λ-DEFINABILITY AND RECURSIVENESS

By S. C. Kleene

1. **Introduction.** In Kleene [2][1] a theory of the definition of functions of positive integers by certain formal means is developed in connection with the study of a system of formal logic.[2] The system of formal logic is shown in Kleene-Rosser [1] to be inconsistent; however, the theory of formal definition remains of interest, both for its use in a new system of formal logic proposed by Church in [3], and for its connection with questions of constructibility and decidability in number theory.[3] Hence it seems desirable to bring together the essentials of the theory, and to develop them from a somewhat new point of view, in which the emphasis is on the connection with the recursive functions. In this presentation, no knowledge of systems of formal logic is presupposed, but use will be made of a few results of the intuitive theory of recursive functions.[4]

It is found convenient here to treat the functions as functions of natural numbers, rather than of positive integers. This change can be regarded as a change merely in the notation.

The theory deals with a class of formulas composed of the symbols $\{$, $\}$, (,), λ, [,] and other symbols f, x, ρ, $\cdots$ called variables or *proper symbols*, where f, x, ρ, $\cdots$ is a given infinite list.

A formula is called *properly-formed* if it is obtainable from proper symbols by zero or more successive operations of combining $\mathbf{M}$ and $\mathbf{N}$ to form $\{\mathbf{M}\}(\mathbf{N})$ or $\lambda\mathbf{x}[\mathbf{M}]$, where $\mathbf{x}$ is any proper symbol. An occurrence of a proper symbol $\mathbf{x}$ in a formula $\mathbf{F}$ is called *bound* or *free* according as it is or is not an occurrence in a properly-formed part of the form $\lambda\mathbf{x}[\mathbf{M}]$. By a free (bound) symbol of $\mathbf{F}$ is meant a proper symbol which occurs in $\mathbf{F}$ as a free (bound) symbol. A formula shall be *well-formed*, if it is properly-formed, and if, for each properly-formed part of the form $\lambda\mathbf{x}[\mathbf{M}]$, where $\mathbf{x}$ is a proper symbol, $\mathbf{x}$ is a free symbol of $\mathbf{M}$.

Heavy-typed letters will henceforth represent undetermined well-formed formulas under the convention that each set of symbols standing apart in which a heavy-typed letter occurs shall stand for a well-formed formula.[5] As abbre-

Received July 1, 1935; presented to the American Mathematical Society, September 13, 1935.

[1] The numbers in brackets refer to the bibliography at the end.

[2] Use is made, directly or indirectly, of Church [1]–[2], Kleene [1], Rosser [1], Curry [1]–[3], Schönfinkel [1].

[3] See Kleene [2] p. 232, Church [4], and Church-Kleene [1].

[4] In writing this paper, I have profited from discussion of the subject with Dr. J. B. Rosser, and I also thank him for assistance with the manuscript.

[5] A detailed analysis of the structure of well-formed formulas, and of the implications of this convention, is given in Kleene [1] §§2, 3. The term "proper symbol" was introduced in place of "variable" in order to save the latter for use in another meaning in connection with the formal logics under consideration.

viations, we shall write $\{\mathbf{F}\}(\mathbf{A}_1, \cdots, \mathbf{A}_n)$ or $\mathbf{F}(\mathbf{A}_1, \cdots, \mathbf{A}_n)$ instead of $\{\cdots\{\mathbf{F}\}(\mathbf{A}_1)\cdots\}(\mathbf{A}_n)$, and $\lambda\mathbf{x}_1\cdots\mathbf{x}_n\cdot\mathbf{M}$ instead of $\lambda\mathbf{x}_1[\cdots\lambda\mathbf{x}_n[\mathbf{M}]\cdots]$. $\mathrm{S}^{\mathbf{x}_1 \cdots \mathbf{x}_n}_{\mathbf{A}_1 \cdots \mathbf{A}_n}\mathbf{M}\,|$ shall denote the result of substituting $\mathbf{A}_i$ for each of the occurrences (if any) of $\mathbf{x}_i$ in $\mathbf{M}$ ($i = 1, \cdots, n$). From time to time we assign individual symbols to stand as abbreviations for particular formulas, indicating this by an arrow $\rightarrow$, as

$$I \rightarrow \lambda f\cdot f, \qquad J \rightarrow \lambda fxyz\cdot f(x, f(z, y)) .$$

We introduce an equivalence relation **A** conv **B**, or **A** is *convertible* into **B**, between well-formed formulas, which is defined to be the relation of least domain which is (1) reflexive, (2) symmetric, and (3) transitive, and has further the properties (4) if **A** conv **B**, then $\{\mathbf{C}\}(\mathbf{A})$ conv $\{\mathbf{C}\}(\mathbf{B})$, $\{\mathbf{A}\}(\mathbf{C})$ conv $\{\mathbf{B}\}(\mathbf{C})$, and $\lambda\mathbf{x}[\mathbf{A}]$ conv $\lambda\mathbf{x}[\mathbf{B}]$, (5) if the proper symbol **y** does not occur in **A**, $\lambda\mathbf{x}[\mathbf{A}]$ conv $\mathrm{S}^{\mathbf{x}}_{\mathbf{y}}\lambda\mathbf{x}[\mathbf{A}]\,|$, and (6) if **x** and the free symbols of **N** are not bound symbols of **M**, $\{\lambda\mathbf{x}[\mathbf{M}]\}(\mathbf{N})$ conv $\mathrm{S}^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}\,|$.[6]

If $\{\mathbf{F}\}(\mathbf{N})$ is interpreted as representing the value of **F** (considered as a function) for **N** as argument, and $\lambda\mathbf{x}[\mathbf{M}]$ as representing the function which **M** is of **x**, then the equivalence relation **A** conv **B** corresponds to a relation of equality in meaning.[7] The analysis of the relation **A** conv **B** given in Church-Rosser [1] can be regarded as furnishing a demonstration of the consistency of the system under these interpretations: A formula **A** which has no part of the form $\{\lambda\mathbf{x}[\mathbf{M}]\}(\mathbf{N})$ is said to be in *normal form*, and to be a normal form of any formula **A** convertible into it. According to Theorem 1, Corollary 2, if **A** has a normal form, the latter is unique to within the choice of symbols used in it as bound variables.[8]

Evidently, a demonstration that **A** conv **B** is given by passing from **A** to **B** by successive substitutions (on individual parts of a formula not immediately following λ) of (a) **C** for **D** (or inversely), where **D** conv **C** is known, and (b) $\mathrm{S}^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}\,|$ for $\{\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$ (or inversely), changing bound variables when necessary to avoid confounding variables that should be distinct or to reach a desired formula.

The substitution $\mathrm{S}^{\mathbf{x}_1 \cdots \mathbf{x}_n}_{\mathbf{N}_1 \cdots \mathbf{N}_n}\mathbf{M}\,|$ for $\{\lambda\mathbf{x}_1\cdots\mathbf{x}_n\cdot\mathbf{M}\}(\mathbf{N}_1, \cdots, \mathbf{N}_n)$ is equivalent to a series of the substitutions (b). Indeed, from the interpretations of $\{\mathbf{F}\}(\mathbf{N})$ and $\lambda\mathbf{x}[\mathbf{M}]$, it follows that the expression which we abbreviate to $\mathbf{F}(\mathbf{N}_1, \cdots, \mathbf{N}_n)$ represents the value of **F** (considered as a function of n variables) for the set of

[6] (1) and the clause "$\{\mathbf{A}\}(\mathbf{C})$ conv $\{\mathbf{B}\}(\mathbf{C})$" of (4) are redundant. The present definition is equivalent to the former one, according to which **A** conv **B** whenever **B** is derivable from **A** by certain rules I–III, the derivation being called a *conversion* (cf. both Church [1] and Kleene [1]).

