Stochastic homogenization and Abelian networks

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Guest Lecture for Cornell MATH 7710

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Physical introduction

Mathematical introduction

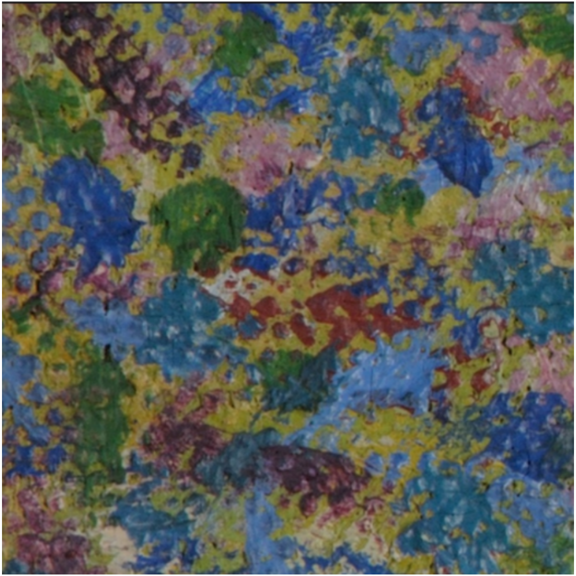
An important tool

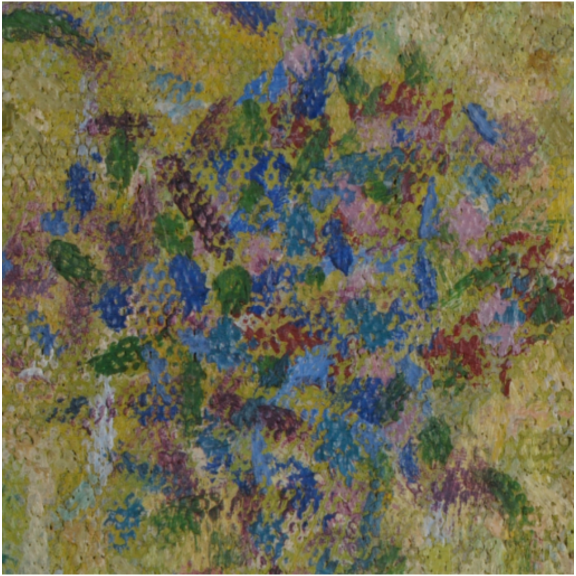
Applications to Abelian networks

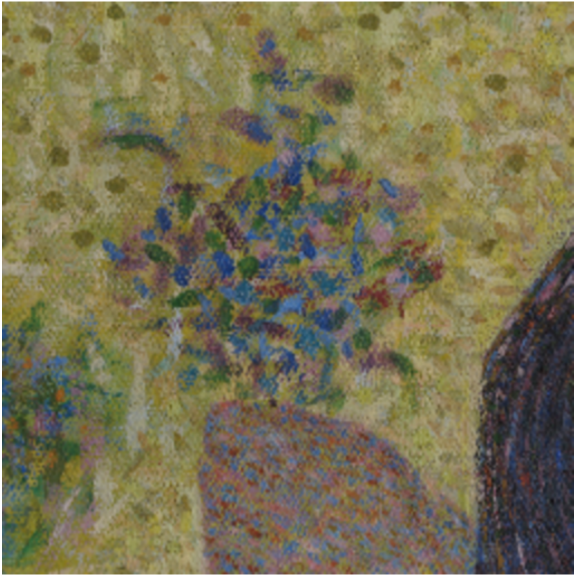
What is homogenization?

