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Embeddings of graphs

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Abstract

We survey some recent results on graphs embedded in higher surfaces or general topological spaces.

1. Introduction

A graph G is *embedded* in the topological space X if G is represented in X such that the vertices of G are distinct elements in X and an edge in G is a simple arc connecting its two ends such that no two edges intersect except possibly at a common end.

Graph embeddings in a broad sense have existed since ancient time. Pretty tilings of the plane have been produced for aestetic or religious reasons (see e.g. [7]). The characterization of the Platonic solids may be regarded as a result on tilings of the sphere. The study of graph embeddings in a more restricted sense began with the 1890 conjecture of Heawood [8] that the complete graph K_n can be embedded in the orientable surface S_g provided Euler's formula is not violated. That is, $g \geqslant \frac{1}{12}(n-3)(n-4)$. Much work on graph embeddings has been inspired by this conjecture the proof of which was not completed until 1968. A complete proof can be found in Ringel's book [12].

Graph embedding problems also arise in the real world, for example in connection with the design of printed circuits. Also, algorithms involving graphs may be very sensitive to the way in which the graphs are represented. A certain graph embedding may be a convenient representation.

Graph embeddings also arise in harmonic analysis on surfaces. In solving the Laplace equation $\Delta u = 0$ on a surface, an approximative solution may be found by considering a discrete version of the equation on an appropriate graph embedded on the surface. Significant theoretical results on the interplay between the two types of problems have been established by Kanai [10].

In these lectures we concentrate on graph embeddings on compact 2-dimensional surfaces. Such embeddings play a central role in the deep theory on minors by Robertson and Seymour [13]. The interplay between minors and embeddings is

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emphasized in the survey [18]. Here we survey some recent graph theoretic investigations on the Jordan curve theorem and some generalizations, the classification of surfaces, polynomial time algorithms for finding the genus of some graphs (i.e. the smallest genus of an orientable surface that admits an embedding of the graph), and the NP-completeness of the graph genus problem (even the triangulation problem) in general. Finally, we comment on tilings of surfaces with applications to symmetry properties of surfaces.

2. The Jordan Curve Theorem

As 2-dimensional surfaces are locally homeomorphic to a disc, the study of higher surfaces begins with the Euclidean plane $\mathbb{R}^2$. Two of the most fundamental results on the Euclidean plane are the Jordan Curve Theorem and the Kuratowski Theorem. Before we state them we need some definitions. A *simple arc* in a Hausdorff topological space X is the image of a continuous 1-1 function $f: [0,1] \to X$. A *simple closed curve* is defined analogously except that now $f(0)=f(1)$. A set $Y \subseteq X$ is said to be *arcwise connected* if, for each pair p, q of elements in Y, there is a simple arc in Y from p to q. An arcwise *connected component* of X is a maximal arcwise connected subset. All graphs considered are finite.

Theorem 2.1 (The Jordan Curve Theorem). *If J is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus J$ has precisely two arcwise connected components.*

Theorem 2.2 (Kuratowski's Theorem). *A graph G can be embedded in $\mathbb{R}^2$ if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

All proofs of the easy part of Theorem 2.2, namely that K_5 and $K_{3,3}$ cannot be embedded in $\mathbb{R}^2$, depend on Theorem 2.1 (see e.g. [15]). Although Theorem 2.1 seems intuitively obvious, it is fascinatingly difficult to prove rigorously from first principles. There are several proofs in the literature. In [16] a simple graph theoretic proof based on the easy part of Kuratowski's theorem is presented and in [17] it is shown how the two theorems are intimately related in more general Hausdorff topological spaces X. We say that a subset Y in X *separates* X if $X \setminus Y$ is not arcwise connected.

Theorem 2.3. *Let X be an arcwise connected Hausdorff topological space which is not a simple closed curve. Suppose that no simple arc separates X. Then the following statements are equivalent:*

(a) *Every simple closed curve separates X.*
(b) *Every simple closed curve separates X into precisely two components.*
(c) *$K_{3,3}$ cannot be embedded into X.*
(d) *Neither K_5 nor $K_{3,3}$ can be embedded into X.*

Theorem 2.3 says, roughly speaking, that the Jordan Curve Theorem holds iff the easy part of Kuratowski's theorem holds. It is somewhat surprising that (a) and (b) are equivalent. The equivalence of (c) and (d) shows that K_5 is redundant in Theorem 2.3 while K_5 is essential in Theorem 2.2. The explanation is that the former theorem characterizes certain topological spaces, while the latter characterizes certain graphs.

