued) The integral evaluates as follows:

$$\begin{aligned}
W_E &= 200\epsilon_0 \int_1^2\int_1^2 \left[-\left(\frac{1}{3}\right)\frac{1}{x^3y^2z^2} - \frac{1}{xy^4z^2} - \frac{1}{xy^2z^4}\right]_1^2 dy\,dz \\
&= 200\epsilon_0 \int_1^2\int_1^2 \left[\left(\frac{7}{24}\right)\frac{1}{y^2z^2} + \left(\frac{1}{2}\right)\frac{1}{y^4z^2} + \left(\frac{1}{2}\right)\frac{1}{y^2z^4}\right] dy\,dz \\
&= 200\epsilon_0 \int_1^2 \left[-\left(\frac{7}{24}\right)\frac{1}{yz^2} - \left(\frac{1}{6}\right)\frac{1}{y^3z^2} - \left(\frac{1}{2}\right)\frac{1}{yz^4}\right]_1^2 dz \\
&= 200\epsilon_0 \int_1^2 \left[\left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{1}{4}\right)\frac{1}{z^4}\right] dz \\
&= 200\epsilon_0(3)\left[\frac{7}{96}\right] = \underline{387\,\mathrm{pJ}}
\end{aligned}$$

b) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube? At $C(1.5, 1.5, 1.5)$ the energy density is

$$w_E = 200\epsilon_0(3)\left[\frac{1}{(1.5)^4(1.5)^2(1.5)^2}\right] = 2.07\times 10^{-10}\ \mathrm{J/m^3}$$

This, multiplied by a cube volume of 1, produces an energy value of $\underline{207\,\mathrm{pJ}}$.

5.1. Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]\ \mathrm{kA/m^2}$:

a) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$: This is found through

$$I = \int\!\!\int_S \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^2\int_0^1 \mathbf{J}\cdot\mathbf{a}_y\Big|_{y=1} dx\,dz = \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$
$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2} = \underline{-1.23\,\mathrm{MA}}$$

b) Find the total current leaving the region $0 < x, x < 1$, $2 < z < 3$ by integrating $\mathbf{J}\cdot\mathbf{dS}$ over the surface of the cube: Note first that current through the top and bottom surfaces will not exist, since $\mathbf{J}$ has no z component. Also note that there will be no current through the $x = 0$ plane, since $J_x = 0$ there. Current will pass through the three remaining surfaces, and will be found through

$$I = \int_2^3 \int_0^1 \mathbf{J} \cdot (-\mathbf{a}_y)\Big|_{y=0} dx\,dz + \int_2^3 \int_0^1 \mathbf{J} \cdot (\mathbf{a}_y)\Big|_{y=1} dx\,dz + \int_2^3 \int_0^1 \mathbf{J} \cdot (\mathbf{a}_x)\Big|_{x=1} dy\,dz$$
$$= 10^4 \int_2^3 \int_0^1 \left[\cos(2x)e^{-0} - \cos(2x)e^{-2}\right] dx\,dz - 10^4 \int_2^3 \int_0^1 \sin(2)e^{-2y}\,dy\,dz$$
$$= 10^4 \left(\frac{1}{2}\right) \sin(2x)\Big|_0^1 (3-2)\left[1 - e^{-2}\right] + 10^4 \left(\frac{1}{2}\right) \sin(2)e^{-2y}\Big|_0^1 (3-2) = \underline{0}$$

c) Repeat part *b*, but use the divergence theorem: We find the net outward current through the surface of the cube by integrating the divergence of **J** over the cube volume. We have

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -10^{-4}\left[2\cos(2x)e^{-2y} - 2\cos(2x)e^{-2y}\right] = \underline{0} \text{ as expected}$$

5.11. Two perfectly-conducting cylindrical surfaces are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc. *Assume the cylinders are both of length l.*

a) Find the voltage and resistance between the cylinders, and **E** in the region between the cylinders, if a conducting material having $\sigma = 0.05\,\text{S/m}$ is present for $3 < \rho < 5$ cm: Given the current, and knowing that it is radially-directed, we find the current density by dividing it by the area of a cylinder of radius ρ and length l:

$$\mathbf{J} = \frac{3}{2\pi\rho l}\mathbf{a}_\rho \text{ A/m}^2$$

Then the electric field is found by dividing this result by σ:

$$\mathbf{E} = \frac{3}{2\pi\sigma\rho l}\mathbf{a}_\rho = \underline{\frac{9.55}{\rho l}\mathbf{a}_\rho \text{ V/m}}$$

The voltage between cylinders is now:

$$V = -\int_5^3 \mathbf{E} \cdot \mathbf{dL} = \int_3^5 \frac{9.55}{\rho l}\mathbf{a}_\rho \cdot \mathbf{a}_\rho d\rho = \frac{9.55}{l} \ln\left(\frac{5}{3}\right) = \underline{\frac{4.88}{l} \text{ V}}$$

Now, the resistance will be

$$R = \frac{V}{I} = \frac{4.88}{3l} = \underline{\frac{1.63}{l}\,\Omega}$$

b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power: We calculate

$$P = \int_v \mathbf{E} \cdot \mathbf{J}\,dv = \int_0^l \int_0^{2\pi} \int_{.03}^{.05} \frac{3^2}{(2\pi)^2\rho^2(.05)l^2}\rho\,d\rho\,d\phi\,dz = \frac{3^2}{2\pi(.05)l} \ln\left(\frac{5}{3}\right) = \frac{14.64}{l} \text{ W}$$

We also find the power by taking the product of voltage and current:

$$P = VI = \frac{4.88}{l}(3) = \frac{14.64}{l} \text{ W}$$

which is in aggreement with the power density integration.

