CHAPTER 2 Systems

UNIT I

Syllabus :

Systems : Definition, Classification : linear and non linear, Time variant and invariant, causal and non-causal, static and dynamic, stable and unstable, invertible.

2.1 System :

Definition :

- A system is defined as an entity that manipulates one or more signals to accomplish a function, thereby producing new signals.
- The block diagram representation of a system is shown in Fig. 2.1.1.

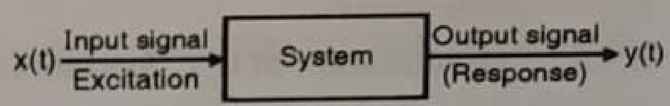


Fig. 2.1.1 : Block diagram representation of a signal

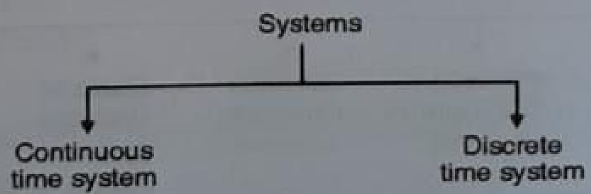
- A system may be defined as a set of elements and functional blocks interconnected to produce an output y (t) in response to an input x (t).
- The input signal x (t) is also called as excitation and y (t) is also called as response of the system.
- The well known examples of continuous time systems around us are : amplifier, filters, and other electronic systems.
- The response y (t) of a system to the excitation x (t), in the time domain depends on the impulse response h (t) of the system. The relation between them is as follows :

$$y(t) = x(t) * h(t) \qquad ...(2.1.1)$$

  where * represents the convolution of x (t) and h (t).
- In the frequency domain, the system response depends on the transfer function of the system.

2.1.1 Types of Systems :

Depending on the type of input signal (continuous time or discrete time) the systems are classified into two types as follows :



Continuous time system :

- A continuous time system is defined as the system which processes a continuous time sig... produce another continuous time signal.
- The CT signal at the input of CT system is denoted by x(t) and the CT response of the syste... denoted by y(t) as shown in Fig. 2.1.1.

Discrete time system :

- A discrete time system is defined as the system which processes a discrete time signal to pro... another discrete time signal.
- The DT signal at the input of DT system is denoted by x (n) and the DT response of the system is denoted by y (n).

  Fig. 2.1.2 shows the block schematic of a D.T. system.

Fig. 2.1.2 : Block schematic of a D.T. ...

2.1.2 Examples of Practical Systems :

There are several types of practical systems. We can classify them into following categories :

1. Communication system
2. Control system
3. Remote sensing
4. Biomedical signal processing
5. Auditory systems.

2.1.3 Classification of C.T. Systems :

The systems are classified as follows :

1. Linear and non-linear systems
2. Time variant and time invariant systems
3. Causal or non-causal systems
4. Stable and unstable systems.
5. Static or dynamic systems.

This classification is made based on various properties of systems.

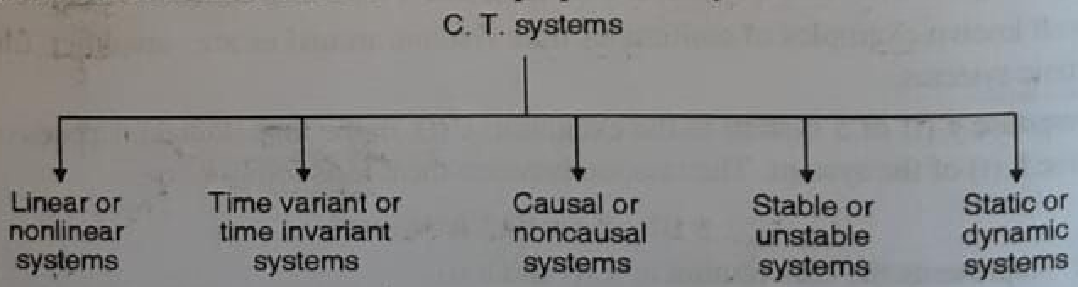


Fig. 2.1.2(a) : Classification of C.T. systems

2.1.4 Classification of D.T. Systems :

Similarly the D.T. systems can be classified as shown in Fig. 2.1.2(b).

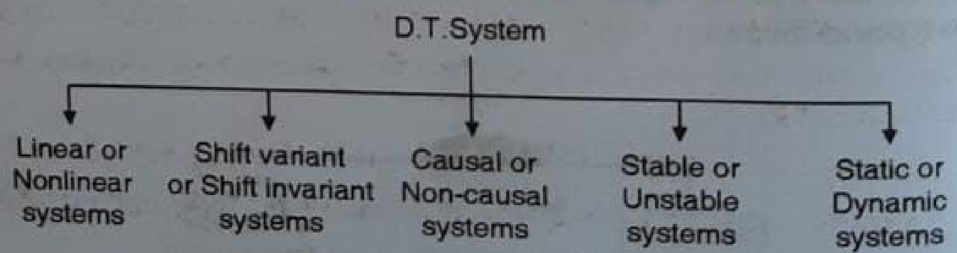


Fig. 2.1.2(b) : Classification of D.T. systems

Ex. 2.2.2 : Determine whether the following system is linear or not. If not give suitable condition for linearity.

$$y(t) = \frac{1}{12} x(t) - \frac{5}{6}$$

Soln. :

Step 1 : Apply zero input to the system. Then,

$$y(t) = -\frac{5}{6}$$

Thus system is initially non-relaxed

Step 2 : Apply individual input to the system.

$$\therefore \quad x_1(t) \xrightarrow{T} \frac{1}{12} x_1(t) - \frac{5}{6}$$

$$\text{and} \quad x_2(t) \xrightarrow{T} \frac{1}{12} x_2(t) - \frac{5}{6}$$

$$\therefore \quad y'(t) = \frac{1}{12} x_1(t) - \frac{5}{6} + \frac{1}{12} x_2(t) - \frac{5}{6}$$

Step 3 : Combine both input and apply to the system.

$$\therefore \quad [x_1(t) + x_2(t)] \xrightarrow{T} \frac{1}{12}[x_1(t) + x_2(t)] - \frac{5}{6}$$

$$\therefore \quad y''(t) = \frac{1}{12} x_1(t) + \frac{1}{12} x_2(t) - \frac{5}{6}$$

Step 4 : Since $y'(t) \neq y''(t)$; the system is non-linear.

Condition for linearity :

For linearity important condition is that if we apply zero input then output must be zero. But as per step I, for zero input, system produces output $= -\frac{5}{6}$. Thus for the system to be linear, the term $-\frac{5}{6}$ should not be present.

Ex. 2.2.3 : $T\{x(n)\} = ax(n) + 6$. Check if the system is linear or nonlinear.

Soln. : The given equation can be written as,

$$T\{x(n)\} = ax(n) + 6$$

Step 1 : If input x (n) is zero then,

$$y(n) = 0 + 6 \qquad \therefore \quad y(n) = 6$$

Thus system is initially non-relaxed.

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$. We will apply these two inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = ax_1(n) + 6$$

and $x_2(n) \xrightarrow{T} y_2(n) = a x_2(n) + 6$

Add these outputs to get $y'(n)$

Note that the given equation contains the coefficients 'a' and 6. So to check the linearity, add the outputs as follows :

$$y'(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$\therefore y'(n) = a_1 [a x_1(n) + 6] + a_2 [a x_2(n) + 6]$$

Here 'a_1' and 'a_2' are arbitrary constants used to check the linearity.

Step 3 : We will add two inputs and then we will apply this signal to the system. Again we arbitrary constants a_1 and a_2.

$$\therefore [a_1 x_1(n) + a_2 x_2(n)] \xrightarrow{T} a[a_1 x_1(n) + a_2 x_2(n)] + 6$$

$$\therefore y''(n) = a[a_1 x_1(n) + a_2 x_2(n)] + 6$$

Note that in this case the function of system is to multiply input by constant 'a' and then The term $a_1 x_1(n) + a_2 x_2(n)$ acts as combined input signal. Thus the output $y''(n)$ is obtained by Equation (2).

Step 4 : Compare Equations (1) and (2),

Since $y'(n) \neq y''(n)$; the system is non-linear.

Note : Whenever the given equation contains some constants and the equation contains add two or more terms, then to check the linearity, use arbitrary constants a_1 and a_2. In othe we are assuming $a_1 = a_2 = 1$.

Ex. 2.2.4 : $y(n) = \cos x(n)$. Check the linearity of the system.

Soln. :

Step 1 : If input $x(n)$ is zero then,

$$y(n) = \cos(0) \qquad \therefore \qquad y(n) = 1$$

Thus system is initially non-relaxed

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these two inputs separately to the Observe that the function of system is to take 'cos' of input signal.

$$\therefore x_1(n) \xrightarrow{T} y_1(n) = \cos x_1(n)$$

$$\text{and } x_2(n) \xrightarrow{T} y_2(n) = \cos x_2(n)$$

Now add two outputs to get $y'(n)$

$$\therefore y'(n) = y_1(n) + y_2(n)$$

$$\therefore y'(n) = \cos x_1(n) + \cos x_2(n)$$

Add these two outputs to get y′ (n),

$$\therefore\ y'(n) = y_1(n) + y_2(n)$$

$$\therefore\ y''(n) = x_1(-n+2) + x_2(-n+2) \quad ...(1)$$

Step 3 : Add two inputs and apply it to the system.

$$\therefore\ x_1(n) + x_2(n) \xrightarrow{T} x_1(-n+2) + x_2(-n+2)$$

$$\therefore\ y''(n) = x_1(-n+2) + x_2(-n+2) \quad ...(2)$$

Step 4 : Since $y'(n) = y''(n)$; the system is linear. **...Ans.**

Ex. 2.2.9 : $y(n) = n\,x^2(n)$ check the linearity :

Soln. :

Step 1 : When input x (n) is zero; output is zero. Thus system is initially relaxed.

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply each input separately to the system.

$$\therefore\ x_1(n) \xrightarrow{T} y_1(n) = n\,x_1^2(n)$$

$$\text{and}\ x_2(n) \xrightarrow{T} y_2(n) = n\,x_2^2(n)$$

Add these outputs to get y′ (n),

$$\therefore\ y'(n) = n\left[x_1^2(n) + x_2^2(n)\right] \quad ...(1)$$

Step 3 : Combine two inputs and apply it to the system.

$$\therefore\ [x_1(n) + x_2(n)] \xrightarrow{T} n\,[x_1(n) + x_2(n)]^2$$

$$\therefore\ y''(n) = n\,[x_1(n) + x_2(n)]^2 \quad ...(2)$$

Step 4 : Compare Equations (1) and (2),

Since $y'(n) \neq y''(n)$; the system is non-linear. **...Ans.**

Ex. 2.2.10 : $y(n) = g(n)\,x(n)$ check the linearity.

Soln. :

Step 1 : When input x (n) = 0; output is zero.

Thus system is initially relaxed.

Step 2 : Apply each input separately to the system.

$$\therefore\ x_1(n) \xrightarrow{T} g(n)\,x_1(n)$$

$$\text{and}\ x_2(n) \xrightarrow{T} g(n)\,x_2(n)$$

$$\therefore\ y'(n) = g(n)\,x_1(n) + g(n)\,x_2(n) \quad ...(1)$$

Step 3 : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} g(n)[x_1(n) + x_2(n)]$$

$$\therefore \quad y''(n) = g(n)\, x_1(n) + g(n)\, x_2(n)$$

Step 4 : Since $y'(n) = y''(n)$; the system is linear.

Ex. 2.2.11 : Check the following system for linearity : **Dec. 12, 4**

$y(t) = x(t) + x(t-100)$

Soln. :

Step 1 : Apply zero input, $y(t) = 0$.

Step 2 : Apply each input separately.

$$\therefore \quad x_1(t) \xrightarrow{T} x_1(t) + x_1(t-100)$$

$$\therefore \quad x_2(t) \xrightarrow{T} x_2(t) + x_2(t-100)$$

$$\therefore \quad y'(t) = x_1(t) + x_1(t-100) + x_2(t) + x_2(t-100)$$

Step 3 : Apply both inputs combinely.

$$\therefore \quad [x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = x_1(t) + x_1(t-100) + x_2(t) + x_2(t-100)$$

Step 4 : Since $y'(t) = y''(t)$; It is linear system.

Example for Practice :

Ex. 2.2.12 : Check whether the following discrete time system is linear or not :

$y(n) = x(n) \cos \omega_0 n.$

Ans. : Linear

Ex. 2.2.13 : Determine whether the following system is linear or not :

$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$

Ans. : Linear

Ex. 2.2.14 : $y(n) = n\, x(n)$ check for the linearity.

Ans. : Linear

Ex. 2.2.15 : $y(n) = x(n - n_0)$, n_0 is constant. Check for the linearity.

Ans. : Linear

Ex. 2.2.16 : $y(n) = x(n) \cdot n(n - n_0)$ check for the linearity.

Ans. : Linear

Ex. 2.2.17 : Examine the following system for linearity,

$y(n) = x(-n)$

Ans. : Linear

Soln. :

$$y''(t) + 3\,y'(t) = 2\,x'(t) + x(t) \quad \text{... Given}$$

(A) Linearity :

Consider there are two inputs $x_1(t)$ and $x_2(t)$. Then if we apply individual input to the system; it satisfies following equations.

$$y_1''(t) + 3\,y_1'(t) = 2\,x_1'(t) + x_1(t) \quad \text{...(1)}$$

$$\text{and} \quad y_2''(t) + 3\,y_2'(t) = 2\,x_2'(t) + x_2(t) \quad \text{...(2)}$$

The linear combinations of Equations (1) and (2) is,

$$y_1''(t) + y_2''(t) + 3\,y_1'(t) + 3\,y_2'(t) = 2\,x_1'(t) + 2\,x_2'(t) + x_1(t) + x_2(t)$$

$$\therefore \quad \frac{d^2}{dt^2} y_1(t) + \frac{d^2}{dt^2} y_2(t) + 3\frac{d}{dt} y_1(t) + 3\frac{d}{dt} y_2(t) = 2\frac{d}{dt}\, x_1(t) + 2\frac{d}{dt}\, x_2(t) + x_1(t) + x_2(t)$$

$$\therefore \quad \frac{d^2}{dt^2}[\,y_1(t) + y_2(t)\,] + 3\frac{d}{dt}[\,y_1(t) + y_2(t)\,] = 2\frac{d}{dt}\,[\,x_1(t) + x_2(t)\,] + x_1(t) + x_2(t) \quad \text{...(3)}$$

This equation can be put in the form of original equation. Thus it is linear system.

(B) Time invariance :

This equation does not contain any term like $t\,y(t)$ or $t\,x(t)$. That means this equation does not vary with time. So it is time invariant system.

Ex. 2.3.4 : Determine whether the following system is time variant or not :

$y(t) = x(t) - 3u(t)$

Soln. :

$$y(t) = x(t) - 3u(t)$$

Step 1 : Delay input by k samples and denote output by $y(t, k)$

$$\therefore \quad y(t, k) = x(t-k) - 3u(t) \quad \text{...(1)}$$

Step 2 : Replace 't' by $t - k$ throughout the equation.

$$\therefore \quad y(t-k) = x(t-k) - 3u(t-k) \quad \text{...(2)}$$

Step 3 : Since $y(t, k) \neq y(t-k)$; the system is time variant.

Ex. 2.3.5 : Determine whether the following system is shift invariant or not :

$T[\,x(n)\,] = e^{x(n)}$

Soln. : We know that $T[\,x(n)\,]$ means $x(n)$ is passed through the system to produce output. Thus given equation can be written as,

$$\therefore \quad y(n) = e^{x(n)} \text{ that means } x(n) \xrightarrow{T} y(n) = e^{x(n)}$$

Step 1 : Delay input by k samples and denote this output by $y(n, k)$

$$\therefore \quad y(n, k) = e^{x(n-k)} \quad \text{...(1)}$$

Step 2 : Replace 'n' by 'n –k' throughout the given equation,

$$\therefore \quad y(n-k) = e^{x(n-k)} \quad \text{...(2)}$$

Step 3 : Compare Equations (1) and (2),

Since $y(n, k) = y(n - k)$; the system is shift invariant

Ex. 2.3.6 : Determine whether the following system is shift invariant or not :

$y(n) = \cos x(n)$

Soln. :

Step 1 : Delay the input by 'k' units and denote this output by $y(n, k)$

$$\therefore \quad y(n, k) = \cos x(n - k)$$

Step 2 : Replace 'n' by 'n – k' throughout the equation.

$$\therefore \quad y(n - k) = \cos x(n - k)$$

Step 3 : Compare Equations (1) and (2)

Since $y(n, k) = y(n - k)$; the system is shift invariant.

Ex. 2.3.7 : Determine whether the following system is shift invariant or not :

$y(n) = x(n) + nx(n - 1)$

Soln. :

Step 1 : Delay input by 'k' units and denote this output by $y(n, k)$

$$\therefore \quad y(n, k) = x(n - k) + nx(n - k - 1)$$

Note that only input $x(n)$ is delayed. That means in the given equation wherever $x(n)$ is pr replace it by $x(n - k)$. You should not replace 'n' present in the second term by $n - k$. That means step do not write the equation as,

$$y(n, k) = x(n - k) + (n - k)\, x(n - k - 1)$$

This is not valid since we are delaying input, $x(n)$ and not only n.

Step 2 : Replace 'n' by 'n – k' throughout the equation.

$$\therefore \quad y(n - k) = x(n - k) + (n - k)\, x(n - k - 1)$$

Step 3 : Compare Equations (1) and (2)

Since $y(n, k) \neq y(n - k)$; the system is shift variant.

Ex. 2.3.8 : Determine whether the following system is shift invariant or not :

$y(n) = x(2n)$

Soln. :

Step 1 : Delay input by k units and denote this output by $y(n, k)$

$$y(n, k) = x[2(n - k)]$$

Step 2 : Replace 'n' by 'n – k' throughout the equation

$$\therefore \quad y(n - k) = x[2(n - k)]$$

Step 3 : Compare Equations (1) and (2)

Since $y(n, k) = y(n - k)$; the system is shift invariant.

Condition for causality in terms of impulse response h (t) :

- The relation between y (t) and x (t) is given by,

$$y(t) = x(t) * h(t)$$

- where * represents convolution and h (t) is the impulse response of the system. The condition for causality in terms of the impulse response is as follows :

$$\text{Condition for causality : } h(t) = 0 \quad \text{for } t < 0 \qquad ...(2.4.2)$$

- This condition states that a C.T. linear time invariant (LTI) system is "causal" if its impulse response h (t) has a zero value for negative values of time.

2.4.2 Causal DT Systems :

Definition :

A DT system is said to be causal system if output at any instant of time depends only on present and past inputs. But the output does not depend on future inputs.

The condition for causality of a D.T. system in terms of its impulse response is as follows :

$$\text{Condition for causality } h(n) = 0 \text{ for } n < 0 \qquad ...(2.4.2(a))$$

Examples :

The output of system depends on present and past inputs that means output, y (n) is a function of x (n), x (n – 1), x (n – 2) ... etc. some examples of causal systems are :

1. $y(n) = x(n) + x(n-1)$ 2. $y(n) = 3\,x(n)$ 3. $y(n) = x(n) + 4\,x(n-1)$

Significance :

Since causal system does not include future input samples; such system is practically realizable. That means such system can be implemented practically. Generally all real time systems are causal systems; because in real time applications only present and past samples are present.

Since future samples are not present; **causal system is memoryless system.**

Anticausal or non-causal DT systems :

Definition :

A DT system is said to be anticausal system if its output depends not only on present and past inputs but also on future inputs.

Examples :

For a noncausal system, output y (n) is function of x (n), x (n – 1), x (n – 2) .... etc. as well as it is function of x (n + 1), x (n + 2) ... etc. some examples of non-causal systems are :

1. $y(n) = x(n) + x(n+1)$
2. $y(n) = B\,x(n+2)$
3. $y(n) = x(n) + nx(n+1)$

Significance :

Since non-causal system contains future samples; a non-causal system is practically not realizable. That means in practical cases it is not possible to implement a non-causal system.

But if the signals are stored in the memory and at a later time they are used by a system the signals are treated as advanced or future signal. Because such signals are already present, before system has started its operation. In such cases it is possible to implement a non-causal system.

Ex. 2.4.1 : Determine if the systems described by following equations are causal or non-causal :

1. $y(n) = \cos x(n)$
2. $y(n) = |x(n)|$
3. $y(n) = x(n) + nx(n-1)$
4. $y(n) = x(n) + nx(n+1)$
5. $y(n) = x(2n)$
6. $y(n) = x(-n+2)$

Soln. :

1. **$y(n) = \cos x(n)$:**

   This is a causal system because the function of system is to obtain cosine value of present input.

2. **$y(n) = |x(n)|$:**

   This is a causal system because output depends on present input.

3. **$y(n) = x(n) + nx(n-1)$:**

   This is a causal system because output depends on present and past input but not on the future input.

4. **$y(n) = x(n) + nx(n+1)$:**

   Here output depends on future input i.e. $x(n+1)$. So this is non-causal system.

5. **$y(n) = x(2n)$:**

   This is non-causal system because output expects future input. This can be verified by putting different values of n.

   At $n = 0 \Rightarrow y(0) = x(0)$ Here present output excepts present input only.

   At $n = 1 \Rightarrow y(1) = x(2)$ Here present output i.e. at $n = 1$, expects future value of input i.e. $x(2)$.

6. **$y(n) = x(-n+2)$:**

   This is non-causal system. This is because at $n = -1$ we get,

   $$y(-1) = x[-(-1)+2] = x[1+2] = x(3)$$

   Thus present output at $n = -1$, expects future input i.e. $x(3)$.

Ex. 2.4.2 : Determine if the system described by the following equation is causal or non causal.

$y(t) = e^{x(t)}$

Soln. : The given equation is,

$$y(t) = e^{x(t)}$$

This is causal system since output depends on present input, $x(t)$.

We have to check the stability of a system by applying bounded input. That means the value of x (t) should be finite (bounded). The value of 'e' is 2.718. So for bounded input, the output y (t) will be bounded. Thus this is stable system.

Ex. 2.5.2 : Determine whether the following discrete-time systems are stable or not :

1. $T[x(n)] = ax(n) + 6$
2. $y(n) = \cos[x(n)]$
3. $y(n) = x(-n)$
4. $y(n) = x(2n)$

Soln. :

1. $T[x(n)] = ax(n) + 6$:

The given equation is,

$$y(n) = ax(n) + 6$$

Here 'a' is some arbitrary constant. So as long as x (n) is bounded the output y (n) is also bounded. Thus it is stable system.

2. $y(n) = \cos[x(n)]$:

For every bounded value of x (n), its cosine value is always bounded (finite). So given system is stable.

3. $y(n) = x(-n)$:

Here x (– n) means folding of input sequence. That means taking mirror image of x (n). But in case of folding operation; the amplitude of signal is not changed. So if input is bounded the output y (n) = x (– n) will be bounded. Thus it is stable system.