Theorem 2.3 shows that a Hausdorff topological space X satisfying the Jordan Curve Theorem must be 'plane like' since X only admits embeddings of plane graphs. Every open connected subset of $\mathbb{R}^2$ satisfies the assumptions in Theorem 2.3 and in [17] is was asked if a topological space X satisfying the requirements of Theorem 2.3 must be homeomorphic to a subset of the plane. However, in order to obtain an affirmative answer, it will be necessary to impose further conditions on X, as the examples below show.

Let X_0 be obtained from the closed half plane $X_1 = \{(x, y) \mid y \geqslant 0\}$ by identifying the points on the x-axis into a single point p_0. Then open discs in $X_1 \backslash \{p_0\}$ and the open sets containing the x-axis form a basis of open sets in X_0. Equipped with this topology, X_0 is not a metric space but it satisfies the conditions of Theorem 2.3.

The *long line* L, which is a standard example in topology, is defined as follows: Let α be the smallest uncountable ordinal and let M consist of all ordinals $<\alpha$. Let M be equipped with the topology arising from its natural partial ordering. Then L is the product space $M \times [0, 1[$. Intuitively, L is obtained by pasting uncountably many intervals $[0, 1]$ together. The *big plane* is the product $L \times L$. It is not difficult to show that $L \times L$ (in fact every open arcwise connected subset of $L \times L$) satisfies the requirements of Theorem 2.3.

We mention a couple of results related to the Jordan Curve Theorem. Clearly, a circle partitions the plane into precisely two regioins. Therefore the following generalizes the Jordan Curve Theorem.

Theorem 2.4 (The Jordan–Schönflies Theorem). *If f is a homeomorphism of a circle in $\mathbb{R}^2$ onto a simple closed curve in $\mathbb{R}^2$, then f can be extended to a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$.*

A simple graph theoretic proof is given in [16].

If F is a closed set in $\mathbb{R}^2$ and Ω is a connected component of $\mathbb{R}^2 \backslash F$, then a point $p \in F$ is *accessible* from Ω if there exists a simple arc J from a point q in Ω to p such that $J \cap F = \{p\}$. Theorem 2.4 shows that each point on a simple closed curve F in $\mathbb{R}^2$ is accessible from both components of $\mathbb{R}^2 \backslash F$. In [19] there is a simple graph theoretic proof of the following result of Schönflies which is a converse of Theorem 2.1.

Theorem 2.5. *If F is a compact set in $\mathbb{R}^2$ such that $\mathbb{R}^2 \backslash F$ has precisely two connected components and each vertex of F is accessible from both components of $\mathbb{R}^2 \backslash F$, then F is a simple closed curve.*

Let us say that a compact set F in $\mathbb{R}^2$ is *well-behaved* if $\mathbb{R}^2 \backslash F$ has finitely many components and each point of F is accessible from at least one component of $\mathbb{R}^2 \backslash F$. Well-behaved compact sets F may have a very complicated structure even if $\mathbb{R}^2 \backslash F$ is connected: They include many of the so-called fractals which have been studied extensively in recent years. Let us say that a compact set F is *very-well-behaved* if each point of F is on the boundary of at least two components of $\mathbb{R}^2 \backslash F$ and is accessible from each such component. The very-well-behaved compact sets have a surprisingly simple structure [19].

Theorem 2.6. *A compact set F is very-well-behaved if and only if F is a bridgeless plane graph.*

Note that Theorem 2.5 is a special case of Theorem 2.6.

3. Higher surfaces

A *surface* is a connected compact Hausdorff topological space which is locally homeomorphic to an open disc (and hence to $\mathbb{R}^2$). A surface S can be constructed as follows: Take a collection of pairwise disjoint triangles (and their interior) in $\mathbb{R}^2$, each of side length 1. Identify each side in each triangle with precisely one side in another triangle. This results in a topological space, and the sides of the triangles form a graph G. Now if S is connected (i.e. G is connected) and S is locally homeomorphic to a disc at each vertex of G (i.e. G is 'locally a wheel'), then S is a surface. We say that S is a *triangulated surface* and that G *triangulates* S.