5.39. A parallel plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_R = 2 + 2 \times 10^6 x^2$, where x is the distance from one plate. If $S = 0.02\,\mathrm{m}^2$, and $d = 1$ mm, find C: Start by assuming charge density ρ_s on the top plate. $\mathbf{D}$ will, as usual, be x-directed, originating at the top plate and terminating on the bottom plate. The key here is that $\mathbf{D}$ *will be constant over the distance between plates*. This can be understood by considering the x-varying dielectric as constructed of many thin layers, each having constant permittivity. The permittivity changes from layer to layer to approximate the given function of x. The approximation becomes exact as the layer thicknesses approach zero. We know that $\mathbf{D}$, which is normal to the layers, will be continuous across each boundary, and so $\mathbf{D}$ is constant over the plate separation distance, and will be given in magnitude by ρ_s. The electric field magnitude is now

$$E = \frac{D}{\epsilon_0 \epsilon_R} = \frac{\rho_s}{\epsilon_0(2 + 2 \times 10^6 x^2)}$$

The voltage beween plates is then

$$V_0 = \int_0^{10^{-3}} \frac{\rho_s\,dx}{\epsilon_0(2 + 2 \times 10^6 x^2)} = \frac{\rho_s}{\epsilon_0} \frac{1}{\sqrt{4 \times 10^6}} \tan^{-1}\left(\frac{x\sqrt{4 \times 10^6}}{2}\right)\Big|_0^{10^{-3}} = \frac{\rho_s}{\epsilon_0} \frac{1}{2 \times 10^3} \left(\frac{\pi}{4}\right)$$

Now $Q = \rho_s(.02)$, and so

$$C = \frac{Q}{V_0} = \frac{\rho_s(.02)\epsilon_0(2 \times 10^3)(4)}{\rho_s \pi} = 4.51 \times 10^{-10}\,\mathrm{F} = \underline{451\,\mathrm{pF}}$$

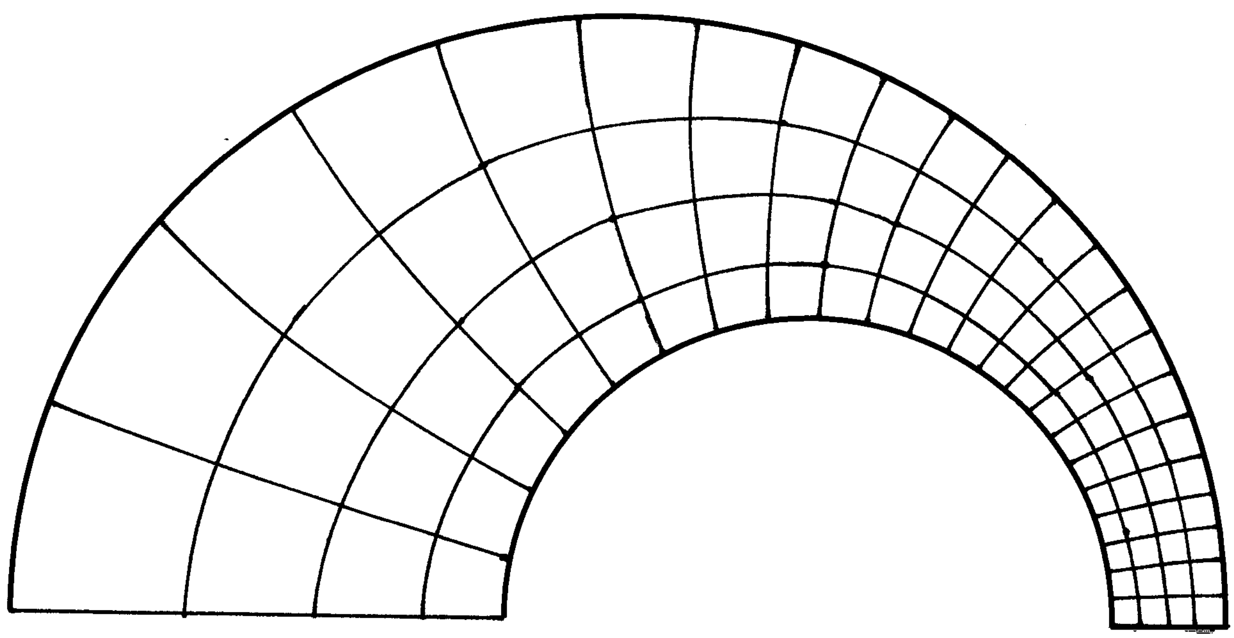
6.3. Construct a curvilinear square map of the potential field between two parallel circular cylinders, one of 4-cm radius inside one of 8-cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2 + b^2 - D^2)/(2ab)\right]}$$

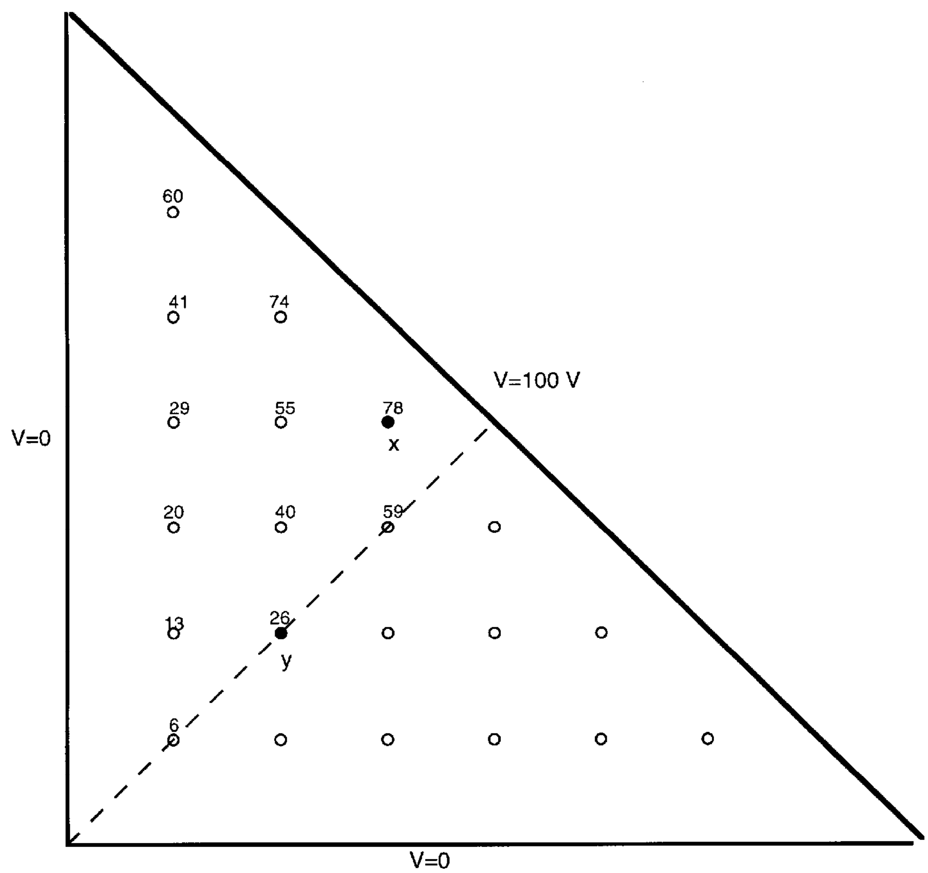
where a and b are the conductor radii and D is the axis separation.

