# مدرسه رياضيات

Combinatorics

Summer 2024

## Introduction

just two lines on however we introduced combinatorics.

## Sum Rule

When you have to perform two tasks and there are a certain number of ways to perform the first task and a certain number of ways to perform the other task such that all ways are mutually exclusive then the sum rule states:

If there are n(A) ways to do A, and distinct from them, n(B) ways to do B, then the number of ways to do A or B is n(A) + n(B).

Suppose the elements of sets A and B represent the ways to do A and B respectively. Then the number of ways to do A or B is the number of elements in the union of A and B. If the sets A and B are disjoint, then the number of ways to do A or B is the sum of the cardinalities of A and B represented as #A + #B (where #X represents the cardinality/number of elements of any set X).

This rule can be generalized to more than two sets. If the elements of the sets  $A_1, A_2, \ldots, A_n$  represent the ways to do n tasks such that the sets are disjoint, then the number of ways to do the n tasks is the sum of the cardinalities of the sets  $A_1, A_2, \ldots, A_n$  represented as  $\#A_1 + \#A_2 + \ldots + \#A_n$  which can be written as  $\sum_{i=1}^n \#A_i$ .

#### Product Rule

When you have to perform two tasks and there are a certain number of ways to perform the first task, and for each of those ways, there are a certain number of ways to perform the second task, then the product rule states:

If there are n(A) ways to do A and n(B) ways to do B, then the number of ways to do A then B is  $n(A) \times n(B)$ . This is true if the number of ways for doing A and B are independent; the number of ways for doing B does not depend on how A is done.

Suppose the elements of sets A and B represent the ways to do A and B respectively. Then the number of ways to do A then B is the cardinality of the cartesian product of A and B,

represented as  $\#(A \times B) = \#A \times \#B$ .

This rule is also generalizable to more than two sets. If the elements of the sets  $A_1, A_2, \ldots, A_n$  represent the ways to do n consecutive tasks such that the number of ways for doing each task is independent of how the previous tasks are done, then the number of ways to do the n tasks is the cardinality of the cartesian product of the sets  $A_1, A_2, \ldots, A_n$  represented as  $\#(A_1 \times A_2 \times \ldots \times A_n) = \#A_1 \times \#A_2 \times \ldots \times \#A_n = \prod_{i=1}^n \#A_i$ .

## Inclusion-Exclusion Principle

The inclusion-exclusion principle is a counting technique that allows us to count the number of unique ways to do any number of tasks. Suppose you have two tasks A and B and you want to count the number of ways to do A or B. The sum rule states that the number of ways to do A or B is n(A) + n(B). However, if the ways to do A and B are not mutually exclusive, then the sum rule overcounts the number of ways to do A or B. The inclusion-exclusion principle corrects this overcounting.

If there are n(A) ways to do A, n(B) ways to do B, and  $n(A \cap B)$  ways to do both A and B, then the number of ways to do A or B is  $n(A) + n(B) - n(A \cap B)$ .

Suppose the elements of sets A and B represent the ways to do A and B respectively. Then the number of ways to do A or B is the number of elements in the union of A and B. If the sets A and B are not disjoint, then the cardinality of the union of A and B is the sum of the cardinalities of A and B minus the cardinality of the intersection of A and B represented as  $\#(A \cup B) = \#A + \#B - \#(A \cap B)$ .

This can be generalized to more than two sets. If the elements of the sets  $A_1, A_2, \ldots, A_n$  represent the ways to do n tasks, then the inclusion-exclusion principle states that:

Let  $A_1, A_2, \ldots, A_n$  be finite sets, then the cardinality of their union is given by:

$$\#(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^n \#A_i - \sum_{1 \le i < j \le n} \#(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \#(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} \#(A_1 \cap A_2 \cap \ldots \cap A_n)$$

#### Division Rule

When you have a task that appears to be done in n ways, but it turns out that for each of the n ways, there are d equivalent ways to do the task, then the division rule states:

If there are n(A) ways to do A, and for each of those ways, there are d equivalent ways to do the task, then the number of distinct ways to do the task is n(A)/d.

Suppose the finite set A represents the ways to do a task. Suppose this set is the union of n disjoint sets  $A_1, A_2, \ldots, A_n$  such that each set  $A_i$  contains d equivalent ways to do the task. Then the number of distinct ways to do the task, that is n, is the cardinality of the set A divided by d represented as #A/d.