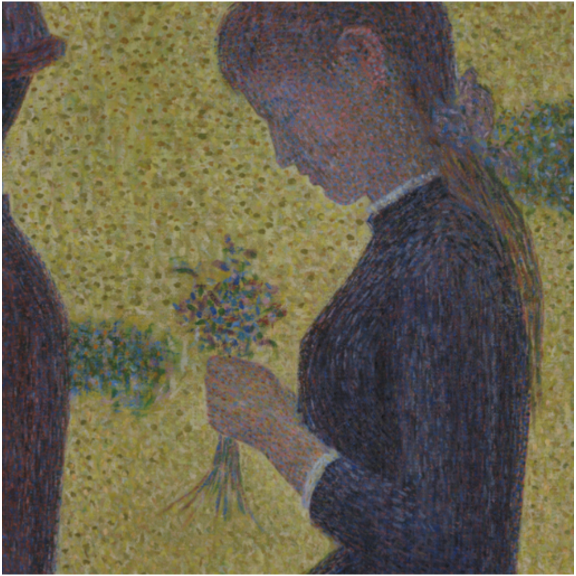
Figure 1: ‘A Sunday Afternoon on the Island of La Grande Jatte’, Georges Seurat

- the world as we know is intrinsically random and composite
- disorder cannot be avoided but can be ‘averaged out’ on large scales
- *homogenization* is understanding disordered, heterogeneous systems by averaged, homogeneous versions









Homogenization in Neo-Impressionist paintings

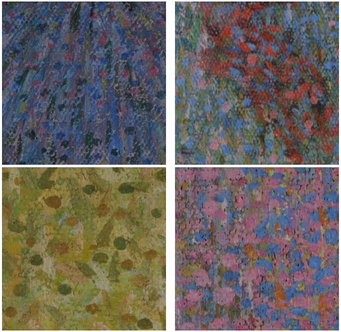


Figure 2: zoomed in squares of Seurat's painting

Homogenization in the real world

- tractable mathematical models of real-life phenomena either implicitly or explicitly assume homogenization
- we will now watch some real-life experiments
- some visuals can be modeled by ‘perfect’ Abelian networks
  - (not claiming Abelian networks should actually be used as physical models!)

Color chromatography

Figure 3: chromatography strip on left, biased iDLA cluster on right.

- biased iDLA clusters were studied by Lucas:
  - 'The Limiting Shape for Drifted Internal Diffusion Limited Aggregation is a True Heat Ball', 2012
- Lucas proved the limit shape is a heat ball and on the way proved an analogous result for the divisible sandpile

Coffee rings

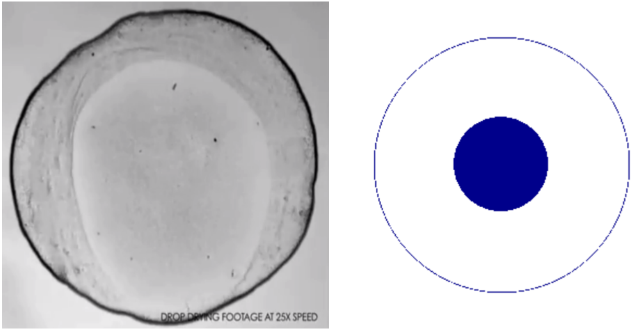


Figure 4: evaporating drop of coffee on left, boundary sandpile on right

- the boundary sandpile model was introduced by Aleksanyan and Shahgholian:
  - ‘Discrete Balayage and Boundary Sandpile’
- an exact formula for the limit of this model was found and studied by Feldman and Smart:
  - ‘A free boundary problem with facets ’

Random walk

Figure 5: trace of simple random walk on $\mathbb{Z}^2$ run for 10^6 steps

- let X_t be a discrete-time simple random walk on $\mathbb{Z}^d$,

$$P(X_t = x | X_{t-1} = y) = \frac{1}{2d} \qquad \text{for } |x - y| = 1$$

- functional central limit theorem tells us that X_t converges (in the appropriate sense) to Brownian motion with identity covariance

Random walk in a periodic environment

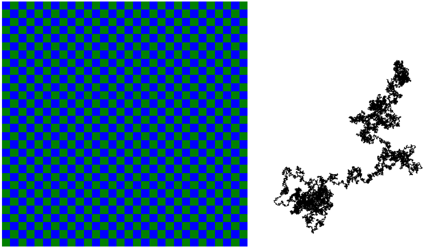


Figure 6: a piece of the periodic environment and random walk in it

- start by periodically coloring $\mathbb{Z}^2$ blue and green
- on blue (green) squares, the random walk has transition probabilities

$$P(X_t = x | X_{t-1} = y) = \begin{cases} \frac{1}{10} & \left(\frac{2}{5}\right) \quad \text{if } x = y + e_1 \\ \frac{1}{10} & \left(\frac{2}{5}\right) \quad \text{if } x = y - e_1 \\ \frac{2}{5} & \left(\frac{1}{10}\right) \quad \text{if } x = y + e_2 \\ \frac{2}{5} & \left(\frac{1}{10}\right) \quad \text{if } x = y - e_2 \end{cases}$$

