Scaling limits of Abelian sandpiles

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Cornell University

Probability Seminar

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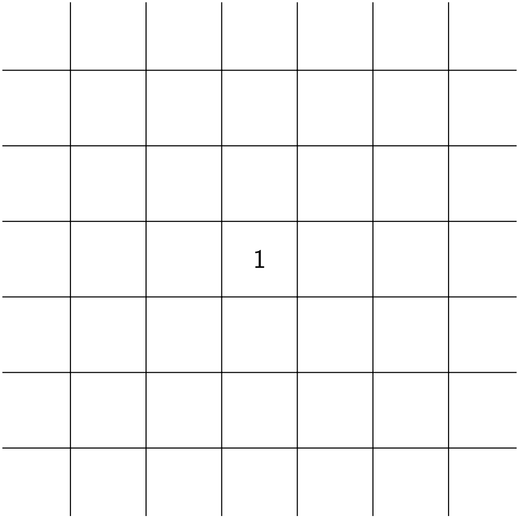
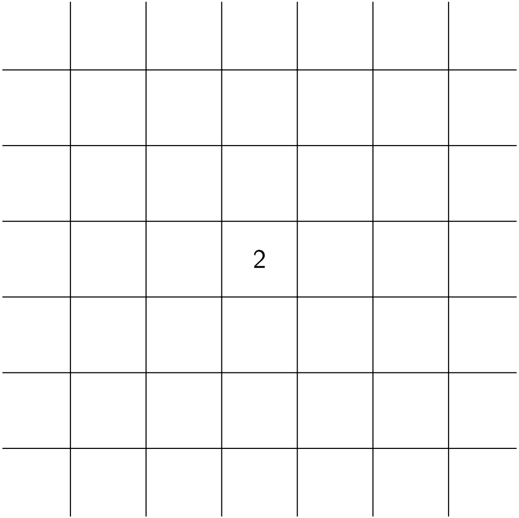
What is the Abelian sandpile?

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

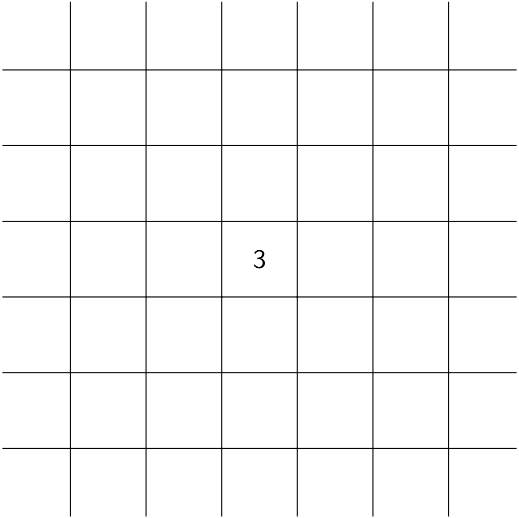
What is the Abelian sandpile?



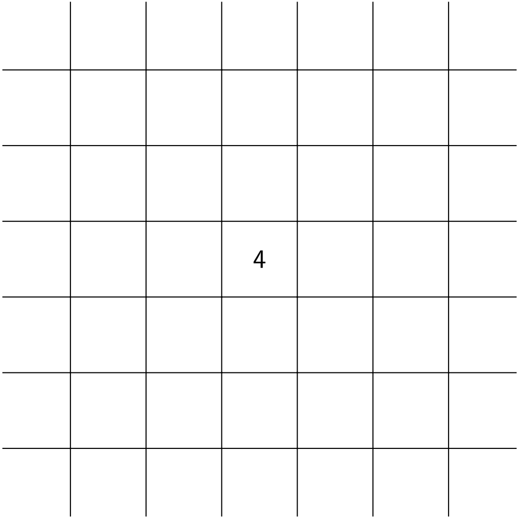
What is the Abelian sandpile?



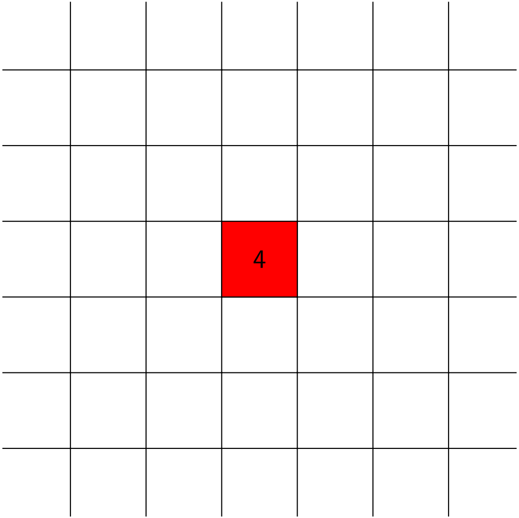
What is the Abelian sandpile?



What is the Abelian sandpile?



What is the Abelian sandpile?



What is the Abelian sandpile?

			1			
		1	0	1		
			1			

What is the Abelian sandpile?

			1			
		1	1	1		
			1			

What is the Abelian sandpile?

			1			
		1	2	1		
			1			

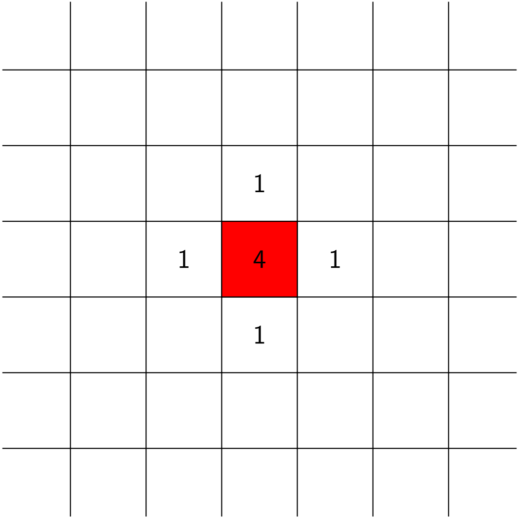
What is the Abelian sandpile?

			1			
		1	3	1		
			1			

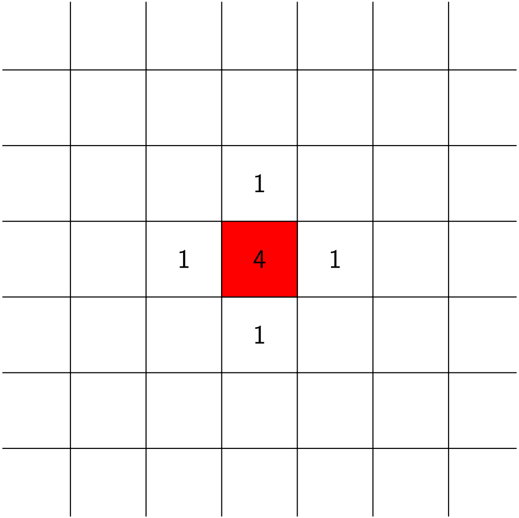
What is the Abelian sandpile?

			1			
		1	4	1		
			1			

What is the Abelian sandpile?



What is the Abelian sandpile?



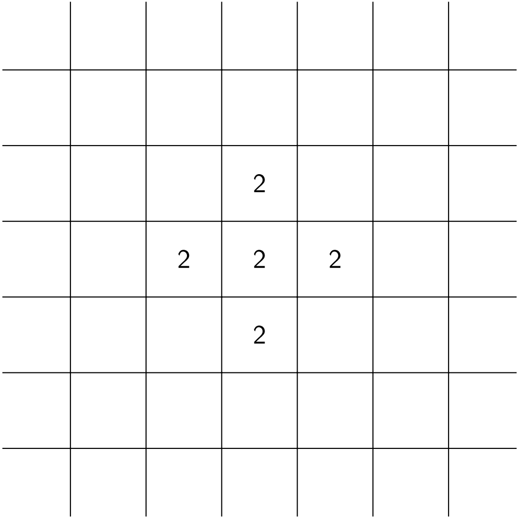
What is the Abelian sandpile?

			2			
		2	0	2		
			2			

What is the Abelian sandpile?

			2			
		2	1	2		
			2			

What is the Abelian sandpile?



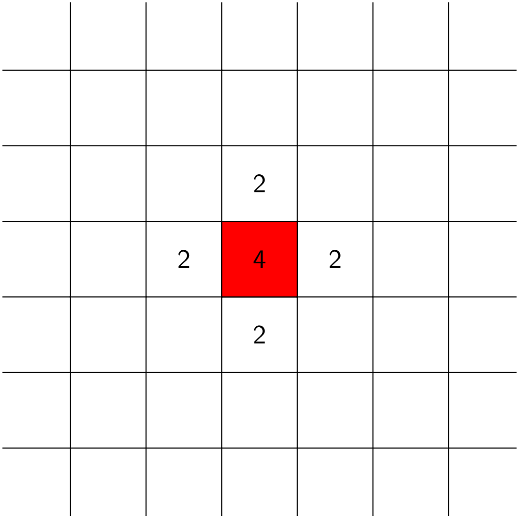
What is the Abelian sandpile?

			2			
		2	3	2		
			2			

What is the Abelian sandpile?

			2			
		2	4	2		
			2			

What is the Abelian sandpile?



What is the Abelian sandpile?

			3			
		3	0	3		
			3			

What is the Abelian sandpile?

			3			
		3	1	3		
			3			

What is the Abelian sandpile?

			3			
		3	2	3		
			3			

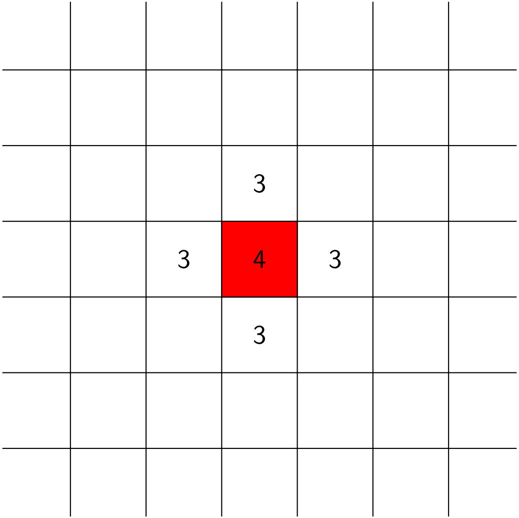
What is the Abelian sandpile?