[7] *A* relation, rather than *the* relation, since, for example, it can be maintained that $\lambda fx\cdot f(x)$ and $\lambda f\cdot f$ have the same meaning.

[8] The notion of the *normal form* of a formula under conversion was originally introduced by Church in lectures at Princeton in the fall of 1931.

arguments $\mathbf{N}_1, \cdots, \mathbf{N}_n$; and the expression which we abbreviate $\lambda \mathbf{x}_1 \cdots \mathbf{x}_n \cdot \mathbf{M}$ represents the function which $\mathbf{M}$ is of $\mathbf{x}_1, \cdots, \mathbf{x}_n$.[9]

We have specified a class of formulas (the well-formed formulas) and an equivalence relation between formulas of this class (the relation of interconvertibility). We bring the natural numbers into relation with this subject-matter by selecting a progression of well-formed formulas

$$\lambda fx \cdot f(x), \qquad \lambda fx \cdot f(f(x)), \qquad \lambda fx \cdot f(f(f(x))), \cdots$$

to "represent" or "be identified with" the natural numbers in our symbolism. This is recognized in the notation by assigning

$$0 \rightarrow \lambda fx \cdot f(x), \qquad 1 \rightarrow \lambda fx \cdot f(f(x)), \qquad 2 \rightarrow \lambda fx \cdot f(f(f(x))), \cdots .$$

It may now happen, for a non-negative integral function $L(x_1, \cdots, x_n)$ of natural numbers, that there are well-formed formulas $\mathbf{L}$ which automatically represent the function $L(x_1, \cdots, x_n)$, on the basis of our equivalence relation and our interpretation of $\mathbf{F}(\mathbf{N}_1, \cdots, \mathbf{N}_n)$. That is, there may be formulas $\mathbf{L}$ such that, whenever $\mathbf{x}_1, \cdots, \mathbf{x}_n$ represent natural numbers $x_1, \cdots, x_n$, respectively, $\mathbf{L}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is convertible into the formula which represents the natural number $L(x_1, \cdots, x_n)$. In this case, we shall say that $L(x_1, \cdots, x_n)$ is (*formally*) *defined* or λ-*defined* by $\mathbf{L}$.

Thus a problem arises: what functions $L(x_1, \cdots, x_n)$ are λ-definable? We have at once that the successor function is λ-definable, since

$$\{\lambda \rho fx \cdot f(\rho(f, x))\} \, (\lambda fx \cdot f(\cdots n+1 \text{ times} \cdots f(x) \cdots)) \text{ conv } \lambda fx \cdot f(\cdots n+2 \text{ times} \cdots f(x) \cdots) \quad (n = 0, 1, 2, \cdots).$$

Accordingly let

$$S \rightarrow \lambda \rho fx \cdot f(\rho(f, x)).$$

The identity function of a natural number is also λ-definable, since the formula which we have called I has the property $I(\mathbf{x})$ conv $\mathbf{x}$.

The problem has arisen from the point of view in which interconvertible formulas are regarded as equivalent. Hence we should consider whether the representations involved are unambiguous from this point of view. Let us call a representation of a class of mathematical entities by well-formed formulas *well-founded* if interconvertible formulas cannot represent different entities of the class. It follows from the above-mentioned consistency proof (Church-Rosser [1]) that the given representation of natural numbers by well-formed formulas is well-founded; this in turn implies that such representation of non-negative integral functions of natural numbers as λ-definition yields is well-founded.[10]

[9] This device for expressing functions of several variables in terms of functions for one variable goes back to Schönfinkel [1].

[10] Non-interconvertible formulas may represent the same entity of a given class, and a formula may represent entities of different classes, e.g., the formulas abbreviated I and 0 both represent the identity function of a natural number, while the latter also represents the natural number 0.

The problem is a special case of the larger problem: what functional relationships among well-formed expressions can be expressed by well-formed formulas?

We shall say, generally, that a function L which associates well-formed formulas with finite ordered sets of well-formed formulas is (*formally*) *defined* or λ-*defined* by **L** if for each finite ordered set $\mathbf{A}_1, \cdots, \mathbf{A}_{n_\mathrm{A}}$ for which L is defined, $\mathbf{L}(\mathbf{A}_1, \cdots, \mathbf{A}_{n_\mathrm{A}})$ is convertible into the value of the function L for the set $\mathbf{A}_1, \cdots, \mathbf{A}_{n_\mathrm{A}}$ of arguments; and we shall understand, by the λ-definition of a function of which the arguments or the values are other mathematical entities, the λ-definition of a function which corresponds under the representation of the mathematical entities by well-formed formulas (in case a representation has been specified).[11]

In this paper we restrict ourselves (except incidentally) to the case of the larger problem in which the independent variables are fixed in number, and range over the natural numbers. The subcase of the problem in which the values are also natural numbers (i.e., the problem first proposed) we treat in §§2, 3 by proving that all recursive functions, in a wide sense of the term recursive, due to Herbrand and Gödel, are λ-definable; and conversely, all λ-definable functions of the type in question are recursive. In §§4, 5 it is shown that, using the term recursive in an extended sense, these results can be generalized (under additional hypotheses) to the case in which the values are any well-formed expressions. The extended sense of the term recursive is obtained by assigning numbers to the values, by the Gödel method, and requiring that they be a recursive function of the arguments in the first sense.

The formulas $\lambda fx \cdot f(x)$, $\lambda fx \cdot f(f(x))$, $\cdots$ were originally coördinated with the positive integers 1, 2, $\cdots$ (Church [2], p. 863). That is a suitable course to follow in developing number-theory (Kleene [2]). In this paper, for technical reasons, we are using instead the correspondence established above between those formulas and the natural numbers 0, 1, $\cdots$. Because of this, the concept of λ-definability of a function is altered. But, for the interpretation of the final results, one can easily go back to the original notion of λ-definability. Since the "natural numbers", "0", "1", $\cdots$ enter into our definitions of λ-definable function and recursive function in the rôle of a progression, it is only necessary to rename them "positive integers", "1", "2", $\cdots$ in those definitions. Or one may use the following relation: A positive integral function $\phi(y_1, \cdots, y_n)$ [well-formed function $\Phi(y_1, \cdots, y_n)$] of positive integers $y_1, \cdots, y_n$ is λ-definable in the original sense if and only if the function $\phi(x_1 + 1, \cdots, x_n + 1) - 1$ [$\Phi(x_1 + 1, \cdots, x_n + 1)$] of natural numbers $x_1, \cdots, x_n$ is λ-definable in the present sense.[12]

[11] The λ-definition of a sequence $\mathbf{A}_0, \mathbf{A}_1, \cdots$ shall mean the λ-definition of the function which $\mathbf{A}_i$ is of i, and the λ-enumeration of a class shall mean the λ-definition of an enumeration (with or without repetitions) of the members of the class.