4. $y(n) = x(2n)$:

If input x (n) is bounded then output y (n) is also bounded. So it is a stable system.

2.6 Static or Dynamic D.T. Systems (Dynamicity Property) :

2.6.1 Static DT Systems :

Definition : It is a DT system in which output at any instant of time depends on input sample at he same time.

Example :

1. $y(n) = 5x(n)$

   Here 5 is constant which multiplies input x (n). But output at n^{th} instant that means y (n) depends on input at the same (n^{th}) time instant x (n). So this is static system.

2. $y(n) = x^2(n) + 5x(n) + 10$

   Here also output at n^{th} instant, y (n) depends on the input at n^{th} instant. So this is static system.

ignificance :

Observe the input-output relations of static system. Output does not depend on delayed x (n – k)] or advanced [x (n + k)] input signals. It depends only on present (n^{th}) input signal. If utput depends on delayed input signals then such signals should be stored in memory to calculate the output at n^{th} instant. This is not required in static systems. Thus for static systems, memo required. So **static systems are memoryless systems.**

2.6.2 Dynamic DT Systems :

Definition :

It is a system in which output at any instant of time depends on input sample at the sam well as at other times. Here other times means, other than the present time instant. It may be pa future time. Note that if x (n) represents input signal at present instant then,

(a) x (n – k); that means delayed input signal is called as **past signal**.

(b) x (n + k); that means advanced input signal is called as **future signal**.

Thus in dynamic systems, output depends on present input as well as past or future input

Examples :

1. $y(n) = x(n) + 5x(n-1)$

Here output at n^{th} instant depends on input at n^{th} instant, x (n) as well as (n – 1) x (n – 1) is previous (past) sample. So the system is dynamic.

2. $y(n) = 3x(n+2) + x(n)$

Here x (n + 2) indicates advanced version of input sample that means it is future sample is dynamic system.

Significance :

Observe input-output relations of dynamic system. Since output depends on past or futu sample; we need a memory to store such samples. Thus **dynamic system has a memory**.

2.6.3 A Static or Dynamic C.T. System :

A C.T. system is static or memoryless if its output depends upon the present input only.

Example :

Voltage drop across a resistor.

It is given by, $v(t) = i(t) \cdot R$

Here the voltage drop depends on the value of current at that instant. So it is static system

On the other hand a C.T. system is dynamic if output depends on present as well as past v

Example :

A current flowing through inductor i (t) is related to applied voltage, v (t) as,

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\, d\tau$$

Here the limits of integration are from $-\infty$ to $+t$. Thus the current at time 't' depends on values of voltage.

2.7 Invertibility :

- A system is said to be **invertible** if the input of the system can be recovered from the system output.
- A set of operations will be needed to recover the input from output which may be viewed as a second system connected in cascade with the given system, such that the output signal of the second system is equal to the input signal of the given system.

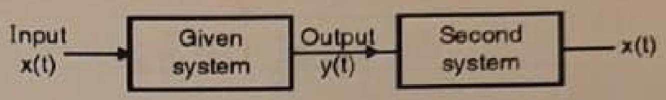


Fig. 2.7.1(a) : Concept of invertibility

- In order to represent the invertibility, mathematically assume that operator H represents a C.T. given system with input signal x(t) producing the output signal y(t).
- Let y(t) be applied to a second C.T. system represented by the operator H^{-1} as shown in Fig. 2.7.1(b). Then the output signal of the second system is given by :

$$\text{Output of second system} = H^{-1}\{y(t)\} = H^{-1}\{H\,x(t)\}$$
$$= H^{-1} H\{x(t)\} = x\{t\}$$

- If we want the output of second system to equal to the original input signal x(t), then it is necessary that

$$H^{-1} H = I \qquad \ldots(2.7.1)$$

Where I denotes the identity operator.

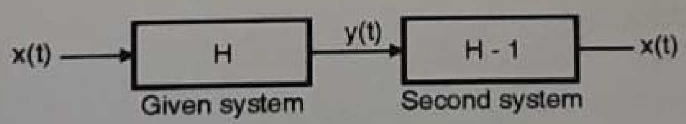


Fig. 2.7.1(b) : Invertibility expressed mathematically

- The operator H^{-1} is called as the **inverse operator** and the second system is called as the **inverse system**.
- However it is important to understand that H^{-1} is not the reciprocal of H. It is inverse of H.
- There has to be a one-to-one mapping between the input and output signals for a system to be invertible.
- That means an invertible system produces distinct outputs for distinct inputs.
- All the discussion done for the C.T. invertible system is applicable to the D.T. systems as well.
- The property of invertibility is very important in designing the **communication systems**. When a signal travels over a communication channel, it gets distorted due to physical characteristics of the channel.
- An **equalizer** can be used to compensate for this distortion and get back the original signal. The equalizer is included on the receiver side.

- An equalizer is an inverse system of the communication channel.

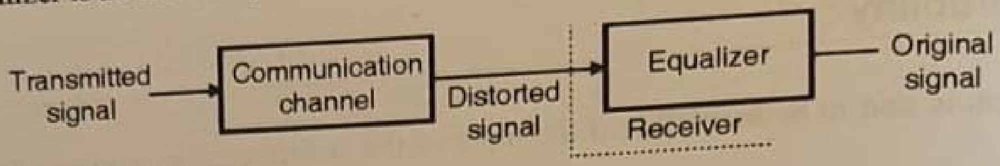


Fig. 2.7.2 : Equalizer is an inverse system of the communication channel

Ex. 2.7.1 : For an inductor the input output relation is as follows :

$$y(t) = \frac{1}{L}\int_{-\infty}^{t} x(t)\,dt$$

Find the operation that represents its inverse system.

Soln. :

$$H = \frac{1}{L}\int_{-\infty}^{t} dt$$

$$\therefore\ H^{-1} = L \times \frac{d}{dt}$$ ...Ans.

This is the operator of the inverse system so that

$$H^{-1} H = I$$

2.7.1 Comparisons :

PU : May 05

University Questions

Q. 1 Compare the following pairs :

1. Causal system and non-causal system
2. Time variant system and time invariant system
3. Static system and dynamic system.

(May 05, 8 Marks)

Causal system and non-causal system :

Sr. No.	Causal system	Non-causal system
1.	A system is said to be causal system if output at any instant of time depends only on present and past inputs. But the output does not depend on future input.	A system is said to be non-causal system if its output depends not only on present and past inputs but also on future inputs.
2.	Causal system is practically realizable.	Non-causal system is practically not realizable.
3.	Condition for causality, h (t) = 0 for t < 0. i.e. h (t) is zero for all negative values of t only.	For non-causal system h (t) ≠ 0 for t < 0 i.e. h (t) exist for negative values of t also.
4.	Example : All real time systems.	Examples : Population growth, Weather broadcasting etc.

(b) $y(n) = \log_{10}(|x(n)|)$:

1. Let $x(n) = x_1(n) + x_2(t)$

$$x_1(n) \xrightarrow{T} \log_{10}(|x_1(n)|)$$

$$x_2(n) \xrightarrow{T} \log_{10}(|x_2(n)|)$$

$$y'(n) = \log(|x_1(n)|) + \log(|x_2(n)|)$$

Now, $[x_1(n) + x_2(n)] \xrightarrow{T} \log_{10}|[x_1(n) + x_2(n)]| = y''(n)$

Since $y'(n) \neq y''(n)$; the system is non-linear.

2. It is memoryless system; because output at any instant depends on input at that instant.
3. Output depends on present input only; so system is causal.
4. The system is stable.
5. $y(n, k) = \log_{10}|x(n-k)|$

and $y(n-k) = \log|x(n-k)|$

Since $y(n, k) = y(n-k)$; it is time invariant system.

Ex. 2.8.3 : A discrete time system is both linear and time invariant. Suppose the output due to an input $x[n] = \delta[n]$ is given in Fig. P. 2.8.3(a), then find the output due to an input :

1. $x[n] = \delta[n-1]$
2. $x[n] = 2\delta[n] - \delta[n-2]$
3. $x[n]$ as depicted in Fig. P. 2.8.3(b).

Dec. 09, 8 Marks

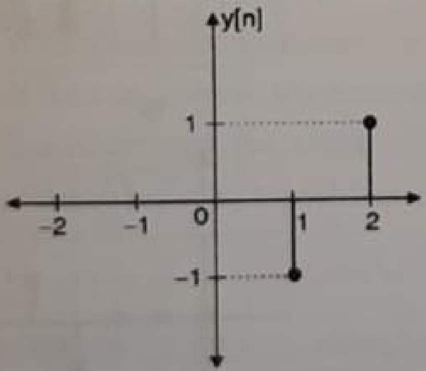


Fig. P. 2.8.3(a)

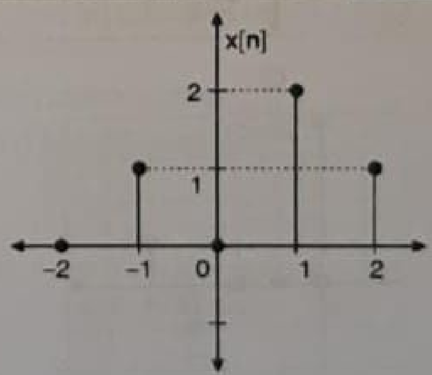


Fig. P. 2.8.3(b)

Soln. :

1. $x(n) = \delta(n-1)$

Since the system is time invariant ; if input is delayed by '1' position then output will be also delayed by '1' position. This output is shown in Fig. P. 2.8.3(c).

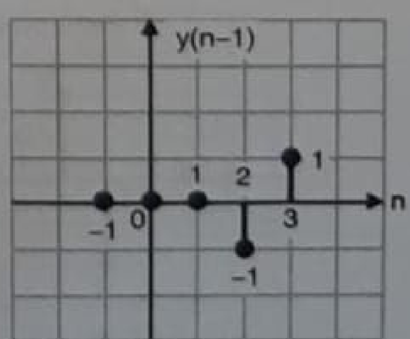


Fig. P. 2.8.3(c)

It contains present and past samples, so it is not memoryless system.

2. The condition of causality is,

$$h(t) = 0 \text{ for } t < 0$$

This signal is only at positive side as shown in Fig. P. 2.8.5. So it is causal system.

3. According to the condition of stability,

$$S = \int_{-\infty}^{\infty} |h(t)| \cdot dt < \infty$$

$$\text{Given} \quad h(t) = e^{-2t} u(t)$$

$$S = \int_{0}^{\infty} |e^{-2t}| \, dt = \left[\frac{e^{-2t}}{2}\right]_0^{\infty} = \left(-\frac{1}{2}\right) [e^{-\infty} - e^{0}]$$

$$\therefore \quad S = \frac{1}{2} < \infty$$

So it is a stable system.

Ex. 2.8.6 : Find the input-output relation of the system shown in Fig. P. 2.8.6. **May 10, 6 Ma[rks]**

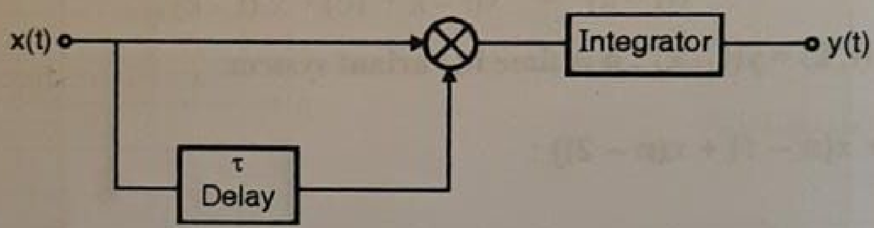


Fig. P. 2.8.6

Soln. :

The signal x (t) is directly applied to multiplier. While x (t) is delayed by τ and then it is appl[ied] to the multiplier. So output of multiplier is,

$x(t) \cdot x(t - \tau)$

The result is integrated to produce output y (t)

$$\therefore y(t) = \int_{-\infty}^{\infty} x(t) \cdot x(t - \tau) \, dt$$

This gives equation of autocorrelation.

Ex. 2.8.7 : Determine if the following system described by :

$y(t) = \sin[x(t + 2)]$

is memoryless, causal, linear, time invariant, stable. **Dec. 10, 7 Mar[ks]**

Soln. :

Given,

$$y(t) = \sin[x(t + 2)]$$

CHAPTER 3 System Analysis

UNIT II

Syllabus :

System modelling : Input output relation, Impulse response, Block diagram, Integro-differential equation, Definition of impulse response, Convolution integral, Convolution sum, Computation of convolution integral using graphical method for unit step to unit step, Unit step to exponential, Exponential to exponential and unit step to rectangular, Rectangular to rectangular only, Computation of convolution sum, Properties of convolution, System interconnection, System properties in terms of impulse response, Step response in terms of impulse response.

3.1 Introduction to System Modeling :

As the name indicates LSI systems are linear and shift (or time) invariant systems. So these systems are also called as linear time invariant (LTI) systems. Generally, in practical cases, majority discrete time systems are linear and time invarient in nature.

These two properties, namely linearity and time invariance are important because of following reasons :

1. Many physical systems can be modelled accurately as LTI systems.
2. We can solve mathematically the equations that model LTI systems for both continuous time and discrete time systems.
3. Many times we model a general signal in terms of linear combinations of basic signals. i.e. $x(n) = x_1(n) + x_2(n) + \dots$

The functions $x_1(n)$, $x_2(n)$ ... are standard functions for which an LTI system response is much easier to find that is the response of x (n). Systems response i.e. output is then the same of the responses of the standard function

$$y(n) = y_1(n) + y_2(n) + y_3(n) + \dots$$

As we will see in the following sections, one of the important characteristic of the unit impulse, i.e. every general signal can be represented as linear combinations of delayed impulses, both in CT (Continious Time) and DT (Discrete Time) system.

This fact, together with the properties (linearity, time invariance) will allow us to develop LTI system in terms of its responses to a unit impulse.

1. Convolution sum in case of discrete time system.
2. Convolution integral in case of continuous time system.

3.2 Block Diagram :

We will discuss the block diagram representation for DT and CT system separately.

(A) For discrete time (DT) system :

First we will study different symbols used to represent the D.T. system.

Symbols used to represent discrete time systems :

The operation of discrete time system can be described by drawing a block schema different building blocks are used to form a complete block schematic. These building block follows :

1. Adder
2. Constant multiplier
3. Signal multiplier
4. Unit delay
5. Unit advance

1. Adder :

As the name indicates, adder block is used to perform addition of two input sequences of an adder is as shown in Fig. 3.2.1(a).

Here $x_1(n)$ and $x_2(n)$ are two input sequences. Adder block adds $x_1(n)$ and $x_2(n)$. The corresponding output is y (n).

$$\therefore \quad y(n) = x_1(n) + x_2(n)$$

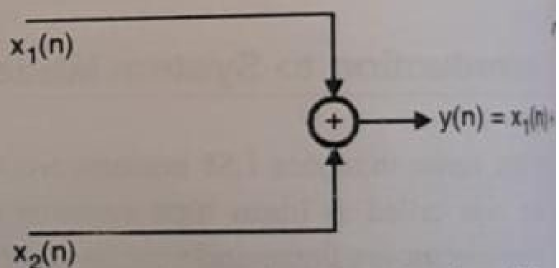


Fig. 3.2.1(a) : An adder

Sometimes instead of addition sign ('+'), sigma (Σ) sign is used to represent the addition. The subtraction operation that means, $y(n) = x_1(n) - x_2(n)$ is represented as shown in Fig. 3.2.1(b).

In case of addition (or subtraction) operation, it is not required to store any one of the inputs in the memory. So this is **memoryless operation.**

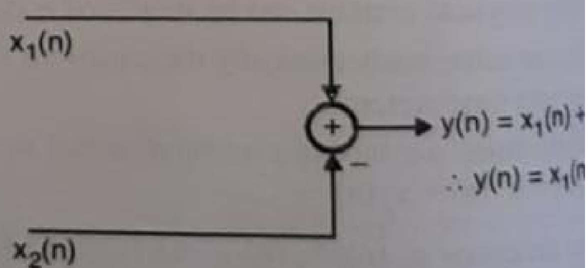


Fig. 3.2.1(b) : Subtraction operation

2. Constant multiplier :

This is used to multiply input sequence by some constant (a). That means this operation is used to change the amplitude of input sample. This is also memoryless operation. It is as shown in Fig. 3.2.2.

Fig. 3.2.2 : Constant multi

Direct form-I structure :

Here y (n – 1) and y (n – 2) denotes delayed output signals while x (n – 1) and x (n – 2) denotes delayed input signals. While y (n) denotes the output of second order LTI system. Implementation of Equation (3.2.1) is shown in Fig. 3.2.6.

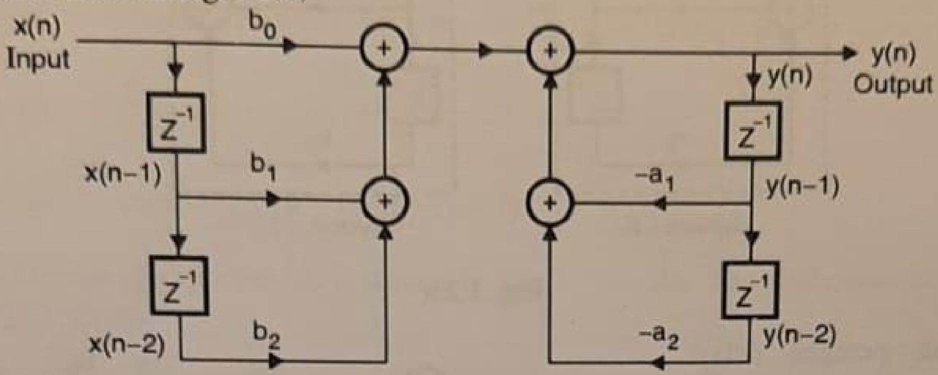


Fig. 3.2.6

Here b_0, b_1 and b_2 are constant multipliers at input side while a_1, a_2 are constant multipliers at output side. Arrow indicates the flow of signal.

Direct form-II structure :

As shown in Fig. 3.2.6 two systems are cascaded. It is shown separately in Fig. 3.2.7.

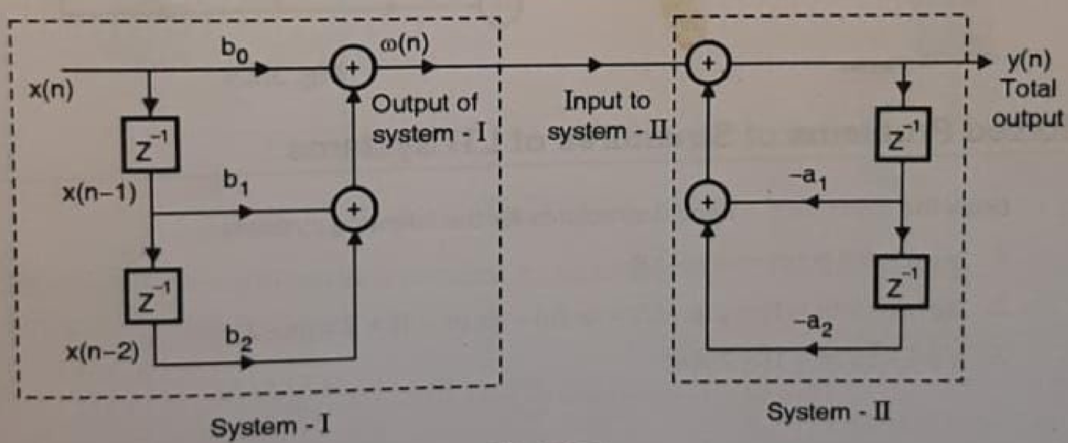


Fig. 3.2.7

Here output of system-I is connected to input of system-II. Since the system is LTI; because of its linearity property we can interchange the positions of these systems. That means we can connect system-II first and then system-I. The output will not be change. Such structure is called as direct form-II structure. It is shown in Fig. 3.2.8.

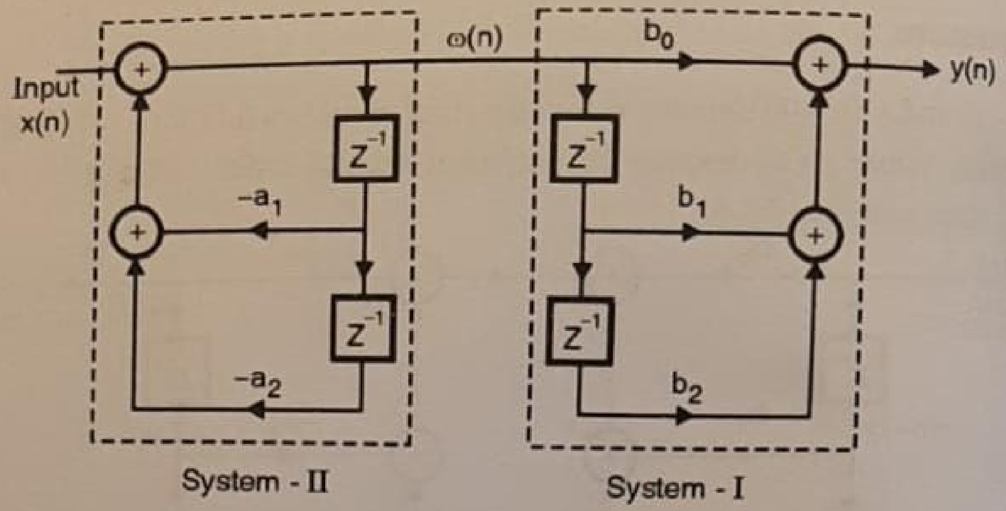


Fig. 3.2.8

In actual practice delay elements are taken common. Thus direct form-II structure by taking delay elements common is as shown in Fig. 3.2.9.

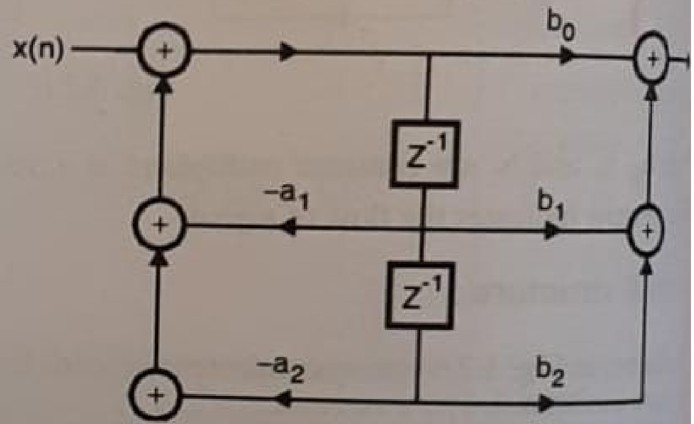


Fig. 3.2.9

3.3 Solved Problems of Structures of LTI Systems :

Ex. 3.3.1 : Draw the direct form – I and II structures for the following systems :

1. $y(n) = 0.5\,[x(n) + x(n-1)]$
2. $3y(n) - 2(n-1) + y(n-2) = 4x(n) - 3x(n-1) + 2x(n-2)$
3. $y(n) - 5y(n-1) = 7\,x(n)$

Soln. :

1. Given, $y(n) = 0.5\,[x(n) + x(n-1)] = 0.5\,x(n) + 0.5\,x(n-1)$

The direct form-I implementation is shown in Fig. P. 3.3.1(a).