			3			
		3	3	3		
			3			

What is the Abelian sandpile?

			3			
		3	4	3		
			3			

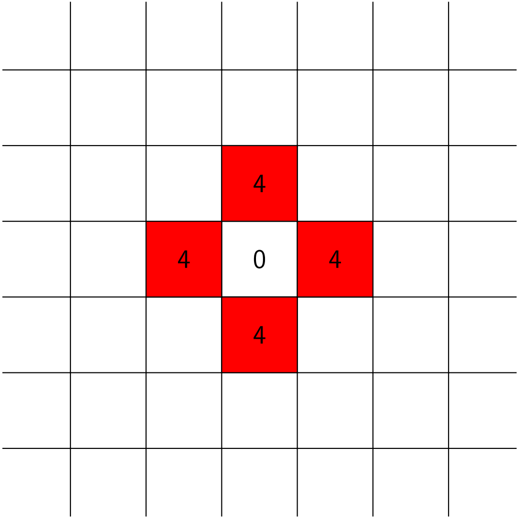
What is the Abelian sandpile?



What is the Abelian sandpile?

			4			
		4	0	4		
			4			

What is the Abelian sandpile?



What is the Abelian sandpile?

			1			
		2	0	2		
	1	0	4	0	1	
		2	0	2		
			1			

What is the Abelian sandpile?

			1			
		2	0	2		
	1	0	4	0	1	
		2	0	2		
			1			

What is the Abelian sandpile?

			1			
		2	1	2		
	1	1	0	1	1	
		2	1	2		
			1			

What is the Abelian sandpile?

			1			
		2	1	2		
	1	1	1	1	1	
		2	1	2		
			1			

What is the Abelian sandpile?

			1			
		2	1	2		
	1	1	2	1	1	
		2	1	2		
			1			

What is the Abelian sandpile?

			1			
		2	1	2		
	1	1	3	1	1	
		2	1	2		
			1			

What is the Abelian sandpile?

			1			
		2	1	2		
	1	1	4	1	1	
		2	1	2		
			1			

What is the Abelian sandpile?

			1			
		2	1	2		
	1	1	4	1	1	
		2	1	2		
			1			

What is the Abelian sandpile?

			1			
		2	2	2		
	1	2	0	2	1	
		2	2	2		
			1			

What is the Abelian sandpile?

			1			
		2	2	2		
	1	2	1	2	1	
		2	2	2		
			1			

What is the Abelian sandpile?

			1			
		2	2	2		
	1	2	2	2	1	
		2	2	2		
			1			

What is the Abelian sandpile?

			1			
		2	2	2		
	1	2	3	2	1	
		2	2	2		
			1			

What is the Abelian sandpile?

			1			
		2	2	2		
	1	2	4	2	1	
		2	2	2		
			1			

What is the Abelian sandpile?

			1			
		2	2	2		
	1	2	4	2	1	
		2	2	2		
			1			

What is the Abelian sandpile?

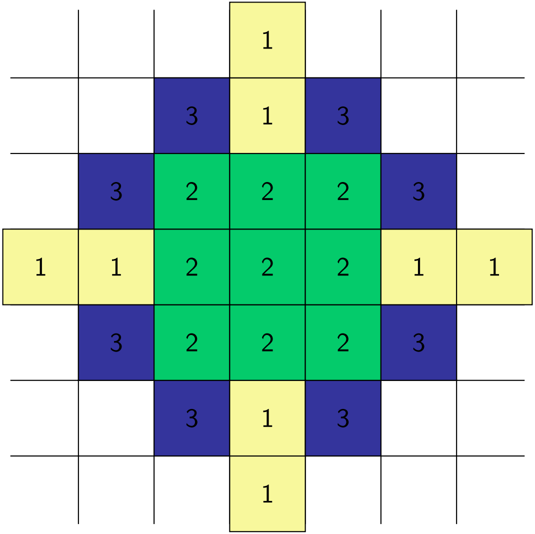
			1			
		2	3	2		
	1	3	0	3	1	
		2	3	2		
			1			

What is the Abelian sandpile?

		1		
	2	3	2	
1	3	0	3	1
	2	3	2	
		1		

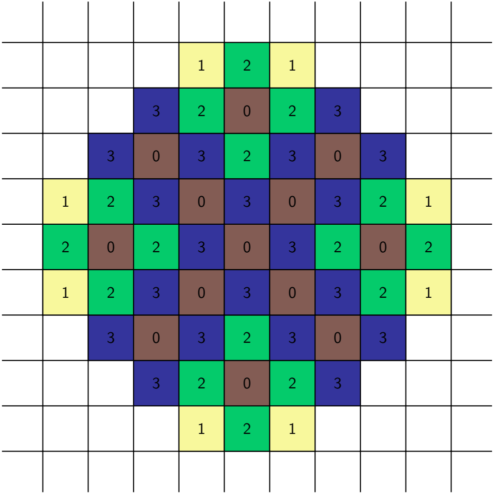
24 grains of sand at the origin

Scaling limit?



50 grains of sand at the origin

Scaling limit?



100 grains of sand at the origin

Scaling limit?

			1	2	3	2	1			
		3	2	3	1	3	2	3		
	3	0	3	2	0	2	3	0	3	
1	2	3	2	2	3	2	2	3	2	1
2	3	2	2	2	3	2	2	2	3	2
3	1	0	3	3	0	3	3	0	1	3
2	3	2	2	2	3	2	2	2	3	2
1	2	3	2	2	3	2	2	3	2	1
	3	0	3	2	0	2	3	0	3	
		3	2	3	1	3	2	3		
			1	2	3	2	1			

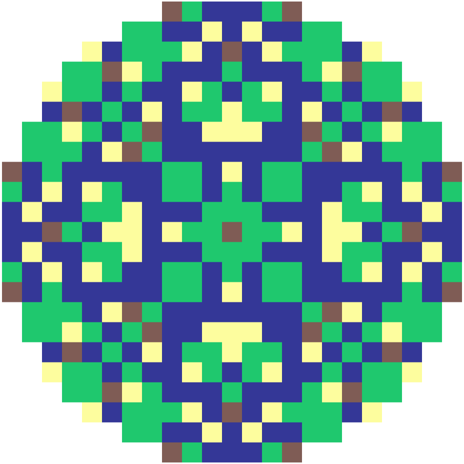
200 grains of sand at the origin

Scaling limit?

		1	3	2	2	1	3	1	2	2	3	1		
	2	1	3	3	3	2	0	2	3	3	3	1	2	
1	1	2	2	0	3	2	3	2	3	0	2	2	1	1
3	3	2	2	3	3	3	3	3	3	3	2	2	3	3
2	3	0	3	0	1	1	3	1	1	0	3	0	3	2
2	3	3	3	1	2	3	3	3	2	1	3	3	3	2
1	2	2	3	1	3	2	1	2	3	1	3	2	2	1
3	0	3	3	3	3	1	0	1	3	3	3	3	0	3
1	2	2	3	1	3	2	1	2	3	1	3	2	2	1
2	3	3	3	1	2	3	3	3	2	1	3	3	3	2
2	3	0	3	0	1	1	3	1	1	0	3	0	3	2
3	3	2	2	3	3	3	3	3	3	3	2	2	3	3
1	1	2	2	0	3	2	3	2	3	0	2	2	1	1
	2	1	3	3	3	2	0	2	3	3	3	1	2	
		1	3	2	2	1	3	1	2	2	3	1		

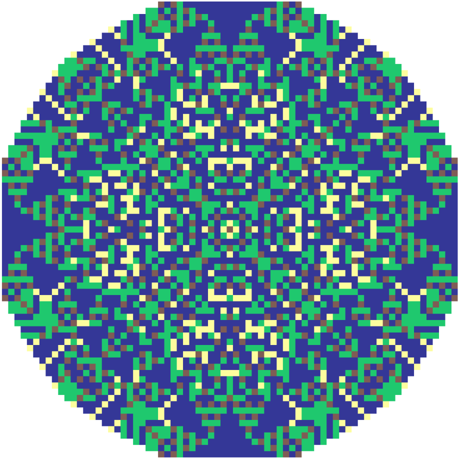
500 grains of sand at the origin

Scaling limit?