[16] contains a graph theoretic proof of Theorem 3.1 below which is the most difficult step in the *Classification Theorem* (Theorem 3.2 below).

Theorem 3.1. *Every surface is homeomorphic to a triangulated surface.*

Proof. The idea behind the proof is simple: For each point p on S there is a neighborhood around p which is homeomorphic to a disc. In that disc we consider a circle C_p whose interior int C_p contains p. As S is compact, S is the union of finitely many of the sets int C_p. If any two of the C_p-s have only finite intersection it is easy to extend the union of the C_p-s to a triangulation of S. In [16] repeated use of the Jordan–Schönflies theorem is used to modify the C_p-s such that any two of them have finite intersection. □

Adding a handle to the sphere S_0 means that we delete two disjoint discs (bounded by circles) and identify their boundaries in such a way that the clockwise orientation in one of them corresponds to the anticlockwise orientation in the other. Adding a *crosscap* to S_0 means that we delete a disc and identify diametrically opposite points on the boundary. If we add g handles (respectively k crosscaps) to S_0 we obtain the surfaces S_g (respectively N_k).

Theorem 3.2 (The Classification Theorem). *Each surface is homeomorphic to precisely one of S_g ($g \geqslant 0$) or N_k ($k \geqslant 1$).*

In view of Theorem 3.1 it is sufficient to prove Theorem 3.2 for triangulated surfaces. A short graph theoretic proof is given in [16]. Since a surface can be viewed as a union of triangles (and their interior), we may embed graphs on surfaces such that all edges are polygonal arcs. A standard argument shows that every graph G which can be embedded on a surface S can be (and will be in what follows) embedded in that nice way. Then there is no topological difficulty in speaking of the *cyclic ordering* of the edges incident with a vertex, and it is clear that the number f of regions (faces) of $S \backslash G$ is finite. Such a face will be called a *2-cell* if it admits no embedding of $K_{3,3}$ (or, equivalently, if it is homeomorphic to a disc). The proof in [15] of Theorem 3.2 includes the following.

Theorem 3.3 (Euler's formula). *If G is a connected graph with n vertices and e edges and f faces on S (which is S_g or N_k), then*

$$n-e+f \geqslant 2-2g \quad (\text{if } S=S_g)$$

or

$$n-e+f \geqslant 2-k \quad (\text{if } S=N_k).$$

Equality holds if every face is a 2-cell.

A simple counting argument shows that Euler's formula implies

$$e \leqslant 3n-6+6g$$

or

$$e \leqslant 3n-6+3k$$

with equality when G triangulates S_g or N_k. Thus such a triangulation has too many edges in order to be embeddable in $S_{g'}$ or $N_{k'}$ when $g' < g$ (or $k' < k$). Therefore, all the surfaces $S_0, S_1, \ldots, N_1, N_2, \ldots$ are non-homeomorphic except that we still have to show that S_g and N_{2g} are not homeomorphic. This can be demonstrated by showing that no S_g contains a Möbius strip (a short proof of which is indicated in [16]) while N_k clearly does. S_g is called *orientable* while N_k is called *non-orientable*. In the following we focus on the orientable surfaces.

4. The rotation principle and the graph genus problem

The *genus* $g(G)$ of a graph G is the smallest g such that G can be embedded in S_g. If G has n vertices and e edges, then $e \leqslant 3n-6+6g$, by Euler's formula, and hence

$$g(G) \geqslant \tfrac{1}{6}(e-3n+6)$$

with equality iff G triangulates an orientable surface. In 1890 Heawood [8] claimed that,

$$g(G)=\lceil \tfrac{1}{6}(e-3n+6\rceil=\lceil \tfrac{1}{12}(n-3)(n-4)\rceil$$

for complete graphs. This claim, which became known as the *Heawood conjecture* was settled by Ringel and Youngs, see [12]. The fact that it took almost 80 years to determine the genus for the complete graphs indicates that the problem for graphs in general is hard. The *graph genus problem* can be formulated as a decision problem as follows: Given a graph G and a natural number k. Is $g(G)\leqslant k$?