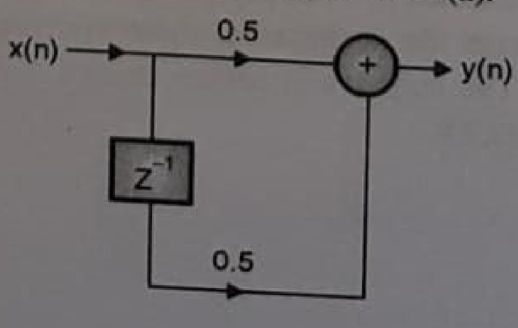
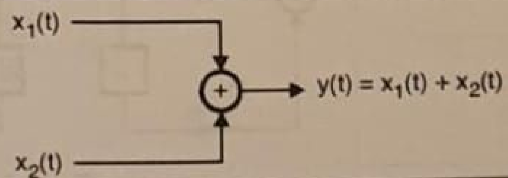
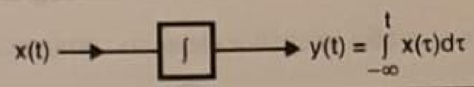


Table 3.3.1

Sr. No.	Name of block	Symbol
1.	Scalar multiplier	
2.	Adder	
3.	Integrator	

Direct form-I structure :

The general difference equation for CT system is given by,

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + b_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$

Here a_0, a_1, b_0 and b_1 are the coefficients.

$y^{(1)}(t) \rightarrow$ Single integration of y (t)

$y^{(2)}(t) \rightarrow$ Double integration of y (t)

$x^{(1)}(t) \rightarrow$ Single integration of x (t)

$x^{(2)}(t) \rightarrow$ Double integration of x (t).

The direct form-I implementation is shown in Fig. 3.3.1.

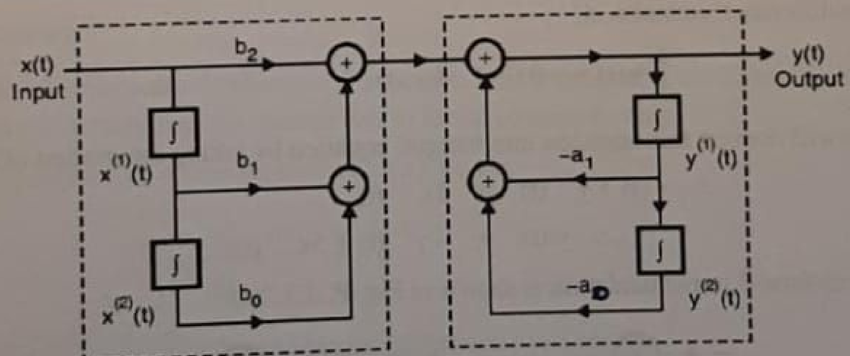


Fig. 3.3.1

This is similar to representation of D.T. system. Only difference is that, in place of delay elements; integrators are used.

Direct-form-II structure :

Similar to D.T. system; we can interchange the position of system-I and system-II, to obtain direct form-II structure. It is shown in Fig. 3.3.2.

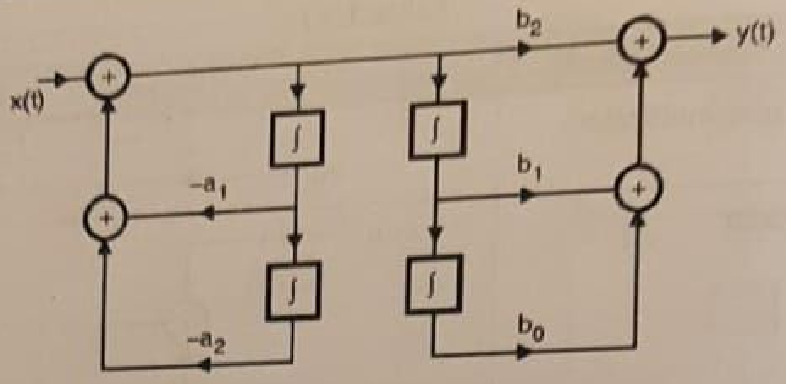


Fig. 3.3.2

We can also take integrator elements common as shown in Fig. 3.3.2(a).

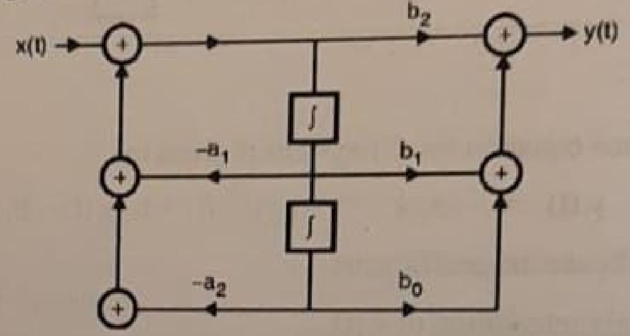


Fig. 3.3.2(a)

Ex. 3.3.2 : Draw direct form-I and II structures for the difference equation,

$$\frac{d}{dt} y(t) + y(t) = 5x(t)$$

Soln. : Given difference equation is,

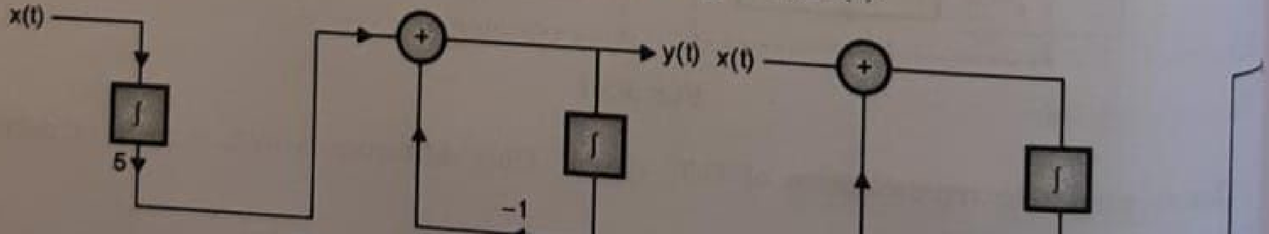
$$\frac{d}{dt} y(t) + y(t) = 5x(t)$$

First we will convert this equation into integral equation by taking integration of both sides

$$\therefore \quad y(t) + y^{(1)}(t) = 5x^{(1)}(t)$$

$$\therefore \quad y(t) = -y^{(1)}(t) + 5x^{(1)}(t)$$

The direct-form-I implementation is shown in Fig. P. 3.3.2(a).



Direct - form - I

(a)

(b)

Fig. P. 3.3.2(b)

By interchanging the positions; direct form II structure is obtained as shown in Fig. P. 3.3.2(b)

$$\therefore \quad Y_{zi}(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t} \qquad ...(4)$$

Calculation of particular solution :

Given input x (t) = u (t). Thus particular solution will be in the form,

$$y_p(t) = k\,u(t) \qquad ...(5)$$

We have to calculate the value of k. Thus putting $y(t) = y_p(t) = k\,u(t)$ and $x(t) = u(t)$ in Equation (1) we get,

$$2\frac{d^2}{dt^2}[k\,u(t)] + 3\frac{d}{dt}[k\,u(t)] + k\,u(t) = u(t)$$

But u (t) = 1

$$\therefore \quad 2\frac{d^2}{dt^2}k + 3\frac{d}{dt}k + k = 1$$

$$\therefore \quad k = 1 \qquad ...(6)$$

Putting this value in Equation (5) we get,

$$y_p(t) = u(t) \qquad ...(7)$$

Calculation of total response :

The total response is given by,

$$y(t) = y_{zi}(t) + y_p(t)$$

$$\therefore y(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t} + u(t) \qquad ...(8)$$

We will use initial conditions to calculate C_1 and C_2.

Thus putting $y(0) = -1$ (for t = 0) we get,

$$-1 = C_1 + C_2 + 1$$

$$\therefore \quad C_1 + C_2 = 2 \qquad ...(9)$$

Differentiating Equation (8) with respect to 't' we get,

$$\frac{dy(t)}{dt} = -\frac{1}{2}C_1 e^{-\frac{1}{2}t} - 1\,C_2 e^{-t}$$

Putting t = 0 we get $\quad \frac{d}{dt}y(0) = -\frac{1}{2}C_1 e^{0} - 1\,C_2 e^{0}$

But we have initial condition $\quad y'(0) = \frac{d}{dt}y(0) = 1$

$$\therefore \quad 1 = -\frac{1}{2}C_1 - C_2$$

$$\therefore \quad +\frac{1}{2} C_1 + C_2 = -1$$

From Equation (9) we get,

$$C_1 = 2 - C_2$$

Putting this value in Equation (10),

$$\frac{1}{2}(2 - C_2) + C_2 = -1$$

$$\therefore \quad 1 - \frac{1}{2} C_2 + C_2 = -1$$

$$\therefore \quad \frac{1}{2} C_2 = -2, \qquad \therefore C_2 = -4$$

$$\text{and} \quad C_1 - 4 = 2, \qquad \therefore C_1 = 6$$

Putting these values in Equation (8) we get,

$$y(t) = 6\,e^{-\frac{1}{2}t} - 4\,e^{-t} + u(t)$$

3.4.1.1 RC Circuit Described by Difference Equation :

Consider RC circuit shown in Fig. 3.4.1

Here x (t) is input voltage applied to RC circuit.

The current flowing through the loop is denoted by y (t).

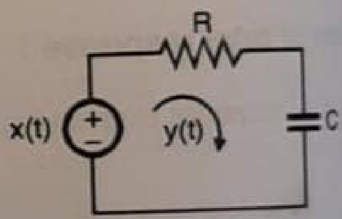


Fig. 3.4.1

Applying kVL we can write,

$$R\,y(t) + \frac{1}{c} \int_{-\infty}^{t} y(\tau)\,d\tau = x(t) \qquad \text{...(3.4.5)}$$

Taking differentiation of both sides we get,

$$R \frac{dy(t)}{dt} + \frac{1}{c} y(t) = \frac{dx(t)}{dt}$$

This is the difference equation of RC circuit.

Here, N = 1, which is order of system.

Now the generalized difference equation is given by,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

In this case N = M = 1

$$\therefore \quad \sum_{k=0}^{1} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{1} b_k \frac{d^k}{dt^k} x(t)$$

Expanding the summation terms we get,

$$Y_{zi}(t) = h(t) = k e^{-at} \quad ...(1)$$

$$\therefore \quad h(t) = k e^{-3/2t} \quad ...(2)$$

But we know that h (0) = 1

Putting t = 0 in Equation (2) we get,

$h(0) = k e^{0} \qquad \therefore \quad k = 1$

Putting this value in Equation (1)

$$h(t) = e^{-3/2t} \quad \text{for } t \geq 0$$

$$\therefore \quad h(t) = e^{-\frac{3}{2}t} u(t)$$

This is the impulse response of system.

Ex. 3.4.3 : Find impulse response of a system described by :

$$y''(t) + 5y'(t) + 6y(t) = x(t)$$

Soln. :

Given, $\quad y''(t) + 5y'(t) + 6y(t) = x(t)$

This is the second order difference equation.

Its characteristic equation is given by,

$$s^2 + 5s + 6 = 0 \quad ...(1)$$

Thus roots are at, $\quad s_1 = -2 \text{ and } s_2 = -3$

$$\therefore \quad a_1 = 2 \text{ and } a_2 = 3$$

Now natural response is given by,

$$h(t) = k_1 e^{-a_1 t} + k_2 e^{-a_2 t}$$

$$\therefore \quad h(t) = k_1 e^{-2t} + k_2 e^{-3t} \quad ...(2)$$

For second order system we have initial conditions,

$$h(0) = 0 \quad \text{and} \quad h'(0) = 1$$

Putting t = 0 in Equation (2) we get,

$$h(0) = k_1 + k_2$$

$$\therefore \quad k_1 + k_2 = 0 \quad ...(3)$$

Differentiating Equation (2) with respect to t we get,

$$h'(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t} \quad ...(4)$$

Putting t = 0 we get,

$$h'(t) = -2k_1 - 3k_2$$

$$\therefore \quad -2k_1 - 3k_2 = 1 \quad ...(5)$$

We will solve Equations (4) and (5) to obtain values of k_1 and k_2.

From Equation (3) we get,

$$k_1 = -k_2$$

Putting this value in Equation (5),

$$+2k_2 - 3k_2 = 1 \qquad \therefore \quad k_2 = -1$$

$$\text{and} \quad k_1 = 1$$

Putting these values in Equation (2) we get,

$$h(t) = e^{-2t} - e^{-3t} \quad \text{for } t \geq 0$$

$$\therefore \quad h(t) = e^{-2t}\,u(t) - e^{-3t}\,u(t)$$

This is impulse response of given system.

3.4.2 DT System Representation :

Note : We will discuss linear convolution later in this chapter

The output of discrete time system can be calculated using equation of linear convolutio

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Here h (n) is the impulse response of system. By the use of h (n); we can calculate outp for any input x (n) using Equation (3.4.14).

Now in this section we will discuss the difference equation which are useful to im (realize) the discrete time system. These difference equations are especially used for desig digital filter.

We discussed that impulse response h (n) is used to calculate output of discrete time Now if the system is causal then impulse response h (n) is zero for negative values of n. So fo LTI system equation of convolution becomes,

$$y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k)$$

Equation (3.4.15) can also be written as,

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

Since output depends on impulse response h (n); the number of samples, present in h (n) important role to determine output y (n). The number of samples present in h (n) is called as le impulse response. Depending upon the length of impulse response; there are two types of disc systems :

1. Finite impulse response (FIR) systems.
2. Infinite impulse response (IIR) systems

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

Expanding the summation we get,

$$y(n) = h(0)\, x(n) + h(1)\, x(n-1) + h(2)\, x(n-2) + ... + h(\infty)\, x(n-\infty)$$

We have discussed earlier that such systems require to store all past inputs. That means it is required to store infinite input terms in the memory. So infinite memory is required. This is not possible practically. Thus non-recursive implementation of causal IIR system is not possible.

The representation of non-recursive system is shown in Fig. 3.4.4(a).

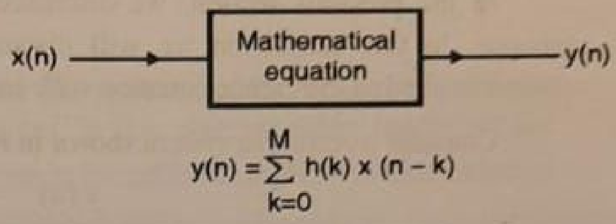


Fig. 3.4.4(a) : Non-recursive system

2. Recursive systems :

A discrete time system in which output y (n) depends on present input, past inputs as well as previous outputs; is called as recursive system.

Let us consider one example. Consider a system represented by block schematic as shown in Fig. 3.4.4(b).

First we will write the equation for system.

1. As shown in Fig. 3.4.4(b); a connection is drawn from output y (n) to the input side. This is a feedback.
2. This feedback signal, y (n) is passed through an unit delay (z^{-1}) block. So the output of delay block is y (n – 1). This is past output compared to y (n).
3. The signal y (n – 1) is multiplied by constant multiplier 'a' to obtain ay (n – 1).
4. An adder is used to add input x (n) and signal ay (n – 1).
5. Now the addition of x (n) and a y (n – 1) produces the output.

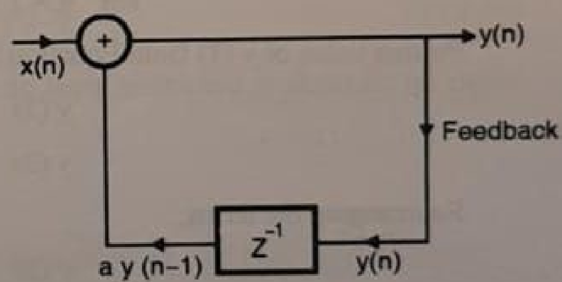


Fig. 3.4.4(b) : Example of recursive system

Thus the output of system is expressed as,

$$y(n) = x(n) + ay(n-1) \quad ...(3.4.23)$$

At instant n = 0 we get,

$$y(0) = x(0) + ay(-1)$$

Here $y(0) \Rightarrow$ present output

$x(0) \Rightarrow$ present input

and $y(-1) \Rightarrow$ past output

Similarly at n = 1 we get,

$$y(1) = x(1) + ay(0)$$

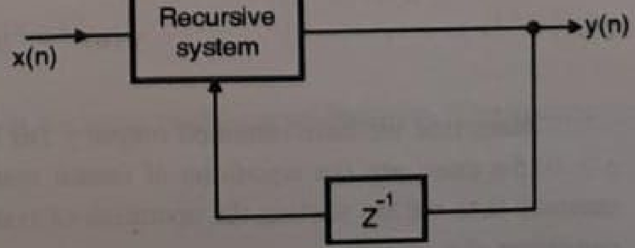


Fig. 3.4.4(c) : Representation of recursive system

Thus y (0) is past output.

So at any instant present output depends on past output. Hence it is recursive system... representation of recursive system is shown in Fig. 3.4.4(c).

3.4.2.1 LTI Systems Characterized by Constant Coefficient Difference Equations :

In the previous section, we discussed that, the behaviour of system depends on its i... response. In this subsection we will discuss how LTI systems are characterized by input-... relations; called as difference equation with constant coefficients.

Consider a recursive system shown in Fig. 3.4.4(b). The equation of output is,

$$y(n) = x(n) + ay(n-1) \quad ...(3...$$

This is first order difference equation. Here 'a' is a constant coefficient. Now output y... calculated by putting different values of n in Equation (3.4.24).

$$\text{For } n = 0 \Rightarrow y(0) = x(0) + ay(-1)$$

$$\text{For } n = 1 \Rightarrow y(1) = x(1) + ay(0)$$

Putting value of y (0) from Equation (1) we get,

$$y(1) = x(1) + a[x(0) + ay(-1)]$$

$$\therefore \quad y(1) = x(1) + ax(0) + a^2 y(-1)$$

Rearranging the terms,

$$y(1) = a^2 y(-1) + ax(0) + x(1)$$

$$\text{For } n = 2 \Rightarrow y(2) = x(2) + ay(1)$$

Putting value of y (1) from Equation (3) we get,

$$y(2) = x(2) + a[a^2 y(-1) + ax(0) + x(1)]$$

$$y(2) = x(2) + a^3 y(-1) + a^2 x(0) + ax(1)$$

Rearranging the terms,

$$y(2) = a^3 y(-1) + a^2 x(0) + ax(1) + x(2)$$

Observe Equations (3) and (5) carefully. Now we can directly write the equation for value...

$$\therefore \quad y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \ldots + ax(n-1) + x(n) \quad ...(3...$$

In terms of summation Equation (3.4.25) can be written as,

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k) \quad ...(3...$$

Note that we have obtained output y (n) by putting only positive values of n. That mea... $n \geq 0$. So these are the equations of causal system. Every equation contains the term y (– 1)... assumed that, we are starting the operation of system at n = 0. Then the term y (– 1) is called as i... **condition of a system.**

$$\sum_{k=0}^{N} a_k y(n-k) = a_0 y(n) + \sum_{k=1}^{N} a_k y(n-k)$$

$$\therefore \quad -\sum_{k=1}^{N} a_k y(n-k) = a_0 y(n) - \sum_{k=0}^{N} a_k y(n-k) \qquad ...(3.4.33)$$

Putting this value in Equation (3.4.32) we get,

$$y(n) = a_0 y(n) - \sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Assume $a_0 = 1$

$$\therefore \quad y(n) = y(n) - \sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$\therefore \quad \sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad ...(3.4.34)$$

We will obtain homogeneous solution by assuming input $x(n) = 0$.

Since $x(n) = 0 \Rightarrow x(n-k) = 0$. Thus Equation (3.4.34) becomes,

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \qquad ...(3.4.35)$$

We will assume that the solution is in the form of exponential.

Thus let, homogeneous solution,

$$[y_h(n)] = \lambda^n \qquad ...(3.4.36)$$

Replace n by $n-k$ $\quad \therefore \quad y_h(n-k) = \lambda^{n-k}$

Putting this value in Equation (3.4.35) we get,

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \qquad ...(3.4.37)$$

Expanding the summation we get,

$$a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + ... + a_N \lambda^{n-N} = 0$$

We have assumed that $a_0 = 1$

$$\therefore \quad \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + ... + a_N \lambda^{n-N} = 0$$

Taking λ^{n-N} common we get,

$$\lambda^{n-N} [\lambda^n \cdot \lambda^{-n+N} + a_1 \lambda^{n-1} \cdot \lambda^{-n+N} + a_2 \lambda^{n-2} \cdot \lambda^{-n+N} + ... + a_N] = 0$$

$$\therefore \quad \lambda^{n-N} [\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + ... + a_N] = 0 \qquad ...(3.4.38)$$

The term inside the bracket is called as characteristic polynomial of the system. This polynomial has 'N' roots denoted by $\lambda_1, \lambda_2 \ldots \lambda_N$. These roots can be real or complex valued.

Case I : Roots are distinct :

When the roots are distinct then we can write the solution as,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n + \ldots + c_N \lambda_N^n \quad \ldots(3.4$$

Here $c_1, c_2 \ldots c_N$ are constants.

Case II : Multiple roots :

In case of multiple roots; the solution can be written as,

$$y_h(n) = c_1 \lambda_1^n + c_2 n \lambda_2^n + c_3 n^2 \lambda_3^n + \ldots + c_N n^N \lambda_N$$

Solved Examples :

Ex. 3.4.4 : Determine the homogeneous solution of the system described by :

$$y(n) - 3y(n-1) - 4y(n-2) = x(n).$$

Soln. :

Step 1 : Assume input $x(n) = 0$

$$\therefore \quad y(n) - 3y(n-1) - 4y(n-2) = 0$$

Step 2 : Assume the homogeneous solution is in the form

$$y_h(n) = \lambda^n$$

Thus we get,

$$y(n-1) = \lambda^{n-1} \text{ and } y(n-2) = \lambda^{n-2}$$

Step 3 : Putting these values in Equation (1) we get,

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

Taking λ^{n-2} common we get,

$$\lambda^{n-2}\left[\lambda^n \cdot \lambda^{-n+2} - 3\lambda^{n-1} \cdot \lambda^{-n+2} - 4\right] = 0$$

$$\therefore \quad \lambda^{n-2}\left[\lambda^2 - 3\lambda - 4\right] = 0$$

$$\therefore \quad \lambda^2 - 3\lambda - 4 = 0$$

Roots of Equation (4) are,

$$\therefore \quad \lambda_1 = -1 \text{ and } \lambda_2 = 4$$

Step 4 : The homogeneous solution is given by,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n$$

$$\therefore \quad y_h(n) = c_1(-1)^n + c_2(4)^n$$

Now we will obtain values of c_1 and c_2.

