Mathematical Reasoning and Argumentation

مدرسه ریاضیات

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Contents

1 Introduction

1.1 Truth value

Definition A truth value is a value that indicates the truth or falsity of a statement or proposition. It is usually represented as ”true” or ”false.”

Example *The sky is blue.* When you look up during the day, the sky is indeed blue. So, the truth value of this statement is *true*. It’s like answering a question with *yes* or *no*.

Example *Cats can fly.* The truth value of this statement is *false* because cats cannot fly. So, a truth value helps us know whether a statement or idea is correct (true) or incorrect (false).

1.2 Statement

Definition A statement or proposition is a sentence that says something that can be either true or false, but not both. It’s like claiming something.

Example *The sun rises in the east.* This is a statement that can be checked to see if it’s true or false.

Example *Elephants are smaller than mice.* This is another statement, and we can check if it’s true or false.

In other words, a statement is any sentence that tells us something about the world and has a truth value, meaning it can be judged as true or false.

1.3 Logic

Definition Logic is a way of thinking that helps us determine if statements or propositions are true or false. It uses rules and principles to analyze arguments and reasoning.

Here's a simple way to think about it:

1. **Reasoning**: Logic helps us think clearly and make good decisions. For example, if you know that "All birds have feathers" and "A robin is a bird," logic helps you conclude that "A robin has feathers."
2. **Rules**: Logic follows specific rules to make sure our thinking is correct. These rules help us avoid mistakes in our reasoning.
3. **Truth**: Logic helps us understand what is true and what is false. It provides a structured way to figure out the truth value of statements or propositions.

Example If we know that *If it rains, the ground gets wet*, and we observe that *It is raining*, we can logically conclude that *The ground will get wet.*

Logic is used in many areas, like math, computer science, and everyday problem-solving, to ensure our conclusions are based on solid reasoning.

2 Connectives

Definition Connectives, also called logical operators, are words or symbols that connect statements to form more complex statements.

They help us understand the relationships between different propositions and determine the overall truth value.

2.1 Negation

Definition Negates a statement, making it the opposite of its original truth value.

Example *It is NOT raining.* If *It is raining* is true, then *It is NOT raining* is false, and vice versa.

2.2 Conjunction

Definition Combines two statements and is true only if both statements are true.

Example *It is raining AND it is cold.* This is true only if both *It is raining* is true and *It is cold* is true.

2.3 Disjunction

Definition Combines two statements and is true if at least one of the statements is true.

Example *It is raining OR it is sunny.* This is true if either *It is raining* is true or *It is sunny* is true, or if both are true.

2.4 Implication

Definition Indicates that if the first statement is true, then the second statement must also be true.

Example *IF it rains, THEN the ground gets wet.* This means if *It rains* is true, then *The ground gets wet* must also be true.

There are different ways of rephrasing an implication. Let's look at each of these with examples to make them clear. Keep in mind an implication has the form *If P, then Q*.

2.4.1 Contrapositive

Definition The contrapositive of the implication *If P, then Q* is *If not Q, then not P*. The contrapositive is always logically equivalent to the original statement, meaning they are either both true or false.

Example *If the ground is not wet (not Q), then it does not rain (not P).*

2.4.2 Converse

Definition The converse of the implication *If P, then Q* is *If Q, then P*. The converse is not necessarily logically equivalent to the original statement.

Example *If the ground gets wet (Q), then it rains (P).* This might not always be true because the ground could get wet for other reasons, like someone watering the garden.

2.4.3 Inverse

Definition The inverse of the implication *If P, then Q* is *If not P, then not Q*. The inverse is also not necessarily logically equivalent to the original statement.

Example *If it does not rain (not P), then the ground does not get wet (not Q)*. This might not always be true for the same reason as the converse—the ground could get wet in other ways.

Statement	**Example**
Original Statement	If P, then Q.
Contrapositive	If not Q, then not P.
Converse	If Q, then P.
Inverse	If not P, then not Q.

Table 1: Different ways of rephrasing an implication

2.5 Tautology

Definition A tautology is a statement that is always true, regardless of the truth values of its components.

Example *All birds have feathers.*

2.6 Contradiction

Definition A contradiction is a statement that is always false, regardless of the truth values of its components.

Example *The sky is blue AND the sky is not blue* is a contradiction because the sky can't be both blue and not blue at the same time.

3 Quantifiers

Definition Quantifiers are symbols or words in logic that express the quantity of specimens in the domain of discourse that satisfy a certain property.