Instead of starting with a surface S and drawing a graph G on S, we start with G and use it as a 'skeleton' for a surface. This fundamental idea, which is called the *Heffter–Edmonds–Ringel rotation principle* is formalized as follows: Let G be a connected graph with vertices $v_1, v_2, \ldots, v_n$. For each $i=1,2,\ldots,n$, let π_i be a cyclic permutation of the edges incident with v_i. π_i will also be called a *clockwise ordering* around v_i. The collection $\Pi=\{\pi_1, \pi_2, \ldots, \pi_n\}$ will be called a *combinatorial embedding* of G. We define a *Π-facial* walk as follows: Let $a_1=v_iv_j$ be any edge. Put $\pi_j(a_1)=a_2=v_jv_k$. Put $a_3=\pi_k(a_2)$ etc. Then the closed sequence W: $v_ie_1v_je_2v_ke_3\cdots$ is a Π-facial walk. In this way each edge is in two Π-facial walks (which may coincide). Let f denote the number of Π-facial walks. Then define the *Π-genus* $g(\Pi, G)$ by the formula

$$n-e+f=2-2g(\Pi, G).$$

We now define an embedding of G in a surface S_g as follows: For each facial walk W of G we consider a convex polygon in the plane with the same number of sides as there are edges in W. We assume that distinct facial walks correspond to polygons which (together with their interior) are disjoint. Then we take the union of these polygons (and their interior) and identify sides corresponding to the same edge. This results in a surface S. One can show that S does not contain a Möbius strip and hence $S=S_g$ for some g. Since G is a 2-cell embedding of S_g Euler's formula implies

$$n-e+f=2-2g.$$

Hence $g=g(\Pi,G)$. So, a combinatorial embedding results in an embedding of G in $S_{g(\Pi,G)}$. Conversely, if G is embedded in S_g, then

$$n-e+f\geqslant 2-2g.$$

We can define an embedding Π simply by letting π_i be the clockwise orientation in S_g around vertex v_i. (This makes sense as S_g may be obtained by pasting triangles in the plane together.) If f' is the number of Π-facial walks, then clearly $f\leqslant f'$. Hence $g(\Pi, G)\leqslant g$. This shows that $g(G)$ is the minimum of all $g(\Pi, G)$ taken over all combinatorial embeddings. Therefore, from now on, 'embedding' will simply mean 'combinatorial embedding' and now it makes sense to speak about the computational complexity of the graph genus problem.

5. NP-completeness of the graph genus problem and the graph triangulation problem

Garey and Johnson [5] asked if the graph genus problem is NP-complete. (For fixed genus, the problem is in P [3, 13].) An affirmative answer was given in [20].

Theorem 5.1. *The graph genus problem is NP-complete.*

If G is a graph, then $\alpha(G)$ is the maximum number of pairwise non-adjacent vertices in G. One of the fundamental NP-complete problems is the following: Given a connected graph G and a natural number k, is $\alpha(G) \geqslant k$? (See [5].) This problem was in [20] reduced to the graph genus problem. For each edge pq in G we delete pq and add instead a cycle of length $3n^2$ (where n is the number of vertices of G) and join it completely to both p and q. It is shown in [20] that the resulting graph G' has genus $e-n+1$. (Intuitively, each of the new cycles outside a fixed spanning tree in G must 'go around a handle'). So, from a genus point of view, G' is easy to handle. But a slight modification of G' results in a complicated graph G''. G'' is obtained from G' by adding a new vertex v and joining v to precisely one vertex of each of the new cycles. It is shown in [20] that

$$g(G'') = e - \alpha(G).$$

As G'' is obtained from G in polynomial time, the graph genus problem is NP-complete.

In 1976 G. Ringel raised another fundamental question on the graph genus: When does a graph G triangulate a surface? This is (at least in some sense) answered in [22].

Theorem 5.2. *The following three problems are NP-complete. Given a graph G: Does G triangulate a surface? Does G triangulate an orientable surface? Does G triangulate a non-orientable surface?*

The proof of the Heawood Conjecture is complicated and is split up into 12 different cases [12]. A simple answer to Ringel's question (which now seems hopeless in view of Theorem 5.2) might give a unified and simpler solution of the Conjecture, since a graph G triangulates an orientable surface if and only if $e = 3n - 6 + 6g(G)$. Hence Theorem 5.2 also implies Theorem 5.1.