Step 5 : From Equation (5) we get,

$$\text{For } n = 0 \Rightarrow y(0) = c_1 + c_2$$

$$\therefore \quad y_h(n) = c_1\left(\frac{1}{6}\right)^n + c_2(-1)^n$$

We will denote $y_h(n)$ by $h(n)$

$$\therefore \quad h(n) = c_1\left(\frac{1}{6}\right)^n + c_2(-1)^n \quad \text{...(5)}$$

Now we will obtain values of c_1 and c_2.

Step 5 : We will apply initial conditions. In this case $x(n) = \delta(n)$ and $y(n) = h(n)$. Putting these values in Equation (1) we get,

$$h(n) = \delta(n) - \frac{5}{6}h(n-1) + \frac{1}{6}h(n-2)$$

$$\text{For } n = 0 \Rightarrow h(0) = \delta(0) - \frac{5}{6}h(-1) + \frac{1}{6}h(-2)$$

Here $\delta(0) = 1$ and since initial conditions are zero thus $h(-1) = 0$ and $h(-2) = 0$

$$\therefore \quad h(0) = 1 - \frac{5}{6}\times 0 + \frac{1}{6}\times 0$$

$$\therefore \quad h(0) = 1$$

$$\text{For } n = 1 \Rightarrow h(1) = \delta(1) - \frac{5}{6}h(0) + \frac{1}{6}h(-1)$$

Here $\delta(1) = 0$, $h(-1) = 0$ and $h(0) = 1$

$$\therefore \quad h(1) = 0 - \frac{5}{6}\times 1 + \frac{1}{6}\times 0$$

$$\therefore \quad h(1) = -\frac{5}{6}$$

Step 6 : Putting $n = 0$ and $n = 1$ in Equation (5) we get,

$$\text{For } n = 0 \Rightarrow c_1\left(\frac{1}{6}\right)^0 + c_2(-1)^0 = h(0)$$

$$\therefore \quad c_1 + c_2 = 1 \quad \text{...(6)}$$

$$\text{For } n = 1 \Rightarrow c_1\left(\frac{1}{6}\right)^1 + c_2(-1)^1 = h(1)$$

$$\therefore \quad c_1 \cdot \frac{1}{6} - c_2 = -\frac{5}{6}$$

$$\therefore \quad \frac{1}{6}c_1 - c_2 = -\frac{5}{6} \quad \text{...(7)}$$

We will obtain values of c_1 and c_2

From Equation (6) $\qquad c_2 = 1 - c_1$

Putting this value in Equation (7) we get,

$$\frac{1}{6}c_1 - 1 + c_1 = -\frac{5}{6}$$

$$\therefore \quad +\frac{7}{6}c_1 = -\frac{5}{6}+1=\frac{1}{6} \qquad \therefore \quad 7c_1 = 1$$

$$\therefore \quad c_1 = \frac{1}{7}$$

$$\text{and} \quad c_2 = 1-c_1, \qquad \therefore \quad c_2 = 1-\frac{1}{7}=\frac{6}{7}$$

$$\therefore \quad c_2 = \frac{6}{7}$$

Putting c_1 and c_2 in Equation (5) we get,

$$\therefore \quad h(n) = \frac{1}{7}\left(\frac{1}{6}\right)^n + \frac{6}{7}(-1)^n$$

This is impulse response of given system.

2. Particular solution of difference equation :

The particular solution is denoted by $y_p(n)$. It is calculated by assuming that the output from at that of input. For example if input $x(n) = u(n)$ then the output is assumed to be $y_p(n)$ = where 'k' is arbitrary constant.

Important rules to assume the solution :

1. If any term in a particular solution (y_p) is also present in homogeneous solution then multiply by n.
2. If such term appears 'r' times in a homogeneous solution then multiply 'y_p' by n^r.

Some commonly used input signals and the particular solution that we have to assume in Table 3.4.1.

Table 3.4.1

Sr. No.	Input x (n)	Particular solution
1.	Constant term (A)	k
2.	u (n)	k u (n)
3.	$A \cdot c^n$	$k c^n$
4.	$A^n \cdot n^c$	$k_0 n^c + k_1 n^{M-1} + \ldots + k_c$
5.	$A \cos \omega_0 n$	$k_1 \cos \omega_0 n + k_2 \sin \omega_0 n$

A particular solution is basically used to calculate the zero state response (y_{zs}) of the system. following steps are used to calculate the forced response of system :

Step 1 : Find the nature of zero input response from the roots of characteristics polynomial $y_h(n)$ or y_{zi}.

Step 2 : Assume the particular solution $y_p(n)$

Step 3 : Determine the coefficients in the zero input response. Then the forced $y_{zs} = y_h(n) + y_p(n)$.

$x(t)$ = input signal

$y(t)$ = output signal

Writing the impulse response of the system in progressive form,

i.e. Impulse response $h(t)$ of an LTI system is defined to be response of system to unit impulse $\delta(t)$ [$\delta(t) \rightarrow$ unit impulse at $t = 0$]

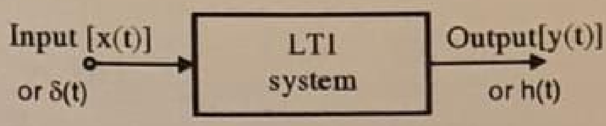


Fig. 3.5.1

By **time invarient property** of LTI system, response to impulse applied at any time "nT" [$\delta(t - t\tau)$ is $h(t - t\tau)$].

It indicates that if unit impulse is delayed by 'τ' then impulse response is also delayed by the same amount.

According to the definition of linear convolution,

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau)\, d\tau$$

In this case we can write,

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Here $h(t - \tau)$ is response of LTI system to input $\delta(t)$.

$$\therefore Y(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

3.5.1 Representation of C.T. Signal in terms of Impulses :

Consider a signal $x(t)$ which will be first approximated as a set of pulse of duration T as shown in Fig. 3.5.2. The function can be then expressed as,

$$x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, P_T(t - tT)$$

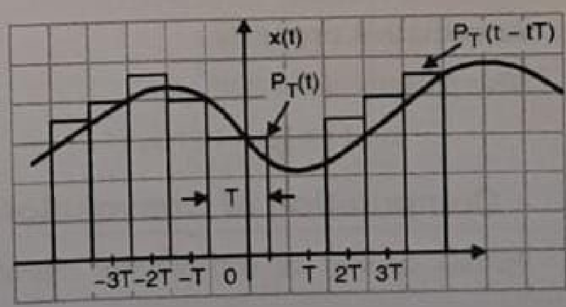


Fig. 3.5.2

Where $x(tT)$ = Is weight of signal, (t is an integer)

At every value of time; weight of signal is multiplied by pulse of duration T.

Note that, this approximation is valid even for small value of T.

The above equation can be written as,

$$\therefore x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, T\, [P_T(t - tT)/T]$$

But $\left[\lim_{T \to 0} P_T (t - tT) / T\right] \to \delta (t - tT)$

$\therefore$ The signal x (t) can be expressed as :

$$\therefore \ x(t) = \sum_{t=-\infty}^{\infty} x(tT)\ \delta(t - tT)$$

Where T = Pulse width

f_s = Sampling frequency = 1/T

T is very small and f_s is high compared to the frequencies present in the signal.

3.5.2 Properties of Convolution Integral : PU : May 05, May 07, Dec. 07, May 10

University Questions

Q. 1	State any two properties of convolution integral.	(May 05
Q. 2	State and explain the properties of convolution integral.	(May 07
Q. 3	State and prove properties of convolution for continuous time domain.	(Dec. 07
Q. 4	State and discuss the properties of convolution.	(May 10
Q. 5	State the properties of convolution integral.	(Dec. 10

Some of the important properties of convolution integral are given below.

(a) **Commutative property :**

If $x_1(t)$ and $x_2(t)$ are continuous time signals then,

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

(b) **Distributive property :**

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

Where $x_3(t)$ is another signal.

(c) **Associative property :**

This property states that,

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

3.6 Computation of Convolution Integral using Graphical Metho

In this section, we will perform convolution integral of two signals with the hel examples.

3.6.1 Unit step to Unit step :

Solved Problems :

Ex. 3.6.1 : Perform the following convolution operation in time domain :

$x_1(t) = x_2(t) = u(t)$ May 07

Soln. : Let $x(t) = u(t)$ and $h(t) = u(t)$

According to the definition of convolution, This equation indicates that we have to perform multiplication of $x(\tau)$ and $h(-\tau)$. Here $h(-\tau)$ is folded version of $h(\tau)$. It is shown in Fig. P. 3.6.2(c).

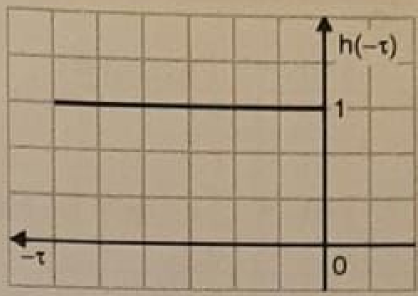


Fig. P. 3.6.2(c)

The result integration of multiplication of $x(\tau)$ and $h(-\tau)$ is shown in Fig. P. 3.6.2(d). Since two waveforms do not overlap; this result is zero.

$$\therefore \quad y(0) = 0 \qquad \text{...(3)}$$

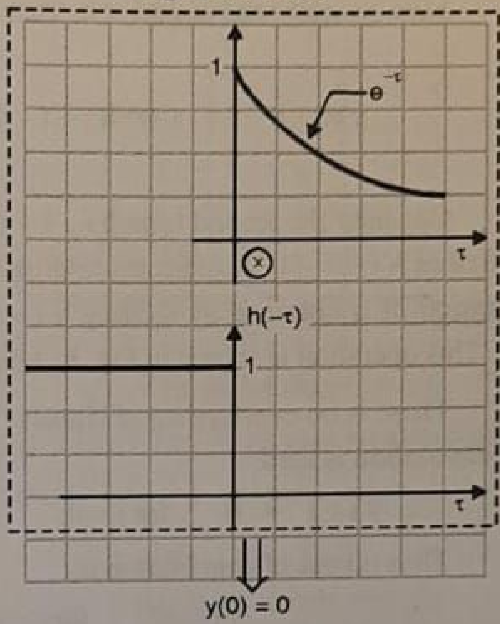


Fig. P. 3.6.2(d)

Now we will consider other possible ranges of 't'. First we will consider $t > 0$. Then Equation (1) becomes.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad \text{...(4)}$$

Note that 't' is positive since it is greater than '0'.

Consider the second term that is $h(t-\tau)$. It can be written as $h(-\tau + t)$

This indicates folded and delay operation. That mean we have to shift the signal $h(-\tau)$ towards right by 't'. This operation is shown in Fig. P. 3.6.2(e).

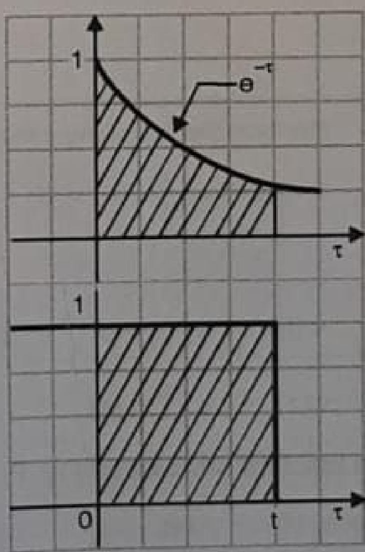


Fig. P. 3.6.2(e)

- Here the value (magnitude) of $h(t-\tau)$ is 1. There is overlapping of two signals in the range '0' to 't'. As per Equation (4) we will get the result of integration, only in the region where two signals overlap. Thus we will make the limits of integration as '0' to 't'.

$$\therefore\ y(t) = \int_0^t x(\tau)\cdot 1\, d\tau$$

$$\therefore\ y(t) = \int_0^t e^{-\tau} d\tau = \left[-1\, e^{-\tau}\right]_0^t = -\left[e^{-t} - e^0\right]$$

$$\therefore\ y(t) = 1 - e^{-t} \qquad \text{... for } t > 0$$

Now consider the range $t < 0$. That means 't' is negative. Thus Equation (1) becomes,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(-t-\tau)\, d\tau.$$

Consider the second term $h(-t-\tau)$. It can be written as $h(-\tau-t)$. It indicates folding and avance operation. It is obtained by shifting $h(-\tau)$ towards left. This operation is shown in Fig. P. 3.6.2(f).

There is no overlapping of signals. Thus result of convolution is zero.

$$\therefore\ y(t) = 0 \quad \text{for } t < 0.$$

Thus output of convolution is,

$$y(t) = \begin{cases} 1 - e^{-t} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}$$

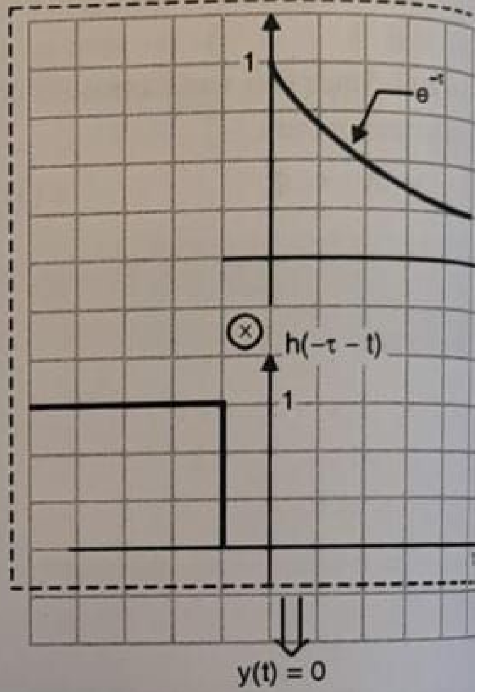


Fig. P. 3.6.2(f)

Ex. 3.6.3 : Perform the following convolution operation of two functions in time domain.

$$x_1(t) = e^{-4t} u(t),\ x_2(t) = u(t-4)$$

Soln. :

Let $x(t) = x_1(t) = e^{-4t} u(t)$,

$$\therefore\ x(\tau) = e^{-4\tau} u(\tau)$$

This function is shown in Fig. P. 3.6.3(a).

Here $x_2(t) = u(t-4)$

Let $h(t) = u(t-4)$

$$\therefore\ h(\tau) = u(\tau - 4)$$

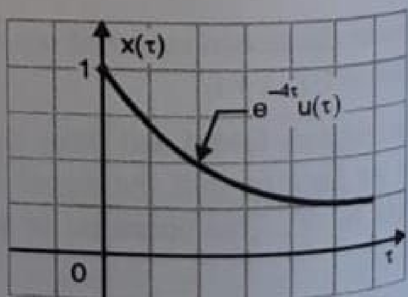


Fig. P. 3.6.3(a)

It indicates unit step delayed by '4'. It is shown in Fig. P. 3.6.3(b).

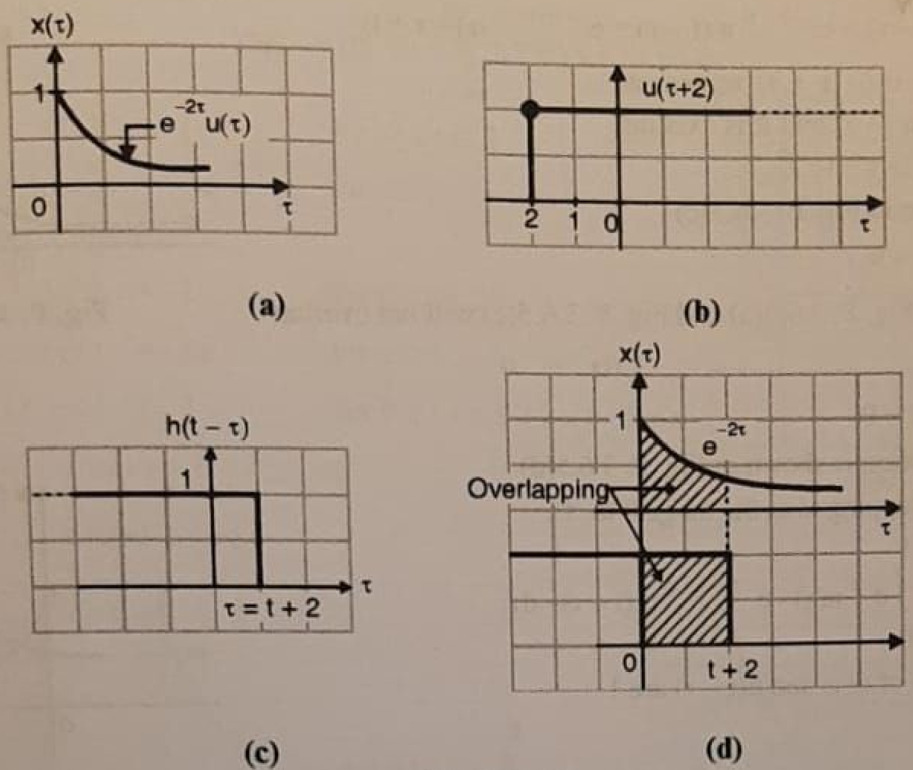


Fig. P. 3.6.4

3.6.3 Exponential to Exponential :

Ex. 3.6.5 : For an LTI system with unit impulse response $h(t) = e^{-2t}u(t)$, determine output to the input $x(t) = e^{-t}u(t)$.

Soln. :

We have,
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Given : $x(t) = e^{-t}u(t) \qquad \therefore\ x(\tau) = e^{-\tau}u(\tau)$

and $h(t) = e^{-2t}u(t) \qquad \therefore\ h(\tau) = e^{-2\tau}u(\tau)$

These signals are as shown in Fig. P. 3.6.5(a) and Fig. P. 3.6.5(b) respectively.

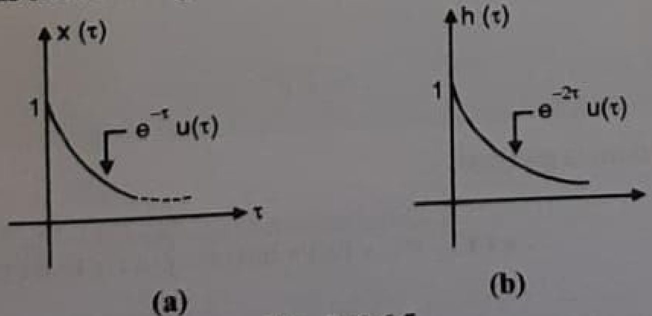


Fig. P. 3.6.5

Now $h(t-\tau) = e^{-2(t-\tau)} u(t-\tau) = e^{-2(-\tau+t)} u(-\tau+t)$

The term $u(-\tau+t)$ represents folded unit step $u(-\tau)$ and it is existing between $\tau = -\infty$ to $\tau = t$

It is shown in Fig. P. 3.6.5(c)

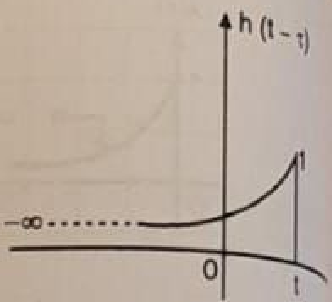


Fig. P. 3.6.5(c)

Case I : When t < 0

For $t < 0$; Fig. P. 3.6.5(a) and Fig. P. 3.6.5(c) will not overlap

$$\therefore \; y(t) = 0$$

Case II : When t > 0

This condition is shown in Fig. P. 3.6.5(d)

Here overlapping is in the range 0 to 't'.

$$\therefore \; y(t) = \int_0^t e^{-\tau} u(\tau) \cdot e^{-2(-\tau+t)} u(t-\tau) \cdot d\tau$$

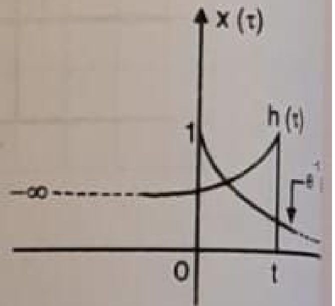


Fig. P. 3.6.5(d)

Amplitude of $u(\tau)$ and $u(t-\tau)$ are 1.

$$\therefore \; y(t) = \int_0^t e^{-\tau} e^{-2(-\tau+t)} d\tau$$

$$\therefore \; y(t) = \int_0^t e^{-\tau} \cdot e^{+2\tau} \cdot e^{-2t} d\tau$$

$$\therefore \; y(t) = e^{-2t} \int_0^t e^{\tau} d\tau = e^{-2t} [e^{\tau}]_0^t$$

$$\therefore \; y(t) = e^{-2t} [e^t - 1]$$

3.6.4 Exponential to Rectangular :

Ex. 3.6.6 : Impulse response of an LTI system is given by :

$$h(t) = \begin{cases} 3e^{-0.5t} & ; \; t \geq 0 \\ 0 & ; \; \text{Otherwise} \end{cases}$$

Find the system output due to the input :

$$x(t) = \begin{cases} 1 & ; \; 0 \leq t \leq 2 \\ 0 & ; \; \text{Otherwise} \end{cases}$$

Also sketch the output.

May 05, 10 M

Soln. : Output of LTI system is given as,

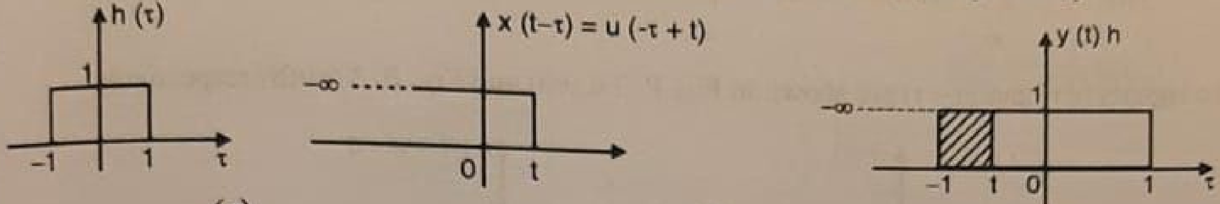
$$\therefore y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$$

According to the definition of convolution.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)^{d\tau} = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Here $x(t-\tau) = u(t-\tau) = u(-\tau + t)$

These two signals are shown in Fig. P. 3.6.8(a) and Fig. P. 3.6.8(b) respectively.



(a) (b) (c)

Fig. P. 3.6.8

Case - I : When $-1 < t < 0$:

This condition is shown in Fig. P. 3.6.8(c)

Here overlapping is in the range -1 to t.

$$\therefore\ y(t) = \int_{-1}^{t} u(\tau)\cdot 1\, d\tau = \int_{-1}^{t} d\tau = [\tau]_{-1}^{t}$$

$$\therefore\ y(t) = t + 1$$

Case - II : When $0 < t < 1$:

It is shown in Fig. P. 3.6.8(d).