Our attempt at the drawing is shown below. Use of the exact expression above yields a capacitance value of $C = \underline{11.5\epsilon_0 \text{ F/m}}$. Use of the drawing produces:

$$C \doteq \frac{22 \times 2}{4}\epsilon_0 = \underline{11\epsilon_0 \text{ F/m}}$$



6.7. Use the iteration method to estimate the potentials at points x and y in the triangular trough of Fig. 6.13. Work only to the nearest volt: The result is shown below. The mirror image of the values shown occur at the points on the other side of the line of symmetry (dashed line). Note that $V_x = \underline{78\text{ V}}$ and $V_y = \underline{26\text{ V}}$.



6.17. A two-wire transmission line consists of two parallel perfectly-conducting cylinders, each having a radius of 0.2 mm, separated by center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_R = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. Calculate:

a) the magnitude of the charge per meter length on each wire: Use

$$C = \frac{\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{\pi \times 3 \times 8.85 \times 10^{-12}}{\cosh^{-1}(1/0.2)} = 3.64 \times 10^{-9}\ \text{C/m}$$

Then the charge per unit length will be

$$Q = CV_0 = (3.64 \times 10^{-11})(100) = 3.64 \times 10^{-9}\ \text{C/m} = \underline{3.64\,\text{nC/m}}$$

b) the battery current: Use

$$RC = \frac{\epsilon}{\sigma} \quad \Rightarrow \quad R = \frac{3 \times 8.85 \times 10^{-12}}{(1.5 \times 10^{-3})(3.64 \times 10^{-11})} = 486\,\Omega$$

Then

$$I = \frac{V_0}{R} = \frac{100}{486} = 0.206\ \text{A} = \underline{206\,\text{mA}}$$

7.3. Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$: Since $\rho_v = 0$, we know that $\nabla^2 V = 0$, and so

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 16e^{2x} + \frac{d^2 f}{dx^2} - 6 = 0$$

Therefore

$$\frac{d^2 f}{dx^2} = -16e^{2x} + 6 \quad \Rightarrow \quad \frac{df}{dx} = -8e^{2x} + 6x + C_1$$

Now

$$E_x = \frac{\partial V}{\partial x} = 8e^{2x} + \frac{df}{dx}$$

and at the origin, this becomes

$$E_x(0) = 8 + \left.\frac{df}{dx}\right|_{x=0} = 0(\text{as given})$$

Thus $df/dx\,|_{x=0} = -8$, and so it follows that $C_1 = 0$. Integrating again, we find

$$f(x, y) = -4e^{2x} + 3x^2 + C_2$$

which at the origin becomes $f(0, 0) = -4 + C_2$. However, $V(0, 0) = 0 = 4 + f(0, 0)$. So $f(0, 0) = -4$ and $C_2 = 0$. Finally, $f(x, y) = \underline{-4e^{2x} + 3x^2}$, and $V(x, y) = 4e^{2x} - 4e^{2x} + 3x^2 - 3y^2 = \underline{3(x^2 - y^2)}$.

7.13. Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100V and the outer at 0V, find:

a) the location of the 20V equipotential surface: From Eq. (16) we have

$$V(\rho) = 100 \frac{\ln(.012/\rho)}{\ln(.012/.005)} \text{ V}$$

We seek ρ at which $V = 20$ V, and thus we need to solve:

$$20 = 100 \frac{\ln(.012/\rho)}{\ln(2.4)} \quad \Rightarrow \quad \rho = \frac{.012}{(2.4)^{0.2}} = \underline{1.01 \text{ cm}}$$

b) $E_{\rho\, max}$: We have

$$E_\rho = -\frac{\partial V}{\partial \rho} = -\frac{dV}{d\rho} = \frac{100}{\rho \ln(2.4)}$$

whose maximum value will occur at the inner cylinder, or at $\rho = .5$ cm:

$$E_{\rho\, max} = \frac{100}{.005 \ln(2.4)} = 2.28 \times 10^4 \text{ V/m} = \underline{22.8 \text{ kV/m}}$$

c) ϵ_R if the charge per meter length on the inner cylinder is 20 nC/m: The capacitance per meter length is

$$C = \frac{2\pi\epsilon_0\epsilon_R}{\ln(2.4)} = \frac{Q}{V_0}$$

We solve for ϵ_R:

$$\epsilon_R = \frac{(20 \times 10^{-9}) \ln(2.4)}{2\pi\epsilon_0(100)} = \underline{3.15}$$

or

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -200r^{-.4}$$

Integrate once:

$$\left(r^2 \frac{dV}{dr}\right) = -\frac{200}{.6} r^{.6} + C_1 = -333.3r^{.6} + C_1$$

or

$$\frac{dV}{dr} = -333.3r^{-1.4} + \frac{C_1}{r^2} = \nabla V \text{ (in this case)} = -E_r$$

Our first boundary condition states that $r^2 E_r \to 0$ when $r \to 0$ Therefore $C_1 = 0$. Integrate again to find:

$$V(r) = \frac{333.3}{.4} r^{-.4} + C_2$$

From our second boundary condition, $V \to 0$ as $r \to \infty$, we see that $C_2 = 0$. Finally,

$$V(r) = \underline{833.3r^{-.4} \text{ V}}$$

b) Now find $V(r)$ by using Gauss' Law and a line integral: Gauss' law applied to a spherical surface of radius r gives:

$$4\pi r^2 D_r = 4\pi \int_0^r \frac{200\epsilon_0}{(r')^{2.4}}(r')^2 dr = 800\pi\epsilon_0 \frac{r^{.6}}{.6}$$

Thus

$$E_r = \frac{D_r}{\epsilon_0} = \frac{800\pi\epsilon_0 r^{.6}}{.6(4\pi)\epsilon_0 r^2} = 333.3 r^{-1.4} \text{ V/m}$$

Now

$$V(r) = -\int_\infty^r 333.3(r')^{-1.4} dr' = \underline{833.3 r^{-.4} \text{ V}}$$

8.1a. Find $\mathbf{H}$ in cartesian components at $P(2,3,4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction:
Applying the Biot-Savart Law, we obtain

$$\mathbf{H}_a = \int_{-\infty}^{\infty} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [2\mathbf{a}_x + 3\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 29)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[2\mathbf{a}_y - 3\mathbf{a}_x]}{4\pi(z^2 - 8z + 29)^{3/2}}$$

Using integral tables, this evaluates as

$$\mathbf{H}_a = \frac{I}{4\pi}\left[\frac{2(2z-8)(2\mathbf{a}_y - 3\mathbf{a}_x)}{52(z^2 - 8z + 29)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{26\pi}(2\mathbf{a}_y - 3\mathbf{a}_x)$$