Random walk in a random environment

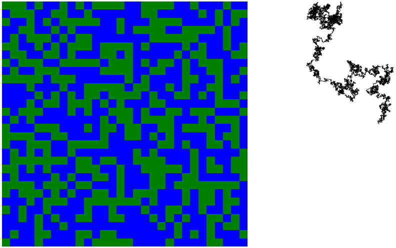


Figure 7: a piece of the random environment and random walk in it

- independently color sites in $\mathbb{Z}^2$ blue or green with probability 1/2
- run random walk with the same transition probabilities as before

Random walk on a random graph

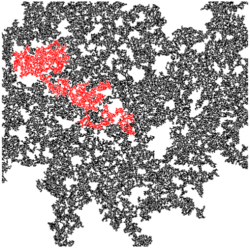


Figure 8: random walk (red) on a supercritical percolation cluster (black) in $\mathbb{Z}^2$

- independently with probability $p = 0.4$ delete sites on $\mathbb{Z}^2$, $\mathcal{C}_\infty$ is the infinite connected component of sites left
- run a blind random walk on $\mathcal{C}_\infty$: $P(X_{t+1} = y | X_t = x) = \frac{1}{4}$ if $|y - x| = 1$ and $y \in \mathcal{C}_\infty$ and $P(X_{t+1} = x | X_t = x) = 1 - \frac{\deg(x)}{4}$

Homogenization

- in every case considered (and more) random walk almost surely converges to Brownian motion:
  - quenched invariance principle
  - Green's function asymptotics
  - local central limit theorem / (parabolic Harnack inequality)
- contributions by many researchers over the years
  - recent, detailed bibliography is given in Section 1.3 of (Dardio and Gu's Quantiative Homogenization ...)
- note that the random/periodic environment may change the covariance of the limit Brownian motion

'See' homogenization with IDLA

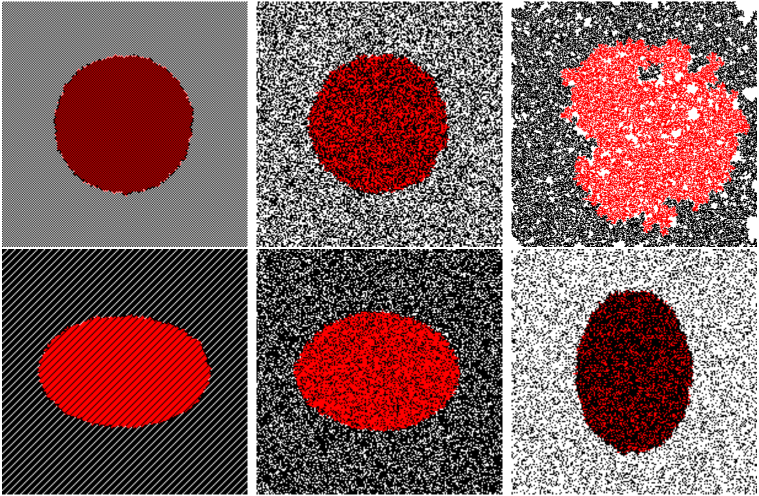


Figure 9: Top row: last three examples. Bottom row: green squares everywhere except when $(x + y)$ divisible by 6, biased random checkerboard $p = 3/4, 1/4$

Random walk and the Laplacian

- generator of simple random walk is the Laplacian, for $f : \mathbb{Z}^d \to \mathbb{R}$ and SRW started at $X_0 = x$,

$$\Delta f(x) = \mathrm{E}(f(X_1) - f(x)) = \frac{1}{2d}\left(\sum_{i=1}^{d}(-2f(x) + f(x \pm e_i)\right)$$