The problem of deciding whether a graph G has a Hamiltonian cycle is another fundamental NP-complete problem [5]. Garey et al. [4] showed that the problem remains NP-complete even for cubic planar graphs. This is used in [22] to show that it is NP-complete to decide if a cubic bipartite graph G has two Hamiltonian cycles whose intersection is a perfect matching. Such two Hamiltonian cycles are called *compatible*. Now suppose G is a cubic bipartite graph with partite sets A and B. Let G' be a copy of G with partite sets A', B'. Form the disjoint union $G \cup G'$, add an edge from each vertex in G to the corresponding vertex in G', add four new vertices x_1, x_2, x'_1, x'_2

such that x_i is joined completely to G, and x_i' is joined completely to G' for $i=1,2$. Then contract all edges between A and A'. It is shown in [22] that the resulting graph triangulates a surface S iff G has two compatible Hamiltonian cycles. Furthermore, S must be orientable. This proves the NP-completeness of the two first questions in Theorem 5.2.

6. The genus of special classes of graphs

As previously mentioned, any triangulation (of an orientable surface) is a minimum genus embedding. If a triangle free graph with n vertices and e edges is embedded in S_g, then $e \leqslant 2n-4+4g$ with equality holding when all facial walks are 4-cycles. A large class of such embeddings were described by Pisanski [11].

Theorem 6.1. *Let G and H be bipartite graphs each of which is the union of r perfect matchings. Then the Cartesian product of G and H has genus $1+pm(r-2)/4$ where p and m are the number of vertices of G and H, respectively.*

Theorem 6.1 generalizes the result of A.T. White (see [6]) describing the genus of the Cartesian product of even cycles.

The methods used in Theorem 6.1 and in the proof of the Heawood conjecture are particularly applicable to graphs with a high degree of symmetry or other special properties. We shall now describe a method which applies to a large class of graphs with no special structure.

We shall classify a simple closed curve J on a surface S according to what happens when we delete J from S. $S\backslash J$ has at most two arcwise connected components. If one of them is homeomorphic to a disc, then J is *contractible*. Otherwise, J is *non-contractible*. There are two types of non-contractible simple closed curves, namely those which separate S and those which do not.

We shall translate these concepts into our combinatorial framework. If C is a cycle in a connected graph G and Π is an embedding of G, then we choose a positive orientation of C and now it makes sense to say that an edge not in C but incident with a vertex in C goes to the *right* or *left* of C. Let $G_r(C,\Pi)$ denote C together with all those paths in G which have no intermediate vertex in common with C and which start with a vertex of C and an edge on the right side of C. We define $C_l(C,\Pi)$ analogously except that we interchange 'right' by 'left'. Now C is *Π-contractible* if $G_r(C,\Pi)\cap G_l(C,\Pi)=C$ and one of $G_r(C,\Pi), G_l(C,\Pi)$ has Π-genus zero. Since the genus is defined by Euler's formula, it can be checked in linear time if a given cycle C is Π-contractible. If $G_r(C,\Pi)\cap G_l(C,\Pi)=C$, and C is not contractible, then C is *non-contractible* and *separating*. Finally, if $G_r(C,\Pi)\cap G_l(C,\Pi)\neq C$, then C is *non-contractible, non-separating*. (Note that when we here refer to separability we are thinking of the surface, not the graph.) The *edgewidth* $\mathrm{ew}(G,\Pi)$ is defined as the length of a shortest non-contractible cycle. If no such cycle exists, we put $\mathrm{ew}(G,\Pi)=\infty$. An embedding Π is an

LEWembedding (large-edge-width-embedding) if all Π-facial walks have length $<\mathrm{ew}(G,\Pi)$. The concept of LEW-embedding is justified by the following [14].