Then with $I = 8$ mA, we finally obtain $\mathbf{H}_a = \underline{-294\mathbf{a}_x + 196\mathbf{a}_y\ \mu\text{A/m}}$

b. Repeat if the filament is located at $x = -1$, $y = 2$: In this case the Biot-Savart integral becomes

$$\mathbf{H}_b = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [(2+1)\mathbf{a}_x + (3-2)\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 26)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[3\mathbf{a}_y - \mathbf{a}_x]}{4\pi(z^2 - 8z + 26)^{3/2}}$$

Evaluating as before, we obtain with $I = 8$ mA:

$$\mathbf{H}_b = \frac{I}{4\pi}\left[\frac{2(2z-8)(3\mathbf{a}_y - \mathbf{a}_x)}{40(z^2 - 8z + 26)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{20\pi}(3\mathbf{a}_y - \mathbf{a}_x) = \underline{-127\mathbf{a}_x + 382\mathbf{a}_y\ \mu\text{A/m}}$$

c. Find $\mathbf{H}$ if both filaments are present: This will be just the sum of the results of parts a and b, or

$$\mathbf{H}_T = \mathbf{H}_a + \mathbf{H}_b = \underline{-421\mathbf{a}_x + 578\mathbf{a}_y\ \mu\text{A/m}}$$

This problem can also be done (somewhat more simply) by using the known result for $\mathbf{H}$ from an infinitely-long wire in cylindrical components, and transforming to cartesian components. The Biot-Savart method was used here for the sake of illustration.

8.23. Given the field $\mathbf{H} = 20\rho^2\,\mathbf{a}_\phi$ A/m:

a) Determine the current density $\mathbf{J}$: This is found through the curl of $\mathbf{H}$, which simplifies to a single term, since $\mathbf{H}$ varies only with ρ and has only a ϕ component:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(20\rho^3\right)\,\mathbf{a}_z = \underline{60\rho\,\mathbf{a}_z \text{ A/m}^2}$$

b) Integrate $\mathbf{J}$ over the circular surface $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction: The integral is:

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 60\rho \mathbf{a}_z \cdot \rho \, d\rho \, d\phi \mathbf{a}_z = \underline{40\pi \text{ A}}$$

c) Find the total current once more, this time by a line integral around the circular path $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$:

$$I = \oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} 20\rho^2 \, \mathbf{a}_\phi \Big|_{\rho=1} \cdot (1) d\phi \mathbf{a}_\phi = \int_0^{2\pi} 20 \, d\phi = \underline{40\pi \text{ A}}$$

8.41. Assume that $\mathbf{A} = 50\rho^2 \mathbf{a}_z$ Wb/m in a certain region of free space.

a) Find $\mathbf{H}$ and $\mathbf{B}$: Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho} \mathbf{a}_\phi = \underline{-100\rho \, \mathbf{a}_\phi \text{ Wb/m}^2}$$

Then $\mathbf{H} = \mathbf{B}/\mu_0 = \underline{-100\rho/\mu_0 \, \mathbf{a}_\phi \text{ A/m}}$.

b) Find $\mathbf{J}$: Use

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) \mathbf{a}_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{-100\rho^2}{\mu_0} \right) \mathbf{a}_z = \underline{-\frac{200}{\mu_0} \mathbf{a}_z \text{ A/m}^2}$$

c) Use $\mathbf{J}$ to find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi < 2\pi$, $z = 0$: The current is

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \frac{-200}{\mu_0} \mathbf{a}_z \cdot \mathbf{a}_z \, \rho \, d\rho \, d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ kA}}$$

d) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$: Have

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} \frac{-100}{\mu_0} \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, (1) d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ kA}}$$

9.3. A point charge for which $Q = 2 \times 10^{-16}$ C and $m = 5 \times 10^{-26}$ kg is moving in the combined fields $\mathbf{E} = 100\mathbf{a}_x - 200\mathbf{a}_y + 300\mathbf{a}_z$ V/m and $\mathbf{B} = -3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$ mT. If the charge velocity at $t = 0$ is $\mathbf{v}(0) = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s:

a) give the unit vector showing the direction in which the charge is accelerating at $t = 0$: Use $\mathbf{F}(t = 0) = q[\mathbf{E} + (\mathbf{v}(0) \times \mathbf{B})]$, where

$$\mathbf{v}(0) \times \mathbf{B} = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z)10^5 \times (-3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z)10^{-3} = 1100\mathbf{a}_x + 1400\mathbf{a}_y - 500\mathbf{a}_z$$

So the force in newtons becomes

$$\mathbf{F}(0) = (2 \times 10^{-16})[(100 + 1100)\mathbf{a}_x + (1400 - 200)\mathbf{a}_y + (300 - 500)\mathbf{a}_z] = 4 \times 10^{-14}[6\mathbf{a}_x + 6\mathbf{a}_y - \mathbf{a}_z]$$

The unit vector that gives the acceleration direction is found from the force to be

$$\mathbf{a}_F = \frac{6\mathbf{a}_x + 6\mathbf{a}_y - \mathbf{a}_z}{\sqrt{73}} = \underline{.70\mathbf{a}_x + .70\mathbf{a}_y - .12\mathbf{a}_z}$$

b) find the kinetic energy of the charge at $t = 0$:

$$\text{K.E.} = \frac{1}{2}m|\mathbf{v}(0)|^2 = \frac{1}{2}(5 \times 10^{-26}\,\text{kg})(5.39 \times 10^5\,\text{m/s})^2 = 7.25 \times 10^{-15}\,\text{J} = \underline{7.25\ \text{fJ}}$$

9.15. A solid conducting filament extends from $x = -b$ to $x = b$ along the line $y = 2$, $z = 0$. This filament carries a current of 3 A in the $\mathbf{a}_x$ direction. An infinite filament on the z axis carries 5 A in the $\mathbf{a}_z$ direction. Obtain an expression for the torque exerted on the finite conductor about an origin located at $(0, 2, 0)$: The differential force on the wire segment arising from the field from the infinite wire is

$$d\mathbf{F} = 3\,dx\,\mathbf{a}_x \times \frac{5\mu_0}{2\pi\rho}\,\mathbf{a}_\phi = -\frac{15\mu_0 \cos\phi\,dx}{2\pi\sqrt{x^2+4}}\,\mathbf{a}_z = -\frac{15\mu_0 x\,dx}{2\pi(x^2+4)}\,\mathbf{a}_z$$