[12] Under the Church representation of the positive integers, the formula of n denotes the operation of applying the n-th power of a function to an argument, and exceedingly simple

2. **Recursive non-negative integral functions.** In the λ-notation, the definition of a function by substitution is immediate:

(1) $\{\lambda \mathbf{x}_1 \cdots \mathbf{x}_n \cdot \mathbf{G}(\mathbf{H}_1(\mathbf{x}_1, \cdots, \mathbf{x}_n), \cdots, \mathbf{H}_m(\mathbf{x}_1, \cdots, \mathbf{x}_n))\}(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ *conv* $\mathbf{G}(\mathbf{H}_1(\mathbf{X}_1, \cdots, \mathbf{X}_n), \cdots, \mathbf{H}_m(\mathbf{X}_1, \cdots, \mathbf{X}_n))$.[13]

When an italic letter denotes a number, the same letter in heavy type shall denote the corresponding formula. Our remark that S λ-defines the successor function of a natural number can now be written thus:

(2) *If* $x + 1 = z$, $S(\mathbf{x})$ *conv* $\mathbf{z}$ $(x = 0, 1, \cdots)$.

In view of the form of $\mathbf{x}$ $(x = 0, 1, \cdots)$:

(3) $\mathbf{x}(\mathbf{F}, \mathbf{A})$ *conv* $\mathbf{F}(\cdots x + 1 \text{ times} \cdots \mathbf{F}(\mathbf{A}) \cdots)$ $\quad (x = 0, 1, \cdots)$.

(4) $\mathbf{x}(I)$ *conv* I $\quad (x = 0, 1, \cdots)$. $\quad$ ($I(\mathbf{A})$ *conv* $\mathbf{A}$.)

By use of (4):

(5) $\{\lambda t \cdot t(I, 0)\}(\mathbf{x})$ *conv* 0 $\quad (x = 0, 1, \cdots)$.

(6) $\{\lambda t_1 \cdots t_n \cdot t_1(I, \cdots, t_n(I, t_i) \cdots)\}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ *conv* $\mathbf{x}_i$ $\quad (x_1, \cdots, x_n = 0, 1, \cdots)$.

λ-definitions of addition, multiplication, and exponentiation, due to Rosser (Kleene [2] pp. 160–164), are possible.

If that representation is extended by adding $\lambda fx \cdot x(f)$ to represent 0, the resulting formal definition of functions of natural numbers is equivalent to the one of this paper in respect to the results we have summarized.

If the Church representation is extended by the natural method of using the class of properly-formed formulas instead of the class of well-formed formulas, modifying suitably the relation *conv*, and letting $\lambda fx \cdot x$ represent 0, simplifications are afforded in the proofs of many theorems, but unfortunately difficulties are introduced in the formal logics in which this theory is used. Rosser has shown that the formal definition (λ-K-definition) under this program is equivalent to λ-definition, when the range of the independent variables is the set of natural numbers, and all the values have the same free symbols. For functions over all well-formed formulas, λ-K-definition is not equivalent to λ-definition, but we conjecture that the equivalence holds for many other significant ranges of the independent variables (such as functions of natural numbers, functions of functions of natural numbers, $\cdots$, with values in the same range, and ordinal numbers represented by well-formed formulas as in Church-Kleene [1]), and fails only for very heterogeneous ranges.

The formal definition which is obtained from that of this paper by using the [λ-δ-]conversion of Church [3] is likewise equivalent to λ-definition, when the range of the independent variables is the set of natural numbers and the values do not contain δ.

[13] Here we assume that $\mathbf{x}_1, \cdots, \mathbf{x}_n$ do not occur in $\mathbf{G}, \mathbf{H}_1, \cdots, \mathbf{H}_m$ as free symbols; and, in general, when a heavy-typed letter represents occurrences of a proper symbol in a formula, we suppose the only occurrences of the symbol in the formula to be those appearing explicitly, unless the contrary is implied by the original convention concerning heavy-type.

For the moment we abbreviate $\{\lambda\rho\sigma\tau fgha\cdot\rho(f, \sigma(g, \tau(h, a)))\}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$, $\{\lambda\rho f\cdot\rho(f, I, I)\}(\mathbf{X})$ to $\mathbf{X}_1$, $\{\lambda\rho f\cdot\rho(I, f, I)\}(\mathbf{X})$ to $\mathbf{X}_2$, $\{\lambda\rho f\cdot\rho(I, I, f)\}(\mathbf{X})$ to $\mathbf{X}_3$.

(7) $[\mathbf{x}, \mathbf{y}, \mathbf{z}]_1$ *conv* $\mathbf{x}$, $[\mathbf{x}, \mathbf{y}, \mathbf{z}]_2$ *conv* $\mathbf{y}$, $[\mathbf{x}, \mathbf{y}, \mathbf{z}]_3$ *conv* $\mathbf{z}$ $(x, y, z = 0, 1, \cdots)$.

If $\mathfrak{F} \to \lambda\rho\cdot[\rho_2, \rho_3, S(\rho_3)]$ and $\mathfrak{J} \to [0, 0, 0]$, then, using (2) and (7):

(8) $\mathfrak{F}(\mathfrak{J})$ *conv* $[0, 0, 1]$, $\mathfrak{F}(\mathfrak{F}(\mathfrak{J}))$ *conv* $[0, 1, 2]$, $\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{J})))$ *conv* $[1, 2, 3]$, $\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathfrak{J}))))$ *conv* $[2, 3, 4]$, $\cdots$.

Hence, letting $P \to \lambda\rho\cdot\rho(\mathfrak{F}, \mathfrak{J})_1$, and using (3), (7) and (8):

(9) *If* $x = z + 1$, $P(\mathbf{x})$ *conv* $\mathbf{z}$ $(x = 1, 2, \cdots)$. $P(0)$ *conv* 0.

Now let $\dot{-} \to \lambda\mu\nu\cdot\nu(P, S(\mu))$, and abbreviate $\{\dot{-}\}(\mathbf{x}, \mathbf{y})$ to $[\mathbf{x}] \dot{-} [\mathbf{y}]$ (omitting brackets when no ambiguity results). By (3) and (9):

(10) *If* $x \geqq y$ *and* $x - y = z$, $\mathbf{x} \dot{-} \mathbf{y}$ *conv* $\mathbf{z}$; *if* $x \leqq y$, $\mathbf{x} \dot{-} \mathbf{y}$ *conv* 0 $(x, y = 0, 1, \cdots)$.

Let $\min \to \lambda xy\cdot y \dot{-} [y \dot{-} x]$.

(11) *If* $x \leqq y$, $\min(\mathbf{x}, \mathbf{y})$ *conv* $\min(\mathbf{y}, \mathbf{x})$ *conv* $\mathbf{x}$ $(x, y = 0, 1, \cdots)$.