Theorem 6.2. *Let Π be an LEW-embedding of a connected graph G. Then $g(\Pi, G) = g(G)$. Furthermore, if G is 3-connected, then G has no other embedding (except the one obtained by reversing all local orientations of Π) of genus $g(G)$.*

It turns out that LEW-embeddings share many properties with planar embeddings as demonstrated in [14]. For example, a classical result of Whitney [25] says that any planar (i.e. genus zero) embedding of a 2-connected graph G can be obtained from any other planar embedding by a sequence of so-called 2-switchings (which means that we reflect a subgraph attached to the rest of the graph by only two vertices). In [14] it is shown that, if a 2-connected graph G has a LEW-embedding Π, then every minimum-genus-embedding of G can be obtained from Π by a sequence of 2-switching of planar subgraphs.

Theorem 6.2 shows that if we want to find a minimum genus embedding of a connected graph, it may be feasible to look for an LEW-embedding. The first question that arises is: Given an embedding Π of a connected graph G, is Π an LEW-embedding? This question can be answered in polynomial time. For a fixed cycle C, we can find $G_r(C,\Pi)$ and $G_l(C,\Pi)$ in linear time and we can thus decide in linear time whether C is contractible or non-contractible. But we still have to find a shortest non-contractible cycle. This can be done by the general algorithm in Theorem 6.3 below.

If $\mathscr{C}$ is a collection of cycles in a graph G we say that $\mathscr{C}$ satisfies the *3-path-condition* if the following holds: If P_1, P_2, P_3 are three internally disjoint paths from p to q in G, and if $P_1 \cup P_2 \in \mathscr{C}$, $P_2 \cup P_3 \in \mathscr{C}$, then $P_1 \cup P_3 \in \mathscr{C}$.

Theorem 6.3. *Let G be a connected graph and $\mathscr{C}$ a collection of cycles such that the cycles not in $\mathscr{C}$ satisfy the 3-path-condition. Suppose further that membership in $\mathscr{C}$ can be tested in polynomial time. Then there exists a polynomial time algorithm for finding a shortest cycle in $\mathscr{C}$.*

Proof. The algorithm is simple: Pick a convex v and grow a breadth — first spanning tree T_v from v. For each edge a not in T_v, let $C(a,v)$ be the unique cycle in $T_v \cup \{a\}$. Then the shortest cycle in $\mathscr{C}$ of the form $C(a,v)$ (where v runs through all vertices of G and a runs through all edges not on T_v) is a shortest cycle in $\mathscr{C}$. $\square$

Theorem 6.3 applies to many classes of cycles, for example odd cycles in a graph and non-balanced cycles in a directed graph (*balanced* means that half of the edges have the same direction). Also, it can be used to find a shortest non-contractible cycle and a shortest non-contractible, non-separating cycle in an embedded graph. If the surface is non-orientable it can be used to find a shortest *Möbius cycle* (i.e. a cycle where left and right interchange, when we transverse the cycle). We have defined combinatorial embeddings only of orientable surfaces, but it is easy to extend the combinatorial framework to include the non-orientable case as well. The results on the properties of LEW-embeddings in [14] also imply the following.

Theorem 6.4. *There exists a polynomial time algorithm for deciding if a 2-connected graph G has an LEW-embedding.*

The *face-width* $\mathrm{fw}(\Pi, G)$ of a Π-embedded graph G is topologically defined as the minimum number of intersections of G with a non-contractible simple closed curve on the surface, Combinatorically, $\mathrm{fw}(\Pi, G)$ is defined as follows: First, subdivide every edge. Then for each facial walk we add a new vertex and join it to all vertices of the facial walk such that G and Π are extended to a triangulation Π' (possibly with multiple edges) H of the same surface. Then $\mathrm{fw}(\Pi, G) = 2\mathrm{ew}(\Pi', H)$. By Theorem 6.3. we can find $\mathrm{ew}(\Pi', H)$ and hence $\mathrm{fw}(\Pi, G)$ in polynomial time.

Robertson and Vitray (see [24]) conjectured that every embedding which is not of minimum genus must have face width at most 2. While this is true for planar graphs [14], it was in [14] shown that toroidal graphs (graphs of genus 1) may have non-minimum-genus embeddings of face-width 4. Then Robertson and Vitray replaced 2 by 10^{10} in their conjecture, but the new conjecture was disproved by Archdeacon [1].