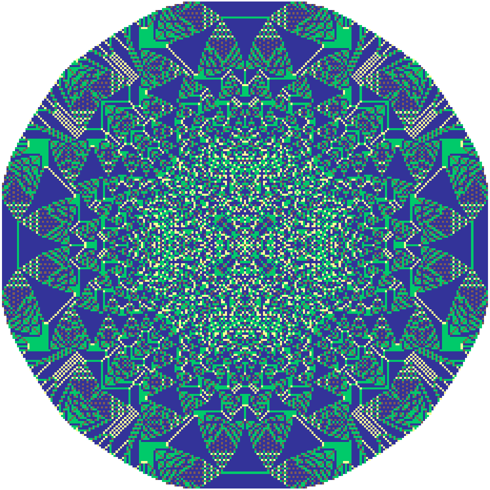
10^3 grains of sand at the origin

Scaling limit



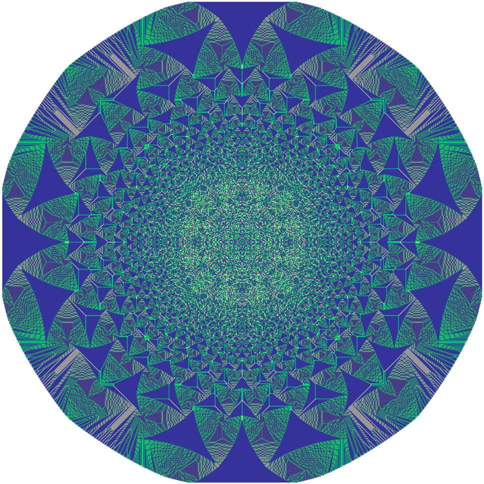
10^4 grains

Scaling limit



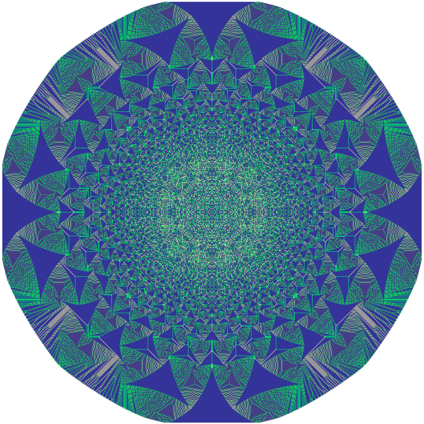
10^5 grains

Scaling limit



10^6 grains

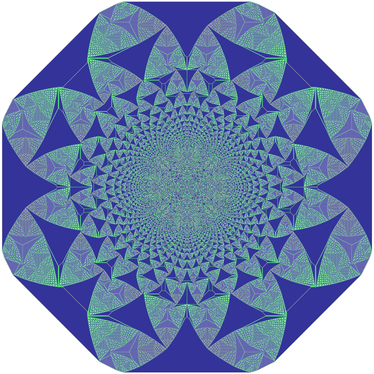
Scaling limit



Theorem (Pegden, Smart 2011)

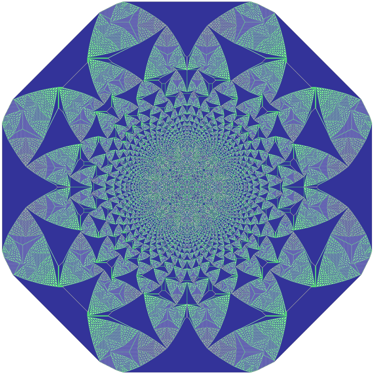
The scaling limit of the single source sandpile exists and is the Laplacian of the solution to an elliptic free boundary problem.

Other scaling limits?



put a large stack of grains at the origin on a background filled with 1s

Other scaling limits?



put a large stack of grains at the origin on any *periodic*, non-explosive, initial background; the scaling limit exists by the Pegden-Smart theorem

Random initial states?

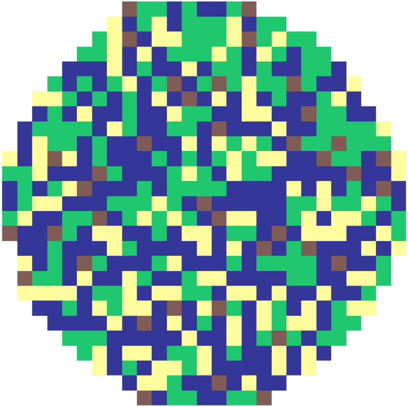
1	0	0	0	0	0	1
0	0	1	0	0	1	1
1	0	0	1	1	1	0
0	1	1	1	0	1	0
0	1	1	1	1	0	0
0	1	1	0	0	0	0
1	0	0	1	1	0	1

Random initial state

1	0	1	2	1	0	1
0	2	3	2	2	3	1
2	2	2	2	3	3	1
2	3	2	1	1	3	2
1	3	3	2	3	2	1
0	3	3	2	2	2	0
1	0	1	3	2	0	1

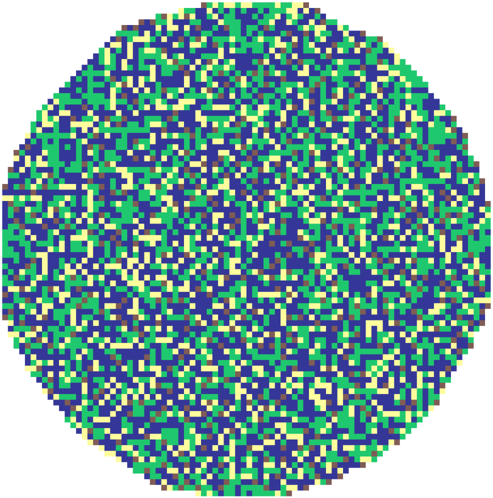
50 grains at the origin

Random initial state



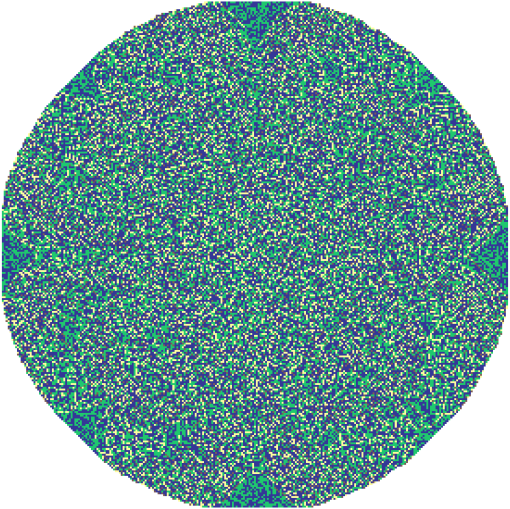
1000 grains at the origin

Random initial state



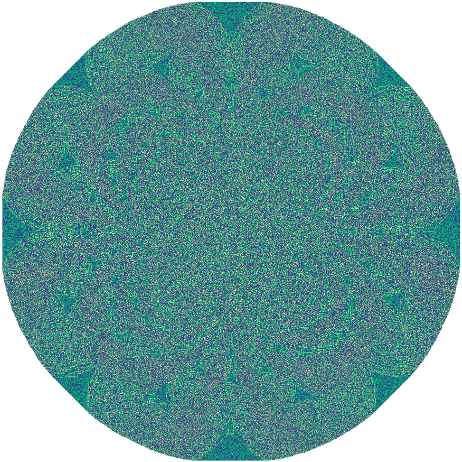
10^4 grains at the origin

Random initial state – scaling limit?



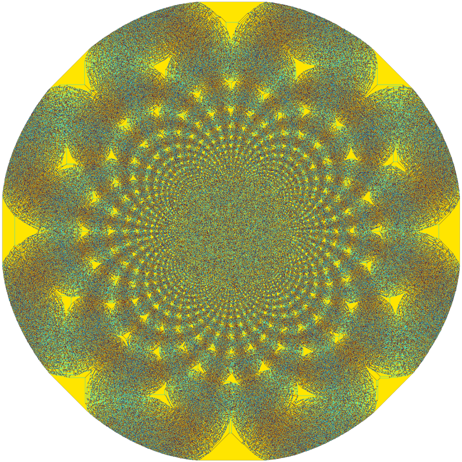
10^5 grains at the origin

Random initial state - scaling limit



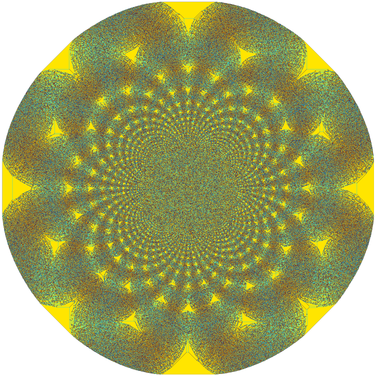
10^6 grains at the origin

Random initial state - scaling limit



$5 \cdot 10^6$ grains at the origin

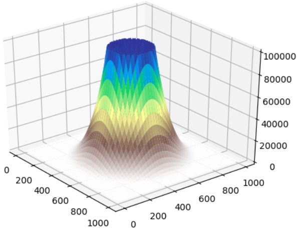
Convergence of the random Abelian sandpile

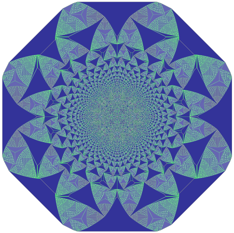


Theorem (B. 2019)

The scaling limit of the single source sandpile on a random background exists and is the Laplacian of the solution to an elliptic free boundary problem.

A foothold





- the *odometer function*, which counts the number of times each square topples when stabilizing, is much easier to study
- the odometer function obeys a discrete PDE, *the least action principle*

Least action principle - (Fey,Levine,Peres 2009)

$$v_n = \inf\{w : \mathbf{Z}^2 \to \mathbf{N} : \Delta w + n\delta_0 + \eta \le 3\}$$
$$s_n = \Delta v_n + \eta + n\delta_0$$

- $\eta : \mathbf{Z}^2 \to \mathbf{Z}$ - initial background
- $n\delta_0$ - stack of n grains at the origin
- $s_n : \mathbf{Z}^2 \to \mathbf{Z}$, $s_n \le 3$ - stable configuration
- $v_n : \mathbf{Z}^2 \to \mathbf{N}$ - number of topples per site when stabilizing
- $\Delta v_n(x) = \sum_{y\sim x} (v_n(y) - v_n(x))$ - graph Laplacian

(More) General Framework

- sample a random background $\eta : \mathbf{Z}^2 \to \mathbf{Z}$ from a distribution which is
  - stationary, ergodic under spatial translations
  - uniformly bounded
  - $\mathbf{E}(\eta(0)) < 2$
- motivating example: $\eta \sim$ Bernoulli$(0, 1)$

Convergence of the Random Abelian Sandpile

Theorem (B. 2019)