We call a formula constructed out of I, J and proper symbols by zero or more successive operations of passing from $\mathbf{F}$ and $\mathbf{A}$ to $\{\mathbf{F}\}(\mathbf{A})$ a *combination*, and the individual occurrences in it of I, J and proper symbols which enter in the course of this construction its *terms*. Let $T \to J(I, I)$, so that T conv $\lambda fx\cdot x(f)$. The reader may verify that

(12) $J(T, \mathbf{A}, \mathbf{F})$ *conv* $\lambda\mathbf{x}\cdot\mathbf{F}(\mathbf{x}, \mathbf{A})$, $J(T, \mathbf{A}, J(I, \mathbf{F}))$ *conv* $\lambda\mathbf{x}\cdot\mathbf{F}(\mathbf{A}(\mathbf{x}))$, $J(T, T, J(I, J(T, T, J(T, A, J(T, \mathbf{F}, J)))))$ *conv* $\lambda\mathbf{x}\cdot\mathbf{F}(\mathbf{x}, \mathbf{A}(\mathbf{x}))$

If $\mathbf{C}$ is the proper symbol $\mathbf{x}$, I is a combination convertible into $\lambda\mathbf{x}\cdot\mathbf{C}$. If $\mathbf{C}$ is a combination of the form $\mathbf{F}(\mathbf{A})$ and has $\mathbf{x}$ as a free symbol, then $\mathbf{x}$ is a free symbol either (a) of $\mathbf{F}$ but not of $\mathbf{A}$, (b) of $\mathbf{A}$ but not of $\mathbf{F}$, or (c) both of $\mathbf{F}$ and of $\mathbf{A}$. In case (a), if $\mathbf{F}^\circ$ is a combination convertible into $\lambda\mathbf{x}\cdot\mathbf{F}$, then by (12) $J(T, \mathbf{A}, \mathbf{F}^\circ)$ is a combination convertible into $\lambda\mathbf{x}\cdot\mathbf{F}^\circ(\mathbf{x}, \mathbf{A})$ and hence into $\lambda\mathbf{x}\cdot\mathbf{F}(\mathbf{A})$ or $\lambda\mathbf{x}\cdot\mathbf{C}$, and similarly in cases (b) and (c). Thus, by induction on the number of terms of $\mathbf{C}$:

(13) *If* x *is a free symbol of the combination* $\mathbf{C}$, *there is a combination* $\mathbf{C}^\circ$ *such that* $\mathbf{C}^\circ$ *conv* $\lambda\mathbf{x}\cdot\mathbf{C}$.

A proper symbol is a combination; if $\mathbf{F}'$ and $\mathbf{A}'$ are combinations convertible into $\mathbf{F}$ and $\mathbf{A}$, respectively, $\mathbf{F}'(\mathbf{A}')$ is a combination convertible into $\mathbf{F}(\mathbf{A})$; if $\mathbf{R}$ has $\mathbf{x}$ as a free symbol, and $\mathbf{R}'$ is a combination convertible into $\mathbf{R}$, then $\mathbf{R}'$ has $\mathbf{x}$ as a free symbol (interconvertible formulas have the same free symbols), and by (13) there is a combination $\mathbf{R}'^\circ$ convertible into $\lambda\mathbf{x}\cdot\mathbf{R}'$ and hence into $\lambda\mathbf{x}\cdot\mathbf{R}$.

Thus, by an induction corresponding to the process of construction of a well-formed formula:

(14) *Given* $\mathbf{A}$, *there is a combination* $\mathbf{A}'$ *such that* $\mathbf{A}'$ *conv* $\mathbf{A}$.[14]

Let $H \to \lambda\sigma\cdot\sigma(I, I, I, I)$. Given a formula $\mathbf{A}$ having no free symbols, there is by (14) a combination $\mathbf{A}'$ convertible into $\mathbf{A}$. $\mathbf{A}'$ has no free symbols, and hence its terms are I's and J's. Let $\mathrm{S}'^{\mathrm{T}}_{y(\mathrm{T})}\,\mathbf{A}'|$ denote the result of replacing each term T of $\mathbf{A}'$ by $y(\mathrm{T})$, and let $\mathbf{C} \to \lambda y\cdot \mathrm{S}'^{\mathrm{T}}_{y(\mathrm{T})}\,\mathbf{A}'|$. Then $\mathbf{C}(I)$ conv $\mathrm{S}'^{\mathrm{T}}_{I(\mathrm{T})}\,\mathbf{A}'|$ conv $\mathbf{A}'$ conv $\mathbf{A}$; and $\mathbf{C}(H)$ conv $\mathrm{S}'^{\mathrm{T}}_{H(\mathrm{T})}\,\mathbf{A}'|$ conv $\mathrm{S}'^{\mathrm{T}}_{I}\,\mathbf{A}'|$ conv I.

(15) *If* $\mathbf{A}$ *has no free symbols, there is a formula* $\mathbf{C}$ *such that* $\mathbf{C}(I)$ *conv* $\mathbf{A}$ *and* $\mathbf{C}(H)$ *conv* I.

Let $B_{-1} \to \lambda pxyz\cdot p(x, z, y)$, $B_0 \to \lambda pxyz\cdot p(y, x, z)$, $B_1 \to \lambda pxyz\cdot p(x, y, z)$.

(16) $B_1(B_{-1})$ *conv* B_{-1}, $B_{-1}(B_1)$ *conv* B_0, $B_{-1}(B_{-1}(B_1))$ *conv* B_1.

We now adopt the notation $-1 \to \lambda fx\cdot x(f)$. Given formulas $\mathbf{A}_{-1}, \mathbf{A}_0, \mathbf{A}_1$ having no free symbols, there are by (15) formulas $\mathbf{C}_i$ such that $\mathbf{C}_i(I)$ conv $\mathbf{A}_i$ and $\mathbf{C}_i(H)$ conv I ($i = -1, 0, 1$). Then $\lambda n\cdot n(B_{-1}, B_1, \lambda abc\cdot b(a(H, c(H))), \mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_{-1})$ has the properties of $\mathbf{F}$ in the following:

(17) *If* $\mathbf{A}_{-1}, \mathbf{A}_0, \mathbf{A}_1$ *have no free symbols, there is a formula* $\mathbf{F}$ *such that* $\mathbf{F}(\mathbf{i})$ *conv* $\mathbf{A}_i$ ($i = -1, 0, 1$).

If $\mathfrak{A} \to \lambda\rho\cdot\rho(0, 1)$, then, using (3) and the relations $0(1)$ conv 1 and $1(0)$ conv 0:

(18) $\mathfrak{A}(\mathbf{n})$ *conv* 1 ($n = 0, 1, \cdots$). $\mathfrak{A}(-1)$ *conv* 0.

If $\mathbf{F}$ has no free symbols, there is by (17) a formula $\mathbf{B}$ such that $\mathbf{B}(-1)$ conv $\mathbf{B}(0)$ conv I and $\mathbf{B}(1)$ conv $\lambda bx\cdot\mathbf{F}(x, \lambda\rho\cdot b(\mathfrak{A}(\rho), b, \rho))$. Then $\lambda\rho\cdot\mathbf{B}(\mathfrak{A}(\rho), \mathbf{B}, \rho)$ has the properties of $\mathbf{L}$ in the following:

(19) *If* $\mathbf{F}$ *has no free symbols, there is a formula* $\mathbf{L}$ *such that* $\mathbf{L}(\mathbf{x})$ *conv* $\mathbf{F}(\mathbf{x}, \mathbf{L})$ ($x = 0, 1, \cdots$) *and* $\mathbf{L}(-1)$ *conv* I.

