Some Binomial Identities

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$P(n,r) :=$ Total number of permutations of r elements out of a set of n distinct elements.

A permutation of k items is an arrangement of k items where order matters

$(1,2,3) \neq (1,3,2)$ ← two distinct permutations of 3 objects.

$$P(n,r) := \underset{1^{st}}{n} \times \underset{2^{nd}}{(n-1)} \times \underset{3^{rd}}{(n-2)} \times \cdots \times \underset{r^{th}}{(n-r+1)} = \frac{n!}{(n-r)!}$$

where $k! = 1 \times 2 \times 3 \times \cdots k$

Eg $P(3,3) = 3 \times 2 \times 1 = 6$

$P(3,1) = 3$

$C(n,r) :=$ The number of ways to pick a combination of r elements out of a set of n elements

A combination is an arrangement in which the order does not matter.

$C(n,r) = \binom{n}{r} :=$ Count number of way to pick r elements out of n where order does not matter.

$$:= \frac{\text{Count number of way } P(n,r) \text{ where order does matter}}{\text{Total Permutations on } r \text{ elements}}$$

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}$$

$\binom{n}{r}$ - binomial coefficients

Some identities involving binomial coefficients

1) $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$ Pascal's Identity

LHS: Choose k+1 elements out of a set of n+1 elements

RHS: Lets say an element $n \in S$ of n+1 elements
Any subset of S of k+1 elements would either

contain n
then since one element out of n+1 is already in the k+1 set
there are $\binom{n}{k}$ choices left for the subset

or not contain n
then this means I have to choose the k+1 set out of n elements
$\Rightarrow \binom{n}{k+1}$

2) $\binom{n}{k} = \binom{n}{n-k}$

$\binom{n}{k}$ is the number of ways of choosing k elements out of n
is the same as choosing what n-k elements to leave out of n
$= \binom{n}{n-k}$

3) $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$

RHS: Number of ways of choosing n elements out of a set of 2n elements

LHS: $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$

Given a set of 2n elements. Divide the set into two sets each of n elements.

Want ways of On set out of 2n set

Count the number of ways of choosing 0 elements from first set and n elements from 2nd set

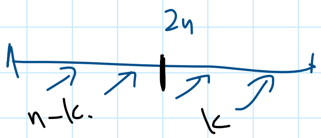
$$\binom{n}{0}\binom{n}{n}$$

Count the number of ways of choosing 1 element from set 1
n-1 elements from set 2

$$\binom{n}{1}\binom{n}{n-1}$$

and so on.

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$



$$= \binom{2n}{n}$$

4) $k\binom{n}{k} = n\binom{n-1}{k-1}$

Consider you want to pick a team of k members out of n players. You also want to pick a captain out of n players. You also want to pick a captain of the team

ways to pick a captain of k team and k out of n team.

LHS: $k\binom{n}{k}$

First pick a k-team out of n players

number of ways $\binom{n}{k}$

pick a captain out of k members

number of ways k

RHS: $n\binom{n-1}{k-1}$

pick a captain out of n players — n ways

since the captain is already chosen

we have to choose k-1 other members out of n-1 players

$\binom{n-1}{k-1}$ ways.

$$\boxed{k\binom{n}{k} = n\binom{n-1}{k-1}}$$

∎

Proof: $2^n = \sum_{k=0}^{n}\binom{n}{k} = \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$

$\binom{n}{k} = \left|\binom{X}{k}\right|$, $|X| = n$

where $\binom{X}{k}$ = The set of all $\underbrace{\text{k-element subsets}}_{\text{k-combination}}$

where $\binom{X}{k}$ = The set of all $\underbrace{\text{k-element subsets}}_{\text{k-combination}}$ of X.

2^n = The total number of subsets of X where $|X|=n$

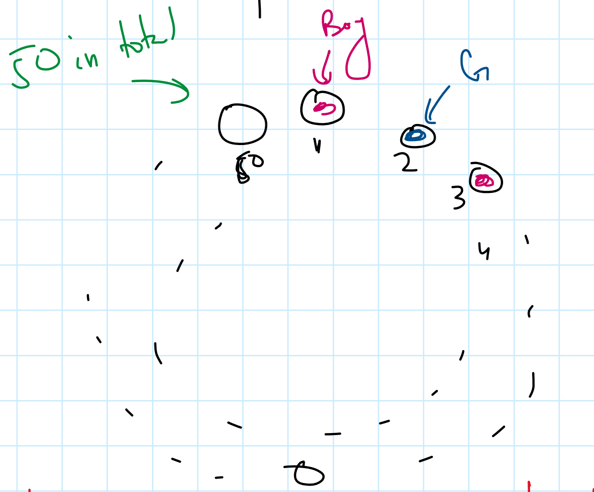
$\quad$ = The total number of subsets of size 1 of X $= \binom{n}{1}$

$\quad$ + The total number of subsets of size 2 of X $= \binom{n}{2}$

$\quad$ + $\vdots$

$\quad$ + The total number of subsets of size n of X $= \binom{n}{n}$

Show that when 25 girls & 25 boys are seated in a circle. There is some person with boys on both sides



Want to show there is some segment in the arrangement with

B G B

or. B B B

Try to construct an arrangement that avoids it

Suppose G in the first place

50^{th} place

B G G

at 50

G G G or B

50th place

B G G
2nd place

G G G or B
50 1st in 2nd place.

Case 1

$\frac{B}{50}$ $\frac{G}{1^{st}}$ $\frac{G}{2^{nd}}$ or

Case 2

$\frac{G}{50^{th}}$ $\frac{G}{1^{st}}$ $\frac{B \text{ or } G}{2^{nd}}$

47 places to cover

Case 1 ~~at most~~ 23 girls | Case 2 23 or 22 girls

~~at least~~ 24 boys | 24 or 25 boys

In both cases we have less girls than boys

∴ some place with BGB or BBB

To avoid BGB or BBB all together, if there is a girl G sitting, one of her neighbors must be another girl.

12 pairs 12 pairs last or sits alone

GG GG G BGB

50

If GGG by symmetry BBB.