The Utilities Problem

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An old, yet ever popular recreational problem is the utilities problem, also known as the water, gas, and electricity problem, which is usually stated as follows:

> Try to install water, gas, and electrical lines from utilities W, G, and E to each of the houses A, B, and C without any line crossing another (FIGURE 1).

A number of other problems are equivalent to this one. For example, if we replace the utilities by three wells, x,y, and z, we get the bad neighbors problem or the houses and wells problem [**8**, p. 14]:

> Three persons live on adjacent lots, each provided with a well. Occasionally one or more of the wells run dry, so each person requires a path to each well. After a while the residents develop strong dislikes for each other and decide to construct their paths in such a way that they will avoid meeting each other on their way to and from the wells. Is such an arrangement possible?

A more violent version is the Corsican vendetta [**2**, pp. 80–81]:

> There are three families such that any member of one family will attempt to kill any member of another family whenever their paths cross. However, the well, the market, and the church are, by tradition, neutral places which are free from violence. Therefore, the families would like to take paths from their homes to the three safe places so that the paths of different families will never cross.

Any problem with this many variations must have roots which go way back in history. In fact, most published references to the problem characterize it as "very ancient." However, the earliest published reference to the problem known to this author dates only as far back as 1917. In that year, the English puzzlist H. E. Dudeney included the utility problem in his book *Amusements in Mathematics* [**3**, Problem 251]. Dudeney himself characterizes the problem as "old as the hills,...much older than electric lighting, or even gas," yet he gives no earlier sources. Sam Loyd, Jr., claimed that his father, an American contemporary of Dudeney, "brought out" the utility problem in 1900 [**1**, p. 142]. Loyd does not claim that his father invented the puzzle,

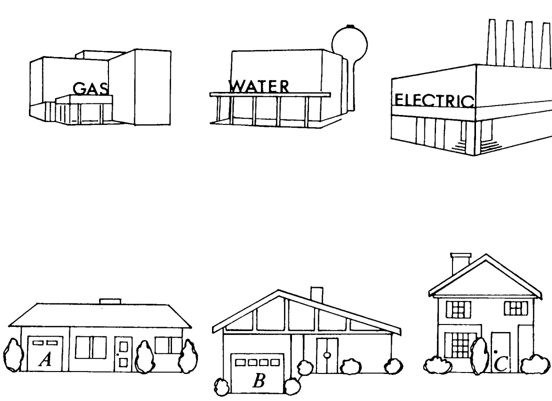


FIGURE 1

Two equivalent problems. In FIGURE 1 three utilities are to be connected to each of three houses, while in FIGURE 2 the like numbered handles of the urns are to be joined. This latter task would be easy if handles were numbered in the same order on both urns.

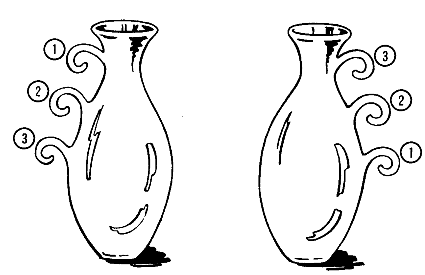


FIGURE 2

and many mathematical historians feel that its origins are at least a century older than that. (Any new information along these lines will be gratefully received.)

Still another problem, whose equivalence with the utility problem is less obvious, concerns a Persian caliph with a beautiful daughter [**6**, pp. 274–276].

> A caliph was so troubled by the large number of suitors, that he decided to set up a competition to determine who was best qualified to marry the girl. In the qualifying round, the aspirants were presented with a picture of two urns, each having three handles numbered 1, 2, and 3 from top to bottom. It was required to join 1 with 1, 2 with 2, and 3 with 3 by curves which do not intersect each other or cross the urns. This task was not so difficult, but a caliph's daughter is not so easily won. The father insisted that any suitor who survived the first round also compete in the finals. This time the same urns were pictured, but the numbers on the handles of the second urn were in reverse order (FIGURE 2).

Relabeling the ends of handles 1 and 3 on the first urn and the end of handle 2 on the second urn as W, G, and E and the other three handles as A, B, and C, this problem is seen to be equivalent to the utility problem provided we consider the urns themselves as paths joining some of the houses and utilities.

One reason that these problems are perennial favorites among mathematical puzzlists is their deceptive simplicity. At first glance they appear easy to solve, yet it can actually be proved that no solution exists. (It is rumored that the caliph's daughter died as an old maid.) You can begin to appreciate the difficulty as soon as you try to connect the houses and utilities. Although any two of the houses can be connected to all three utilities with no lines crossing, you run into difficulties when you try to connect the third house. Persons who claim to have solved the problem usually resort to a semantic loophole. The owner of one house, for example, can permit a utility company to run its line through his basement on the way to another house. Of course this trick will not work if we think of the houses and utilities as being represented by single points.