Given formulas $\mathbf{G}$ and $\mathbf{H}$ having no free symbols, choose $\mathbf{K}$ by (17) so that $\mathbf{K}(0)$ conv $\lambda yf\cdot y(f(-1), \mathbf{G})$ and $\mathbf{K}(1)$ conv $\lambda yfx_2 \cdots x_n\cdot\mathbf{H}(P(y), f(P(y), x_2, \cdots, x_n), x_2, \cdots, x_n)$, and let $\mathbf{F} \to \lambda y\cdot\mathbf{K}(\min(y, 1), y)$. Then the $\mathbf{L}$ given by (19) for this $\mathbf{F}$ satisfies the following:

(20) *If* $\mathbf{G}$ *and* $\mathbf{H}$ *have no free symbols, there is a formula* $\mathbf{L}$ *such that*

$$\mathbf{L}(0, \mathbf{x}_2, \cdots, \mathbf{x}_n) \text{ conv } \mathbf{G}(\mathbf{x}_2, \cdots, \mathbf{x}_n) \text{ and } \mathbf{L}(S(\mathbf{y}), \mathbf{x}_2, \cdots, \mathbf{x}_n) \text{ conv}$$
$$\mathbf{H}(\mathbf{y}, \mathbf{L}(\mathbf{y}, \mathbf{x}_2, \cdots, \mathbf{x}_n), \mathbf{x}_2, \cdots, \mathbf{x}_n)\ (y, x_2, \cdots, x_n = 0, 1, \cdots).$$

Choose $\mathbf{K}$ by (17) so that $\mathbf{K}(0)$ conv $\lambda fyr\cdot r(y, f(-1), y)$ and $\mathbf{K}(1)$ conv $\lambda fyr\cdot f(r(S(y)), S(y), r)$, let $\mathbf{F} \to \lambda x\cdot\mathbf{K}(\min(x, 1))$, and choose $\mathbf{L}$ by (19) for this $\mathbf{F}$. Then $\mathbf{L}(\mathbf{x}, \mathbf{y}, \mathbf{r})$ conv $\mathbf{L}(\mathbf{r}(S(\mathbf{y})), S(\mathbf{y}), \mathbf{r})$ ($x = 1, 2, \cdots$) and, if $\mathbf{r}(\mathbf{y})$ conv

[14] This theorem derives from Rosser [1], and the present proof of it from Church [3].

z, where z is a natural number, $\mathbf{L}(0, \mathbf{y}, \mathbf{r})$ conv **y**. Hence, letting $\mathfrak{e}_n \to \lambda r x_1 \cdots x_n \cdot \mathbf{L}(r(x_1, \cdots, x_n, 0), 0, r(x_1, \cdots, x_n))$:

(21) *If* **r** *λ-defines a non-negative integral function* $\rho(x_1, \cdots, x_n, y)$ *of natural numbers such that* $(x_1, \cdots, x_n)(Ey)[\rho(x_1, \cdots, x_n, y) = 0]$, then $\mathfrak{e}_n(\mathbf{r})$ *λ-defines* $\epsilon y[\rho(x_1, \cdots, x_n, y) = 0]$.[15]

According to Kleene [3] IV every function of natural numbers recursive in the general Herbrand-Gödel sense (see [3] Def. 2a or Def. 2b) is expressible in the form $\psi(\epsilon y[\rho(x_1, \cdots, x_n, y) = 0])$, where $\psi(y)$ and $\rho(x_1, \cdots, x_n, y)$ are primitive recursive ([3] Def. 1) and $(x_1, \cdots, x_n)(Ey)[\rho(x_1, \cdots, x_n, y) = 0]$. In view of (1), (2), (5), (6) and (20),[16] every primitive recursive function is λ-definable; and therefore, from (21), every general recursive function is λ-definable.

(22) *Every non-negative integral function of natural numbers which is recursive in the Herbrand-Gödel sense is λ-definable.*

(19) constitutes a schema for circular definition. Given any set of conditions of dependence of an entity $L(x)$ on the variable natural number x and on L itself, if the set can be expressed in the λ-notation by a formula **F**, a formula **L** satisfying the conditions in terms of the equivalence relation **A** conv **B** can be found.[17] To do this it need not be known that the conditions actually determine a function $L(x)$. Further analysis of this situation (Kleene [2] §18) shows that to each problem of a large class, which includes many famous unsolved problems (such as the Fermat problem and the 4-color problem), there is a formula **P** such that *whether* **P** *has a normal form* is an equivalent problem.

3. **λ-definable non-negative integral functions.** We now set up a representation of the well-formed formulas by natural numbers, by the Gödel method. The symbols which occur in well-formed formulas we number thus:[18]

$$\lambda \ldots 1;\ \{,\ (,\ [\ \ldots 11;\ \},\ ),\ ]\ \ldots 13;\ \text{the } i\text{-th proper symbol} \ldots p_{i+6}$$

(p_i = the i-th prime number), and we order numbers to formulas (considered as finite sequences of symbols), finite sequences of formulas, etc., on the basis of the correspondence $n_1, n_2, \cdots, n_k$ to $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ between finite sequences of numbers and individual numbers $(n_1, n_2, \cdots, n_k > 0)$. Using the methods

[15] Read $(x_1, \cdots, x_n)$ "for all $x_1, \cdots, x_n$", (Ey) "there is a y", $\epsilon y[R(y)]$ "the least y such that $R(y)$ (0 if there is no such y)".

[16] A formula which λ-defines a non-negative integral function of natural numbers has no free symbols.

[17] (17) can be used in the selection of **F**, if cases are distinguished in the form of the dependence of $L(x)$ on x and L. (4) and the clause $\mathbf{L}(-1)$ conv I of (19) can be used whenever under a given case $L(x)$ is independent of either or both of x and L. These devices are illustrated in the proofs of (20), (21) and (24).

[18] The distinction in well-formed formulas of three species of parentheses is unessential, since the species of each parenthesis can be determined from its situation.

and notations of Gödel [1] pp. 179–182,[19] and starting from 1–5, 7–10 of his list (p. 182) and 6, 11–18 of Kleene [3], we define the additional primitive recursive functions and relations 19–42:

19. $Z(0) = 2\cdot 3^{17}\cdot 5^{11}\cdot 7\cdot 11^{19}\cdot 13^{11}\cdot 17^{11}\cdot 19^{17}\cdot 23^{13}\cdot 29^{11}\cdot 31^{19}\cdot 37^{13}\cdot 41^{13}\cdot 43^{13}$,
$Z(n+1) = Su\; Z(n) \begin{pmatrix} 11+4n \\ 2^{11}\cdot 3^{17}\cdot 5^{13}\cdot 7^{11}\cdot 11^{19}\cdot 13^{13} \end{pmatrix}$.
$Z(0), Z(1), \cdots$ are the numbers corresponding to the formulas 0, 1, $\cdots$

20. $\mathrm{Num}(x) \equiv (En)[n \leqq x \;\&\; x = Z(n)]$.
x corresponds to one of the formulas 0, 1, $\cdots$.

21. $Z^{-1}(x) = \epsilon n[n \leqq x \;\&\; x = Z(n)]$.
If x corresponds to $\lambda fx\cdot f(\cdots n+1 \text{ times} \cdots f(x) \cdots)$, $Z^{-1}(x)$ is n.

22. $PS(x) \equiv \mathrm{Prim}(x) \;\&\; x > 13$.
x is a *proper symbol.*

23. $PFR(x) \equiv (n)\{0 < n \leqq l(x) \rightarrow (Ev)[v \leqq x \;\&\; PS(v) \;\&\; n\; Gl\; x = R(v)] \vee (Ep, q)[0 < p, q < n \;\&\; \{n\; Gl\; x = E(p\; Gl\; x)*E(q\; Gl\; x) \vee \{n\; Gl\; x = R(1)*[p\; Gl\; x]*E(q\; Gl\; x) \;\&\; (Ev)[v \leqq x \;\&\; PS(v) \;\&\; p\; Gl\; x = R(v)]\}\}]\} \;\&\; l(x) > 0$.
x is a sequence of *formulas* of which each is either a *proper symbol* or is compounded out of the preceding ones by the operations { } ( ) and λ [ ].

