A Comparative Survey of Solutions to Russel's Paradox

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***Abstract*—This comparative study examines various solutions to Russel's Paradox, a well-known problem in set theory first identified by Bertrand Russell in 1901. The paradox arises from the question of whether a set can be a member of itself. This study compares and contrasts the different solutions proposed by various mathematicians and logicians, including Theory of Types, Zermelo-Fraenkel set theory, von Neumann-Bernays-Gödel set theory, and Paraconsistent set theory and Fuzzy set theory. The study also examines the pros and cons of each of these proposed solutions and suggests the reason why Zermelo- Frankael Set theory seems to be the simplest and most-suited solution to Russel's paradox as compared to others.**

***Index Terms*—Logic, Bertrand Russell, Paradox, Set Theory**

I. Introduction

Russell's Paradox, named after the famed Bertrand Russell, is a paradox that focuses on naïve set theory and how the unrestricted comprehension principle leads to contradictions and illogical results. This paper will first help the reader build intuition for the paradox and then explain its working principle in Section 2. Afterward, the paper will explore possible solutions to the paradox and propose the selection of the best-suited solution, and then conclude our findings.

II. Background

A simpler version of Russell's paradox is the Barber's paradox. Consider a town where it is mandatory for all men to shave. There is a barber in the town who shaves all those who do not shave themselves. Now the question arises, does the barber shaves himself [17]? Consider the scenario where the barber does shave himself, then this contradicts our definition of the barber's job that is, the barber becomes someone who shaves himself and, therefore, the barber should not shave himself [17]. Alternatively, suppose the barber does not shave himself. If this were the case then the barber is in the class of people who do not shave themselves, therefore, the barber should shave himself [17]. So if the barber shaves himself then he should not shave himself, and if the barber does not shave himself then he should shave himself [17]. This leads to a paradoxical result no matter what option is chosen by the barber and serves to be a great analogy to Russell's Paradox.

This seemingly pointless problem actually becomes a very foundational question in mathematics if we generalize this to the notion of sets.

Consider the set of all sets that do not contain themselves, let $R = \{x | x \notin x\}$. Now the question is does that set, R, contain itself? Let us say $R \in R$, then according to R definition it only contains those sets that do not contain themselves and as R contains itself then R should not contain itself, therefore, $R \notin R$, so we first have $R \in R \Rightarrow R \notin R$. Now let us say R does not contain itself $R \notin R$, then according to the definition of R, it contains all those sets that do not contain themselves, as such, R should contain itself, $R \in R$. So we have that $R \notin R \Rightarrow R \in R$.

This leads us to the conclusion that $R \in R \Leftrightarrow R \notin R$. [1]

This is a paradoxical result as the statement says that R contains itself if and only if R does not contain itself. The bigger problem is that the line of reasoning used to reach this conclusion itself is valid and seems logically sound using the naive set theory axioms. Adding on, the comprehension of this set itself is very similar to that used by Cantor to prove his famous theorem. As such, the question now is, where does the flaw in this reasoning even lie, if there is any? In the following part of this paper we will look into some proposed solutions to this paradox and the attempts that have been made to find where the issue lies in the line of reasoning used.

III. Existing Solutions

A. Theory of Types

Russell and others in his era singled out that the violation of the vicious circle principle is the root cause of most of

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the famous self-referential paradoxes including his own [3]. The vicious circle principle violation refers to an object being defined using its own self or its properties. Russell proposed the theory of types during 1902-1908 in Principia Mathematica as a fix to paradoxes of such circular nature. Under this theory, the main objects were types rather than sets which are arranged hierarchically. The hierarchy begins from the simplest type that describes simple objects/individuals thus constituting the first logical type [4]. These individuals cannot be propositions or predicates. Higher or more abstract types are constructed from the lower-level types. For example, the second logical type consists of all such functions and propositions which operate on objects of type 1. Similarly, propositions or functions whose terms take on values from objects of type 2 or less constitute the third logical type. Formally, type structure as defined in Principia is the following [2]

1) i is the type of individuals/objects
2) $()$ is the type of propositions
3) if $A_1, \ldots A_n$ are types then $(A_1, \ldots, A_n)$ is the type of n-ary relations over objects of respective types $A_1, \ldots, A_n$

With these foundations in place, it is not possible to create a predicate of the form $R(R)$ where R quantifies over objects of the same type as itself [2]. The type of R should be of the form (A) as defined in the above 3 axioms and the arguments of R should be of type A. However, that is not the case here. Here, the arguments and the proposition itself are of the same logical type hence such a proposition cannot be defined and Russell's paradox is avoided. In the simplest function analogy, Russell's paradox results when the function or proposition itself is allowed to be the value of one of its terms/parameters. Therefore, arguments of a function cannot be the function itself.