- *There exists a unique, compactly supported* $\bar{s} : \mathbf{R}^2 \to [0, 3]$ *and* $\bar{v} : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}^+$ *so that almost surely*

$$n^{-1} v_n([n^{1/2} x]) \to \bar{v} \textit{ locally uniformly}$$

$$s_n([n^{1/2} x]) \to \bar{s} \textit{ weakly-*}$$

and, away from the origin, weakly

$$\bar{s}(x) = \Delta \bar{v}(x) + \mathbf{E}(\eta(0)).$$

- $\bar{v}$ *is the unique viscosity solution to the elliptic obstacle problem*

$$\bar{v} := \min\{w \in C(\mathbf{R}^2 \setminus \{0\}) | w \geq 0 \ D^2 w \in \bar{\Gamma}_\eta\},$$

where $\bar{\Gamma}_\eta$ *is nonrandom and downwards closed.*

Convergence of the (non-random) Abelian Sandpile

Theorem (Pegden-Smart 2011)

- *There exists a unique, compactly supported* $\bar{s} : \mathbf{R}^2 \to [0,3]$ *and* $\bar{v} : \mathbf{R}^2 \backslash \{0\} \to \mathbf{R}^+$ *so that almost surely*

$$n^{-1} v_n([n^{1/2} x]) \to \bar{v} \text{ locally uniformly}$$

$$s_n([n^{1/2} x]) \to \bar{s} \text{ weakly-*}$$

and away from the origin, weakly

$$\bar{s}(x) = \Delta \bar{v}(x).$$

- $\bar{v}$ *is the unique viscosity solution to the elliptic obstacle problem*

$$\min\{v \in C(\mathbf{R}^2 \backslash \{0\}) : v \geq 0 \ D^2 v \in \bar{\Gamma}_0\}$$

$$\bar{\Gamma}_0 = \{M \in S^2 \text{ so that there exists } u : \mathbf{Z}^2 \to \mathbf{Z}$$

$$\Delta u \leq 3 \text{ and } u(y) \geq \tfrac{1}{2} y^T M y + o(|y|^2))\}$$

What is $\bar{\Gamma}_\eta$?

- can characterize $\bar{\Gamma}_\eta$ as the closure of the following set

  $\Gamma_\eta = \{$rational $M \in S^2$ so that $\Delta q_M + \eta$ is quadratically stabilizable$\}$

- random (but not-random!) analogue of

$$\Gamma_0 = \{\text{rational } M \in S^2 \text{ so that } \Delta q_M \text{ is stabilizable}\}$$

- where $q_M(x) = \lfloor \frac{1}{2}(x^T M x) \rfloor$

What is $\bar{\Gamma}_0$?

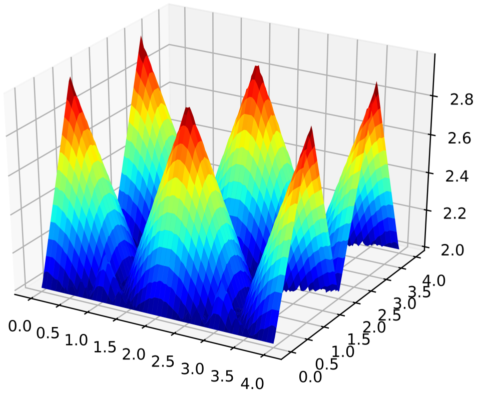
$$\Gamma_0 = \{\text{rational } M \in S^2 \text{ so that } \Delta q_M \text{ is stabilizable}\}$$

- Γ_0 is downwards closed in the semidefinite matrix order, so is fully specified by its boundary, $\partial\Gamma_0$
- can look at $\partial\Gamma_0$ with a computer algorithm
- parameterize $M \in \mathbf{S}^2$ by

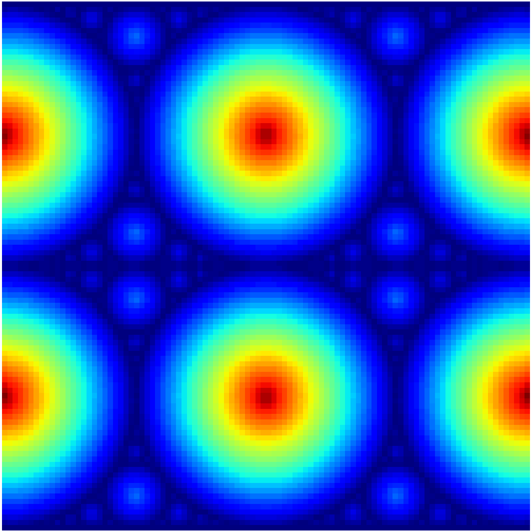
$$M(a,b,c) = \frac{1}{2}\begin{bmatrix} c-a & b \\ b & c+a \end{bmatrix}$$

  and view $\partial\Gamma_0$ as a surface in $\mathbf{R}^3$

What is $\bar{\Gamma}_0$?



What is $\bar{\Gamma}_0$?



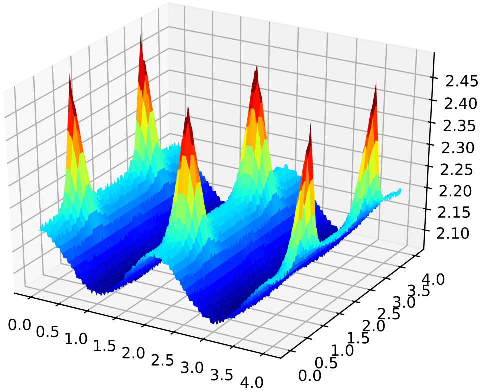
$\partial\bar{\Gamma}_0$ is an Appolonian circle packing (Levine, Pegden, Smart 2017)

What is $\bar{\Gamma}_\eta$?

$\Gamma_\eta = \{$rational $M \in S^2$ so that $\Delta q_M + \eta$ is quadratically stabilizable$\}$

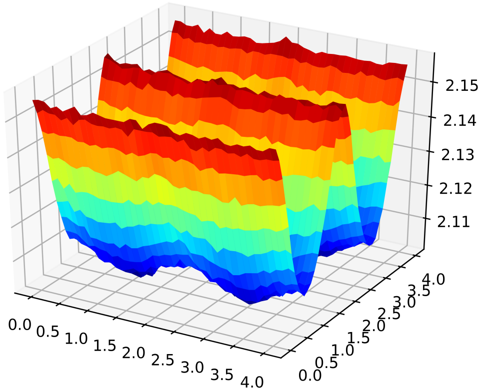
- will depend on the distribution of η
- we believe (but can't prove!) that the same algorithm as before can show us $\partial\bar{\Gamma}_\eta$

What is $\bar{\Gamma}_\eta$?



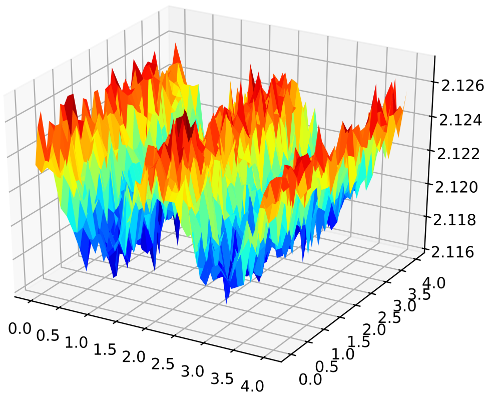
$\eta \sim$ Bernoulli$(0, 1, 1/2)$

What is $\bar{\Gamma}_\eta$?



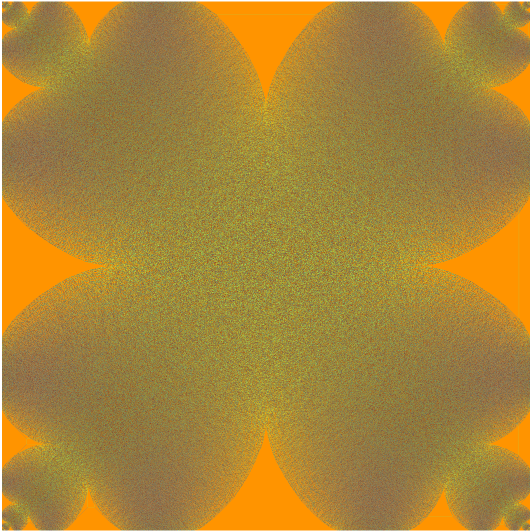
$\eta \sim$ Bernoulli$(-1, 1, 1/2)$

What is $\bar{\Gamma}_\eta$?



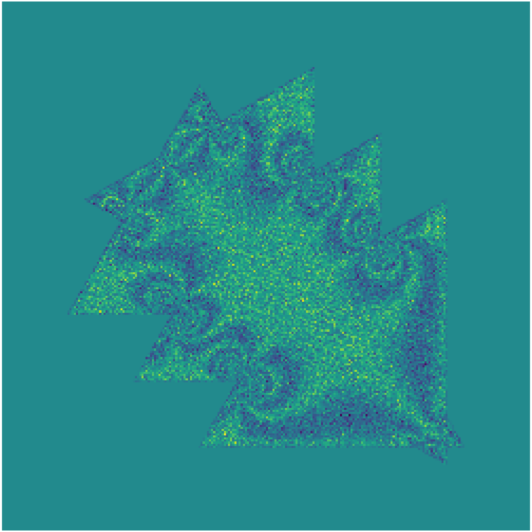
$\eta \sim$ Bernoulli$(-2, 2, 1/2)$

Convergence of the Random Abelian Sandpile



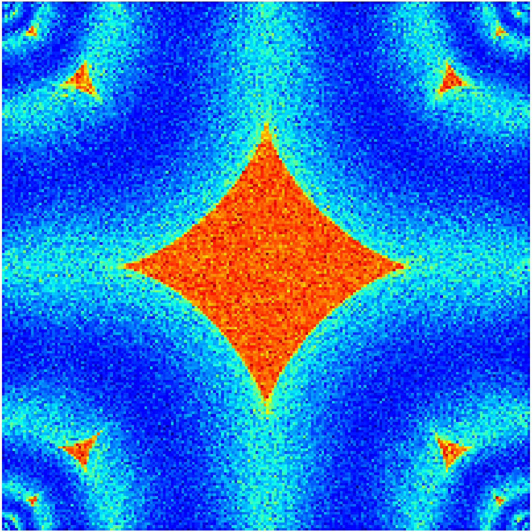
Dirichlet problem on square domain $\eta \sim$ Bernoulli$(3, 4, 1/2)$