24. $PF(x) \equiv (En)\{n \leqq (Pr[l(x)^2])^{x\cdot l(x)^2} \;\&\; PFR(n) \;\&\; x = [l(n)]\; Gl\; n\}$.
x is a *properly-formed formula.*[20]

25. v Geb $n, x \equiv PS(v) \;\&\; PF(x) \;\&\; (Ea, b, c)[a, b, c \leqq x \;\&\; x = a*R(1)*R(v)*E(b)*c \;\&\; PF(b) \;\&\; l(a) < n \leqq l(a) + l(b) + 4]$.
The *proper symbol* v is *bound* at the n-th point of the *properly-formed formula* x.

26. v Fr $n, x \equiv PS(v) \;\&\; PF(x) \;\&\; v = n\; Gl\; x \;\&\; n \leqq l(x) \;\&\; \overline{v \text{ Geb } n, x}$.
The *proper symbol* v occurs as a *free symbol* at the n-th point of the *properly-formed formula* x.

27. v Geb $x \equiv (En)[n \leqq l(x) \;\&\; v = n\; Gl\; x \;\&\; v$ Geb $n, x]$.
The *proper symbol* v occurs in the *properly-formed formula* x as a *bound symbol.*

28. v Fr $x \equiv (En)[n \leqq l(x) \;\&\; v$ Fr $n, x]$.
The *proper symbol* v occurs in the *properly-formed formula* x as a *free symbol.*

29. $WF(x) \equiv PF(x) \;\&\; (n)[n < l(x) \;\&\; (n+1) Gl\; x = 1 \rightarrow (Ep, q, r)\{p, q, r \leqq x \;\&\; x = p*R(1)*R[(n+2) Gl\; x]*E(q)*r \;\&\; l(p) = n \;\&\; PF(q) \;\&\; [(n+2)\; Gl\; x]$ Fr $q\}]$.
x is a *well-formed formula.*

[19] The possibility of defining number-theoretic functions by means of recursion was expounded in Skolem [1]. In that paper Skolem also showed that restricted existence and restricted generality (the restriction by an upper bound) can be expressed by recursive functions.

[20] Cf. Gödel [1] p. 183, footnote 35.

30. $x \text{ Imr } y \equiv WF(x) \ \& \ \{x = y \vee (Ep, q, r, s, t)[p, q, s, t \leqq x \ \& \ r \leqq y \ \& \ x = p{*}R(1){*}R(q){*}E(s){*}t \ \& \ PS(q) \ \& \ WF(s) \ \& \ PS(r) \ \& \ \overline{r \text{ Occ } s} \ \& \ y = p{*}R(1){*}R(r){*}E(S(s, q, R(r))){*}t] \vee (Ep, q, r, s, t)[p, q, r, s, t \leqq x \ \& \ x = p{*}E(R(1){*}R(q){*}E(r)){*}E(s){*}t \ \& \ PS(q) \ \& \ WF(r) \ \& \ WF(s) \ \& \ \overline{q \text{ Geb } r} \ \& \ (u)[u \leqq s \ \& \ u \ Fr \ s \rightarrow \overline{u \text{ Geb } r}] \ \& \ y = p{*}S(r, q, s){*}t]\}.$

$x \text{ Imc } y \equiv x \text{ Imr } y \vee y \text{ Imr } x.$

x Imr y (x Imc y) corresponds to the relation obtained from **A** conv **B** by omitting (2) and (3) (omitting (3)) in the definition of the latter.

31. $EC(x, m) = \theta(R(x), m)$ for the $\theta(z, m)$ given by Kleene [3] I when $\phi(n, x, y) = \epsilon z[z \leqq n + x \ \& \ \{(x \text{ Imc } n \ \& \ z = n) \vee (\overline{x \text{ Imc } n} \ \& \ z = x)\}].$

$EC(x, 0), EC(x, 1), \cdots$ is an enumeration (with repetitions) of the numbers y *convertible* into x (if x is *well-formed*).

Now let L be a given non-negative integral function of n natural numbers, and **L** a formula which λ-defines L, i.e., a formula such that, for each set $x_1, \cdots, x_n$ of natural numbers, $\mathbf{L}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ conv $\lambda fx \cdot f(\cdots m+1 \text{ times} \cdots f(x) \cdots)$ when $m = L(x_1, \cdots, x_n)$ and (by Church-Rosser [1], Thm. 1, Cor. 2) only then. If l denotes the correspondent of **L** under our representation of well-formed formulas by natural numbers,

$$A(x_1, \cdots, x_n) = E(\cdots E(l){*}E(Z(x_1)) \cdots){*}E(Z(x_n))$$

is the correspondent of $\mathbf{L}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ (8, 10, 19). Hence, if $z_{x_1 \cdots x_n}$ denotes the correspondent of the formula $\lambda fx \cdot f(\cdots m+1 \text{ times} \cdots f(x) \cdots)$, there are y's such that $EC(A(x_1, \cdots, x_n), y) = z_{x_1 \cdots x_n}$ (31). For those and only those y's Num $(EC(A(x_1, \cdots, x_n), y))$ holds (20). Hence

$$(x_1, \cdots, x_n)\ (Ey) \text{ Num } (EC(A(x_1, \cdots, x_n), y))$$

and

$$Z^{-1}(EC(A(x_1, \cdots, x_n), \epsilon y[\text{Num } (EC(A(x_1, \cdots, x_n), y))])) = Z^{-1}(z_{x_1 \cdots x_n}) = L(x_1, \cdots, x_n) \ (21) .$$

Using Kleene [3]V, the expression on the left is seen to be recursive. Thus:

(23′) *Every λ-definable non-negative integral function of natural numbers is recursive in the Herbrand-Gödel sense.*[21]

4. **Recursive well-formed functions.** Let L be a function of a fixed number n of natural numbers $x_1, \cdots, x_n$, of which the values $L_{x_1 \cdots x_n}$ are well-formed formulas. Let $\lambda(x_1, \cdots, x_n)$ be the function which corresponds to L under our representation of formulas by numbers, i.e., the function which the correspondent of $L_{x_1 \cdots x_n}$ is of $x_1, \cdots, x_n$. We call L recursive if $\lambda(x_1, \cdots, x_n)$

[21] This result was first announced by Church.

is recursive in the Herbrand-Gödel sense. This definition agrees with the former one, when the values of L are formulas representing natural numbers, in view of the recursiveness of Z and Z^{-1} (19, 21).

In order that L be λ-definable, it is necessary that all the values $L_{x_1 \cdots x_n}$ have the same set $\mathbf{z}_1, \cdots, \mathbf{z}_m$ of free symbols. If L is recursive, the function L' whose values are the expressions $\lambda \mathbf{z}_1 \cdots \mathbf{z}_m \cdot L_{x_1 \cdots x_n}$ (which contain no free symbols) is recursive, since

$$\lambda'(x_1, \cdots, x_n) = R(1)*R(s_1)*E(\cdots R(1)*R(s_m)*E(\lambda(x_1, \cdots, x_n)) \cdots),$$

where $s_1, \cdots, s_m$ are the numbers corresponding to the symbols $\mathbf{z}_1, \cdots, \mathbf{z}_m$, respectively. Moreover, if L' is λ-defined by $\mathbf{L}'$, then L is λ-defined by

$$\lambda \mathbf{x}_1 \cdots \mathbf{x}_n \cdot \mathbf{L}'(\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{z}_1, \cdots, \mathbf{z}_m).$$

These remarks reduce the problem of this section (to prove (25)) to the special case in which the values of L contain no free symbols.