B. Zermelo-Fraenkel Set Theory

Another solution to the Russell's paradox is Zermelo-Fraenkel (ZF) Set Theory. This theory builds upon naïve set theory, so firstly, let us look at that, and more specifically the Naïve Comprehension Principle. As per the Naïve Comprehension principle, for every formula $A(x)$ there exists a set y such that y has objects that satisfy formula $A(x)$ [5]. Mathematically, there exists a set y

$$y = \{x : A(x)\} \tag{1}$$

i.e., $x \in y$ iff $A(x)$ holds true for x. This led to Russel's paradox when $A(x) = x \notin x$ [5]

$$y = \{x : x \notin x\} \tag{2}$$

i.e., $x \in y$ if $x \notin x$

When $x = y$, Equation 2 shows that $y \in y$ if $y \notin y$ which is a contradiction [5]. Thus we see that the birth of Russell's paradox is from the Naïve Comprehension principle [5]. Zermelo-Fraenkel proposed the Principle of Separation (Aussonderungsaxiom) instead of the Naïve Comprehension principle [6]. As stated below, there exists a set y [5]:

$$y = \{x : x \in b \wedge A(x)\} \tag{3}$$

i.e., $x \in y$ if $x \in b$ and $A(x)$ holds true
This means that for every formula $A(x)$ and every set b, there is a set y that has elements that satisfy formula $A(x)$ and that are in b [5].

It is to be noted that using the principle of separation, the size of the set is being restricted as there has to be another set that contains all the elements of the set that we form [6]. Thus the size of set y would not be greater than set b [6]. In order to evaluate how ZF set theory eliminates Russell's paradox, let us substitute $A(x) = x \notin x$ in Equation 3

$$y = \{x : (x \in b) \wedge (x \notin x)\} \tag{4}$$

i.e., $x \in y$ if $x \in b$ and $x \notin x$
When $x = y$, Equation 4 shows that $y \in y$ iff $y \in b$ and $y \notin y$ [5]
Note that this solves the problem as when we want y to be a set of all sets then as per the ZF, y does not exist because there can be no such set b that contains y (y should contain all sets, even b) [5].

C. Von Neumann–Bernays–Gödel Set Theory

Looking at a third solution, Von Neumann–Bernays–Gödel (NBG) is yet another set theory extension proposed by John von Neuman, Paul Bernays, and Kurt Gödel designed to eliminate Russell's paradox in a slightly different way than Zermelo Fraenkel's Set Theory. Under the axiomatic formulations of NBG, there are two constructs: Sets and Classes. All objects are classes. Classes that are members of other classes are sets. If a class is not a member of some other class i.e, it is not a set, then it is a proper class. Axioms from ZF are generalized to NBG by using classes in place of sets. There exists an additional axiom in NBG, the Axiom of *Limitation of Size*, which eliminates Russell's paradox entirely by claiming that some classes are too big to be sets. The Axiom of Limitation of Size dictates that a class A is a set if and only if there is no bijection between A and the class V of all sets [9]. Equivalently, A is a set if and only if it is smaller than V i.e, there exists no function that can map A to V. It is important to consider that V here, is the set-theoretic universe- a proper class of all sets without itself being a member of any other class. Thus, under this formulation, R, the class of all sets which are not members of themselves is too big of a class to be a set, as a bijection can be defined between R and V. Thus NBG's solution is to restrict R to being a proper class instead of a set [7]. This may also be perceived in the following manner. Let $R = x : x \notin x$. Let us assume that $R \in R$. Then by the definition of R which says its members should not contain themselves, $R \notin R$. Assume that $R \notin R$. This was previously contradictory in naïve set theory because by definition R had to contain R. However in the NBG axiomatic system, this is allowed because R is allowed to be a proper class and none of its members will contain themselves. In this manner, a contradiction is avoided. In addition to solving the paradox, the Axiom of Limitation of Size is so powerful, that it can replace or imply all set-existence axioms except for the Axiom of Power Set and the Axiom of Infinite Set, which is an indication of how large proper classes in NBG are, and how small the sets are compared to those classes [8].