Convergence of the Random Abelian Sandpile



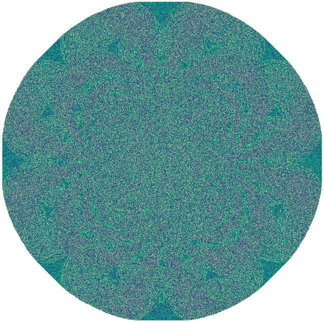
Dirichlet problem on stingray domain $\eta \sim \text{Bernoulli}(3, 5, 1/2)$

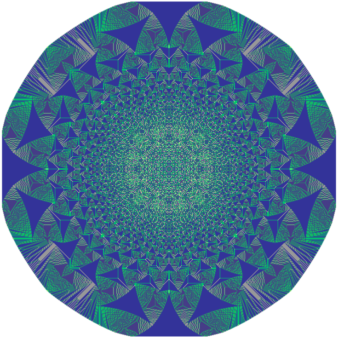
Convergence of the Random Abelian Sandpile



recurrent equivalent to Bernoulli$(0, 1, 1/2)$ on square domain

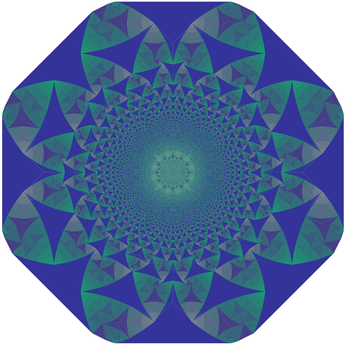
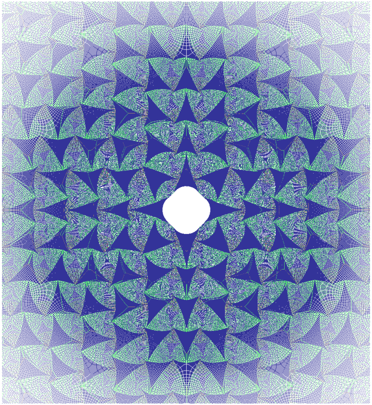
What is it?





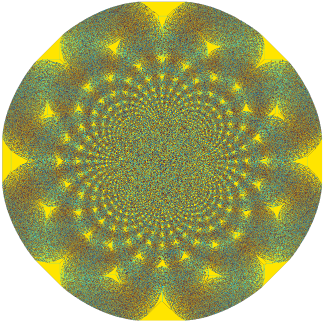
the scaling limit exists, but can we find a closed-form solution?

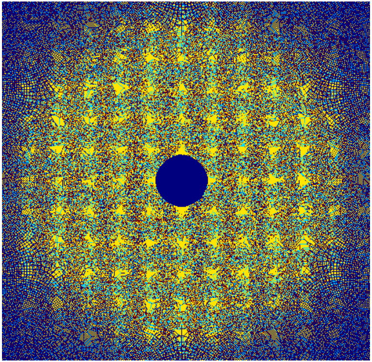
Ostojic's heuristic

conformally map the single-source sandpile by $1/z^2$

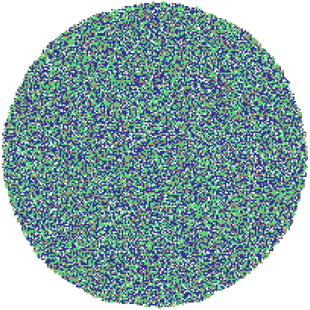
Ostojic's heuristic





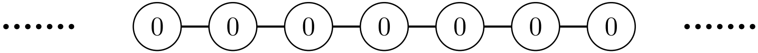
conformally map the random single-source sandpile by $1/z^2$

Stochastic sandpile

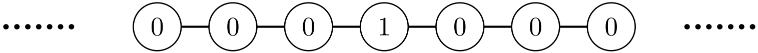


another difficult, open problem: the scaling limit of the single-source sandpile with stochastic toppling rules appears to be a Euclidean ball

An attempt to make the problem easier

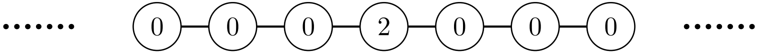


Single-source sandpile on $\mathbb{Z}^1$

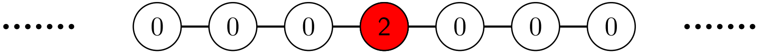


1 grain

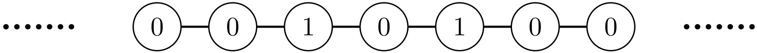
Single-source sandpile on $\mathbb{Z}^1$



Single-source sandpile on $\mathbb{Z}^1$

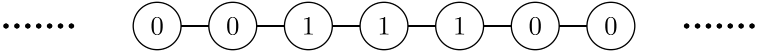


Single-source sandpile on $\mathbb{Z}^1$



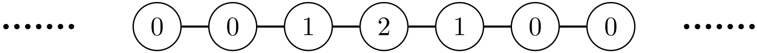
2 grains

Single-source sandpile on $\mathbf{Z}^1$

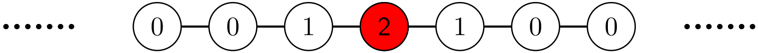


3 grains

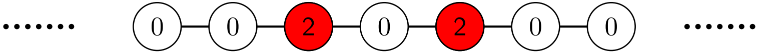
Single-source sandpile on $\mathbb{Z}^1$



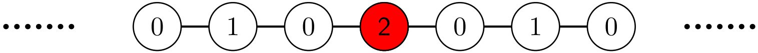
Single-source sandpile on $\mathbb{Z}^1$



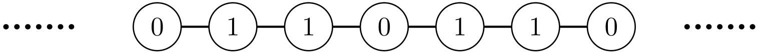
Single-source sandpile on $\mathbf{Z}^1$



Single-source sandpile on $\mathbb{Z}^1$

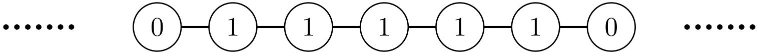


Single-source sandpile on $\mathbf{Z}^1$



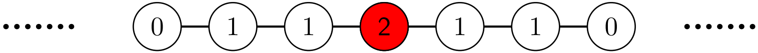
4 grains

Single-source sandpile on $\mathbf{Z}^1$

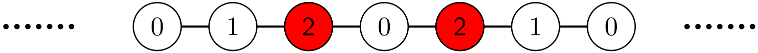


5 grains

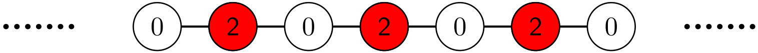
Single-source sandpile on $\mathbb{Z}^1$



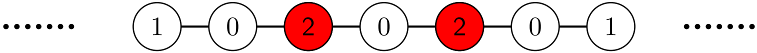
Single-source sandpile on $\mathbf{Z}^1$



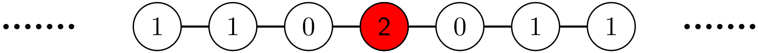
Single-source sandpile on $\mathbb{Z}^1$



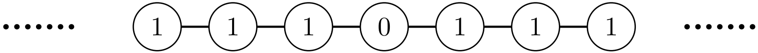
Single-source sandpile on $\mathbb{Z}^1$



Single-source sandpile on $\mathbf{Z}^1$

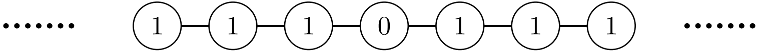


Single-source sandpile on $\mathbb{Z}^1$



6 grains

Single-source sandpile on $\mathbf{Z}^1$



Proposition

The stabilization of $n\delta_0$ on $\mathbf{Z}^1$ has a simple closed form: if n is odd,

$$\begin{aligned} s_n(x) &= 1 \textit{ if } x \in [-\lfloor n/2 \rfloor, \lfloor n/2 \rfloor] \\ &= 0 \textit{ otherwise,} \end{aligned}$$

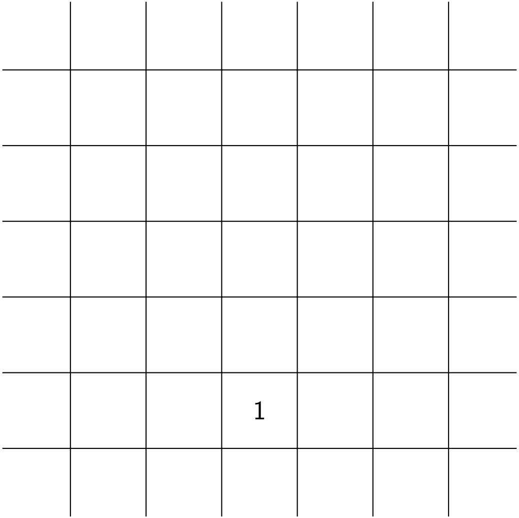
if n is even,

$$\begin{aligned} s_n(x) &= 1 \textit{ if } x \in [-n/2, 1] \cup [1, n/2] \\ &= 0 \textit{ otherwise.} \end{aligned}$$

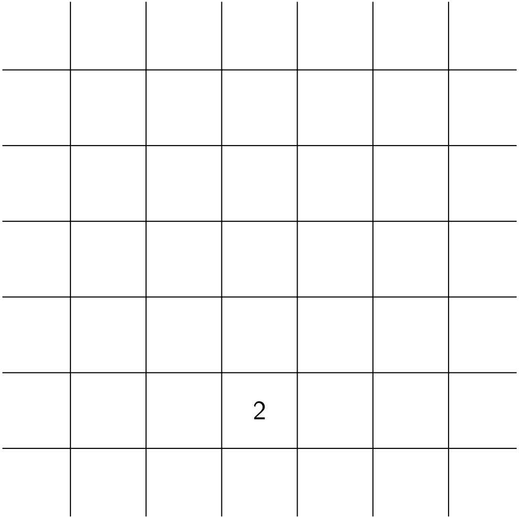
A second attempt to make the problem easier