So now the differential torque about the $(0, 2, 0)$ origin is

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = x\,\mathbf{a}_x \times -\frac{15\mu_0 x\,dx}{2\pi(x^2+4)}\,\mathbf{a}_z = \frac{15\mu_0 x^2\,dx}{2\pi(x^2+4)}\,\mathbf{a}_y$$

The torque is then

$$\mathbf{T} = \int_{-b}^{b} \frac{15\mu_0 x^2\,dx}{2\pi(x^2+4)}\,\mathbf{a}_y = \frac{15\mu_0}{2\pi}\,\mathbf{a}_y \left[x - 2\tan^{-1}\left(\frac{x}{2}\right)\right]_{-b}^{b}$$

$$= \underline{(6 \times 10^{-6})\left[b - 2\tan^{-1}\left(\frac{b}{2}\right)\right]\,\mathbf{a}_y\quad \text{N}\cdot\text{m}}$$

9.33. A toroidal core has a square cross section, 2.5 cm $< \rho <$ 3.5 cm, -0.5 cm $< z <$ 0.5 cm. The upper half of the toroid, $0 < z < 0.5$ cm, is constructed of a linear material for which $\mu_R = 10$, while the lower half, -0.5 cm $< z < 0$, has $\mu_R = 20$. An mmf of 150 A $\cdot$ t establishes a flux in the $\mathbf{a}_\phi$ direction. For $z > 0$, find:

a) $H_\phi(\rho)$: Ampere's circuital law gives:

$$2\pi\rho H_\phi = NI = 150 \quad \Rightarrow \quad H_\phi = \frac{150}{2\pi\rho} = \underline{23.9/\rho\ \text{A/m}}$$

b) $B_\phi(\rho)$: We use $B_\phi = \mu_R\mu_0 H_\phi = (10)(4\pi \times 10^{-7})(23.9/\rho) = \underline{3.0 \times 10^{-4}/\rho\ \text{Wb/m}^2}$.

c) $\Phi_{z>0}$: This will be

$$\Phi_{z>0} = \int\!\!\int \mathbf{B} \cdot d\mathbf{S} = \int_0^{.005} \int_{.025}^{.035} \frac{3.0 \times 10^{-4}}{\rho}\,d\rho dz = (.005)(3.0 \times 10^{-4}) \ln\left(\frac{.035}{.025}\right)$$
$$= \underline{5.0 \times 10^{-7}\ \text{Wb}}$$

d) Repeat for $z < 0$: First, the magnetic field strength will be the same as in part a, since the calculation is material-independent. Thus $H_\phi = \underline{23.9/\rho\ \text{A/m}}$. Next, B_ϕ is modified only by the new permeability, which is twice the value used in part a: Thus $B_\phi = \underline{6.0 \times 10^{-4}/\rho\ \text{Wb/m}^2}$. Finally, since B_ϕ is twice that of part a, the flux will be increased by the same factor, since the area of integration for $z < 0$ is the same. Thus $\Phi_{z<0} = \underline{1.0 \times 10^{-6}\ \text{Wb}}$.

e) Find Φ_{total}: This will be the sum of the values found for $z < 0$ and $z > 0$, or $\Phi_{\text{total}} = \underline{1.5 \times 10^{-6}\ \text{Wb}}$.

Supplemental Solutions for E-text

10.1. In Fig. 10.4, let $B = 0.2\cos 120\pi t$ T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by $I(t)$ is negligible. Find:

a) $V_{ab}(t)$: Since B is constant over the loop area, the flux is $\Phi = \pi(0.15)^2 B = 1.41 \times 10^{-2}\cos 120\pi t$ Wb. Now, $emf = V_{ba}(t) = -d\Phi/dt = (120\pi)(1.41 \times 10^{-2})\sin 120\pi t$. Then $V_{ab}(t) = -V_{ba}(t) = \underline{-5.33\sin 120\pi t \text{ V}}$.

b) $I(t) = V_{ba}(t)/R = 5.33\sin(120\pi t)/250 = \underline{21.3\sin(120\pi t) \text{ mA}}$

10.9. A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 Ω per meter length. The loop lies in the $z = 0$ plane with its corners at $(0,0,0)$, $(0.25,0,0)$, $(0.25,0.25,0)$, and $(0,0.25,0)$ at $t = 0$. The loop is moving with velocity $v_y = 50$ m/s in the field $B_z = 8\cos(1.5\times 10^8 t - 0.5x)$ μT. Develop a function of time which expresses the ohmic power being delivered to the loop: First, since the field does not vary with y, the loop motion in the y direction does not produce any time-varying flux, and so this motion is immaterial. We can evaluate the flux at the original loop position to obtain:

$$\Phi(t) = \int_0^{.25}\int_0^{.25} 8\times 10^{-6}\cos(1.5\times 10^8 t - 0.5x)\,dx\,dy$$
$$= -(4\times 10^{-6})\left[\sin(1.5\times 10^8 t - 0.13x) - \sin(1.5\times 10^8 t)\right] \text{ Wb}$$

Now, $emf = V(t) = -d\Phi/dt = 6.0\times 10^2\left[\cos(1.5\times 10^8 t - 0.13x) - \cos(1.5\times 10^8 t)\right]$, The total loop resistance is $R = 125(0.25+0.25+0.25+0.25) = 125\,\Omega$. Then the ohmic power is

$$P(t) = \frac{V^2(t)}{R} = \underline{2.9\times 10^3\left[\cos(1.5\times 10^8 t - 0.13x) - \cos(1.5\times 10^8 t)\right] \text{ Watts}}$$

10.17. The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$, $0 < z < 0.06$ m in free space is given by $\mathbf{E} = C\sin(12y)\sin(az)\cos(2\times 10^{10}t)\,\mathbf{a}_x$ V/m. Beginning with the $\nabla\times\mathbf{E}$ relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than zero: In this case we find

$$\nabla\times\mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y - \frac{\partial E_z}{\partial y}\mathbf{a}_z$$
$$= C\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\cos(2\times 10^{10}t) = -\frac{\partial\mathbf{B}}{\partial t}$$

Then

$$\mathbf{H} = -\frac{1}{\mu_0}\int \nabla\times\mathbf{E}\,dt + C_1$$
$$= -\frac{C}{\mu_0(2\times 10^{10}}\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\sin(2\times 10^{10}t) \text{ A/m}$$

where the integration constant, $C_1 = 0$, since there are no initial conditions. Using this result, we now find

$$\nabla\times\mathbf{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\mathbf{a}_x = -\frac{C(144+a^2)}{\mu_0(2\times 10^{10})}\sin(12y)\sin(az)\sin(2\times 10^{10}t)\,\mathbf{a}_x = \frac{\partial\mathbf{D}}{\partial t}$$