Dudeney's solution to the problem in 1917 is illustrated in FIGURE 3. However, he published the problem again in 1926 [**4**, Problem 156; reprinted in **5**, Problem 413]. This time he remarked that he had been receiving an average of ten letters per month from correspondents asking about it. He also gave a proof "for the first time in a book" that there is "no solution without any trick."

Suppose we connect houses A and B with W, G, and E. By changing the location of B, we have an equivalent network in the shape of a rhombus with one diagonal as shown in FIGURE 4. This divides the plane into three regions, and, no matter where house C is placed, it will be in some region and one of the utilities W, G, or E will be outside that region. It will be impossible

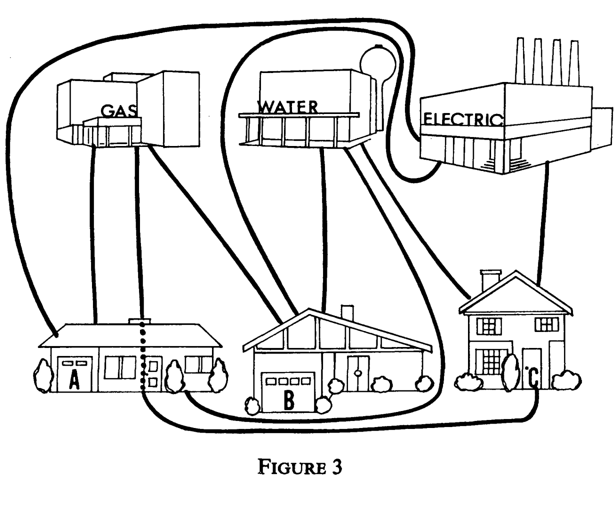


FIGURE 3

A trick solution to the utilities problem (FIGURE 3) suggests a proof that no legitimate solution is possible. FIGURE 4 represents three utilities (center row) connected to two houses (top and bottom). No matter where the third house is located, it will be impossible to join it with all three utilities without any lines crossing.

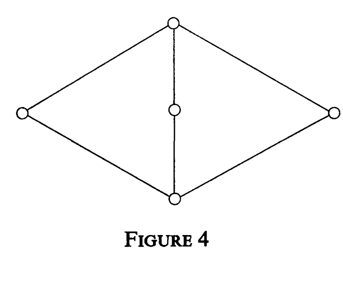
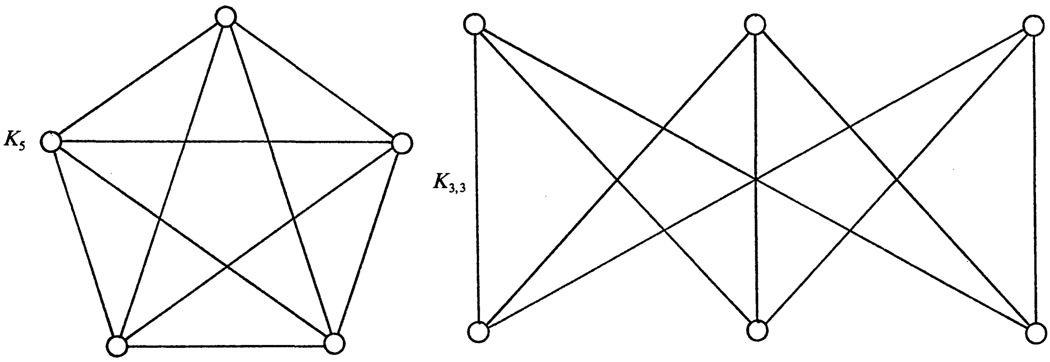


FIGURE 4

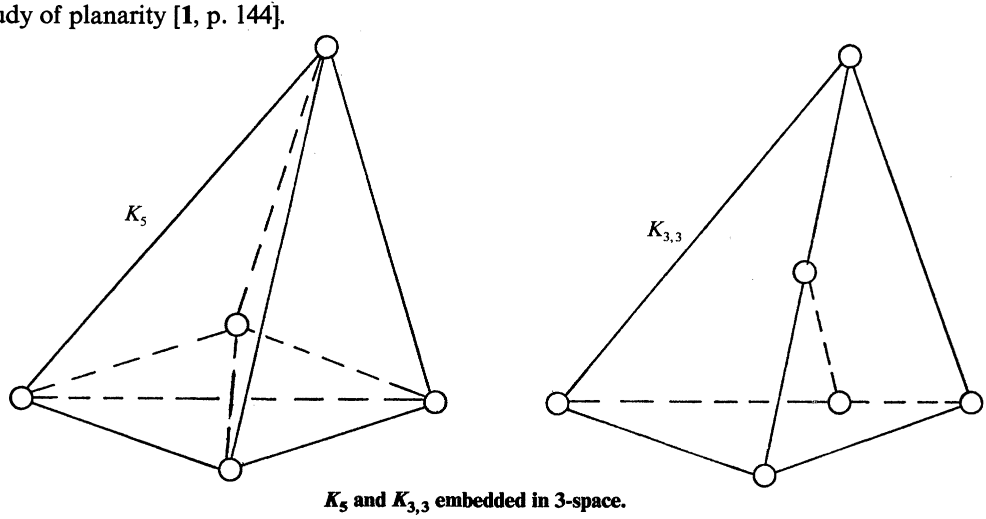
to connect C to that utility without crossing the boundary of the region, and the boundary consists of other utility lines. This proof is based on the Jordan curve theorem, which was proved in 1905.