In the following i and j denote the numbers corresponding to the formulas I and J, respectively:

32. $CR(x) \equiv (n)\{0 < n \leqq l(x) \rightarrow n \, Gl \, x = i \vee n \, Gl \, x = j \vee (Ep, q)[0 < p, q < n \;\&\; n \, Gl \, x = E(p \, Gl \, x)*E(q \, Gl \, x)]\} \;\&\; l(x) > 0.$

x is a sequence of *formulas* of which each is either I or J or is compounded out of the preceding ones by the operation { } ( ).

33. $\text{Comb}\,(x) \equiv (En)\{n \leqq (Pr[l(x)^2])^{x \cdot l(x)^2} \;\&\; CR(n) \;\&\; x = [l(n)] \, Gl \, n\}.$
x is a *combination.*

34. $C(x) = EC(x, \epsilon y\{[WF(x) \;\&\; \text{Comb}\,(EC(x, y))] \vee [\overline{WF(x)} \;\&\; y = 0]\})$.[22]
If x is *well-formed,* $C(x)$ is a *combination convertible* into x.

35. $D(x) \equiv (Ep, q)[p, q \leqq x \;\&\; x = E(p)*E(q) \;\&\; WF(p) \;\&\; WF(q)].$
x corresponds to a formula of the form $\{\mathbf{P}\}\,(\mathbf{Q})$.

36. $M_1(x) = \epsilon p[p \leqq x \;\&\; WF(p) \;\&\; (Eq)[q \leqq x \;\&\; x = E(p)*E(q)]].$
$M_2(x) = \epsilon q[q \leqq x \;\&\; WF(q) \;\&\; (Ep)[p \leqq x \;\&\; x = E(p)*E(q)]].$

If x corresponds to the formula $\{\mathbf{P}\}\,(\mathbf{Q})$, $M_1(x)$ and $M_2(x)$ correspond to $\mathbf{P}$ and $\mathbf{Q}$, respectively.

37. $I(x) \equiv x = i.$
x corresponds to the formula I.

By the λ-definition of a relation we mean the λ-definition of the representing function of the relation (i.e., the function which is 0 or 1 according as the relation holds or not). Since a recursive relation is one of which the representing function is recursive, recursive relations among natural numbers, as well as recursive functions, are λ-definable (by (22)).

Accordingly, let $\mathbf{C}$, $\mathbf{D}$, $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{I}$ be formulas which λ-define C, D, M_1, M_2, I,

[22] By (14) and Kleene [3] V, this function is recursive, which is sufficient for our purpose. Actually, it is primitive recursive, by Gödel [1] IV, since a primitive recursive bound for y is given implicitly by the proofs of (14) and the property of $EC(x, m)$.

respectively. Using (17), choose a formula $\mathfrak{N}$ such that $\mathfrak{N}(0)$ conv $\lambda xf \cdot x(f(-1))$, $\mathfrak{N}(1)$ conv $\lambda xf \cdot x(f(-1), J)$, and a formula $\mathfrak{K}$ such that $\mathfrak{K}(0)$ conv $\lambda xf \cdot f(\mathbf{M}_1(x), f(\mathbf{M}_2(x)))$, $\mathfrak{K}(1)$ conv $\lambda xf \cdot \mathfrak{N}(\mathbf{I}(x), x, f)$; and let $\mathfrak{B} \rightarrow \lambda xf \cdot \mathfrak{K}(\mathbf{D}(x), x, f)$. By (19), there is a formula $\mathfrak{G}$ such that $\mathfrak{G}(\mathbf{x})$ conv $\mathfrak{B}(\mathbf{x}, \mathfrak{G})$ and $\mathfrak{G}(-1)$ conv I. Then $\mathfrak{G}(\mathbf{y})$ conv I if y corresponds to I, $\mathfrak{G}(\mathbf{y})$ conv J if y corresponds to J, and $\mathfrak{G}(\mathbf{y})$ conv $\mathfrak{G}(\mathbf{M}_1(\mathbf{y}), \mathfrak{G}(\mathbf{M}_2(\mathbf{y})))$ if y corresponds to a formula of the form $\{\mathbf{P}\}$ $(\mathbf{Q})$. Hence, if y corresponds to a combination $\mathbf{Y}$ whose terms are I's and J's, $\mathfrak{G}(\mathbf{y})$ conv $\mathbf{Y}$. If x corresponds to a formula $\mathbf{X}$ having no free symbols, $C(x)$ corresponds to a combination $\mathbf{Y}$ of I's and J's convertible into $\mathbf{X}$. Hence, letting $G \rightarrow \lambda x \cdot \mathfrak{G}(\mathbf{C}(x))$:

(24) *If the number x corresponds to a formula $\mathbf{X}$ having no free symbols, $G(\mathbf{x})$ conv $\mathbf{X}$.*

Now, if the function L is recursive, and if the values $L_{x_1 \cdots x_n}$ contain no free symbols, there is a formula $\mathbf{l}$ which λ-defines $\lambda(x_1, \cdots, x_n)$ (by (22)), and then $\lambda x_1 \cdots x_n \cdot G(\mathbf{l}(x_1, \cdots, x_n))$ λ-defines L. Passing to the general case by the means we have indicated:

(25) *If the function L of n natural numbers having well-formed formulas as values is recursive (i.e., if the corresponding numerical function is recursive in the Herbrand-Gödel sense), and if all the values have the same free symbols, then L is λ-definable.*

We are now in a position to infer the λ-definability of various sequences of well-formed formulas from the theory of recursive functions. We give several examples, each accompanied by a definition displaying the recursiveness of the corresponding numerical function $\lambda(x)$.[23] a, f, $\cdots$ stand for the numbers corresponding to $\mathbf{A}$, $\mathbf{F}$, $\cdots$, respectively.

(26) *The sequence $\mathbf{A}_0, \cdots, \mathbf{A}_{k-1}, \mathbf{F}(0), \mathbf{F}(1), \cdots$ is λ-definable (if $\mathbf{A}_0, \cdots, \mathbf{A}_{k-1}, \mathbf{F}$ have the same free symbols).*

$$\lambda(x) = \epsilon y[(x = 0 \,\&\, y = a_0) \vee \cdots \vee (x = k - 1 \,\&\, y = a_{k-1}) \vee (x \geqq k \,\&\, y = E(f) * E(Z(x \dot{-} k))].$$

(27) *The sequence $\mathbf{A}_0, \cdots, \mathbf{A}_{k-1}, \mathbf{F}(0, \mathbf{A}_0, \cdots, \mathbf{A}_{k-1}), \mathbf{F}(1, \mathbf{A}_1, \cdots, \mathbf{A}_k), \cdots$, where $\mathbf{A}_i$ denotes the $(i + 1)$-th member, is λ-definable (if $\mathbf{A}_0, \cdots, \mathbf{A}_{k-1}$ have the same free symbols, and the free symbols of $\mathbf{F}$ are free symbols of $\mathbf{A}_0$).*

$$\lambda(0) = a_0, \cdots, \lambda(k - 1) = a_{k-1}, \lambda(k + x) = E(\cdots E(E(f) * E(Z(x))) * E(\lambda(x)) \cdots) * E(\lambda(x + [k - 1])).$$

(28) *The set of formulas derivable from $\mathbf{A}(0), \mathbf{A}(1), \cdots$ by zero or more successive operations of passing from $\mathbf{M}$ and $\mathbf{N}$ to $\mathbf{R}(0, \mathbf{M}, \mathbf{N}), \mathbf{R}(1, \mathbf{M}, \mathbf{N}), \cdots$ is λ-enumerable (if the free symbols of $\mathbf{R}$ are free symbols of $\mathbf{A}$).*

[23] Here are used known recursive functions and relations, the methods of Gödel [1], Kleene [3] V, direct recursive definition by equations.