10.17. (continued) Now

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \int \frac{1}{\epsilon_0} \nabla \times \mathbf{H}\, dt + C_2 = \frac{C(144 + a^2)}{\mu_0 \epsilon_0 (2 \times 10^{10})^2} \sin(12y) \sin(az) \cos(2 \times 10^{10} t)\, \mathbf{a}_x$$

where $C_2 = 0$. This field must be the same as the original field as stated, and so we require that

$$\frac{C(144 + a^2)}{\mu_0 \epsilon_0 (2 \times 10^{10})^2} = 1$$

Using $\mu_0 \epsilon_0 = (3 \times 10^8)^{-2}$, we find

$$a = \left[\frac{(2 \times 10^{10})^2}{(3 \times 10^8)^2} - 144 \right]^{1/2} = \underline{66}$$

11.7. The phasor magnetic field intensity for a 400-MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}$ A/m. Knowing that the maximum amplitude of $\mathbf{E}$ is 1500 V/m, find β, η, λ, v_p, ϵ_R, μ_R, and $\mathbf{H}(x, y, z, t)$: First, from the phasor expression, we identify $\beta = \underline{25\ \text{m}^{-1}}$ from the argument of the exponential function. Next, we evaluate $H_0 = |\mathbf{H}| = \sqrt{\mathbf{H} \cdot \mathbf{H}^*} = \sqrt{2^2 + 5^2} = \sqrt{29}$. Then $\eta = E_0/H_0 = 1500/\sqrt{29} = \underline{278.5\ \Omega}$. Then $\lambda = 2\pi/\beta = 2\pi/25 = .25$ m $= \underline{25\ \text{cm}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{25} = \underline{1.01 \times 10^8\ \text{m/s}}$$

Now we note that

$$\eta = 278.5 = 377 \sqrt{\frac{\mu_R}{\epsilon_R}} \quad \Rightarrow \quad \frac{\mu_R}{\epsilon_R} = 0.546$$

And

$$v_p = 1.01 \times 10^8 = \frac{c}{\sqrt{\mu_R \epsilon_R}} \quad \Rightarrow \quad \mu_R \epsilon_R = 8.79$$

We solve the above two equations simultaneously to find $\epsilon_R = \underline{4.01}$ and $\mu_R = \underline{2.19}$. Finally,

$$\begin{aligned} \mathbf{H}(x, y, z, t) &= Re\left\{(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x} e^{j\omega t}\right\} \\ &= \underline{2\cos(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_y + 5\sin(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_z\ \text{A/m}} \end{aligned}$$

11.15. A 10 GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a non-magnetic material for which

a) $\epsilon'_R = 1$ and $\epsilon''_R = 0$: In a non-magnetic material, we would have:

$$\alpha = \omega \sqrt{\frac{\mu_0 \epsilon_0 \epsilon'_R}{2}} \left[\sqrt{1 + \left(\frac{\epsilon''_R}{\epsilon'_R} \right)^2} - 1 \right]^{1/2}$$

and

$$\beta = \omega \sqrt{\frac{\mu_0 \epsilon_0 \epsilon'_R}{2}} \left[\sqrt{1 + \left(\frac{\epsilon''_R}{\epsilon'_R} \right)^2} + 1 \right]^{1/2}$$

With the given values of ϵ'_R and ϵ''_R, it is clear that $\beta = \omega\sqrt{\mu_0 \epsilon_0} = \omega/c$, and so $\lambda = 2\pi/\beta = 2\pi c/\omega = 3 \times 10^{10}/10^{10} = \underline{3\ \text{cm}}$. It is also clear that $\alpha = \underline{0}$.

11.15. (continued)

b) $\epsilon'_R = 1.04$ and $\epsilon''_R = 9.00 \times 10^{-4}$: In this case $\epsilon''_R/\epsilon'_R << 1$, and so $\beta \doteq \omega\sqrt{\epsilon'_R}/c = 2.13 \text{ cm}^{-1}$. Thus $\lambda = 2\pi/\beta = \underline{2.95 \text{ cm}}$. Then

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\omega\epsilon''_R}{2}\frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\epsilon'_R}} = \frac{\omega}{2c}\frac{\epsilon''_R}{\sqrt{\epsilon'_R}} = \frac{2\pi \times 10^{10}}{2 \times 3 \times 10^8}\frac{(9.00 \times 10^{-4})}{\sqrt{1.04}}$$
$$= \underline{9.24 \times 10^{-2} \text{ Np/m}}$$

c) $\epsilon'_R = 2.5$ and $\epsilon''_R = 7.2$: Using the above formulas, we obtain

$$\beta = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3 \times 10^{10})\sqrt{2}}\left[\sqrt{1 + \left(\frac{7.2}{2.5}\right)^2} + 1\right]^{1/2} = 4.71 \text{ cm}^{-1}$$

and so $\lambda = 2\pi/\beta = \underline{1.33 \text{ cm}}$. Then

$$\alpha = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3 \times 10^{8})\sqrt{2}}\left[\sqrt{1 + \left(\frac{7.2}{2.5}\right)^2} - 1\right]^{1/2} = \underline{335 \text{ Np/m}}$$

11.29. Consider a left-circularly polarized wave in free space that propagates in the forward z direction. The electric field is given by the appropriate form of Eq. (80).

a) Determine the magnetic field phasor, $\mathbf{H}_s$:
We begin, using (80), with $\mathbf{E}_s = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$. We find the two components of $\mathbf{H}_s$ separately, using the two components of $\mathbf{E}_s$. Specifically, the x component of $\mathbf{E}_s$ is associated with a y component of $\mathbf{H}_s$, and the y component of $\mathbf{E}_s$ is associated with a negative x component of $\mathbf{H}_s$. The result is

$$\mathbf{H}_s = \underline{\frac{E_0}{\eta_0}\left(\mathbf{a}_y - j\mathbf{a}_x\right)e^{-j\beta z}}$$

b) Determine an expression for the average power density in the wave in W/m^2 by direct application of Eq. (57): We have

$$\mathcal{P}_{z,avg} = \frac{1}{2}Re(\mathbf{E}_s \times \mathbf{H}^*_s) = \frac{1}{2}Re\left(E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} \times \frac{E_0}{\eta_0}(\mathbf{a}_y - j\mathbf{a}_x)e^{+j\beta z}\right)$$
$$= \underline{\frac{E_0^2}{\eta_0}\mathbf{a}_z \text{ W/m}^2} \quad \text{(assuming } E_0 \text{ is real)}$$