D. Paraconsistent Set Theory

Paraconsistent set theory is undoubtedly the most interesting yet intuitive solution to the Russel's paradox. In classical logic, a proposition γ can be true or false but not both or neither. Russel's Paradox violates the principle of non-contradiction by simply being a statement which is both true and false. Paraconsistent logic provides a fresh perspective on this matter by proposing Paraconsistent set theory. This modification of the classical set theory allows for inconsistent statements to coexist by relaxing the principle of non contradiction thereby allowing full comprehension [13]. There are several paraconsistent models that have been designed on the top of paraconsistent logic that are undergoing research and refinement. These models ensure that while inconsistent statements are allowed, they do not compromise the mathematical system by ensuring non-triviality [13]. The key motivation for putting forward this theory is to not let a handful of contradictory statements affect the entire, seemingly working, mathematical system [13]. On the flip side, in traditional logic, one must either reject such inconsistent theories as irrational or embrace a seemingly mystical viewpoint. However, with the development of paraconsistent set theory models, there is now a middle ground where inconsistencies can be acknowledged without resorting to mysticism, as a rational representation of the contradictory nature of our world [14].

E. Fuzzy Set Theory

Fuzzy sets were first proposed by L.A. Zadeh in 1965. Defined as sets whose membership criteria is not binary and is rather defined by a continuum of grades, fuzzy sets have the capability of eradicating the Russel's paradox [16]. As a result, fuzzy sets have a continuous membership function, $f(x)$ that ranges between 0 and 1, representing the degree of membership of an element in the set. The closer the value of the membership function is to 1, the higher the degree of membership of that element in the set [16]. This function resembles a probability mass or probability density function of a random variable however, the key distinguishing factor is that fuzzy sets have no statistical relevance [16]. The primary aim for these sets is to capture the vagueness and impreciseness that exists in the real world and the natural language. Russell's paradox can be avoided by defining it as a set that has a membership function with a value of 0.5. This way a set can be simultaneously a part of the set R and not be a part of R [15]. Fuzzy set theory can also solve paradoxes of linguistic nature such as the sorites paradox. With vast applications in the domain of artificial intelligence, control systems, decision making, image processing, and natural language processing, fuzzy logic and sets can also be applied to mathematical systems to remove inconsistencies caused by Russel's paradox and other paradoxes of similar nature.

IV. Discussion

In addition to the solutions mentioned in the previous sections, there were other attempts in history to remove set-theoretic Russell-like paradoxes from Mathematics. Theories like category theory and homotopy theory were not included in this paper as they were complicated, extensive, and were devised not to just remove the paradox but to achieve more. Among the three proposed solutions we discussed in the previous section, the choice of the best solution (in our belief) involves a comparative analysis of all three theories based on their positive points and drawbacks. Conclusively, Zermelo-Frankael Set theory appears to be the simplest and best solution for Russell's paradox for the following reasons:

- There are a few drawbacks to adopting type theory as the best solution. Firstly, since type theory rejects all forms of self-reference, self-referring propositions that are non-paradoxical yet meaningful like 'This sentence consists of six words' hold no meaning within the type theory system at all [4]. More specifically type theory rejects all forms of impredicative definitions. An impredicative definition is a definition that quantifies or generalizes over the set that contains the thing being defined. The issue is that type theory bans all impredicative definitions, which is problematic as impredicative definitions are widely used in mathematics and computer science, one example of an impredicative definition is a the definition of the greatest lower bound which is defined as follows: For a set X, the greatest lower bound $GLB(X) = y$ if and only if, $\forall x \in X, y \leq x$ and if there exists z such that $\forall x \in X, z \leq x$ then $z \leq y$. This is a valid definition widely used in mathematics but type theory rejects all such definitions. In contrast, ZF set theory is not as restrictive as type theory.
- Another compelling issue pertaining to type theory is that for mathematics to work properly, there has to exist some procedure for making statements about all properties of an object, say x. However, since a proposition claiming something regarding all properties of x is itself a property of x, it is not allowed within the type theory universe [4]. Russell attempted to solve this problem with the Axiom of Reducibility which as suggested by literature was a controversial axiom subject to much criticism by Zermelo, Weiner, and others [10]. The Axiom of Infinity within type theory was another axiom that Russell himself was doubtful about and he wrote 'much of this theory is still inchoate, confused, and obscure' [11]. With self-reference forbidden and Russell's doubt about the axioms of his own theory, the theory of types did not seem to be the best fit to solve the problem at hand.
- Law of non-contradiction is an important law of logic and majority of mathematics, logic, physics and computer

science relies of this law. The relaxation of of this law as suggested by paraconsistent models of set theory is quite problematic, the law can't be disregarded as a whole so the question becomes which contradictory statements are valid and which contradictory statements do we remove? Do we exhaustively find all such paradoxical statements and then decide if we relax non contradiction on them or not? We cannot reasonable exhaustively identify all paradoxical statements. Furthermore we first stumble on to a paradoxical statement and then decide if this should be added to the class of statements for which we relax the law of non-contradiction, which isn't the desirable way to eliminate paradoxes, we need to come up with a model then rejects the paradox before it occurs.