- in particular

$$M_n = f(X_n) - f(X_0) - \sum_{k=1}^{n-1} \Delta f(X_k)$$

  is a mean-zero martingale
- this suggests studying *discrete-harmonic* functions, $\Delta f = 0$

Harmonic functions and invariance principles

- denote our (quenched) random environment by ω, random walk by $X_n^{(\omega)}$, and generator by $\Delta^{(\omega)}$
- $\Delta \ell_p = 0$ for $\ell_p(x) = p \cdot x$, $p \in \mathbb{R}^d$ but *no guarantee* $\Delta^{(\omega)} \ell_p = 0$
- however, (in cases considered before), every ω-harmonic function of linear growth is of the form

$$\ell_p^{(\omega)}(x) = c + p \cdot x + \phi^{(\omega)}(p),$$

  where the *corrector* $\phi^{(\omega)}(p)$ is of sublinear growth
- this + martingale functional CLT is used to establish the quenched invariance principle
  - for details see Biskup's survey 'Recent progress on the random conductance model'

Homogenization of PDEs

- lots of physical systems, (including Abelian networks!) can be described by solutions, $u : \mathbb{Z}^d \to \mathbb{R}$ to microscopic systems with disorder ω

$$F^{(\omega)}(D^2 u, Du, u, x) = 0$$

- homogenization occurs if scaled solutions $\bar{u}_n$ converge to $\bar{u} : \mathbb{R}^d \to \mathbb{R}$ which solve

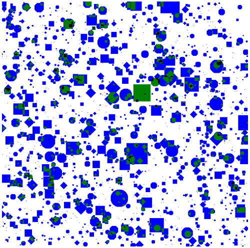
$$\bar{F}(D^2 \bar{u}, D\bar{u}, \bar{u}) = 0$$

- different F require different proofs and scalings but there are some general theories
  - convex integral functionals - includes Laplacian (Armstrong, Smart 2014), (Dal Maso, Modica 1986)
  - fully nonlinear, non-divergence form (Armstrong Smart 2013), (Caffarelli, Souganidis, Wang 2005)
  - hamilton-jacobi (Armstrong, Cardaliaguet 2015), (Lions, Souganidis 2005)

Defining random

- let Ω denote the set of all bounded functions on $\mathbb{Z}^d$, $\omega : \mathbb{Z}^d \to \mathbb{R}$
- endow Ω with the σ-algebra $\mathcal{F}$ generated by translations $\{\omega \to \omega(x) : x \in \mathbb{Z}^d\}$
- denote action of translation in probability space by $T_y\omega(z) = \omega(y + z)$
- take probability measure P on $(\Omega, \mathcal{F})$ with the following properties
  - stationary: for all $E \in \mathcal{F}$, $y \in \mathbb{Z}^d$, $P(T_y E) = P(E)$
  - ergodic: $E = \cap_{y \in \mathbb{Z}^d} T_y E$ implies $P(E) \in \{0, 1\}$

Examples of stationary ergodic fields



- homogeneous poisson point process
- iid, (e.g. random checkerboard)
- periodic: any fixed background ω satisfying $\omega(x+p) = \omega(x)$ for all $x \in \mathbb{Z}^d$, some fixed period $p \in \mathbb{Z}^d$
  - not stationary with respect to $\mathbb{Z}$
  - is stationary with respect $p\mathbb{Z}$ translations (which is enough)
- Abelian sandpile examples:
  - uniform recurrent sandpile on $\mathbb{Z}^d$
  - terminal state of an iid stabilizable sandpile on $\mathbb{Z}^d$

Multiparameter subadditive ergodic theorem - setup

- $\mathcal{U}_0$ family of bounded subsets of $\mathbb{Z}^d$, $\mathcal{L}$ set of bounded Lipschitz domains
- $f : \mathcal{U}_0 \to \mathbb{R}$ is *subadditive* if

$$f(A) \leq \sum_{j=1}^{k} f(A_j)$$

  for pairwise disjoint $A_1, \ldots, A_k$, $A = \cup_{j=1}^{k} A_j$
- $\mathcal{M}_C$ is the collection of subadditive functions $f : \mathcal{U}_0 \to \mathbb{R}$ with

$$0 \leq f(A) \leq C|A| \qquad \text{for every } A \in \mathcal{U}_0$$

- a *subadditive process* is a function $f : \Omega \to \mathcal{M}_C$,

Multiparameter subadditive ergodic theorem

Proposition (Akcoglu Krengel 1981)