12.3. A uniform plane wave in region 1 is normally incident on the planar boundary separating regions 1 and 2. If $\epsilon''_1 = \epsilon''_2 = 0$, while $\epsilon'_{R1} = \mu^3_{R1}$, and $\epsilon'_{R2} = \mu^3_{R2}$, find the ratio $\epsilon'_{R2}/\epsilon'_{R1}$ if 20% of the energy in the incident wave is reflected at the boundary. There are two possible answers: We begin with the reflection coefficient:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\mu_{R2}/\epsilon'_{R2}} - \sqrt{\mu_{R1}/\epsilon'_{R1}}}{\sqrt{\mu_{R2}/\epsilon'_{R2}} + \sqrt{\mu_{R1}/\epsilon'_{R1}}} = \frac{\sqrt{\mu_{R2}/\mu^3_{R2}} - \sqrt{\mu_{R1}/\mu^3_{R1}}}{\sqrt{\mu_{R2}/\mu^3_{R2}} + \sqrt{\mu_{R1}/\mu^3_{R1}}} = \frac{\mu_{R1} - \mu_{R2}}{\mu_{R1} + \mu_{R2}}$$

But we are given that $|\Gamma| = 0.20$, so that $\Gamma = \pm\sqrt{.20}$, or

$$\frac{\mu_{R2}}{\mu_{R1}} = \frac{1 \mp \sqrt{.20}}{1 \pm \sqrt{.20}} = 0.38 \text{ or } 2.62 \quad \Rightarrow \quad \frac{\epsilon'_{R2}}{\epsilon'_{R1}} = \left(\frac{\mu_{R2}}{\mu_{R1}}\right)^3 = \underline{0.05 \text{ or } 18.0}$$

12.11. A 150 MHz uniform plane wave in normally-incident from air onto a material whose intrinsic impedance is unknown. Measurements yield a standing wave ratio of 3 and the appearance of an electric field minimum at 0.3 wavelengths in front of the interface. Determine the impedance of the unknown material: First, the field minimum is used to find the phase of the reflection coefficient, where

$$z_{min} = -\frac{1}{2\beta}(\phi + \pi) = -0.3\lambda \quad \Rightarrow \quad \phi = 0.2\pi$$

where $\beta = 2\pi/\lambda$ has been used. Next,

$$|\Gamma| = \frac{s-1}{s+1} = \frac{3-1}{3+1} = \frac{1}{2}$$

So we now have

$$\Gamma = 0.5e^{j0.2\pi} = \frac{\eta_u - \eta_0}{\eta_u + \eta_0}$$

We solve for η_u to find

$$\eta_u = \eta_0(1.70 + j1.33) = \underline{641 + j501\ \Omega}$$

12.21. A right-circularly polarized plane wave in air is incident at Brewster's angle onto a semi-infinite slab of plexiglas ($\epsilon'_R = 3.45$, $\epsilon''_R = 0$, $\mu = \mu_0$).

a) Determine the fractions of the incident power that are reflected and transmitted: In plexiglas, Brewster's angle is $\theta_B = \theta_1 = \tan^{-1}(\epsilon'_{R2}/\epsilon'_{R1}) = \tan^{-1}(\sqrt{3.45}) = 61.7°$. Then the angle of refraction is $\theta_2 = 90° - \theta_B$ (see Example 12.9), or $\theta_2 = 28.3°$. With incidence at Brewster's angle, all p-polarized power will be transmitted — only s-polarized power will be reflected. This is found through

$$\Gamma_s = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}} = \frac{.614\eta_0 - 2.11\eta_0}{.614\eta_0 + 2.11\eta_0} = -0.549$$

where $\eta_{1s} = \eta_1 \sec\theta_1 = \eta_0 \sec(61.7°) = 2.11\eta_0$,
and $\eta_{2s} = \eta_2 \sec\theta_2 = (\eta_0/\sqrt{3.45})\sec(28.3°) = 0.614\eta_0$. Now, the reflected power fraction is $|\Gamma|^2 = (-.549)^2 = .302$. Since the wave is circularly-polarized, the s-polarized component represents one-half the total incident wave power, and so the fraction of the *total* power that is reflected is $.302/2 = 0.15$, or $\underline{15\%}$. The fraction of the incident power that is transmitted is then the remainder, or $\underline{85\%}$.

b) Describe the polarizations of the reflected and transmitted waves: Since all the p-polarized component is transmitted, the reflected wave will be entirely s-polarized (linear). The transmitted wave, while having all the incident p-polarized power, will have a reduced s-component, and so this wave will be right-elliptically polarized.

13.1. The parameters of a certain transmission line operating at 6×10^8 rad/s are $L = 0.4\ \mu$H/m, $C = 40$ pF/m, $G = 80$ mS/m, and $R = 20\ \Omega/m$.

a) Find γ, α, β, λ, and Z_0: We use

$$\begin{aligned}\gamma &= \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{[20 + j(6\times10^8)(0.4\times10^{-6})][80\times10^{-3} + j(6\times10^8)(40\times10^{-12})]} \\ &= \underline{2.8 + j3.5\ \text{m}^{-1}} = \alpha + j\beta\end{aligned}$$

Therefore, $\alpha = \underline{2.8\ \text{Np/m}}$, $\beta = \underline{3.5\ \text{rad/m}}$, and $\lambda = 2\pi/\beta = \underline{1.8\ \text{m}}$. Finally,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{20 + j2.4\times10^2}{80\times10^{-3} + j2.4\times10^{-2}}} = \underline{44 + j30\ \Omega}$$

13.1. (continued)

b) If a voltage wave travels 20 m down the line, by what percentage is its amplitude reduced, and by how many degress is its phase shifted? First,

$$\frac{V_{20}}{V_0} = e^{-\alpha L} = e^{-(2.8)(20)} = 4.8 \times 10^{-25} \text{ or } \underline{4.8 \times 10^{-23} \text{ percent!}}$$

Then the phase shift is given by βL, which in degrees becomes

$$\phi = \beta L \left(\frac{360}{2\pi}\right) = (3.5)(20)\left(\frac{360}{2\pi}\right) = \underline{4.0 \times 10^3 \text{ degrees}}$$

13.21. A lossless line having an air dielectric has a characteristic impedance of 400 Ω. The line is operating at 200 MHz and $Z_{in} = 200 - j200\ \Omega$. Use analytic methods or the Smith chart (or both) to find: (a) s; (b) Z_L if the line is 1 m long; (c) the distance from the load to the nearest voltage maximum: I will use the analytic approach. Using normalized impedances, Eq. (13) becomes