Here again overlapping is in the same range -1 to t.

$$\therefore\ y(t) = t + 1$$

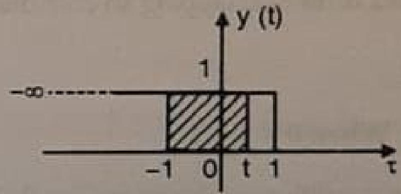


Fig. P. 3.6.8(d)

Case - III : When $t > 1$:

It is shown in Fig. P. 3.6.8(e).

For any value of t which is greater than 1 there is always overlapping from -1 to $+1$.

$$\therefore\ y(t) = \int_{-1}^{1} 1\cdot d\tau = [\tau]_{-1}^{1}$$

$$\therefore\ y(t) = 2$$

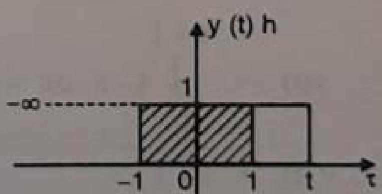


Fig. P. 3.6.8(e)

3.6.6 Rectangular to Rectangular :

Ex. 3.6.9 : Obtain convolution integral of :

$x(t) = 1$ for $-1 \le t \le 1$

$h(t) = 1$ for $0 \le t \le 2$

Soln. : We can write,

$$x(\tau) = 1 \quad \text{for} - 1 \le \tau \le 1$$

$$\text{and} \quad h(\tau) = 1 \quad \text{for } 0 \le \tau \le 2$$

According to equation of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

The signals $h(\tau)$ and $x(-\tau)$ are shown in Fig. P. 3.6.9(a) and Fig. P. 3.6.9(b) respectively.

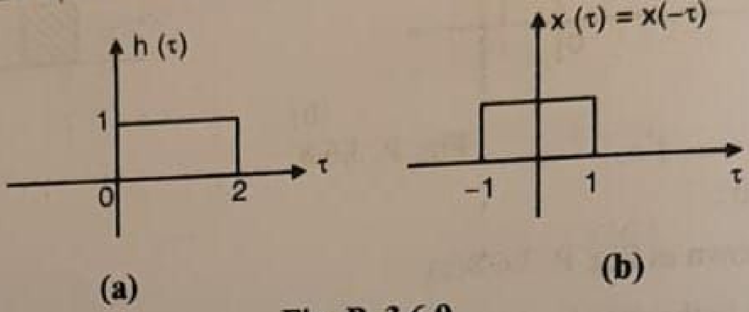


Fig. P. 3.6.9

Case - I : When t < – 1

This condition is shown in Fig. P. 3.6.9(c).

There is no overlapping so convolution is zero.

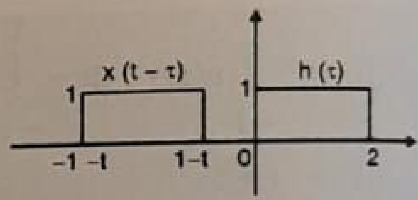


Fig. P. 3.6.9(c)

Case - II : When 0 < t < 2

This condition is shown in Fig. P. 3.6.9(d).

The overlapping is in the range 0 to t + 1

$$y(t) = \int_{0}^{t+1} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore \ y(t) = \int_{0}^{t+1} 1 \cdot 1 \cdot d\tau = [\tau]_0^{t+1}$$

$$\therefore \ y(t) = t + 1$$

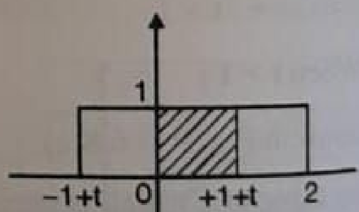


Fig. P. 3.6.9(d)

Case - II : When t > 2

This condition is shown in Fig. P. 3.6.9(e).

Overlapping is in the range $-1+t$ to 2.

$$\therefore \ y(t) = \int_{-1+t}^{2} 1 \cdot 1 \cdot d\tau = [\tau]_{-1+t}^{2} = 2 + 1 - t = 3 - t$$

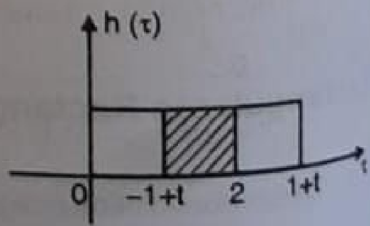


Fig. P. 3.6.9(e)

Case (v) When $3 < t < 4$:

This condition is shown in Fig. P. 3.6.10(h).

Here overlapping is in the range $-2+t$ to 2.

$$\therefore \quad y(t) = \int_{-2+t}^{2} (-1) \times (1)\, d\tau = -[2+2-t]$$

$$\therefore \quad y(t) = t-4$$

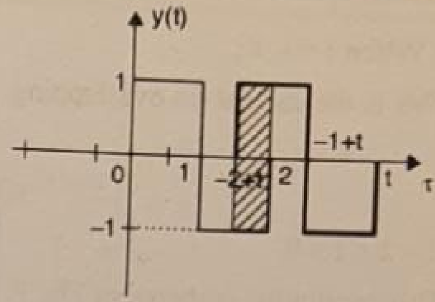


Fig. P. 3.6.10(h)

Case (vi) When $t > 4$:

There will not be any overlapping

$$\therefore \quad y(t) = 0$$

The result of convolution is as follows.

$$y(t) = \begin{cases} 0 & \text{For } t<0 \\ -t & \text{For } 0<t<1 \\ t-2 & \text{For } 1<t<2 \\ 8-3t & \text{For } 2<t<3 \\ t-4 & \text{For } 3<t<4 \\ 0 & \text{For } t>4 \end{cases}$$

Ex. 3.6.11 : Evaluate the convolution integral fo, input x(t) and impulse response h(t) shown in Fig. P. 3.6.11(a) and Fig. P. 3.6.11(b). **Dec. 11, 12 Marks**

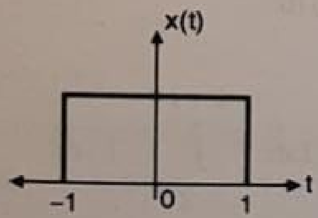


Fig. P. 3.6.11(a)

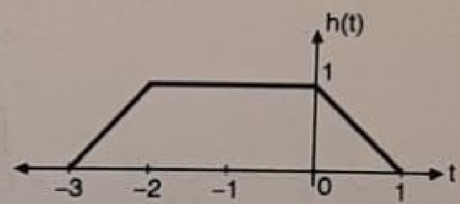


Fig. P. 3.6.11(b)

Soln. : According to the equation of convolution integral.

$$u(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad \text{....(1)}$$

The signals $x(-\tau)$ and $h(\tau)$ are as shown in Fig. P. 3.6.11(c) and P. 3.6.11(d) respectively.

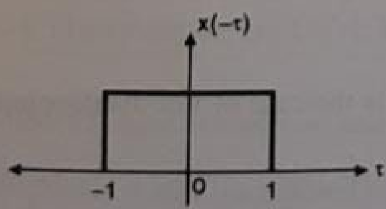


Fig. P. 3.6.11(c)

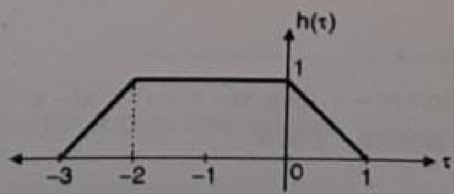


Fig. P. 3.6.11(d)

Case 1 : When $t < -4$:

This is the case of no overlapping. Because signal $x(-\tau)$ will be shifted towards, left by amount $t < -4$.

$$\therefore \quad y(t) = 0$$

Case 2 : $-2 < t < 0$

This conditions is shown in Fig. P. 3.6.11(e).

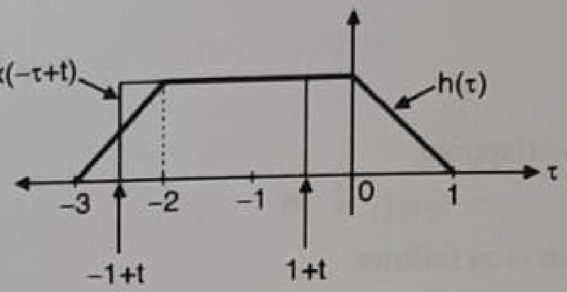


Fig. P. 3.6.11(e)

Here overlapping is from $-1+t$ to $1+t$.

The equation of $h(\tau)$ for this range is,

$$h(\tau) = \begin{cases} \tau + 3 & \text{for} \quad (-1+t) < \tau < -2 \\ 1 & \text{for} \quad -2 < \tau < 1+t \end{cases}$$

$$\therefore \ y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-1+t}^{-2} (\tau+3)\cdot 1\, d\tau + \int_{-2}^{1+t} 1\cdot 1\cdot d\tau$$

$$= \left[\frac{\tau^2}{2} + 3\tau\right]_{-1+t}^{2} + [\tau]_{-2}^{1+t}$$

$$= \left[\left(\frac{4}{2}+6\right) - \left(\frac{(-1+t)^2}{2} + 3(-1+t)\right)\right] + [1+t+2]$$

$$= 8 - \left[\frac{(-1+t)^2}{2} - 3 + 3t\right] + 3 + t$$

$$\therefore \ y(t) = 8 - \frac{(-1+t)^2}{2} + 3 - 3t + 3 + t = 14 - 2t - \frac{(-1+t)^2}{2}$$

Case 3 : $t = -1$

When $t = -1$ then $x(-\tau + t) = x(-\tau - 1)$ and this is the case of full overlapping. So range of integration is from -2 to 0.

$$\therefore \quad y(t) = \int_{-2}^{0} 1\cdot 1\cdot d\tau = [\tau]_{-2}^{0} = 2$$

Decomposition of input sequence x (n) :

The given input sequence is,

$$x(n) = \{1, 2, \underset{\uparrow}{1}, 2, 1\}$$

Its range is from – 2 to +2. Now decomposition means obtaining a particular sample from given sequence x(n). Suppose we want to obtain sample at n = – 2 that means x (– 2). Then take the multiplication of x (n) and δ (n + 2). We know that the value of δ (n + 2) is one only at n = – 2, while for all other values of n, it is zero. Thus if we take multiplication of x (n) and δ (n + 2) then we will get the sample x (– 2). Because all other terms become zero, after multiplication. This is shown in Fig. 3.7.1(e).

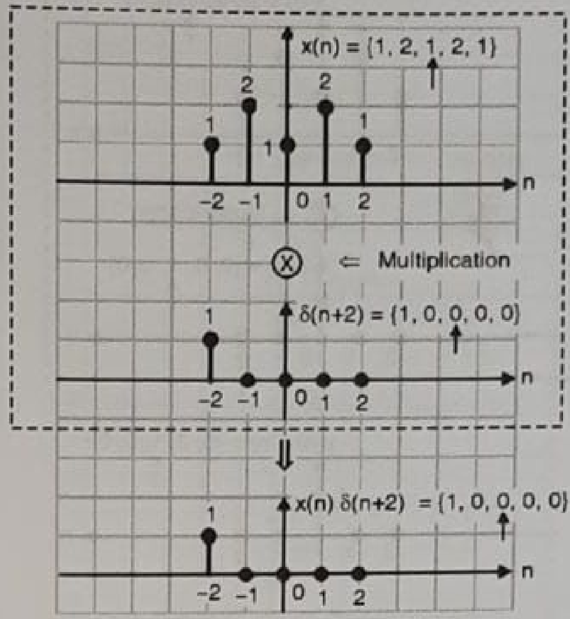


Fig. 3.7.1(e) : Decomposed input signal sample at n = – 2

Similarly other decomposed input signal samples are obtained.

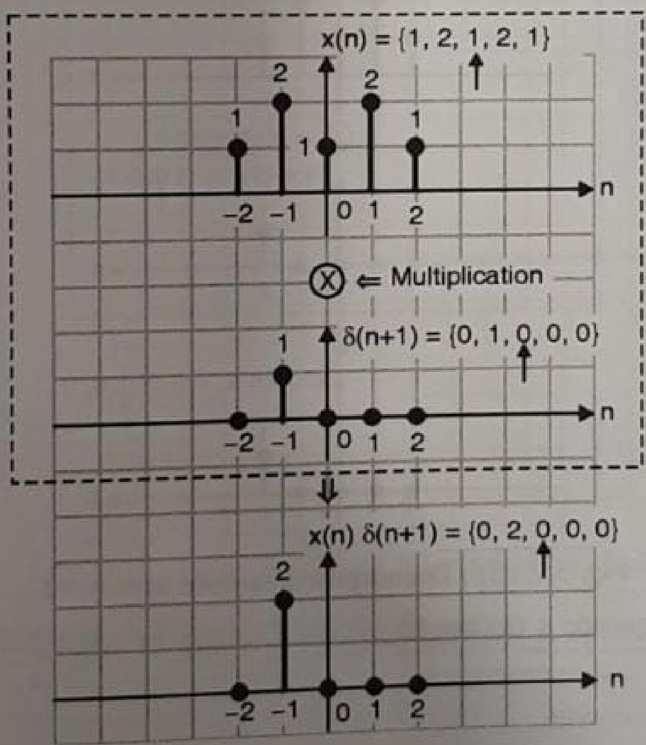


Fig. 3.7.1(f) : Decomposed sample at n = – 1

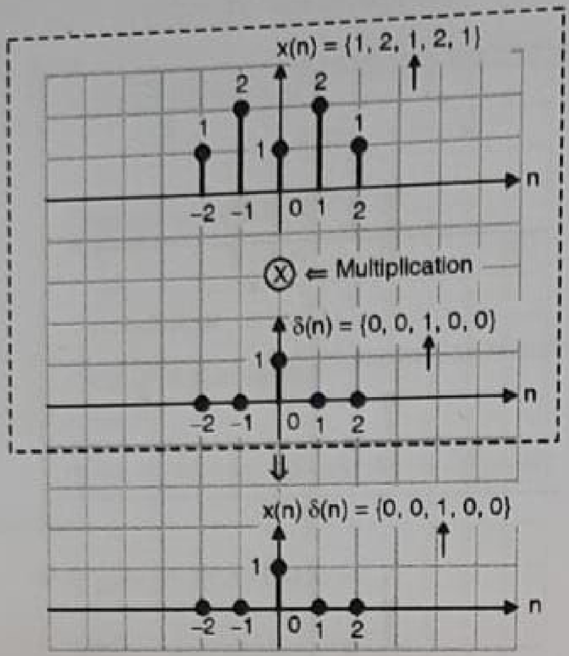


Fig. 3.7.1(g) : Decomposed sample at n = 0

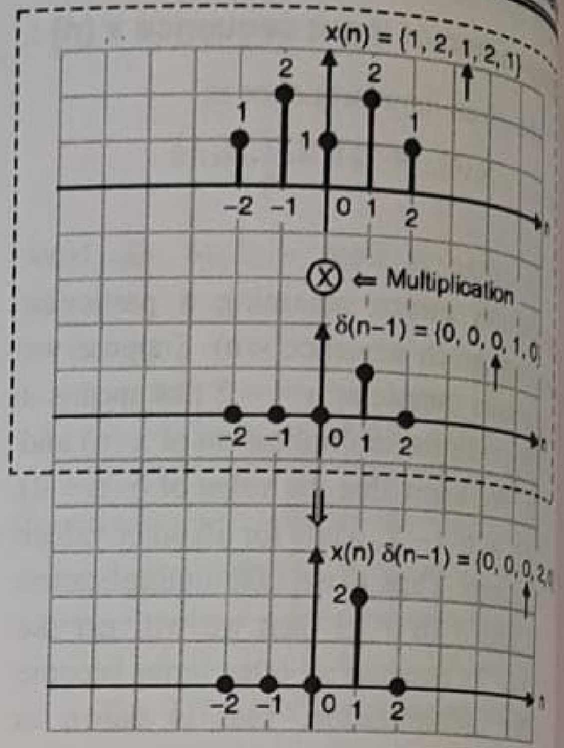


Fig. 3.7.1(h) : Decomposed sample at n = +1

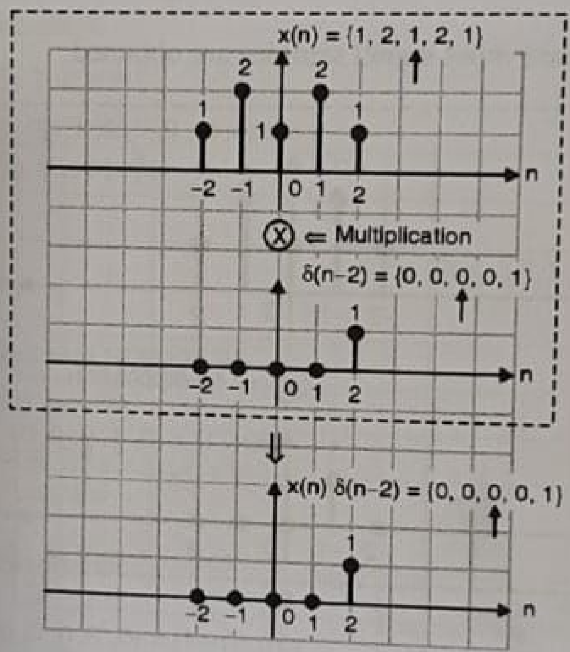


Fig. 3.7.1(i) : Decomposed sample at n = + 2

Now the original sequence x (n) can be obtained simply by adding all decomposed samples as shown in Fig. 3.7.2.

Mathematically we can write,

$$x(n) = x(n) \cdot \delta(n+2) + x(n) \cdot \delta(n+1) + x(n) \cdot \delta(n) + x(n) \cdot \delta(n-1) + x(n) \cdot \delta(n-2) \quad ...(3.7.6)$$

Observe every decomposed sample. We get a decomposed sample only at n = k. For example sample x (− 2) is obtained by multiplication of x (n) and δ (n + 2) as shown in Fig. 3.7.1(e). This sample is present at n = k = − 2. Thus for simplicity replacing x (n) by x (k) in R.H.S. of Equation (3.7.6) we get, We will have to calculate output for different values of 'n'

For n = 0 we get,
$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) \quad ...(3.9.2)$$

Note that $h(-k)$ indicates folding of $h(k)$.

For n = 1 we get,
$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(1-k) \quad ...(3.9.3)$$

Here the term $h(1-k)$ can be written as $h(-k+1)$. Thus Equation (3.9.3) becomes,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k+1) \quad ...(3.9.4)$$

Here $h(-k+1)$ indicates shifting of folded signal $h(-k)$. It indicates that $h(-k)$ is delayed by '1' sample. Similarly for other values of 'n' output $y(n)$ is calculated.

Thus different operations involved in the calculation of linear convolution are as follows :

1. Folding operation : It indicates folding of sequence $h(k)$.
2. Shifting operation : It indicates time shifting of $h(-k)$ eg. $h(-k+1)$.
3. Multiplication : It indicates multiplication of $x(k)$ and $h(n-k)$.
4. Summation : It indicates addition of all product terms obtained because of multiplication of $x(k)$ and $h(n-k)$.

Solved Problems :

Ex. 3.9.1 : For the following signals, determine and sketch convolution $y(n)$ graphically :

$$x(n) = \begin{cases} \frac{1}{3}n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } h(n) = \begin{cases} 1 & -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Soln. : First we will write sequences $x(n)$ and $h(n)$ by putting different values of n.

Sequence x (n) :

Here range of $x(n)$ is from $n = 0$ to $n = 6$ and $x(n) = \frac{1}{3}n$

For $n = 0 \Rightarrow x(0) = \frac{1}{3} \times 0 = 0$

For $n = 1 \Rightarrow x(1) = \frac{1}{3} \times 1 = \frac{1}{3}$

For $n = 2 \Rightarrow x(2) = \frac{1}{3} \times 2 = \frac{2}{3}$

For $n = 3 \Rightarrow x(3) = \frac{1}{3} \times 3 = \frac{3}{3}$

For $n = 4 \Rightarrow x(4) = \frac{1}{3} \times 4 = \frac{4}{3}$

For $n = 5 \Rightarrow x(5) = \frac{1}{3} \times 5 = \frac{5}{3}$

For $n = 6 \Rightarrow x(6) = \frac{1}{3} \times 6 = \frac{6}{3}$

Thus x (n) can be written as,

$$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6)\}$$

$$\therefore \; x(n) = \left\{ \underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3} \right\}$$

Now h (n) is given as,

$$h(n) = 1 \quad \text{for } -2 \le n \le 2$$
$$= 0 \quad \text{otherwise}$$

Thus range of h (n) is from – 2 to +2

$$\therefore \; h(n) = \{h(-2), h(-1), h(0), h(1), h(2)\}$$

$$\therefore \; h(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$$

Now the sequences x (k) and h (k) are obtained by replacing 'n' by 'k'.

$$\therefore x(k) = \left\{ \underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3} \right\} \quad ...(3)$$

$$\text{and } h(k) = \{1, 1, \underset{\uparrow}{1}, 1, 1\} \quad ...(4)$$

The linear convolution of x (k) and h (k) is given by,

$$y(n) = x(k) * h(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad ...(5)$$

Range of 'n' :

Lower limit of y (n) i.e. $y_l = x_l + h_l = 0 + (-2) = -2$ (Addition of lower range of x (n) and h (n))

Higher limit of y (n) i.e. $y_h = x_h + h_h = 6 + 2 = 8$ (Addition of upper range of x (n) and h (n))

Thus range of n is from -2 to $+8$.

Range of 'k' :

Range of 'k' is similar to x (k). Thus range of 'k' is from 0 to 6.

Now using Equation (5) we will obtain n for different values of n from – 2 to 8.

$$y(n) = \sum_{k=0}^{6} x(k)\, h(n-k) \quad ...(6)$$

Calculation of y (– 2) :

Putting n = – 2 in Equation (6) we get,

$$y(-2) = \sum_{k=0}^{6} x(k)\, h(-2-k) \quad ...(7)$$

Step 1 : First plot x (k) as shown in Fig. P. 3.9.1(a) :

Plot of h (k) is shown in Fig. P. 3.9.1(b). From this obtain h (– k) which is folded version of h (k) as shown in Fig. P. 3.9.1(c). Observe that h (k) and h (– k) are same in this case.

Note : Arrow always indicates the sample at n = 0. Thus arrow should be marked at sample y (0).

The sequence y (n) is graphically represented as shown in Fig. P. 3.9.1(f).