The utilities problem is especially significant because of its role in the characterization of nonplanar graphs. During the 1920's, a number of mathematicians were searching for criteria that would characterize whether or not a particular graph can be drawn in a plane without intersecting itself. The first such criterion was announced by the Polish mathematician Kazimierz Kuratowski in 1929 and published in 1930. He proved that a graph is planar if it contains no subgraph equivalent to one of the graphs K_5 or $K_{3,3}$ (FIGURE 5). Two American mathematicians, Orrin Frink and P. A. Smith, independently arrived at the same result approximately six months after Kuratowski.



The fundamental non-planar graphs, K_5 and $K_{3,3}$.
FIGURE 5

The graph K_5 is a pentagon with all of its diagonals, while $K_{3,3}$ is precisely the graph obtained as a result of making all of the connections required in the utilities problem. Showing that $K_{3,3}$ is nonplanar is equivalent to showing that the utilities problem has no solution. Interestingly enough, Kuratowski did not publish a detailed proof that these two graphs are nonplanar. Instead, he drew the tetrahedral forms of these two graphs (FIGURE 6) and, in that time-honored mathematical tradition, asserted that "obviously neither graph is homeomorphic to a graph located in the plane" [**7**, p. 272]. Undoubtedly Kuratowski was more concerned with the converse—that these two examples essentially characterize all nonplanar graphs. There is also reason to believe that Kuratowski was not familiar with the utilities problem when he began his study of planarity [**1**, p. 144].

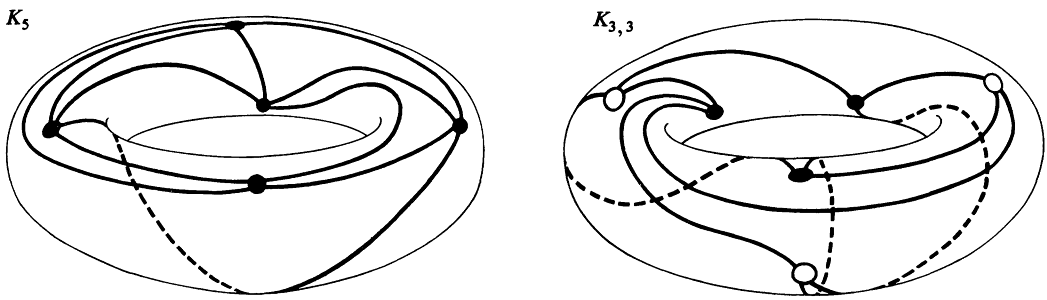


K_5 and $K_{3,3}$ embedded in 3-space.

FIGURE 6

Another proof that the graphs K_5 and $K_{3,3}$ cannot be planar follows from the Euler-Descartes formula $V-E+F=2$, where V, E, and F represent the numbers of vertices, edges, and faces of a finite, connected, planar graph. (The exterior region is also counted as a face.) In the case of the two graphs in question, it is difficult to count the faces directly, since they overlap. However, it is clear from FIGURE 6 that all the faces of K_5 are triangles, while all the faces of $K_{3,3}$ are quadrilaterals. In general, for any planar graph in which every face is an n-gon, the total number of edges is given by $E=n(V-2)/(n-2)$. To see this, note that each edge belongs to exactly two faces. So if each face has exactly n edges, then $nF=2E$. The result follows by substitution in the Euler Descartes formula $V-E+F=2$.

Thus for K_5, we would have $E=3(5-2)/(3-2)=9$. But, by actual count, $E=10$. Similarly, for $K_{3,3}$ we would have $E=4(6-2)/(4-2)=8$, but, by actual count, $E=9$. Hence neither K_5 nor $K_{3,3}$ can be drawn in a plane without any crossing of edges, and they cannot be drawn on a sphere either. However, as FIGURE 7 shows, both graphs can be drawn on a torus. This may be good news for utility companies of the future who wish to lay out their lines on a torus-shaped orbiting space station.



K_5 and $K_{3,3}$ drawn on a torus.
FIGURE 7

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An Uncharacteristic Proof of the Spectral Theorem

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Linear algebra textbooks usually prove the spectral theorem near the end of the book, following discussion of eigenvalues and eigenvectors. This sequence makes it appear that the