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{z_L \cos(\beta L) + j\sin(\beta L)}{\cos(\beta L) + jz_L \sin(\beta L)}\right] = \left[\frac{z_L + j\tan(\beta L)}{1 + jz_L \tan(\beta L)}\right]$$

Solve for z_L:

$$z_L = \left[\frac{z_{in} - j\tan(\beta L)}{1 - jz_{in}\tan(\beta L)}\right]$$

where, with $\lambda = c/f = 3 \times 10^8/2 \times 10^8 = 1.50$ m, we find $\beta L = (2\pi)(1)/(1.50) = 4.19$, and so $\tan(\beta L) = 1.73$. Also, $z_{in} = (200 - j200)/400 = 0.5 - j0.5$. So

$$z_L = \frac{0.5 - j0.5 - j1.73}{1 - j(0.5 - j0.5)(1.73)} = 2.61 + j0.173$$

Finally, $Z_L = z_L(400) = \underline{1.05 \times 10^3 + j69.2\ \Omega}$. Next

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{6.45 \times 10^2 + j69.2}{1.45 \times 10^3 + j69.2} = .448 + j2.64 \times 10^{-2} = .449\angle 5.9 \times 10^{-2} \text{ rad}$$

Now

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + .449}{1 - .449} = \underline{2.63}$$

Finally

$$z_{max} = -\frac{\phi}{2\beta} = -\frac{\lambda\phi}{4\pi} = -\frac{(5.9 \times 10^{-2})(1.50)}{4\pi} = -7.0 \times 10^{-3} \text{ m} = \underline{-7.0 \text{ mm}}$$

13.37. In the transmission line of Fig. 13.17, $R_L = Z_0 = 50\ \Omega$. Determine and plot the voltage at the load resistor and the current in the battery as functions of time by constructing appropriate voltage and current reflection diagrams: Referring to the figure, closing the switch launches a voltage wave whose value is given by Eq. (50):

$$V_1^+ = \frac{V_0 Z_0}{R_g + Z_0} = \frac{50}{75}V_0 = \frac{2}{3}V_0$$

13.37. (continued) We note that $\Gamma_L = 0$, since the load impedance is matched to that of the line. So the voltage wave traverses the line and does not reflect. The voltage reflection diagram would be that shown in Fig. 13.18a, except that no waves are present after time $t = l/v$. Likewise, the current reflection diagram is that of Fig. 13.19a, except, again, no waves exist after $t = l/v$. The voltage at the load will be just $V_1^+ = (2/3)V_0$ for times beyond l/v. The current through the battery is found through

$$I_1^+ = \frac{V_1^+}{Z_0} = \frac{V_0}{75} \text{ A}$$

This current initiates at $t = 0$, and continues indefinitely.

14.1. A parallel-plate waveguide is known to have a cutoff wavelength for the $m = 1$ TE and TM modes of $\lambda_{c1} = 0.4$ cm. The guide is operated at wavelength $\lambda = 1$ mm. How many modes propagate? The cutoff wavelength for mode m is $\lambda_{cm} = 2nd/m$, where n is the refractive index of the guide interior. For the first mode, we are given

$$\lambda_{c1} = \frac{2nd}{1} = 0.4 \text{ cm} \quad \Rightarrow \quad d = \frac{0.4}{2n} = \frac{0.2}{n} \text{ cm}$$

Now, for mode m to propagate, we require

$$\lambda \leq \frac{2nd}{m} = \frac{0.4}{m} \quad \Rightarrow \quad m \leq \frac{0.4}{\lambda} = \frac{0.4}{0.1} = 4$$

So, accounting for 2 modes (TE and TM) for each value of m, we will have a total of $\underline{8}$ modes.

14.9. A rectangular waveguide has dimensions $a = 6$ cm and $b = 4$ cm.

a) Over what range of frequencies will the guide operate single mode? The cutoff frequency for mode mp is, using Eq. (54):

$$f_{c,mn} = \frac{c}{2n}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{p}{b}\right)^2}$$

where n is the refractive index of the guide interior. We require that the frequency lie between the cutoff frequencies of the TE_{10} and TE_{01} modes. These will be:

$$f_{c10} = \frac{c}{2na} = \frac{3 \times 10^8}{2n(.06)} = \frac{2.5 \times 10^9}{n}$$

$$f_{c01} = \frac{c}{2nb} = \frac{3 \times 10^8}{2n(.04)} = \frac{3.75 \times 10^9}{n}$$

Thus, the range of frequencies over which single mode operation will occur is

$$\underline{\frac{2.5}{n} \text{ GHz} < f < \frac{3.75}{n} \text{ GHz}}$$

b) Over what frequency range will the guide support *both* TE_{10} and TE_{01} modes and no others? We note first that f must be greater than f_{c01} to support both modes, but must be less than the cutoff frequency for the next higher order mode. This will be f_{c11}, given by

$$f_{c11} = \frac{c}{2n}\sqrt{\left(\frac{1}{.06}\right)^2 + \left(\frac{1}{.04}\right)^2} = \frac{30c}{2n} = \frac{4.5 \times 10^9}{n}$$

The allowed frequency range is then

$$\underline{\frac{3.75}{n} \text{ GHz} < f < \frac{4.5}{n} \text{ GHz}}$$

14.25. A dipole antenna in free space has a linear current distribution. If the length is 0.02λ, what value of I_0 is required to:

a) provide a radiation-field amplitude of 100 mV/m at a distance of one mile, at $\theta = 90°$: With a linear current distribution, the peak current, I_0, occurs at the center of the dipole; current decreases linearly to zero at the two ends. The average current is thus $I_0/2$, and we use Eq. (84) to write:

$$|E_\theta| = \frac{I_0 d\eta_0}{4\lambda r}\sin(90°) = \frac{I_0(0.02)(120\pi)}{(4)(5280)(12)(0.0254)} = 0.1 \quad \Rightarrow \quad I_0 = \underline{85.4 \text{ A}}$$

b) radiate a total power of 1 watt? We use

$$\mathcal{P}_{avg} = \left(\frac{1}{4}\right)\left(\frac{1}{2}I_0^2 R_{rad}\right)$$

where the radiation resistance is given by Eq. (86), and where the factor of 1/4 arises from the average current of $I_0/2$: We obtain:

$$\mathcal{P}_{avg} = 10\pi^2 I_0^2 (0.02)^2 = 1 \quad \Rightarrow \quad I_0 = \underline{5.03 \text{ A}}$$