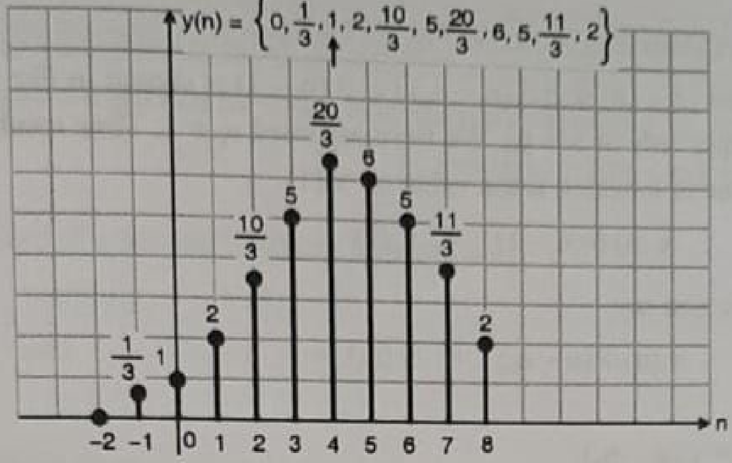


Fig. P. 3.9.1(f) : Result of linear convolution

Ex. 3.9.2 : Prove and explain graphically the difference between relations :

1. $x(n)\,\delta(n - n_0) = x(n_0)$ 2. $x(n) * \delta(n - n_0) = x(n - n_0)$

Soln. :

1. $\mathbf{x(n)\,\delta(n - n_0) = x(n_0)}$

Here x (n) is input sequence. Let the input sequence be given by,

$$x(n) = \{1, 1, \underset{\uparrow}{2}, 1, 1\}$$

This sequence is plotted as shown in Fig. P. 3.9.2(a).

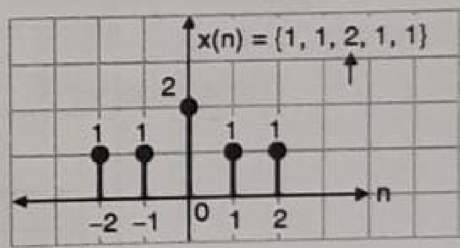
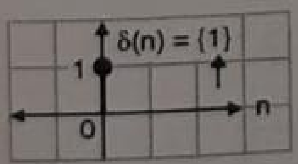


Fig. P. 3.9.2(a) : Sequence x (n)

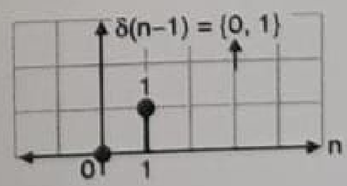
We know that δ (n) represent unit impulse. It is defined as,

$$\begin{aligned} \delta(n) &= 1 && \text{for } n = 0 \\ &= 0 && \text{elsewhere} \end{aligned}$$

Sequence δ (n) is shown in Fig. P. 3.9.2(b).



(b) Unit impulse, δ (n)



(c) Sequence δ (n – 1)

Fig. P. 3.9.2

Now δ (n – n_0) indicates delay of δ (n) by no units. So sequence δ (n – n_0) can be represented as,

$$\begin{aligned} \delta(n - n_0) &= 1 && \text{for } n = n_0 \\ &= 0 && \text{for } n \neq n_0 \end{aligned}$$

For example, say $n_0 = 1$. Thus we get,

$$\delta(n-1) = 1 \quad \text{for } n = 1$$
$$= 0 \quad \text{for } n \neq n_0$$

The sequence $\delta(n-1)$ is plotted as shown in Fig. P. 3.9.2(c).

Now multiplication of $x(n)$ and $\delta(n-n_0)$ i.e. $\delta(n-1)$ is shown in Fig. P. 3.9.2(d). Here we ... getting the resultant sample only at $n = n_0$. We have assumed $n_0 = 1$. Thus result of multiplication is ... sample at $n = 1$. This value is 1; which is the value of $x(1)$.

$$\therefore \quad x(n)\,\delta(n-1) = x(1)$$

Thus in general we can say,

$$x(n)\,\delta(n-n_0) = x(n_0)$$

Hence proved.

2. $x(n) * \delta(n-n_0) = x(n-n_0)$

Here '*' indicates convolution. So we have to calculate convolution of $x(n)$ and $\delta(n-n_0)$ ...
According to the definition of convolution,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \quad ...(1)$$

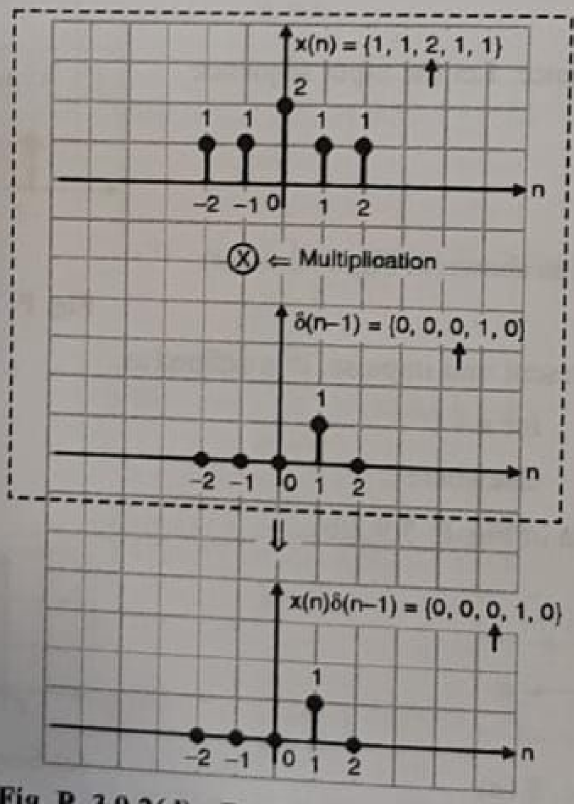


Fig. P. 3.9.2(d) : Product of $x(n)\,\delta(n-n_0)$

Let $\quad h(n) = \delta(n-n_0) \quad ...(2)$

Now assume two sequence as follows,

$$y(2) = \sum_{k=-1}^{1} x(k)\, h(2-k) = 1 \qquad ...(7)$$

The output sequence y (n) is,

$$y(n) = \{ y(-1), y(0), y(1), y(2)\}$$

$$\therefore\ y(n) = \{0, \underset{\uparrow}{1}, 1, 1\}$$

This sequence is plotted in Fig. P. 3.9.2(i).

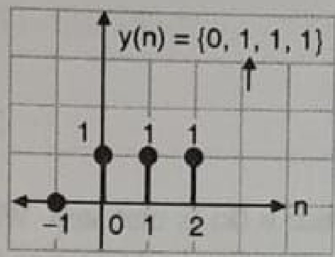


Fig. P. 3.9.2(i) : Output y (n)

Now the output sequence y (n) can also be written as,

$$\therefore\ y(n) = \{\underset{\uparrow}{1}, 1, 1\}$$

Observe graph of x (k) shown in Fig. P. 3.9.2(e) and graph of y (i) shown in Fig. P. 3.9.2(i). It shows that output y (n) is obtained by delaying x (k) by '1' sample.

$$\therefore\ y(n) = x(k-1)$$

But x (k) is same as x (n)

$$\therefore\ y(n) = x(n-1)$$

Initially we have assumed $n_0 = 1$

$$\therefore\ y(n) = x(n-n_0)$$

$$\therefore\ y(n) = x(n) * \delta(n-n_0) = x(n-n_0)$$

Hence proved.

Note : This indicates that the convolution of input sequence x (n) with delayed unit impulse is equivalent to delay of input sequence by corresponding samples (n_0).

Ex. 3.9.3 : Use discrete convolution to find the response to the input $x(n) = a^n u(n)$ of the LTI system with impulse response $h(n) = b^n u(n)$.

Soln. :

Given : $x(n) = a^n u(n)$ and $h(n) = b^n u(n)$.

Sequences x (k) and h (k) can be directly written as,

$x(k) = a^k u(k)$ and $h(k) = b^k u(k)$

According to definition of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(1)$$

$$\text{We have } h(k) = b^k u(k) \qquad ...(2)$$

Sequence h (n – k) is obtained by replacing k by n – k in Equation (2)

$$\therefore\ h(n-k) = b^{n-k}\, u(n-k)$$

Thus Equation (1) becomes,
$$y(n) = \sum_{k=-\infty}^{\infty} a^k u(k) \cdot b^{n-k} u(n-k)$$

$$\therefore\ y(n) = \sum_{k=-\infty}^{\infty} a^k \cdot b^{n-k} u(k)\, u(n-k)$$

Sequence u (k) is unit step. While u (n – k) = u (– k + n) indicates delay of u (– k) by 'n' samples. This sequence is obtained as shown in Fig. P. 3.9.3.

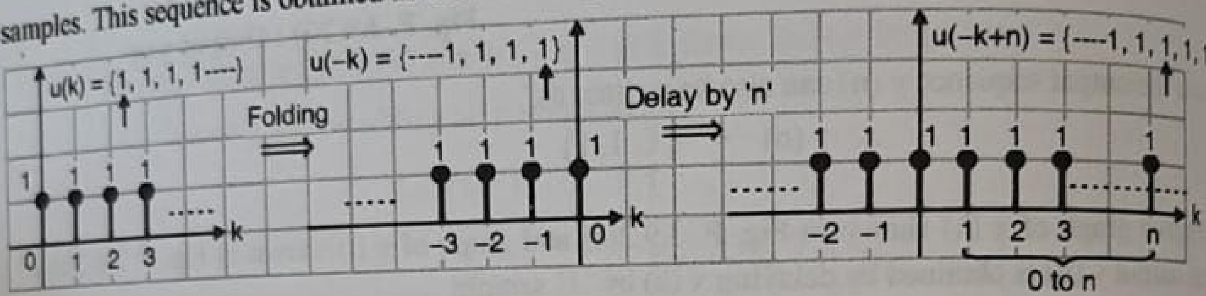


Fig. P. 3.9.3 : Sequence u ((n – k)

Now the product of u (k) and u (n – k) is shown in Fig. P. 3.9.3(a).

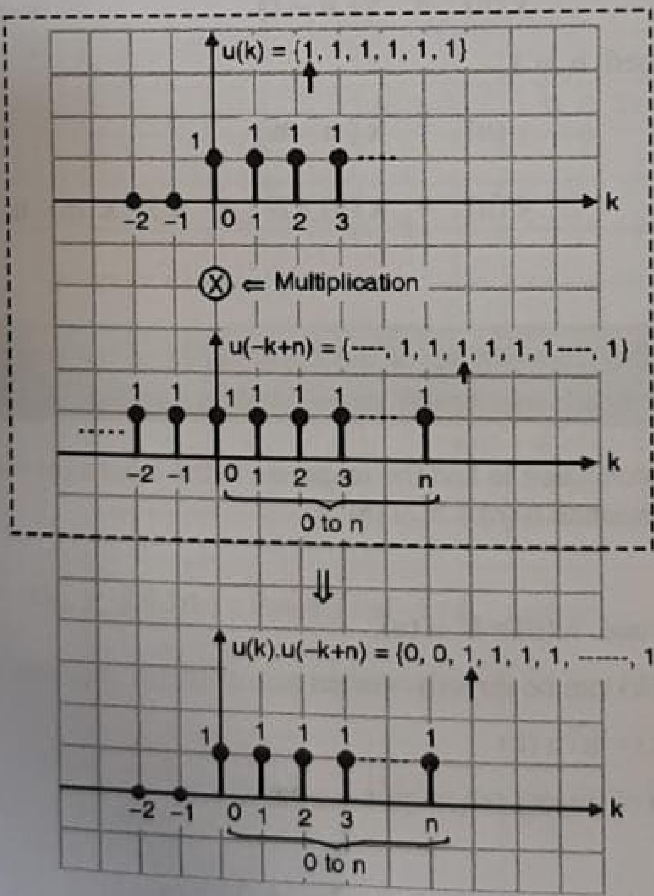


Fig. P. 3.9.3(a) : Product of u (k) and u (n – k)

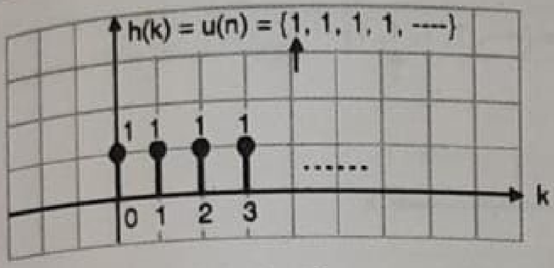
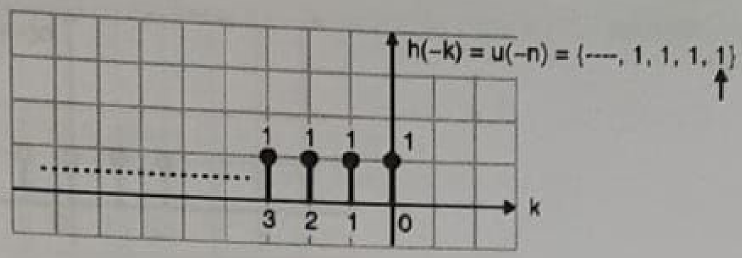


Fig. P. 3.9.4(b) : Sequence $h(k) = u(n)$



(c) : Sequence $h(-k) = u(-n)$

Fig. P. 3.9.4

According to Equation (4) y (0) is obtained by multiplying x (k) by h (– k) and then adding all product terms. This is shown in Fig. P. 3.9.4(d).

$$\therefore \quad y(0) = 2$$

Calculation of y (1) :

Putting n = 1 in Equation (3) we get,

$$y(n) = \sum_{k=0}^{2} x(k)\, h(1-k) \qquad \ldots(5)$$

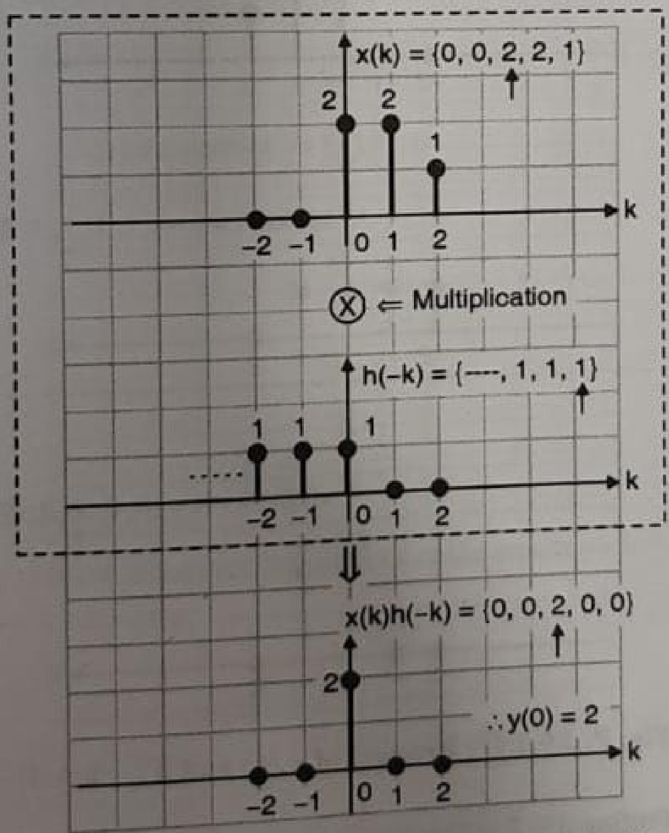


Fig. P. 3.9.4(d) : Product of x (k) and h (– k)

Here h (1 – k) is same as h (– k + 1). It indicates delay of h (– k) by '1' sample. This sequence is shown in Fig. P. 3.9.4(e).

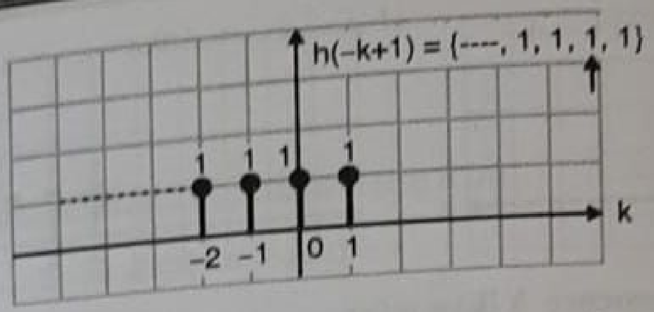


Fig. P. 3.9.4(e) : Sequence h (− k + 1)

The multiplication of x (k) and h (− k + 1) is shown in Fig. P. 3.9.3(f).

Now y (1) is obtained by adding all product terms.

$$\therefore \; y(1) = (2 \times 1) + (2 \times 1) = 2 + 2 = 4$$

$$\therefore \; y(1) = 4$$

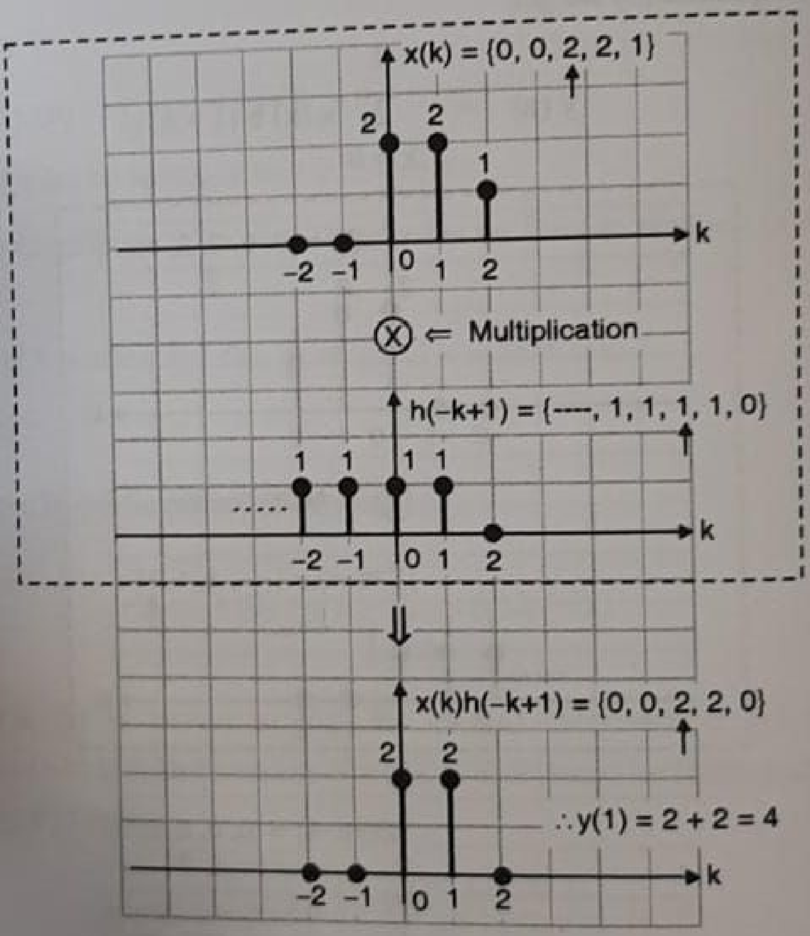


Fig. P. 3.9.4(f) : Product of x (k) and h (− k + 1)

Calculation of y (2) : Putting n = 2 in Equation (3) we get,

$$y(2) = \sum_{k=0}^{2} x(k)\, h(2-k)$$

$$\therefore \; y(n) = \{ \underset{\uparrow}{2}, 4, 5, 5, 5, \ldots \} \quad \ldots(8)$$

General expression : The given sequence x (n) is ,

$$\therefore \; x(n) = \{ \underset{\uparrow}{2}, 2, 1\} \text{ and } h(n) = u(n) = \{ \underset{\uparrow}{1}, 1, 1, 1, \ldots\}$$

$$\therefore \; x(0) = 2 \qquad \therefore \; h(0) = u(0) = 1$$

$$x(1) = 2 \qquad h(1) = u(1) = 1$$

and $$x(2) = 1 \qquad h(2) = u(2) = 1$$

$$h(3) = u(3) = 1 \ldots$$

Now observe Equations (6) and (7) carefully. We can write the generalized expression to obtain Equation (8) as follows,

$$y(n) = x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots \quad \ldots(9)$$

Let us verify this equation.

To calculate y (0) put n = 0 in Equation (9).

$$\therefore \; y(0) = x(0)\,u(0) + x(1)\,u(-1) + x(2)\,u(-2)$$

$$\therefore \; y(0) = (2 \times 1) + (2 \times 0) + (1 \times 0) = 2$$

This is because u (0) = 1, u (– 1) = 0 and u (– 2) = 0. This can be observed from unit step shown in Fig. P. 3.9.4(k).

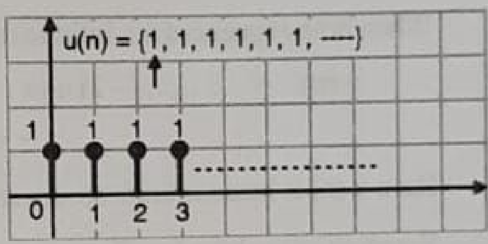


Fig. P. 3.9.4(k) : Unit step u (n)

Similarly putting n = 1 in Equation (9) we get,

$$\therefore \; y(1) = x(0)\,u(1) + x(1)\,u(1) + x(2)\,u(-1)$$

$$\therefore \; y(1) = (2 \times 1) + (2 \times 1) + (1 \times 0) = 2 + 2 + 0$$

$$\therefore \; y(1) = 4$$

Similarly other values can be verified.

Ex. 3.9.5 : Obtain linear convolution of two discrete time signals given as :

$x(n) = u(n)$

$h(n) = a^n u(n), a < 1$

May 07

Soln. :

Given x (n) = u (n) i.e. unit step which is shown in Fig. P. 3.9.4(k). h (n) = a^n u (n) is an exponential signal. Here a^n is exponential signal which is multiplied by u (n). We know that u (n) is present only for positive values of 'n'. That means from n = 0 to n = ∞. So this multiplication indicates that a^n is present for positive values of 'n'. It will not affect the magnitude of a^n, because magnitude of u (n) is 1. This sequence a^n is shown in Fig. P. 3.9.5(a).

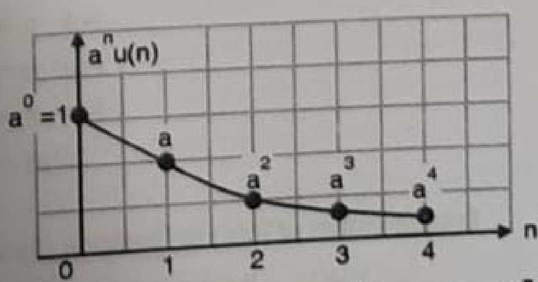


Fig. P. 3.9.5(a) : Exponential sequence a^n

Recall the equation of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

In this equation h (n – k) is delayed version of h (k). Instead of that we can delay x (k) and ... h (k) as it is.

$$\therefore \; y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

This is commutative property of convolution which will be discussed later in this chapter ... will use Equation (2) in this example because it is easy to delay u (n) than $a^n u(n)$.

Now we have derived generalized expression of convolution for finite sequence with an infi... sequence. This equation is,

$$y(n) = x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots$$

When both sequences are infinite then Equation (3) can be easily modified as,

$$y(n) = \ldots\ldots x(-2)\,u(n+2) + x(-1)\,u(n+1) + x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots$$

Here x (n) is some infinite sequence and h (n) = u (n).

Equation (4) is based on Equation (1). In this case input x (k) is kept as it is and h (k) is delay... But if we want to use Equation (2) to obtain linear convolution then,

$$y(n) = \ldots\ldots h(-2)\,u(n+2) + h(-1)\,u(n+1) + h(0)\,u(n) + h(1)\,u(n-1) + h(2)\,u(n-2) + \ldots$$

In Equation (5) we are taking x (n) = u (n).