Let $f : \Omega \to \mathcal{M}_C$ be a subadditive process. There exists an event Ω_0 of full probability and a constant $0 \leq a \leq C$ so that for every $\omega \in \Omega_0$ and $W \in \mathcal{L}$,

$$\lim_{n\to\infty} \frac{f(nW \cap \mathbb{Z}^d, \omega)}{|nW \cap \mathbb{Z}^d|} = a$$

- important details
  - limit is the same no matter what Lipschitz set we choose: balls and cubes are easiest to work with
  - convergence occurs *simultaneously* for all $W \in \mathcal{L}$

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Applications to Abelian networks

Divisible sandpile - review

- begin with some collection of sand on a locally finite, connected, undirected graph G $\eta : G \to \mathbb{R}$
- initialize odometer $v_0 = 0$ on G and let

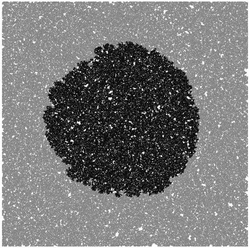
$$v_{t+1}(x) = v_t(x) + \frac{(s_t(x) - 1)^+}{\deg(x)}$$
$$s_{t+1} = s_t + \Delta(v_{t+1} - v_t)$$

Proposition (Levine, Murugan, Peres, Ugurcan 2015)

If $\sum \eta < \infty$, $v := \lim_{t\to\infty} v_t$ *is well-defined, finite, and*

$$v = \min\{f : G \to \mathbb{R}^+ : \Delta f + \eta \leq 1\}.$$

Divisible sandpile on the supercritical percolation cluster



- condition on the origin being in a supercritical percolation cluster and put n grains of sand at the center
- let s_n be terminal *divisible* sandpile on the cluster

Theorem (Shellef 2010, Duminil-Copin, Lucas, Yadin, Yehudayoff 2013)

Almost surely, s_n converges to a Euclidean ball

Random divisible sandpile on $\mathbb{Z}^d$

- $\eta : \mathbb{Z}^d \to \mathbb{R}$ initial background sampled from stationary, ergodic field
- $B_n = \{x \in \mathbb{Z}^d \| |x|_2 \leq n\}$ - discrete ball
- v_n odometer for η on B_n with dissipating boundary conditions,

$$v_n = \min\{f : \bar{B}_n \to \mathbb{R}^+ : \Delta f + \eta \leq 1 \text{ on } B_n \}.$$

- $v_n^{(avg)}$ odometer for $\mathrm{E}(\eta(0))$ on B_n with dissipating boundary conditions

Proposition

$$\frac{1}{n^2} \sup_{x \in B_n} |v_n(x) - v_n^{(avg)}(x)| \to 0$$

Discrete potential theory

- $S_n^{(x)}$ SRW started at $x \in \mathbb{Z}^d$
- $\tau_n = \min\{t \geq 0 : S_t \notin B_n\}$

$$g_n(x,y) = \frac{1}{2d} \operatorname{E} \sum_{n=0}^{\tau_n - 1} 1\{S_n^{(x)} = y\}$$

- fix $\delta > 0$ and $x \in B_n$ estimates on g_n (from Lawler, Limic)
  - exit-time:

$$\sum_{y \in B_n} g_n(x,y) = O(n^2)$$

$$\sum_{z \in B_{\delta n}} g_n(x, x+z) = \delta O(n^2)$$

  - difference for $\max(|x|,|y|) < (1-\delta)n^2$

$$|g_n(x,y) - g_n(x, y+e_i)| = O(|x-y|^{1-d}) + O_\delta(n^{2-2d})$$

Use g_n to construct odometers

- encode the initial condition with

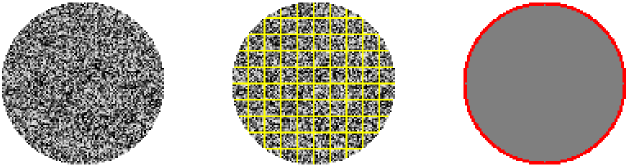
$$r_n(x) := \sum_{y \in B_n} g_n(x,y)\eta(y)$$
$$d_n(x) := \sum_{y \in B_n} g_n(x,y)\,\mathsf{E}(\eta(0))$$