If Π is an embedding of a graph G in the projective plane and J is a non-contractible simple closed curve intersecting G in $\mathrm{fw}(\Pi, G)$ vertices, then we may 'cut' the projective plane along J, thereby transforming it to a closed disc. G and Π are transformed into a planar embedding Π' of a graph H. It is easy to see that we can transform H and Π' into an embedding Π'' of G on an orientable surface with $\lfloor \frac{1}{2}\mathrm{fw}(\Pi, G) \rfloor$ handles. Huneke, Richter and Robertson proved, surprisingly, that Π'' is always minimum genus embedding. Combining this with Theorem 6.3 we get the following.

Theorem 6.5. *There exists a polynomial time algorithm for finding a minimum genus embedding of a graph embedded in the projective plane.*

7. Tilings and symmetry properties of surfaces

The set of homeomorphisms of a fixed surface S onto itself is clearly an infinite group. The torus can be embedded in $\mathbb{R}^2$ such that the rotations of $2\pi/k, 4\pi/k, 6\pi/k, \ldots$ around a fixed axis form a homeomorphism group (i.e. a subgroup of the full homeomorphism group) of order k. Thus the torus S_1 has infinitely many finite homeomorphism groups. The same holds for the plane S_0, the projective plane N_1 and the Klein bottle N_2, but for no other surface.

Theorem 7.1 (Hurwitz' theorem). *Each finite homeomorphism group of S_g $(g \geq 2)$ has order at most $168(g-1)$.*

The study of homeomorphism groups of S_g reduces to a graph problem by the following result of Tucker [23].

Theorem 7.2. *Let A be a finite group of homeomorphisms of S_g. Then there is some Cayley graph G of A which can be embedded on S_g such that each isomorphism of G induced by a left multiplication of an element of A can be extended to a homeomorphism in A of S_g onto itself.*

Theorems 7.1 and 7.2 suggest that perhaps there are only finitely many Cayley graphs of each fixed genus $\geqslant 2$. However, Wormald came up with an infinite family of Cayley graphs of genus 2 (see [6, p. 303]). Then Tucker conjectured that there are only finitely many Cayley graphs of each fixed genus $\geqslant 3$. This conjecture was extended to vertex-transitive graphs by Babai (see [6, p. 303]). This conjecture was verified independently by Babai [2] and the present author [21].

Theorem 7.3. *For each fixed $g \geqslant 3$ every vertex-transitive graph of genus g has less than $10^{10}g$ vertices.*

The proofs in [2, 21] are different and have different applications. They both imply Hurwitz' theorem (except for the multiplicative constant), also a non-orientable version which we have not seen explicitly in the literature.

Theorems 7.1 and 7.2 show that the double-torus S_2 has a remarkable property: It is the only orientable surface which has only finitely many homeomorphism groups, but infinitely many minimum genus embeddings of Cayley graphs. [21] contains a list of graphs which include all (but finitely many) vertex-transitive graphs of each fixed genus (including those on the double-torus).

The proof in [21] of Theorem 7.3 is based on investigations of tilings of surfaces. A *(d, m)-tiling* of a surface S is a d-regular graph G (i.e. a graph where all vertices have degree d) embedded on S such that all facial walks are cycles of length m. If G has n vertices and $S = S_g$ or N_k, then Euler's formula implies that

$$\left(d\left(1-\frac{2}{m}\right)-2\right)n=4(g'-1)$$

where $g' = g$ or $g' = k/2$.

For fixed $g' \neq 1$ there are only finitely many possibilities for G. For $g' = g = 0$ the possibilities can easily be worked out and we get the classical characterization of the Platonic solids. The cases $g = 1$ (the torus) and $k = 2$ (the Klein bottle) are particularly interesting. For fixed d and m, there are only finitely many possibilities for G except in the three cases $(d, m) = (3, 6), (4, 4)$ or $(6, 3)$. Such a tiling is called *regular* if it satisfies the following: If $d = 4$ and v is any vertex of the tiling, then the four facial 4-cycles containing v together contain nine vertices. If $d = 3$, then the girth of G is 6, and if $d = 6$, then G is locally planar, i.e., any vertex together with its six facial 3-cycles induce a planar subgraph. In [21] all regular tilings of the torus or the Klein bottle are characterized.

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