Time Hierarchy Theorem for Quantum Turing machine

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2. Results

1 Preliminaries

2 Results

Quantum Turing machine

A QTM is a 7-tuple such that:

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$$

- Q is a finite set of states
- Σ is the set of input alphabets
- Γ is the set of tape alphabets, where $\Gamma \in \Sigma$.
- $\delta : Q \times \Sigma \times Q \times \Sigma \times \{L, R\} \to \widetilde{\mathbb{C}}$
- q_0 is the start state, q_a is the accept state and q_r is the reject state, where $q_0, q_a, q_r \in Q$.

Time evolution

Definition: For a countable set D, let $l_2(D)$ be the space of all complex values functions on D bounded by the l_2 norm.

$$l_2(D) = \{x : D \to \mathbb{C} | \sqrt{\sum_{i \in D} x(i)x^*(i)} < \infty\}$$

The inner product $\langle a|b\rangle = \sum_{i \in D} a^*(i)b(i)$

Let C_M denote the set of all configurations of a QTM M. Computation of M is performed in the inner-product space $H_M = l_2(C_M)$ with the basis $\{|c\rangle \, | c \in C_M\}$.

Time evolution

The transition function gives us mapping $a : C_M \times C_M \to \mathbb{C}$.
The time evolution $U_M : H_M \to H_M$ operator is then defined as:

1. If $|c\rangle$ is a basis state then

$$U_M |c\rangle = \sum_{c' \in C_M} a(c, c') |c'\rangle$$

2. If $|\psi\rangle = \sum_{c \in C_M} \alpha_c |c\rangle$ is a super position then

$$U_M |\psi\rangle = \sum_{c \in C_M} \alpha_c U_M |c\rangle$$

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Complexity classes

Let $t : \mathbb{N} \to \mathbb{N}$ be time constructible function.
$\mathbf{EQTIME}(t(n)) = \{L \subseteq \Sigma^* |$ There exists a QTM M such that M decides L with zero error and M runs in $t(n)$ time $\}$
$\mathbf{BQTIME}(t(n)) = \{L \subseteq \Sigma^* |$ There exists a QTM M such that
$x \in L \implies Prob(M\ accepts\ x) > \frac{2}{3}$ and
$x \notin L \implies Prob(M\ rejects\ x) > \frac{2}{3}$ and M runs in $t(n)$ time $\}$

$$\mathbf{EQP} = \bigcup_{c \in \mathbb{Z}^+} \mathbf{EQTIME}(n^c)$$

$$\mathbf{BQP} = \bigcup_{c \in \mathbb{Z}^+} \mathbf{BQTIME}(n^c)$$

$$\mathbf{P} \subseteq \mathbf{EQP} \subseteq \mathbf{BQP} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP}$$

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Deterministic Time Hierachy

Let $t : \mathbb{N} \rightarrow \mathbb{N}$ be a time constructible function

$$\mathbf{DTIME}(t(n)) \subset \mathbf{DTIME}(o(t(n)log(t(n))))$$

$$\mathbf{NTIME}(t(n)) \subset \mathbf{NTIME}(o(t(n+1))$$

Universal Quantum Turing machine

Deterministic universal TM: There exixts a universal Turing machine U such that on input $\langle M, w\epsilon\rangle$ where M is a QTM and $w \in \Sigma^*$, U simulates $M(w)$ and is T is the time M takes to decide w then U runs in $TlogT$ time.
Quantum universal TM: There exixts a universal Quantum Turing machine U such that on input $\langle M, w, T, \epsilon\rangle$ where M is a QTM and $w \in \Sigma^*$, $T \in \mathbb{N}$, $\epsilon \in \mathbb{R}$ where $\epsilon > 0$. U simulates $M(w)$ for T time steps with precision ϵ and U runs in time polynomial to $|M|$, $|W|$, T and $1/\epsilon$.

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Main result

Let $t : \mathbb{N} \to \mathbb{N}$ be a time constructible function

$$\textbf{EQTIME}(t(n)) \subset \textbf{EQTIME}(o(nt(n)2^{t(n)} log(t(n)) + nlog(t(n)))$$