They help us talk about *how many* things satisfy a given condition. There are two main types of quantifiers:

3.1 Universal Quantifier

Definition The universal quantifier, denoted by the symbol $\forall$, expresses that a statement is true for all elements in a set or domain. It means *for all* or *for every*.

Example $\forall x \in \mathbb{N}, x > 0$ reads as *for all x in the set of natural numbers, x is greater than 0.* This statement is true because all natural numbers are greater than 0.

3.2 Existential Quantifier

Definition The existential quantifier, denoted by the symbol $\exists$, expresses that a statement is true for at least one element in a set or domain. It means *there exists* or *there is*.

Example $\exists x \in \mathbb{N}, x > 0$ reads as *There exists an x in the set of natural numbers such that x is greater than 0.* This statement is true because there are natural numbers greater than 0.

3.3 Negation of Universal Quantifier

The negation of a universal quantifier is an existential quantifier with a negated statement. Consider the statement *All crows are black.* The negation of this statement is *Not all crows are black*, which can be rewritten as *There exists a crow that is not black.*

3.4 Negation of Existential Quantifier

The negation of an existential quantifier is a universal quantifier with a negated statement. Consider the statement *Some crows are black.* The negation of this statement is *No crows are black*, which can be rewritten as *For all crows, they are not black.*

4 Proofs

Proofs are logical arguments that demonstrate the truth of a statement or proposition. They are used in mathematics, logic, and other fields to establish that something is true based on previously known facts, axioms, and logical reasoning. Here's a breakdown of what proofs are and how they work:

Key Components of a Proof:

- **Axioms or Premises**: These are the starting points or assumptions that are accepted as true without proof. In mathematics, axioms are basic truths about numbers and operations.
- **Theorems or Propositions**: These are statements or claims that we want to prove. A theorem is often a significant result, while a proposition is usually a less important but still valid result.
- **Logical Reasoning**: This involves using rules of logic to connect axioms and previously proven theorems to the statement we want to prove.
- **Conclusion**: The end result of the proof, showing that the theorem or proposition is true.

4.1 Direct Proof

This type of proof involves directly showing that a statement is true by using logical steps from the premises to the conclusion.

Example Prove that if n is an even number, then n^2 is even.

Proof. Assume n is even. Then $n = 2k$ for some integer k. So, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even. □

4.2 Proof by Cases

Proof by cases involves breaking a proof into several distinct cases and proving that the statement holds in each case.

Example Prove that for any integer n, n^2 is either a multiple of 4 or one more than a multiple of 4.

Proof. Consider the integer n. We need to consider two cases based on the parity (whether n is even or odd) of n.

Case 1: n is even. If n is even, we can write $n = 2k$ for some integer k, then $n^2 = (2k)^2 = 4k^2$. Here, n^2 is a multiple of 4.

Case 2: n is odd. If n is odd, we can write $n = 2k + 1$ for some integer k. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Here, n^2 is one more than a multiple of 4.

Since in both cases n^2 is either a multiple of 4 or one more than a multiple of 4, the statement is proved. □

4.3 Proof by Contrapositive

Proof by contrapositive involves proving the contrapositive of the statement. The contrapositive of *If P, then Q* is *If not Q, then not P*. The contrapositive is logically equivalent to the original statement.

Example Prove that if n^2 is even, then n is even.

Proof. We will prove the contrapositive:

If n is not even, then n^2 is not even.

Assume n is odd, then $n = 2k + 1$ for some integer k.

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Here, n^2 is of the form $2m + 1$ for some integer m, which means n^2 is odd.

Since we have shown that *If n is odd, then n^2 is odd*, we have proven the contrapositive. Thus, the original statement *If n^2 is even, then n is even* is also true. □

4.4 Proof by Contradiction

This type of proof involves assuming the opposite of what you want to prove and showing that this assumption leads to a contradiction.

Example Prove that there is no largest prime number.

Proof. Assume there is a largest prime number, p. Consider the number $N = p! + 1$. N is not divisible by any prime number less than or equal to p, which means N is either prime itself or divisible by a prime larger than p. This contradicts the assumption that p is the largest prime. □

4.5 Proof by Induction

This type of proof is used to prove statements about all natural numbers. It involves two steps: the base case and the inductive step.

Example Prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Proof. **Base case:** For $n = 1$, the sum is $1 = \frac{1(1+1)}{2}$, which is true.

Inductive step: Assume the formula is true for some $n = k$. Then the sum of the first k numbers is $\frac{k(k+1)}{2}$. For $n = k + 1$, the sum is

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Thus, the formula holds for $k + 1$. □