- Unlike fuzzy set theory, ZF relies on classical logic. There are various weaknesses of fuzzy logic in comparison to classical logic, one of the bigger weakness is that a lot of results and proofs in fuzzy logic themselves uses classical logic, so the system cannot fully replace classical logic. Furthermore many mathematical results relies on binary truth value, the law of non contradiction and excluded middle are one of the main pillars of mathematics. Using fuzzy logic and therefore fuzzy set theory just introduce more inconsistency and confusion in our system.
- In contrast to Zermelo-Fraenkel axioms which only use one mathematical entity i.e, sets, Von Neumann–Bernays–Gödel set theory introduces a new mathematical construct - class, in addition to sets. This creates a layer of abstraction and nonhomogenity with a need to specify which entity should an axiom quantify over and is applicable to. This does not only require extra formal work but is also a bit unintuitive to understand and explain. ZF has the ability to eliminate the paradox with just one type of entity. Thus ZF is simpler and has less potential to confuse an average reader.
- Axioms of Zermelo-Fraenkel seem natural and intuitive enough to construct the set-theoretic universe. The need and definition of each axiom, especially the one designed to eliminate Russell's paradox, is explicit and clear. To a person with no mathematical background, we may just explain the axiom schema of separation by explaining that all sets have to be a subset of some other set.
- Even though a formal proof of the consistency of Zermelo-Fraenkel's axioms can not be produced, the philosophical non-rigorous justifications of Zermelo-Fraenkel axioms seem good and thorough enough to be acceptable and understandable [12]. It has been a long time since the theory has been formulated and no inconsistencies have been produced since, thus giving mathematicians and logicians confidence to proceed with the theory.
- Out of all of the three set theories, Zermelo-Fraenkel set theory has matured the most over time and has been the most prevalent choice of mathematicians and philosophers to this day. The primary reason is because of its simplicity and less-restrictive nature compared to other proposed set theories. With ample existing material, a good amount of critique, and enough work done in this domain, it would be a wise decision to use this theory as the best solution, as such, we propose ZF Set Theory as our 'best solution' to solve the paradox.
- Although there are several controversies with some axioms of ZF, specifically the axiom of choice which is equivalent to contradictory statements such as Zorn's lemma and Well-ordering theorem, or have paradoxes of its own one famous example being the Banach–Tarski paradox. But these problems have their own solutions in ZF and other systems. Also mathematicians have grown used to and accepted these controversial results of ZF over time and many have come up with their justification for them.

V. Conclusion

To conclude, it is essential to understand Russell's paradox due to the fact that it draws attention to the inherent inconsistencies that might occur when attempting to describe ideas like "set" in a manner that is both rigorous and understandable. The paradox is also responsible for inspiring the creation of formal systems of mathematics such as ZF and NBG set theory, which are intended to circumvent inconsistencies of this self-referential nature. Moreover, it shows that several seemingly harmless assumptions regarding sets might lead to inconsistencies in conclusions. Axiomatic set theory, which is the core of contemporary mathematics and was developed as a direct result of the paradox, is now considered the standard. Furthermore, the results of applying Russell's Paradox have quite far-reaching effects. In a nutshell, it teaches us that we cannot automatically assume that everything that we observe actually exists in the real world. This has significant ramifications for how we comprehend the very nature of reality itself. It also has significant repercussions for the field of mathematics. In particular, it teaches us that we cannot presuppose that all mathematical things actually exist in the real world. This indicates that we cannot take it as a given that the numbers we employ in our computations genuinely exist in the world outside of our imaginations, and as such, has significant repercussions for the field of physics. In particular, what this entails is that we can't assume it as a given that the physical laws are correct throughout all potential universes; rather, we need to test this hypothesis. Because of this, a number of physicists have come to the conclusion that the rules of physics are only the result of human perception, and that they may not, in fact, be accurate representations of the world. As such, Russell's paradox is one of the biggest paradoxes, with far-reaching effects even beyond the field of mathematics.

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