The given equation of h (n) is,

$$h(n) = a^n u(n)$$

$$\therefore h(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Sequence h (n) can be written by putting values of n in Equation (7). That means,

$$\text{For } n = 0 \Rightarrow h(n) = h(0) = a^0 = 1$$

$$\text{For } n = 1 \Rightarrow h(n) = h(1) = a^1 = a$$

$$\text{For } n = 2 \Rightarrow h(n) = h(2) = a^2$$

$$\text{For } n = 3 \Rightarrow h(n) = h(3) = a^3 \ldots\ldots$$

$$\therefore h(n) = \{\underset{\uparrow}{1}, a, a^2, a^3 \ldots\}$$

Putting values of h (n) in Equation (5) we get,

$$\therefore \; y(n) = 1\,u(n) + a\,u(n-1) + a^2 u(n-2) + a^3 u(n-3) + \ldots\ldots$$

Step 2 : Range of 'n' for y (n) is,

Lowest index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 3 + 3 = 6$

Thus range of y (n) is from y (0) to y (6).

Now range of 'k' is same as x (k).

Thus range of 'k' is from k = 0 to k = 3.

Step 3 : Putting the value of 'k' in Equation (1) we get,

$$y(n) = \sum_{k=0}^{3} x(k)\,h(n-k) \quad \text{...(2)}$$

Calculation of y (0) :

Putting n = 0 in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} x(k)\,h(0-k) = \sum_{k=0}^{3} x(k)\,h(-k)$$

Expanding the summation by putting values of 'k' we get,

$$y(0) = x(0)\,h(-0) + x(1)\,h(-1) + x(2)\,h(-2) + x(3)\,h(-3)$$

$$\therefore\ y(0) = (1\times1) + (1\times0) + (1\times0) + (1\times0) \text{ ... as } h(-1) = h(-2) = h(-3) = 0$$

$$\therefore\ y(0) = 1$$

Calculation of y (1) :

Putting n = 1 in Equation (2) we get,

$$y(1) = \sum_{k=0}^{3} x(k)\,h(1-k)$$

Expanding the summation we get,

$$y(1) = x(0)\,h(1) + x(1)\,h(0) + x(2)\,h(-1) + x(3)\,h(-2)$$

$$\therefore\ y(1) = (1\times1) + (1\times1) + (1\times0) + (1\times0).$$

$$\therefore\ y(1) = 1 + 1 \quad \text{.... as } h(-1) = h(-2) = 0$$

$$\therefore\ y(1) = 2$$

Similarly we can calculate the remaining values as follows :

$$y(2) = \sum_{k=0}^{3} x(k)\,h(2-k) = 3$$

$$y(3) = \sum_{k=0}^{3} x(k)\,h(3-k) = 4$$

$$y(4) = \sum_{k=0}^{3} x(k)\,h(4-k) = 3$$

$$y(5) = \sum_{k=0}^{3} x(k)\,h(5-k) = 2$$

$$y(6) = \sum_{k=0}^{3} x(k)\,h(6-k) = 1$$

Step 4 : The result of convolution y (n) is given by,

$$y(n) = \{y(0), y(1), y(2), y(3), y(4), y(5), y(6)\}$$

$$\therefore\ y(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 3, 2, 1\}$$

Note that arrow is marked at the sample y (0).

Ex. 3.9.7 : Compute the linear convolution of following :

x (n) = 1 and h (n) = {2, 1, 2, 1}

Soln. :

Step 1 : The sequences x (k) and h (k) can be written as,

$$x(k) = \{\underset{\uparrow}{1}\} \text{ and } h(k) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$\therefore\ x(0) = 1 \qquad \therefore\ h(0) = 2$$

$$h(1) = 1$$

$$h(2) = 2$$

$$h(3) = 1$$

Note that arrow is not present in the given sequences of x (n) and h (n). So by default it is at the (starting) position.

Step 2 : Range of 'n' for y (n) is,

Lowest index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 0 + 3 = 3$

Thus range of y (n) is from y (0) to y (3).

Step 3 : According to definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$y(1) = h(1)x(1) + h(0)x(1)$$

$$y(2) = h(2)x(0) + h(1)x(1) + h(0)x(2)$$

$$y(3) = h(2)x(1) + h(1)x(2)$$

and $$y(4) = h(2)x(2)$$

After calculating all these values; write down the result of convolution as,

$$y(n) = \{\underset{\uparrow}{y(0)}, y(1), y(2), y(3), y(4)\}$$

Ex. 3.9.8 : Compute the convolution $y(n) = x(n) * h(n)$

where $x(n) = \{1, 1, \underset{\uparrow}{0}, 1, 1\}$ and $h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$

Dec. 10, 7 Marks

Soln. : Here x (n) contains '5' samples and h (n) has '4' samples. To make the length of x (n) and h (n) same; rewrite the sequence h (n) as follows :

$$h(n) = \{1, -2, -3, \underset{\uparrow}{4}, 0\}$$

We can add zeros at the end or at the beginning of sequence. This will not affect the given sequence. This method of adding zeros, to adjust the length of sequence is called as **zero-padding.**

Range of 'n' : The range of 'n' for y (n) is calculated as follows :

Lowest index of y (n) $\Rightarrow y_l = x_l + h_l = -2 + (-3) = -5$

and **Highest index of y (n)** $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 5) to y (3). Now using tabulation method the convolution is obtained as shown in Fig. P. 3.9.8.

	x(n)				
h(n)	1	1	0	1	1
1					
-2					
-3					
4					
0					

Fig. P. 3.9.8(a) : Matrix of x (n) and h (n)

	1	1	0	1	1
1	1 × 1	1 × 1	1 × 0	1 × 1	1 × 1
– 2	– 2 × 1	– 2 × 1	– 2 × 0	– 2 × 1	– 2 × 1
– 3	– 3 × 1	– 3 × 1	– 3 × 0	– 3 × 1	– 3 × 1
4	4 × 1	4 × 1	4 × 0	4 × 1	4 × 1
0	0 × 1	0 × 1	0 × 0	0 × 1	0 × 1

Fig. P. 3.9.8(b) : Convolution of x (n) and h (n)

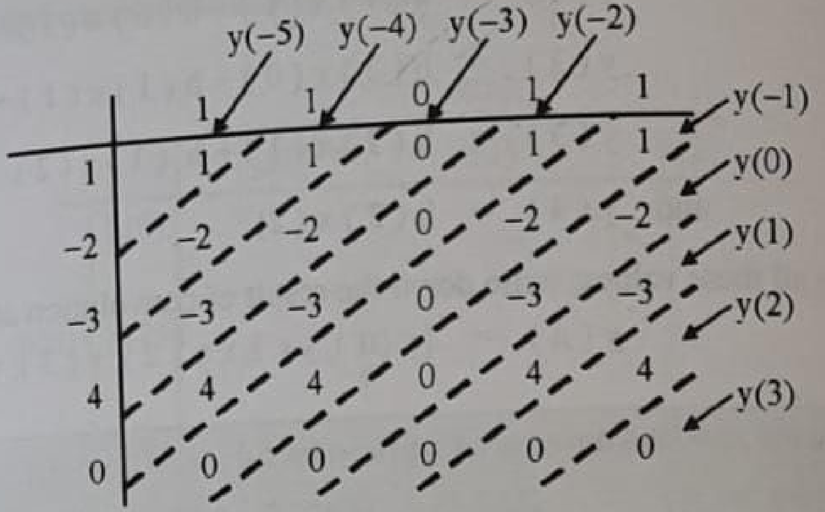


Fig. P. 3.9.8(c) : Diagonally separation of elements

Different values of y (n) are calculated by adding corresponding elements as follows :

$$y(-5) = 1$$
$$y(-4) = -2+1 = -1$$
$$y(-3) = -3-2+0 = -5$$

$y(-2) = 4-3+0+1 = 2$ $\quad$ $y(-1) = 0+4+0-2+1 = 3$

$y(0) = 0+0-3-2 = -5$ $\quad$ $y(1) = 0+4-3 = 1$

$y(2) = 0+4 = 4$ $\quad$ $y(3) = 0$

Thus result of convolution y (n) is ,

$$y(n) = \{y(-5),\ y(-4),\ y(-3),\ y(-2),\ y(-1),\ \underset{\uparrow}{y(0)},\ y(1),\ y(2),\ y(3)\}$$

$$\therefore\ y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4, 0\}$$

Neglecting the last '0' term we have,

$$y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4\}$$

Ex. 3.9.9 : Compute $y(n) = x(n) * h(n)$

If $x(n) = h(n) = \{\underset{\uparrow}{1}, 2, -1, 3\}$

Soln. :

Range of n :

Lowest range of y (n) ⇒ $y_l = x_l + h_l = 0 + 0 = 0$

Highest range of y (n) ⇒ $y_h = x_h + h_h = 3 + 3 = 6$

Using tabular method; the convolution is obtained as shown in Fig. P. 3.9.9.

Soln. :

Given $x(n) = \{2, \underset{\uparrow}{3}, 1, 4\}$

and $h(n) = \{-1, 2, \underset{\uparrow}{3}\} = \{-1, 2, \underset{\uparrow}{3}, 0\}$

Here range of 'n' in y (n) is n = – 3 to 2.

We will use tabular method to obtain

$y(n) = x(n) * h(n)$

	2	3	1	4
	2	6	2	8
–1	–2	–3	–1	–4
2	4	6	2	8
3	6	9	3	12
0	0	0	0	0

Fig. P. 3.9.13

$$\therefore \quad y(n) = \{-2, 1, 11, \underset{\uparrow}{7}, 11, 12, 0\}$$

3.9.4 Multiplication Method of Linear Convolution :

This is another easy method to obtain convolution of two sequences. This method is similar to multiplication of multidigit numbers. Write down the two sequences, x (n) and h (n) and obtain the multiplication by usual method. The result of multiplication will be equal to the convolution of two sequences. We will solve one example by this method.

Ex. 3.9.14 : Obtain linear convolution of following sequences :

$x(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$ and $h(n) = \{2, 2, -\underset{\uparrow}{1}, 1\}$

Soln. :

First we will decide the range of 'n' for y (n).

Lowest index $\Rightarrow y_l = x_l + h_l = -1 + (-2) = -3$

Highest index $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 3) to y (3)

We will solve this problem using multiplication method.

	x(n) :-				1	2	1	2
Multiplication	h(n) :- ⊗				2	2	−1	1
					1	2	1	2 ⇐ x(n) is multiplied by (1)
Addition → ⊕				−1	−2	−1	−2	0 ⇐ x(n) is multiplied by (−1); Zero digit similar to normal multiplication
Addition → ⊕			2	4	2	4	0	0 ⇐ x(n) is multiplied by (2); Two zero digits similar to normal multiplication
Addition → ⊕		2	4	2	4	0	0	0 ⇐ x(n) is multiplied by (2); Three zero digits similar to normal multiplication
		2	6	5	5	5	−1	2 ⇐ These values are obtained by adding all digits in corresponding columns

This result of multiplication is same as linear convolution. We know that range of y (n) is y (− 3) to y (3). So the first term from L.H.S. represents the value of y (− 3). Now mark the arrow at the position of y (0). Thus output y (n) can be written as,

$$y(n) = \{2, 6, 5, \underset{\uparrow}{5}, 5, -1, 2\}$$

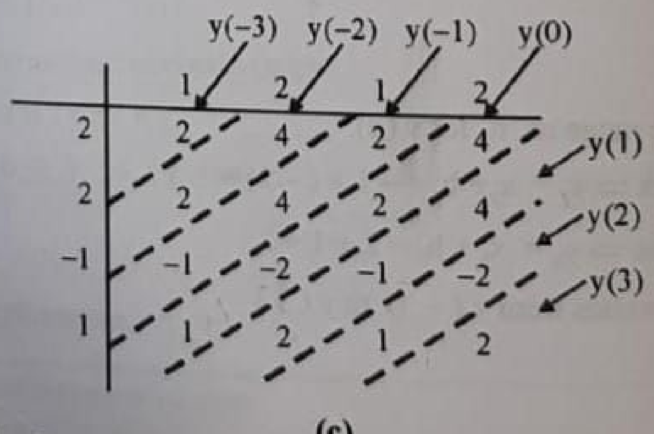
Let us verify this result using tabulation method shown in Fig. P. 3.9.14.

(a)

h(n) \ x(n)	1	2	1	2
2				
2				
−1				
1				

(b)

	1	2	1	2
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 ×
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 ×
− 1	(− 1 × 1)	(− 1 × 2)	(− 1 × 1)	(−1 ×
1	(1 × 1)	(1 × 2)	(1 × 1)	(1 ×



(c)

Fig. P. 3.9.14 : Convolution of $x(n) = \{1, 2, 1, \underset{\uparrow}{2}\}$ and $h(n) = \{2, 2, -\underset{\uparrow}{1}, 1\}$

Thus using tabulation method we get,

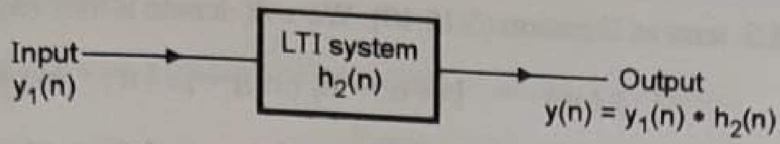


Fig. 3.10.2(b) : $y(n) = y_1(n) * h_2(n)$

Thus to implement Equation (3.10.1) we have to connect (cascade) two systems shown in Figs. 3.10.1 and 3.10.2(a). This cascade connection is shown in Fig. 3.10.2(c).

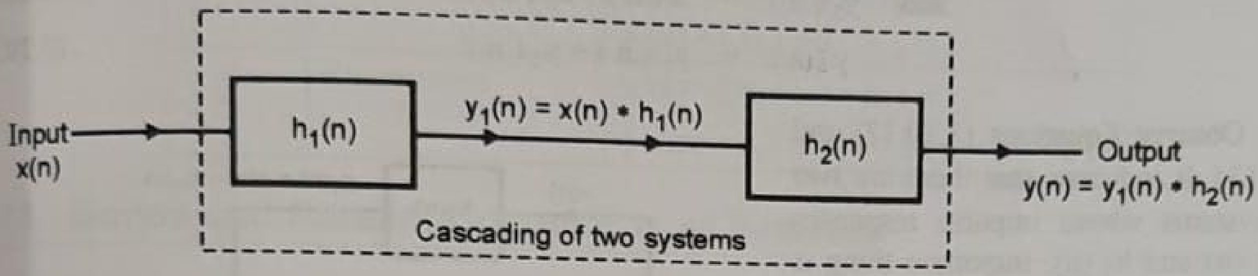


Fig. 3.10.2(c) : $y(n) = [x(n) * h_1(n)] * h_2(n)$

Now consider the R.H.S. term $x(n) * [h_1(n) * h_2(n)]$. We can design one LTI system whose impulse response $h(n)$ is the convolution of $h_1(n)$ and $h_2(n)$. That means convolution of two impulse responses. Thus $h(n) = h_1(n) * h_2(n)$.

$$\therefore \quad x(n) * [h_1(n) * h_2(n)] = x(n) * h(n) \qquad \text{...(3.10.9)}$$

The diagrammatic representation of Equation (3.10.9) is shown in Fig. 3.10.2(d).

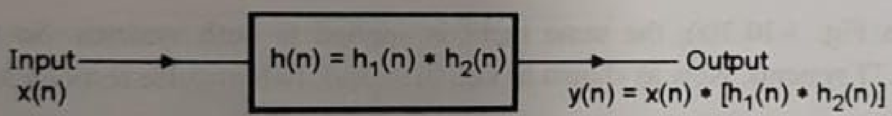


Fig. 3.10.2(d)

Meaning of associative property :

This property indicates that we can replace cascade combination of LTI systems by a single system whose impulse response is convolution of individual impulse responses. It is represented in Fig. 3.10.2(e).

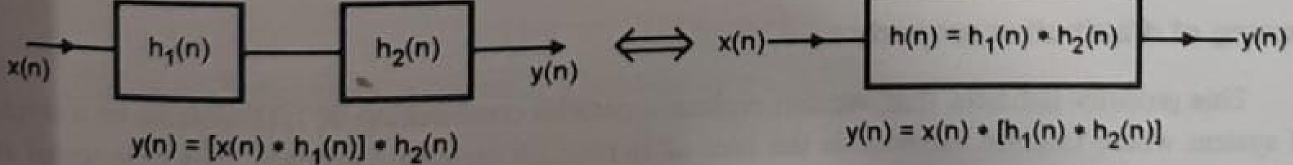


Fig. 3.10.2(e) : Meaning of associative property

3. Distributive property :

Statement : It states that linear convolution is distributive. That means

$$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)] \qquad \text{...(3.10.10)}$$

Proof : Consider the R.H.S. term of Equation (3.10.10). We will denote it by y (n).

$$\therefore \quad y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)] \qquad ...(3.10.11)$$

Every bracket term in Equation (3.10.11) indicates the convolution operation. Thus we denote output of each convolution as follows :

$$\therefore \quad y_1(n) = x(n) * h_1(n) \qquad ...(3.10.12)$$

$$\text{and} \quad y_2(n) = x(n) * h_2(n) \qquad ...(3.10.13)$$

$$\therefore \quad y(n) = y_1(n) + y_2(n) \qquad ...(3.10.14)$$

Observe Equations (3.10.12) and (3.10.13). It indicates that there are two LTI systems whose impulse responses are h_1 (n) and h_2 (n). Important thing is that, the same input signal x (n) is applied to both systems. While Equation (3.10.14) shows that we have to add the outputs of these two LTI system. This process is represented diagrammatically as shown in Fig. 3.10.3(a).

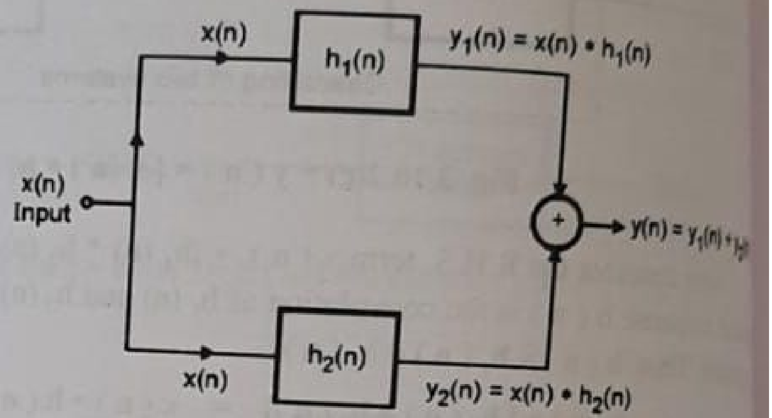


Fig. 3.10.3(a) : $y(n) = y_1(n) + y_2(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

As shown in Fig. 3.10.3(a), the same input is applied to both systems. So this is a parallel connection of two LTI systems. Now as shown in Fig. 3.10.3(a); two impulse responses are added.

$$\therefore \quad h(n) = h_1(n) + h_2(n) \qquad ...(3.10.15)$$

Thus we can design a single LTI system having impulse response $h(n) = h_1(n) + h_2(n)$.

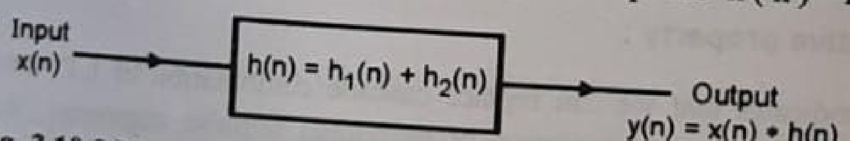


Fig. 3.10.3(b) : $y(n) = x(n) * h(n) = x(n) * [h_1(n) + h_2(n)]$

Meaning of distributive property :

This property indicates that, we can replace a parallel combination of LTI systems by a single LTI system whose impulse response is the sum of individual impulse responses. It is represented as shown in Fig. 3.10.3(c).

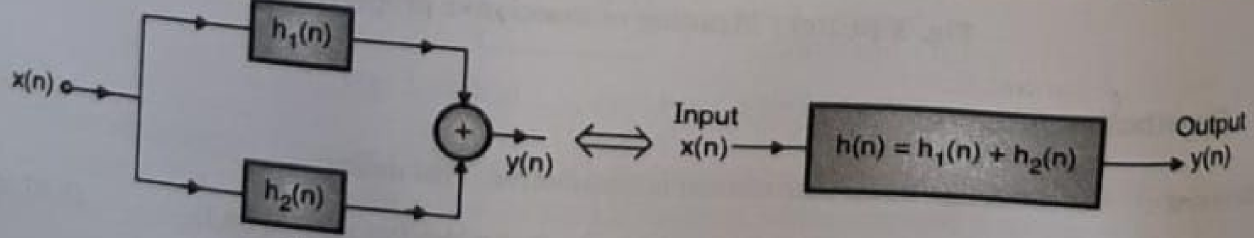


Fig. 3.10.3(c) : Meaning of distributive property

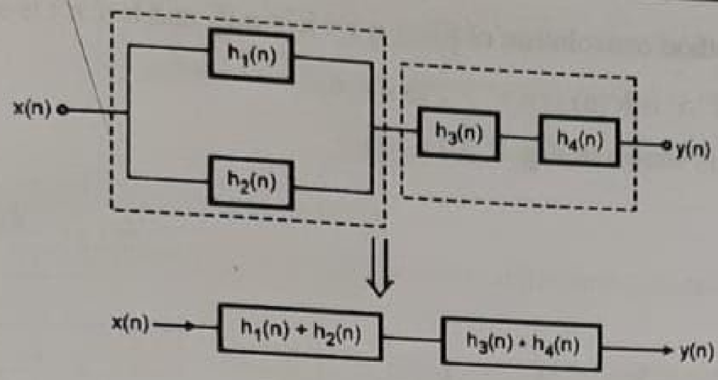


Fig. P. 3.11.1(a)

But $h_4(n) = \delta(n)$ which is unit impulse. The convolution of any function with respect to unit pulse results the same function.

$$\therefore \quad h_3(n) * h_4(n) = h_3(n)$$

Thus the block schematic can be drawn as shown in Fig. P. 3.11.1(b).

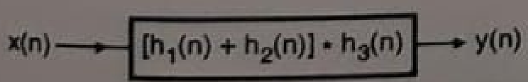
Now two blocks are in series so we can draw the block diagram as shown in Fig. P. 3.11.1(c).

$$\text{Thus} \quad h(n) = [h_1(n) + h_2(n)] * h_3(n)$$

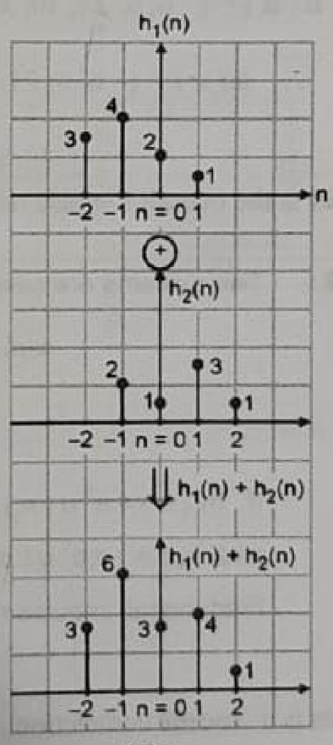
Now we will calculate h (n).