$\lambda(x) = \theta(R(E(a)*E(Z(0))), x)$, where $\theta(z, m)$ is chosen by Kleene [3] I taking

$$\phi(n, x, y) = \epsilon z\Big[\Big\{n \mid 2 \;\&\; z = E\Big(E\Big(E(r)*E\Big(Z\Big(\Big[\frac{n}{2}\Big]\Big)\Big)\Big)*E(x)\Big)*E(y)\Big\} \vee \Big\{n+1 \mid 2 \;\&\; z = E(a)*E\Big(Z\Big(\Big[\frac{n+1}{2}\Big]\Big)\Big)\Big\}\Big].$$

38. $EW(0) = i$ (the number corresponding to I).
$EW(x+1) = \epsilon y\{EW(x) < y \leqq Z(x) \;\&\; WF(y) \;\&\; (p)[p \leqq y \rightarrow \overline{p \text{ Fr } y}]\}$.
$EW(0)$, $EW(1)$, $\cdots$ is an enumeration of the *well-formed formulas* with no *free symbols.*

(29) *The class of well-formed formulas (having a given set of free symbols) is* λ-*enumerable.*

For the case of no free symbols, $\lambda(x)$ is the function $EW(x)$ which precedes; if $\mathbf{L}$ λ-enumerates the class for this case, $\lambda \mathbf{x}\cdot\mathbf{L}(\mathbf{x}, \mathbf{z}_1, \cdots, \mathbf{z}_m)$ λ-enumerates it when the set of free symbols is $\mathbf{z}_1, \cdots, \mathbf{z}_m$.

39. $NF(x) \equiv WF(x) \;\&\; (p, q, r, s, t)\{p, q, r, s, t \leqq x \;\&\; WF(q) \;\&\; WF(r) \;\&\; WF(s) \rightarrow x \neq p*E(R(1)*q*E(r))*E(s)*t\}$.
x is a *well-formed formula* in *normal form.*

40. $ENF(x)$ (defined in the same nanner as $EW(x)$ replacing $WF(x)$ by $NF(x)$).
$ENF(0)$, $ENF(1)$, $\cdots$ is an enumeration of the *well-formed formulas* with no *free symbols* in *normal form.*

41. $EN(x) = EC(ENF(1 \; Gl \; \text{Dy}(x)), 2 \; Gl \; \text{Dy}(x))$.
$EN(0)$, $EN(1)$, $\cdots$ is an enumeration of the *well-formed formulas* with no *free symbols* which have *normal forms.*

(30) *The class of well-formed formulas (having a given set of free symbols) which have normal forms is* λ-*enumerable.*[24]

This follows from 41 (or 40) in the same manner as (29) from 38.

5. **λ-definable well-formed functions.** In the extension of the notion of recursiveness to functions L of which the values are any well-formed formulas, the point of view in which interconvertible formulas are regarded as equivalent is compromised. Every well-formed formula λ-defines $2^{\aleph_0}$ functions L of n natural numbers, each corresponding to a different numerical function $\lambda(x_1, \cdots, x_n)$. Since the power of the class of recursive numerical functions is $\aleph_0$, not all functions L λ-definable by a given $\mathbf{L}$ are recursive. In order to prove a theorem like (23′), there must be added to the hypothesis of λ-definability a condition on the form of the values $L_{x_1 \cdots x_n}$ of L which selects from the formulas in which $\mathbf{L}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is convertible that one which is $L_{x_1 \cdots x_n}$. A condition of this sort

[24] This theorem is due to Church.

which can be used here to replace the condition of representing a natural number is that of being in normal form, supplemented by a convention which removes the ambiguity in the normal form of a formula: a formula shall be in *principal normal form* if it is in normal form and the symbol following the n-th occurrence of λ is the n-th proper symbol (in the given list) which is not a free symbol of the formula.

42. $PNF(x) \equiv NF(x) \;\&\; (p, q, r)\{p, q, r \leqq x \;\&\; x = p*R(1)*R(q)*r \rightarrow$
$q = \epsilon s[s \leqq x \;\&\; PS(s) \;\&\; \overline{s \operatorname{Fr} x} \;\&\; \overline{s \operatorname{Geb} p}]\}.$
x is a *well-formed formula* in *principal normal form.*

If all the values $L_{x_1 \cdots x_n}$ are in principal normal form, and $A(x_1, \cdots, x_n)$ is chosen as in the proof of (23′), we find that $\lambda(x_1, \cdots, x_n) = EC(A(x_1, \cdots, x_n), \epsilon y[PNF(EC(A(x_1, \cdots, x_n), y))])$, which is recursive in the Herbrand-Gödel sense, since $(x_1, \cdots, x_n)(Ey)PNF(EC(A(x_1, \cdots, x_n), y))$.

(31′) *Every λ-definable function of n natural numbers of which the values are well-formed formulas in principal normal form is recursive* (*i.e.*, *the corresponding numerical function is recursive in the Herbrand-Gödel sense*).

BIBLIOGRAPHY

ALONZO CHURCH, [1] *A set of postulates for the foundation of logic*, Ann. of Math., vol. 33 (1932), pp. 346–366. [2] A second paper under the same title, Ann. of Math., vol. 34 (1933), pp. 839–864. [3] *A proof of freedom from contradiction*, Proc. Nat. Acad. Sciences, vol. 21 (1935), pp. 275–281. [4] *An unsolvable problem of elementary number theory*, Am. Jour. Math. (to appear).

ALONZO CHURCH AND S. C. KLEENE, [1] *Formal definitions in the theory of ordinal numbers*, Funda. Math. (to appear).

ALONZO CHURCH AND J. B. ROSSER, [1] *Some properties of conversion*, Trans. Am. Math. Soc. (to appear).

H. B. CURRY, [1] *An analysis of logical substitution*, Am. Jour. Math., vol. 51 (1929), pp. 363–384. [2] *Grundlagen der kombinatorischen Logik*, Am. Jour. Math., vol. 52 (1930), pp. 509–536, 789–834. [3] *Some additions to the theory of combinators*, Am. Jour. Math., vol. 54 (1932), pp. 551–558.

KURT GÖDEL, [1] *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatsh. f. Math. u. Phys., vol. 38 (1931), pp. 173–198.

S. C. KLEENE, [1] *Proof by cases in formal logic*, Ann. of Math., vol. 35 (1934), pp. 529–544. [2] *A theory of positive integers in formal logic*, Am. Jour. Math., vol. 57 (1935), pp. 153–173, 219–244. [3] *General recursive functions of natural numbers*, Math. Ann. (to appear).

S. C. KLEENE AND J. B. ROSSER, [1] *The inconsistency of certain formal logics*, Ann. of Math., vol. 36 (1935), pp. 630–636.

J. B. ROSSER, [1] *A mathematical logic without variables*, Ann. of Math., vol. 36 (1935), pp. 127–150 and this journal, vol. 1 (1935), pp. 328–355.

M. SCHÖNFINKEL, [1] *Über die Bausteine der mathematischen Logik*, Math. Ann., vol. 92 (1924), pp. 305–316.

TH. SKOLEM, [1] *Begründung der elementaren Arithmetik durch die rekurrierende Denkweise ohne Anwendung scheinbarer Veränderlichen mit unendlichem Ausdehnungsbereich*, Videnkapsselskapets Skrifter, 1923, I. Mat.-naturv. Kl., No. 6, pp. 1–38.

PRINCETON UNIVERSITY.