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

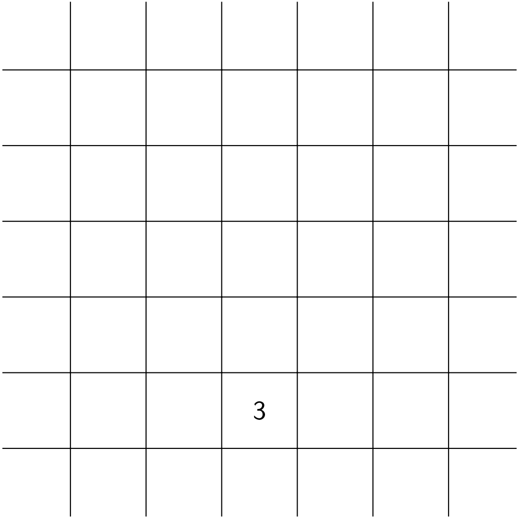
A second attempt to make the problem easier



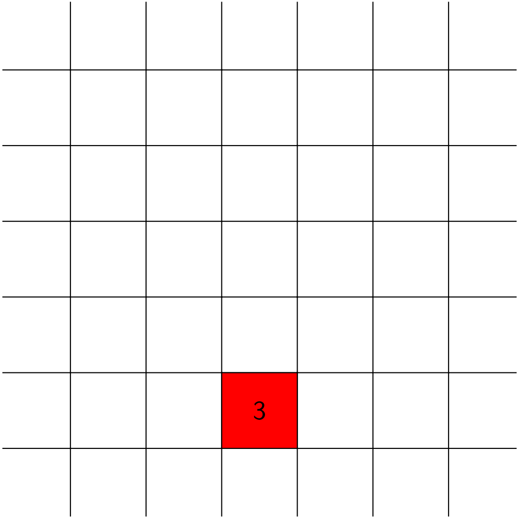
Single-source oriented sandpile



Single-source oriented sandpile



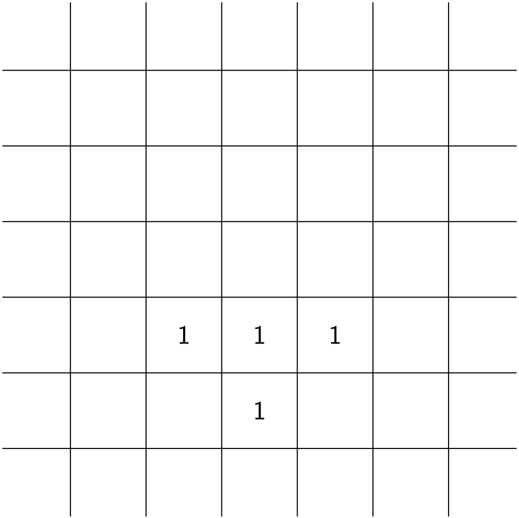
Single-source oriented sandpile



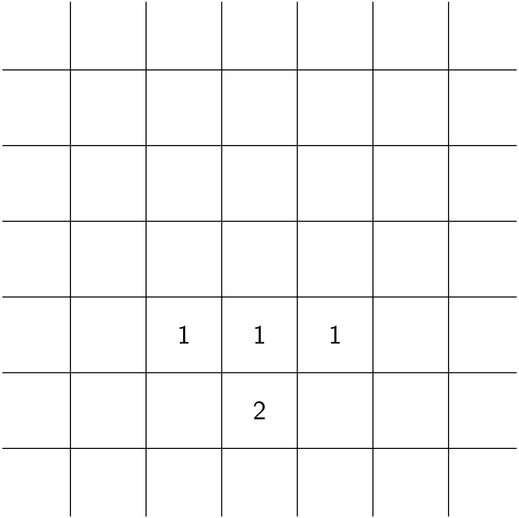
Single-source oriented sandpile

		1	1	1		
			0			

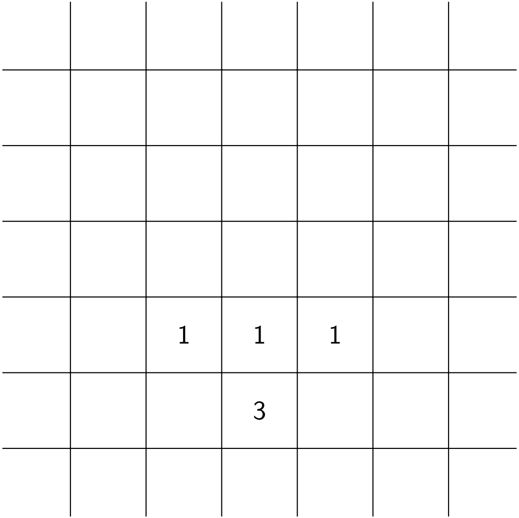
Single-source oriented sandpile



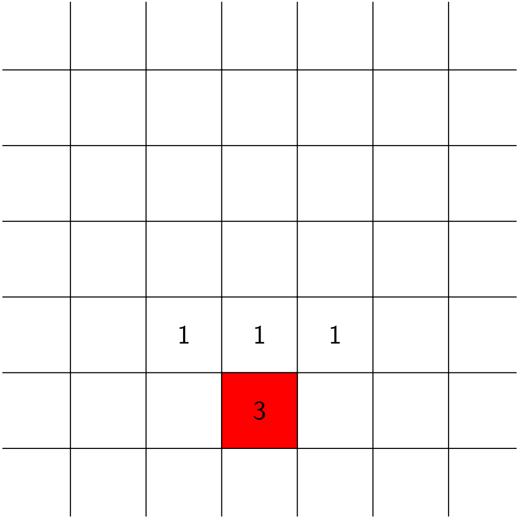
Single-source oriented sandpile



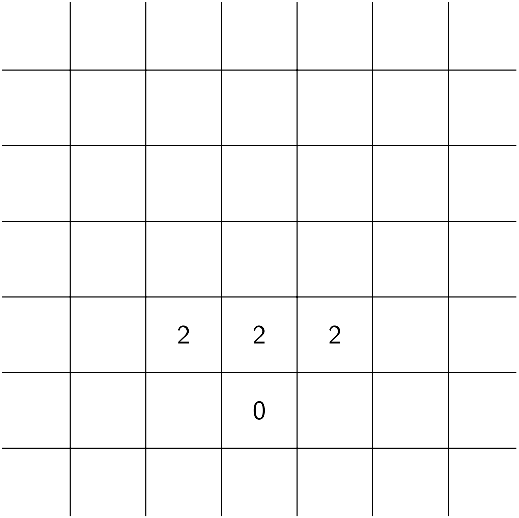
Single-source oriented sandpile



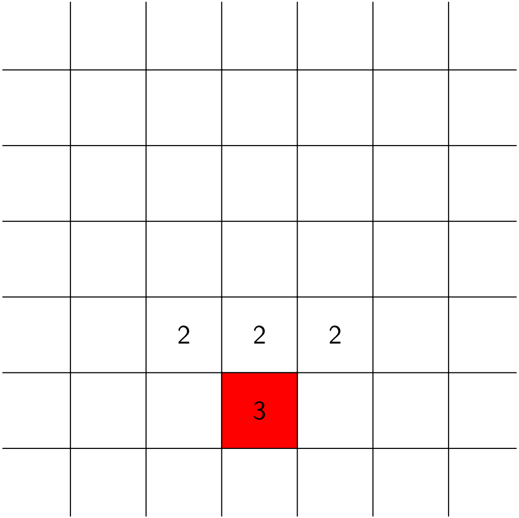
Single-source oriented sandpile



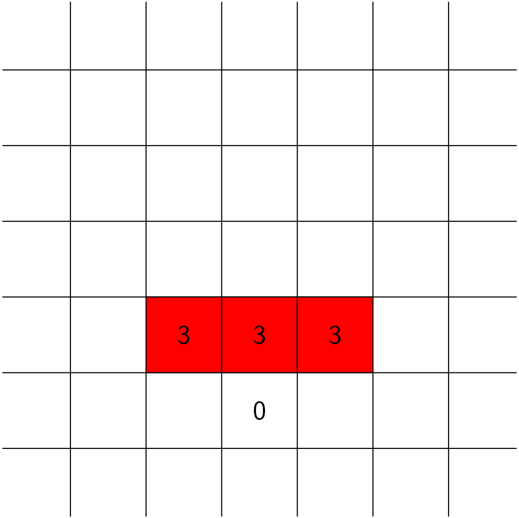
Single-source oriented sandpile



Single-source oriented sandpile



Single-source oriented sandpile



Single-source oriented sandpile

	1	2	3	2	1	
		0	0	0		
			0			

Single-source oriented sandpile

		1	1	1		
	1	2	0	2	1	
		0	0	0		
			0			

Single-source oriented sandpile

		1	1	1		
	1	2	0	2	1	
	1	0	1	0	1	
	2	1	0	1	2	
		0	0	0		
			0			

18 grains at the origin

Single-source oriented sandpile

		1	1	1		
	1	2	0	2	1	
	1	0	1	0	1	
	2	1	0	1	2	
		0	0	0		
			0			

18 grains at the origin

Single-source oriented sandpile

			1	1	1			
		1	2	0	2	1		
	1	2	1	1	1	2	1	
	1	0	2	0	2	0	1	
	1	0	0	1	0	0	1	
		0	0	0	0	0		
			1	1	1			
				0				