The sequence $h_1(n) + h_2(n)$ is shown in Fig. P. 3.11.1(d).

(b)



(c)



(d)

Fig. P. 3.11.1

$$\therefore \quad h_1(n) + h_2(n) = \{3, 6, \underset{\uparrow}{3}, 4, 1\}$$

and we have $h_3(n) = \{-\underset{\uparrow}{2}, 4, 1\} = \{-\underset{\uparrow}{2}, 4, 1, 0, 0\}$

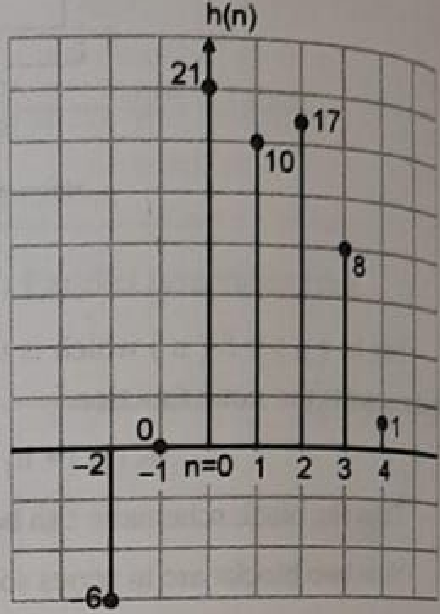
Using tabular method convolution of $[h_1(n) + h_2(n)]$ and $h_3(n)$ is obtained as follows.

Here the range of 'n' is h (n) is n = − 2 to n = 6.

The plot of h (n) is shown in Fig. P. 3.11.1(f).

	3	6	3	4	1
−2	−6	−12	−6	−8	−2
4	12	24	12	16	4
1	3	6	3	4	1
0	0	0	0	0	0
0	0	0	0	0	0

(e)



(f)

Fig. P. 3.11.1

$\therefore \quad h(n) = \{-6, 0, \underset{\uparrow}{21}, 10, 17, 8, 1, 0, 0\}$

$\therefore \quad h(n) = \{-6, 0, \underset{\uparrow}{21}, 10, 17, 8, 1\}$

This is the overall impulse response of the given system.

Ex. 3.11.2 : Two systems are cascaded as shown in Fig. P. 3.11.2.

$x(n) \rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow y(n)$

Fig. P. 3.11.2

If $h_1(n) = a^n u(n)$, $\quad a < 1$

and $h_2(n) = b^n u(n)$, $\quad b < 1$

Find overall impulse response of the system.

Soln. :

This is a cascade connection. Thus overall impulse response is,

$$h(n) = h_1(n) * h_2(n)$$

Here $h_1(n) = a^n u(n)$ and $h_2(n) = b^n u(n)$

Thus h (n) is obtained by performing convolution of $a^n u(n)$ and $b^n u(n)$. We have already obtained this convolution in Ex. 3.9.3.

$$h_T(t) = \{[h_2(t) - h_1(t)] * [(h_3(t) * h_4(t)] + \delta(t)\} \quad ...(1)$$

Consider the term $h_3(t) * h_4(t)$

$$\therefore \quad h_3(t) * h_4(t) = \delta(t-1) * e^{-3(t+2)} u(t+2) \quad ...(2)$$

We have the standard identity

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

$$\therefore \quad h_3(t) * h_4(t) = e^{-3(t+2)} \cdot u(t+2) * \delta(t-1) = e^{-3(t+1)} \cdot u(t+1)$$

Putting this value in Equation (1) we get,

$$h_T(t) = \{[e^{-2t} u(t) - \delta(t-1)] * [e^{-3(t+1)} \cdot u(t+1) + \delta(t)]\}$$

Using distributive property we get,

$$h_T(t) = e^{-2t} u(t) * e^{-3(t+1)} \cdot u(t+1) + e^{-2t} u(t) * \delta(t) - e^{-3(t+1)} \cdot u(t+1) * \delta(t-1) - \delta(t) * \delta(t-1)$$

Now we have, $x(t) * \delta(t) = x(t)$.

$$\therefore \quad h_T(t) = [e^{-2t} u(t) * e^{-3(t+1)} \cdot u(t+1)] + e^{-2t} u(t) - e^{-3t} u(t) - \delta(t)$$

This is overall impulse response of system.

Ex. 3.11.5 : Find the expression for impulse response relating the input x(t) to the output y(t) in terms of impulse for each subsystem for the LTI system shown in Fig. P. 3.11.5.

Dec. 12, 4 Marks

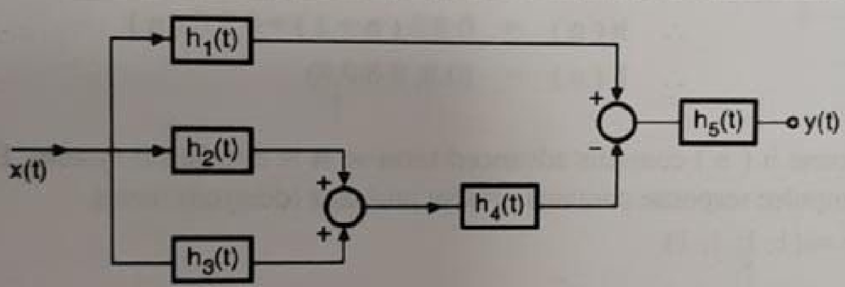


Fig. P. 3.11.5

Soln. : Here $h_2(t)$ and $h_3(t)$ are in parallel. So these blocks can be combined by performing addition, that means $h_2(t) + h_3(t)$.

This block is in series with $h_4(t)$. So it can be combined by performing convolution. That means, $[h_2(t) + h_3(t)] * h_4(t)$.

This block is in parallel with $h_1(t)$. Thus system response is,

$h_1(t) - \{[h_2(t) + h_3(t)] * h_4(t)\}$

Now this block is in series with $h_5(t)$. Thus overall impulse response is,

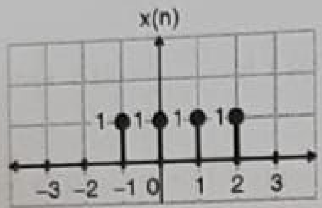
$$h_T(t) = [h_1(t) - \{[h_2(t) + h_3(t)] * h_4(t)\}] * h_5(t)$$

Ex. 3.11.6 : Consider an LTI system with input and output related by $y(n) = 0.8[x(n+1) + x(n)]$

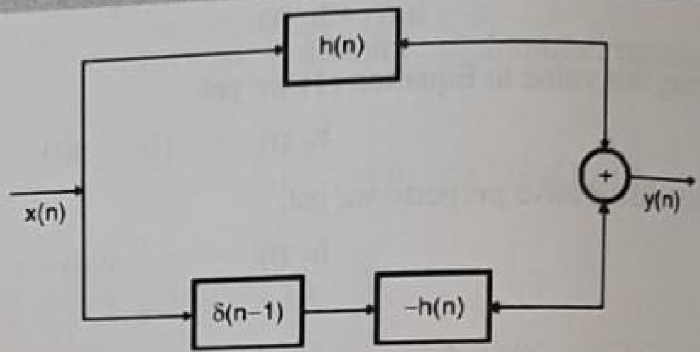
1. Find impulse response h (n)
2. Is this system causal ? Why ?

3. Determine the system response y (n) for input shown in Fig. P. 3.11.6(a).
4. Consider interconnections of LTI system shown in Fig. P. 3.11.6(b). Find impulse response of the total system.
5. Solve for overall response of the total system for the same input x (n).

May 10, 16



(a)



(b)

Fig. P. 3.11.6

Soln. :

1. Given $y(n) = 0.8\,x(n+1) + 0.8\,x(n)$

Impulse response of a system means the response (output) of a system when unit impulse, δ(is applied at input.

$$\therefore \quad h(n) = 0.8\,\delta(n+1) + 0.8\,\delta(n)$$

$$\therefore \quad h(n) = (0.8,\ 0.8\,\underset{\uparrow}{0.8})$$

2. Impulse response h (n) contains advanced term so it is noncausal system. Because a system causal if its impulse response contains present and past (delayed) terms.

3. Given $x(n) = \{1, \underset{\uparrow}{1}, 1, 1\}$

We have $h(n) = \{0.8, \underset{\uparrow}{0.8}\}$

Now response of system is,

$$y(n) = x(n) * h(n)$$

Here lower limit for y (n) is $-1 + (-1) = -2$

	1	1	1	1
0.8	0.8	0.8	0.8	0.8
0.8	0.8	0.8	0.8	0.8
0	0	0	0	0
0	0	0	0	0

$$\therefore y(n) = \{0.8, \underset{\uparrow}{1.6}, 1.6, 1.6, 0.8\}$$

We discussed that for the system to be causal we want only present and past inputs. We do not want future inputs. Thus second bracket in Equation (3.12.4) should be zero to make the system causal.

Here x (n + 1), x (n + 2) ... are future inputs. We do not want to make input to the system zero. So the second bracket is zero if we adjust h (– 1), h (– 2), h (– 3) ... to be equal to zero. Then in this case we will have only first bracket terms. That means we will have only present and past input terms. And then the system will be causal.

Thus for causality,

$$h(-1) = h(-2) = h(-3) = \ldots = 0 \quad \ldots(3.12.5)$$

That means,

$$h(n) = 0 \quad \text{for } n < 0 \quad \ldots(3.12.6)$$

Similarly for C.T. system, the condition for causality is $h(t) = 0$ for $t < 0$.

Modification of convolution equation :

We have the equation of linear convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$\text{OR} \quad y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

We have derived that, the LTI system is causal then $h(n) = h(k) = 0$ for n (or k) < 0. Thus if LTI system is causal then the limits of summation will be from $k = 0$ to $+\infty$; because for negative values of 'k', h (k) is zero. Thus for causal LTI system the equation of linear convolution is,

$$y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k) \quad \text{OR} \quad y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

And for C.T. system we get,

$$y(t) = \int_{0}^{\infty} x(\tau)\,h(t-\tau)\,d\tau \quad \text{or} \quad y(t) = \int_{0}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

3.12.2 Memory Less (Static) and With Memory (Dynamic) :

A system is memoryless if its outputs at any time depends only on the value of the input at that same time.

Discrete-time LTI system is memory less or static if

$$h(n) = 0 \quad \text{For } n \neq 0.$$

In this case the impulse response has the form,

$$h(n) = k\,\delta(n)$$

where $k = h(0)$ is a constants, and convolution sum reduces to,

$$y(n) = k\,x(n)$$

Dynamic or with memory : A system is with memory or dynamic if systems present outputs depends on present input as well as input at other time.

Discrete-time LTI system is with memory or dynamic if

$$h(n) \neq 0 \quad \text{For } n \neq 0.$$

If a discrete time LTI system has an impulse response h (n) that is not zero for $n \neq 0$ the system has memory.

For C.T. system to be static its impulse response must be in the form,

$$h(t) = c\delta(t)$$

3.12.3 Stable Systems :

Earlier we have discussed the stability condition for discrete time system. It states that a dis time system is stable if bounded input produces bounded output. Here the meaning of word "boun is; input and output signals should have some finite magnitude. The magnitude of these signals sh not be infinity. Now in this section we will derive the stability condition applicable to LTI systems terms of unit sample response.

3.12.3.1 Proof of Stability Criteria For LTI (or LSI) Systems in Terms of U Impulse Response :

PU : May 07, Dec. 07, Dec

University Questions		
Q. 1	Explain how we test BIBO stability of DTLTI system using impulse response.	(May 07, 4 Marks
Q. 2	Prove condition of stability in terms of impulse response of LTI system.	(Dec. 07, 4 Marks
Q. 3	Derive expression for stability of system in terms of impulse response.	(Dec. 08, 4 Marks

Statement : LTI system is stable if its impulse response is absolutely summable.

Proof : Consider a linear time invarient system having impulse response h (n). Let x (n) be the input applied to this system. Now according to the definition of convolution; the output of such system is expressed as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ...(3.12.7)$$

Equation (3.12.7) can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad ...(3.12.8)$$

Now first we will define bounded sequence. A discrete time sequence is said to be bounded if the absolute value of every element is less than some finite number. Thus input x (n) will be bounded if | x (n) | is less than some finite number. Let us denote this finite number by M_x. Thus for input signal to be bounded.

$$|x(n)| \leq M_x \qquad ...(3.12.9)$$

Here M_x is a finite number; so definitely its value should be less than infinity. Thus Equation (3.12.9) can be written as,

$$|x(n)| \leq M_x < \infty \qquad ...(3.12.10)$$

Now taking absolute value of both sides of Equation (3.12.8) we get,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \right| \qquad ...(3.12.11)$$

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad ...(1)$$

Given equation is $h(n) = a^n$ for $n \geq 0$ and n even and $h(n) = 0$ otherwise. That means sequence $h(n)$ is present from $n = 0$ to $n = \infty$ and for even values of n only. So limits of summation will change from $n = 0$ to $n = \infty$. We will also replace $h(k)$ by $h(n)$. Thus Equation (1) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} |a^n| \qquad ...(2)$$

Since 'n' is even, to satisfy this condition we will put $n = 2p$. Here value of p is 0, 1, 2 .... ∞. So 'n' is always even for any value of 'p'. Thus Equation (2) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{p=0}^{\infty} |a^p|$$

$$= \sum_{p=0}^{\infty} |a|^p$$

Expanding the summation we get,

$$\sum_{k=-\infty}^{\infty} |h(k)| = |a|^0 + |a|^1 + |a|^2 + |a|^3 + \ldots\ldots$$

$$= 1 + |a| + |a|^2 + |a|^3 + \ldots. \qquad ...(3)$$

This is geometric series. For geometric series we have,

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \ldots. = \frac{1}{1-a} \quad \text{if } |a| < 1$$

Thus Equation (3) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \frac{1}{1-|a|} \quad \text{if } |a| < 1$$

This condition (if $|a| < 1$) indicates that the given series converges to $\frac{1}{1-|a|}$ if and only if $|a| < 1$. Thus for the system to be stable $|a| < 1$ is the condition. Otherwise the system will be unstable.

3.12.4 Invertible Systems and Deconvolution :

A linear time invariant system generates output $y(n)$ by taking convolution of input $x(n)$ and its impulse response $h(n)$. That means $y(n) = x(n) * h(n)$. But in many applications it is required to generate the input signal from its output $y(n)$. This system is called as inverse system.

Invertibility of LTI system :

A system is said to be invertible if input of a system can be recovered from the output of system. To achieve the same input consider two system connected in series; where second system is used to recover the input. The block schematic is shown in Fig. 3.12.1.

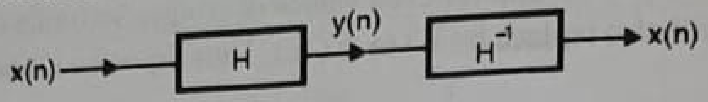


Fig. 3.12.1

Here H represents discrete time system and produces y (n) when x (n) is the input. Then y (n) is applied to the second system. This is also discrete time system denoted by H^{-1}. Here H^{-1} denotes inverse operation.

$$\therefore \quad y(n) = H[x(n)]$$

$$\text{and} \quad H^{-1}[y(n)] = H^{-1}\{H[x(n)\} = HH^{-1}[x(n)]$$

$$\text{But } HH^{-1} = 1$$

$$\therefore \quad H^{-1}y(n) = x(n)$$

Here operator H^{-1} is the inverse operator and the associated system is called as inverse system.

3.12.4.1 Solved Problems on Invertible Systems :

Ex. 3.12.2 : Which of the following systems are invertible ? If invertible find the inverse.

1. $y(t) = x(t-n)$
2. $y(t) = x(2t)$
3. $y(n) = n\,x(n)$

Soln. :

1. Given equation is, $y(t) = x(t-n)$

It indicates that signal x (t) is delayed by 'n' samples. So **it is invertible system.** We can design another system in which x (t) can be made advanced by 'n' samples. Thus inverse system is,

$$Z(t) = y(t+n)$$

This is shown in Fig. P. 3.12.2(a).

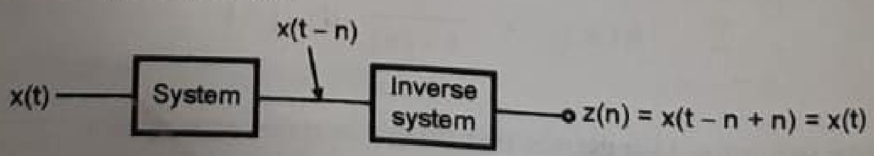


Fig. P. 3.12.2(a)

2. Given, $y(t) = x(2t)$

Input is x (t); so this system indicates that output is obtained by compressing input by the factor 2. There can be another system which will expand input by 2. Thus **it is invertible system** and inverse system is,

3.12.5.1 Solved Problem on System Response :

Ex. 3.12.4 : Impulse response of DT-LTI system is given by $h(n) = n\left(\frac{1}{2}\right)^n u(n)$.

1. Determine whether the system is stable or not.
2. Justify whether the system is causal or anticipatory.

Soln. :

1. We have the condition for stability,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad ...(1)$$

Given equation is,

$$h(n) = n\left(\frac{1}{2}\right)^n u(n) \qquad \therefore \qquad h(k) = k\left(\frac{1}{2}\right)^k u(k)$$

Thus condition for stability can be written as,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{\infty} k\left(\frac{1}{2}\right)^k u(k) \qquad ...(2)$$

Here $u(n) = u(k)$ = Unit step = 1 for $k = 0$ to ∞. Thus changing the limits of summation we get,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k \qquad ...(3)$$

We have the standard summation formula,

$$\therefore \quad \sum_{k=0}^{\infty} k\,a^k = \frac{a}{(1-a)^2} \quad \text{if} \; |a| < 1.$$

$$\therefore \quad \sum_{k=-\infty}^{\infty} |h(k)| = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2 < \infty.$$

So the given system is stable.

2. The given equation is,

$$h(n) = n\left(\frac{1}{2}\right)^n u(n).$$

Multiplication by $u(n)$ indicates that $h(n)$ is valid only for positive values of n. That means for the range $n = 0$ to ∞.

So this system is causal.

Ex. 3.12.5 : Impulse response of the RC circuit is given as $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$ **Dec. 06, 6 Marks**

Find the unit step response of the circuit.

Soln. :

Impulse response is the response of circuit when applied input is unit impulse $\delta(t)$, And unit step response means output of system when applied input is $u(t)$.

Now step response is related to impulse response as,

$$s(t) = \int_{\infty}^{t} h(\tau)\, d\tau$$

$$\text{Give, } h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$\therefore \; h(\tau) = \frac{1}{RC} e^{-\tau/RC} u(\tau)$$

Due to multiplication of $u(\tau)$ the limits of integration will be from 0 to t

$$= \int_{0}^{t} \frac{1}{RC} e^{-\tau/RC} d\tau$$

$$= \frac{1}{RC} \left[\frac{e^{-\tau/RC}}{-\frac{1}{RC}} \right]_0^t = -[e^{-t/RC} - e^{o}]$$

$$\therefore \; s(t) = 1 - e^{-t/RC}$$

Ex. 3.12.6 : Find the impulse response of the following systems : **Dec. 09, 6 Marks**

1. $y[n] = \frac{1}{4} \sum_{K=0}^{3} x[n-K]$

Represent your answer in terms of u[n].

2. $y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$.

Soln. :

1. $$\text{Given } y(n) = \frac{1}{4} \sum_{K=0}^{3} x[n-K]$$

Impulse response of system, h (n) means output of system when applied input is $x(n) = \delta(n)$.

$$\therefore \; h(n) = \frac{1}{4} \sum_{K=0}^{3} \delta[n-K]$$

$$\therefore h(n) = \frac{1}{4} [\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)]$$

Soln. :

1. According to the condition of stability,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

$$\text{Given, } h(t) = e^{-5|t|}$$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} e^{-5|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{5t}\, dt + \int_{0}^{\infty} e^{-5t}\, dt$$

$$= \left[\frac{1}{5}\cdot e^{5t}\right]_{-\infty}^{0} + \left[\frac{-1}{5}\cdot e^{-5t}\right]_{0}^{\infty} = \left[\frac{1}{5}-0\right]+\left[0+\frac{1}{5}\right]$$

$$= \frac{2}{5} < \infty$$

Output is bounded; so system is stable.

2. **Given :**

$$h(t) = e^{4t}\, u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{4t}\, u(t)|\, dt$$

$$= \int_{0}^{\infty} e^{4t}\, dt = \left[\frac{e^{4t}}{4}\right]_{0}^{\infty} = \infty$$

Since output is not bounded; System is unstable.

Ex. 3.12.10 : Find the step response of the first order recursive system with impulse response : $h[n] = \left(\frac{1}{3}\right)^n \cdot u[n]$. **Dec. 11, 8 Marks**

Soln. : The step response of D.T. system is given by,

$$y(n) = \sum_{k=-\infty}^{n} h(k) \qquad ...(1)$$

$$\text{Given } h(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$\therefore \quad h(k) = \left(\frac{1}{3}\right)^k u(k) \qquad ...(2)$$

Due to multiplication by u(k) ; limits in equation becomes, k = 0 to n.

$$\therefore \quad y(n) = \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k$$

We have standard summation formula.

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{for} \quad a \neq 1 \\ N_2 - N_1 + 1 & \text{for} \quad a = 1 \end{cases}$$

Here $a = \frac{1}{3}, N_1 = 0, N_2 = n$

$$\therefore \quad y(n) = \frac{\left(\frac{1}{3}\right)^0 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} = \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{\frac{2}{3}}$$

$$\therefore \quad y(n) = \frac{3}{2}\left[1 - \left(\frac{1}{3}\right)^n \cdot \left(\frac{1}{3}\right)\right]$$

$$\therefore \quad y(n) = \frac{3}{2} - \frac{1}{2} \cdot \left(\frac{1}{3}\right)^n$$

This is the step response of system.

Review Questions

Q. 1 Define convolution integral and state its properties.

Q. 2 Obtain the equations of impulse response for the first and second order systems.

Q. 3 Derive the equation of linear convolution.

Q. 4 Explain the properties of convolution sum.

Q. 5 How any arbitrary signal can be expressed in terms of summation of weighted unit impulses ?

Q. 6 Explain how we test BIBO stability of DTLTI system using impulse response ?

Q. 7 Define the term 'step response'. How it can be expressed in terms of impulse response for CT system ?

Q. 8 State and prove the condition of stability.

Q. 9 Prove that DTLTI system is causal if h (n) = 0 for n < 0.

Q. 10 Explain invertibility of DTLTI system.

□□□