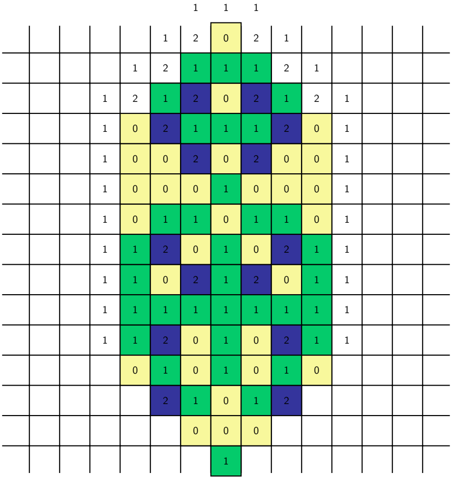
30 grains at the origin

Single-source oriented sandpile

				1	1	1				
			1	2	0	2	1			
		1	2	1	1	1	2	1		
		1	0	2	0	2	0	1		
		1	0	0	1	0	0	1		
		1	1	1	0	1	1	1		
		1	1	0	2	0	1	1		
			2	1	0	1	2			
				1	1	1				
					2					

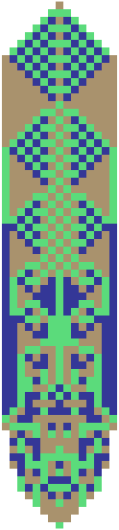
50 grains at the origin

Single-source oriented sandpile



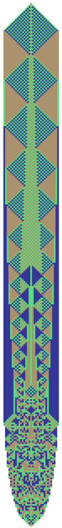
100 grains at the origin

Single-source oriented sandpile



10^3 grains at the origin

Single-source oriented sandpile



10^4 grains at the origin

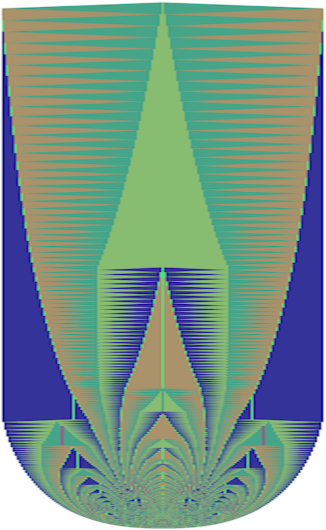
Single-source oriented sandpile

10^5 grains at the origin

Single-source oriented sandpile

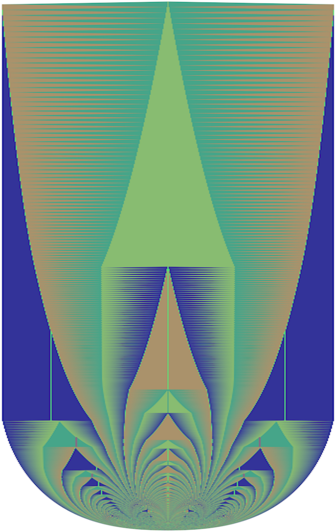
10^6 grains at the origin

Convergence of the oriented sandpile?



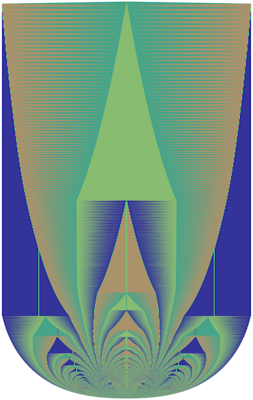
scale parabolically - $n^{1/3}$ in the horizontal direction, $n^{2/3}$ in the vertical direction

Convergence of the oriented sandpile



10^7 grains at the origin

Convergence of the oriented sandpile



Theorem (B. 2019+)

The scaling limit of the single-source oriented sandpile exists and is the Laplacian of the solution to a parabolic free boundary problem.

(More) General Framework

- sample a random background $\eta : \mathbf{Z} \times \mathbf{Z}^+ \to \mathbf{Z}$ from a distribution which is
  - stationary, ergodic under spatial-temporal translations
  - uniformly bounded
  - $\mathbf{E}(\eta(0)) \leq 0$
- this includes nonrandom backgrounds!

The Abelian sandpile heat equation

Theorem (B. 2019+)

- *There exists a unique, compactly supported* $\bar{s} : \mathbf{R}^2 \to [0, 2]$ *and* $\bar{v} : \{\mathbf{R} \times \mathbf{R}^+\} \backslash \{0, 0\} \to \mathbf{R}^+$ *so that almost surely*

  $n^{-2/3} v_n([n^{1/3} x, n^{2/3} t]) \to \bar{v}$ *locally uniformly away from the origin*

  $$s_n([n^{1/3} x, n^{2/3} t]) \to \bar{s} \text{ weakly-*}$$

  and, away from the origin, weakly

  $$\bar{s}(x) = (c_1 \Delta - c_2 \partial_t)\, \bar{v}(x) + \mathbf{E}(\eta(0)),$$

  where c_1, c_2 *are positive constants.*

- $\bar{v}$ *is the unique viscosity solution to the parabolic obstacle problem*

  $$\bar{v} := \min\{w \in C(\{\mathbf{R} \times \mathbf{R}^+\} \backslash \{0, 0\}) | w \geq 0 \text{ and } (D^2 w, \partial_t w) \in \bar{\Sigma}_\eta\},$$

  where $\bar{\Sigma}_\eta$ *is nonrandom and downwards closed.*

Proof outline

discrete adaption of the program of Lin-Smart (2015) for stochastic homogenization of uniformly parabolic equations

1. show convergence of $\bar{v}_n$ along subsequences
   - equicontinuity follows from the parabolic Harnack inequality
   - uniform boundedness follows from a combinatorial argument
2. find a subadditive quantity μ
   - show it controls the sandpile
   - show that it is nice
   - implicitly define $\bar{\Sigma}_\eta$ with μ and the subadditive ergodic theorem
3. conclude that every subsequential limit solves PDE defined by $\bar{\Sigma}_\eta$

Parabolic Monge-Ampère

- for $A \subset \mathbf{Z} \times \mathbf{Z}^+$, $v : \bar{A} \to \mathbf{R}$, let

$$\partial^-(v, A) = \{(p, h) \in \mathbf{R}^2 \text{ so that for some } (x_0, t_0) \in A$$
$$v(x_0, t_0) + p \cdot (x - x_0) - h \leq v(x, t) : \text{ for all } (x, t) \in \bar{A}\}$$

  denote the parabolic subdifferential set of v in A
- the subadditive quantity is

$$\mu(A) = \sup\{|\partial^-(v, A)|\},$$

  supremum is taken over all $v : \bar{A} \to \mathbf{Z}$ with $\tilde{\Delta} v + \eta \leq 2$ in A
  - monotone, stationary by construction
  - convergence of $\frac{\mu(A_n)}{|A_n|}$ then follows by multiparameter subadditive ergodic theorem of Akcoglu and Krengel
- through appropriate perturbations of μ can define Σ_η
  - key tools: Alexandrov-Bakelman-Pucci-Krylov-Tso estimate for fully nonlinear parabolic equations and Dhar's burning algorithm for the sandpile

What is $\bar{\Sigma}_0$?

- the set $\bar{\Sigma}_\eta$ is downwards closed so it suffices to characterize its boundary, $\partial\bar{\Sigma}_\eta$
- parameterize $(D^2 w, \partial_t w)$ by

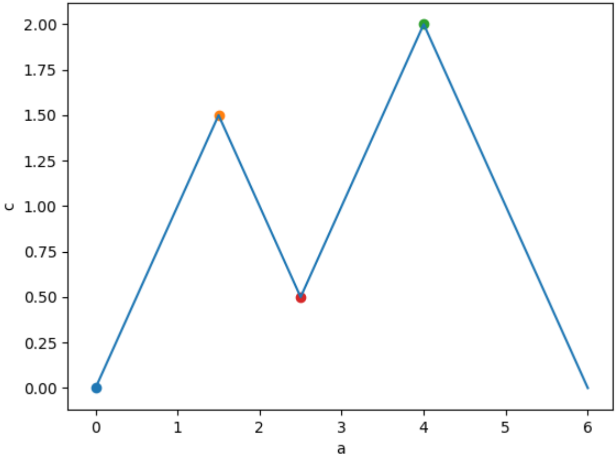
$$M(a,c) = \left(\frac{c-a}{4}, \frac{c+a}{6}\right)$$

- we can show that

$$\partial\bar{\Sigma}_\eta \supseteq \mathcal{R}_\eta := \{M(a,c) : \exists u : \mathbf{Z} \times \mathbf{Z}^+ \to \mathbf{Z}\colon\ 0 \leq \tilde{\Delta} u(x,t) + \eta \leq 2 \\ \text{and } u(x,t) = (\tfrac{c-a}{4})x^2 - (\tfrac{c+a}{6})t + o(x^2) + o(t)\},$$

  where $\tilde{\Delta}$ is the oriented toppling operator
- note if $M(a,c) \in \mathcal{R}_\eta$ then $M(a \pm 6, c) \in \mathcal{R}_\eta$ using the integer-valued function $w(x,t) = -x^2 - x(x+1)/2 - t$

What is $\bar{\Sigma}_0$?



What is $\bar{\Sigma}_0$?

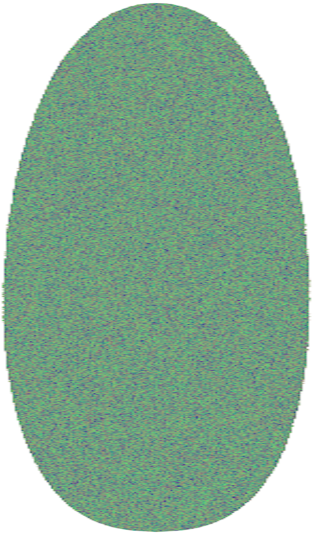
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

2	2	2	2
2	2	2	2
2	2	2	2
2	2	2	2

2	1	2	1
1	2	1	2
2	1	2	1
1	2	1	2

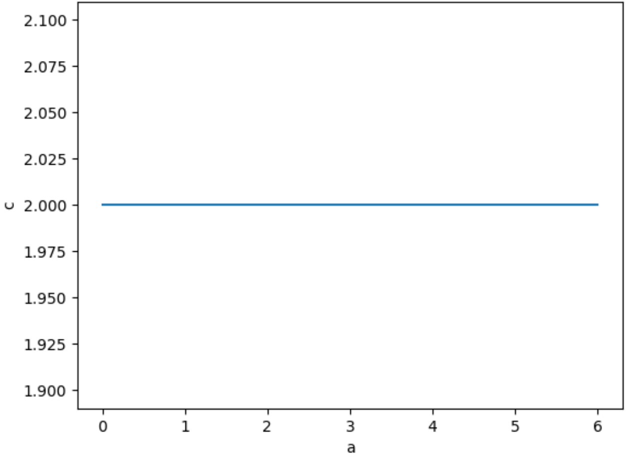
1	0	1	0
0	1	0	1
1	0	1	0
0	1	0	1

What is $\bar{\Sigma}_\eta$?



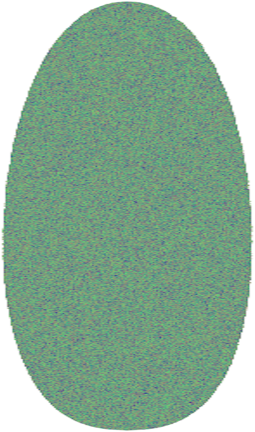
$\eta \sim \text{Uniform}(0, -1, -2)$

What is $\bar{\Sigma}_\eta$?



$\eta \sim \text{Uniform}(0, -1, -2)$

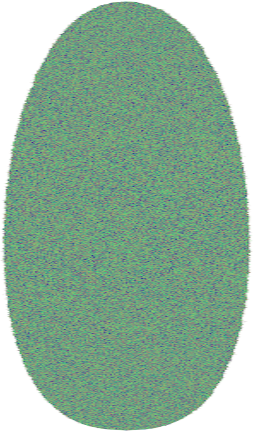
No patterns



Theorem (B. 2019+)

The scaling limit of the single-source directed sandpile on the random background $\eta \sim$ Uniform(0,-1,-2) is nonrandom and constant in a heat-ball.

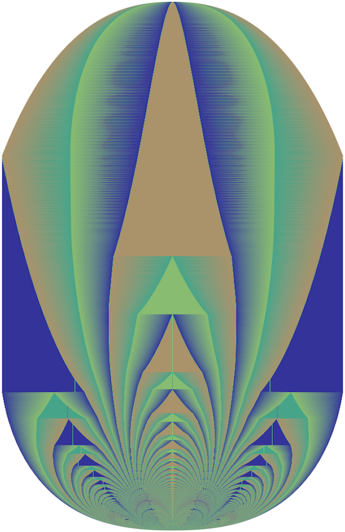
Stochastic directed sandpile



Conjecture

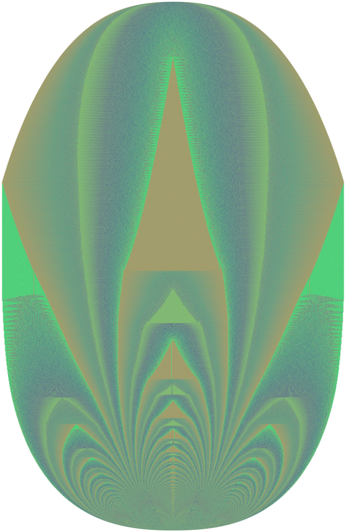
The scaling limit of the stochastic directed sandpile is nonrandom and constant in a heat-ball.

Other random backgrounds?



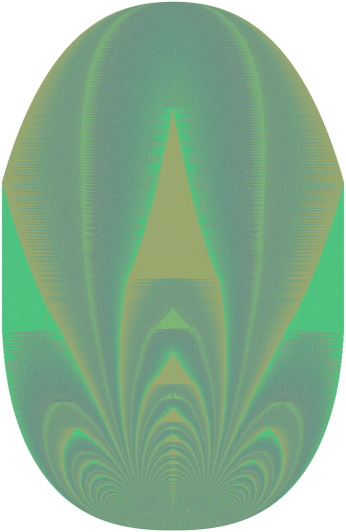
$\eta \sim$ Bernoulli(0,-1,p), $p = 0$, or -1 everywhere

Other random backgrounds?



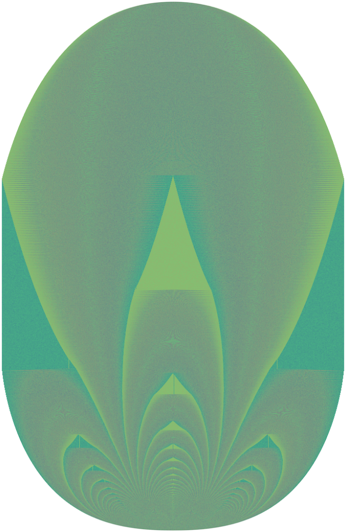
$\eta \sim$ Bernoulli(0,-1,1/8)

Other random backgrounds?



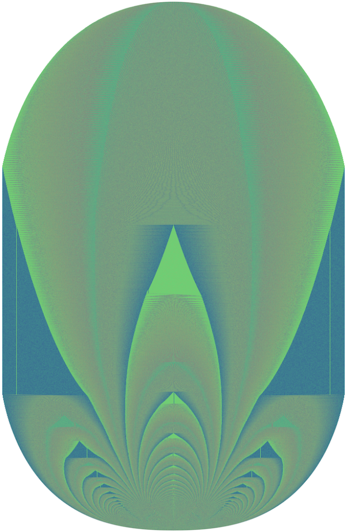
$\eta \sim$ Bernoulli(0,-1,1/4)

Other random backgrounds?



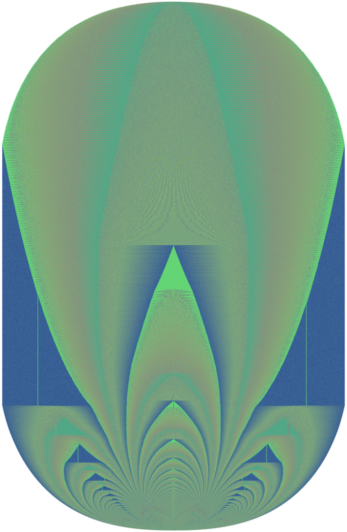
$\eta \sim$ Bernoulli(0,-1,1/2)

Other random backgrounds?



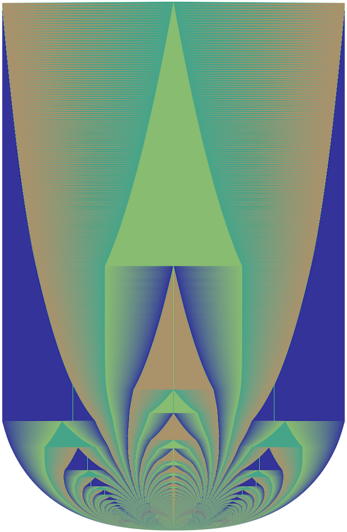
$\eta \sim$ Bernoulli(0,-1,3/4)

Other random backgrounds?



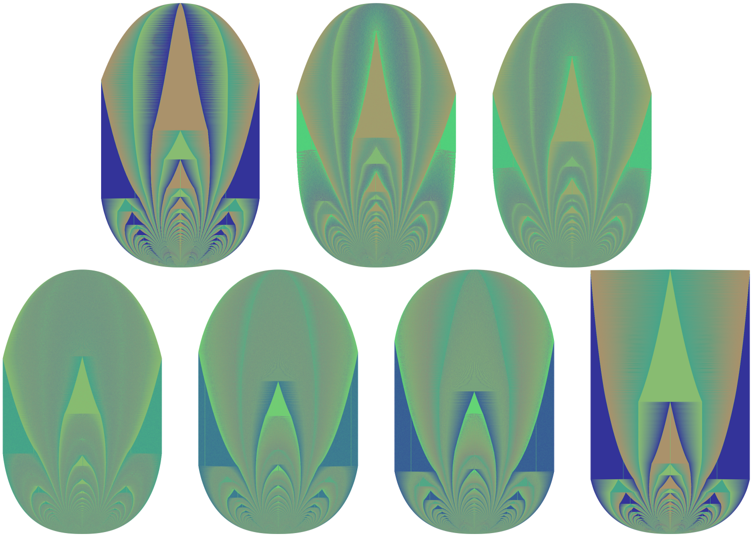
$\eta \sim$ Bernoulli(0,-1,7/8)

Other random backgrounds?



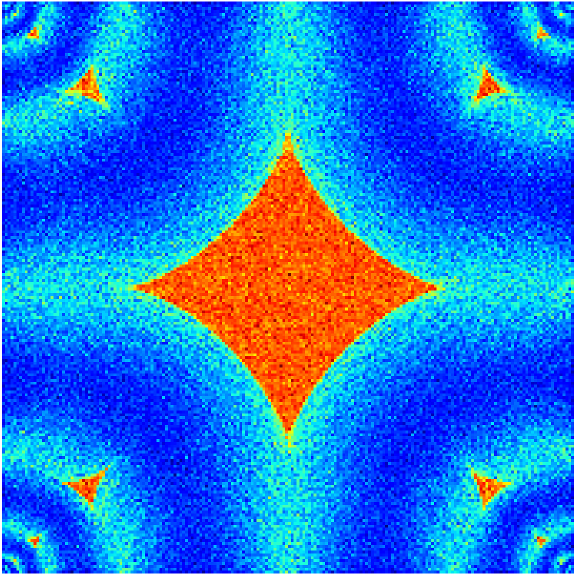
$\eta \sim$ Bernoulli(0,-1,1)

Other random backgrounds?



$\eta \sim$ Bernoulli(0,-1,p) $\quad p = 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1$

Thank you for listening!



arXiv:1909.07849