- can check

$$\Delta r_n(x) = -\eta(x)$$
$$\Delta d_n(x) = -\,\mathsf{E}(\eta(0))$$

  and $r_n = d_n = 0$ on ∂B_n
- by the least action principle, it suffices to show r_n and d_n are close

Closeness proof



- $(r_n - d_n)$ can be though of as a procedure to turn the random initial condition into deterministic one
  - rearrange grains within each subcube to match mean
  - if there are not enough grains to do this, transport mass from boundary
- ergodic theorem - don't need to bring in too many grains
- exit time estimate - cost of bringing in some grains is small
- difference estimate - cost of rearranging is small

Single-source Abelian sandpile

Theorem (B. 2019)

Let v_n be the odometer for an Abelian sandpile started with n chips at the origin on a stationary, ergodic, random background $\eta_{\min} \leq \eta \leq \eta_{\max} = 2d - 2$. Almost surely, as $n \to \infty$,

$$\bar{v}_n \to \bar{w} + G$$

where G is the Green's function in $\mathbb{R}^d$ and $\bar{w} \in C(\mathbb{R}^d)$ is the unique viscosity solution of the obstacle problem,

$$\bar{w} := \inf\{\bar{w} \in C(\mathbb{R}^d) | \bar{w} \geq -G \text{ and } \bar{F}_\eta(D^2(\bar{w} + G)) \leq 0\},$$

for deterministic operator $\bar{F}_\eta$

Single-source sandpile - $\mathbb{Z}^2$

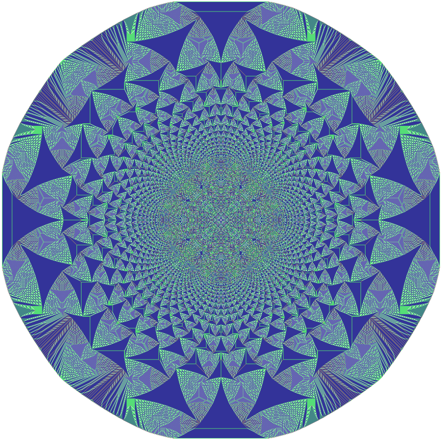


Figure 10: 10^6 chips at origin on $\mathbb{Z}^2$ with toppling rule
$\Delta v(x,y) = -4v(x,y) + v(x+1,y) + v(x-1,y) + v(x,y+1) + v(x,y-1)$

Single-source sandpile - $\mathbb{Z}^2$

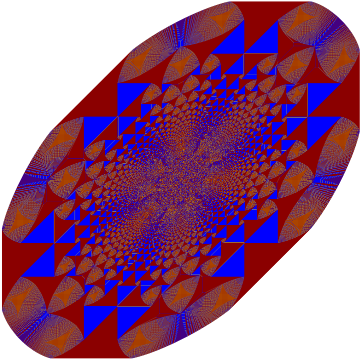


Figure 11: 2^{22} chips at origin on $\mathbb{Z}^2$ with toppling rule
$\Delta v(x, y) = -3v(x, y) + v(x + 1, y) + v(x, y - 1) + v(x - 1, y + 1)$.
facet is on symmetry axis.

Single-source sandpile - $\mathbb{Z}^2$

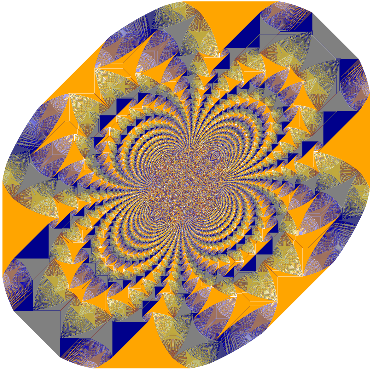


Figure 12: 2^{22} grains at the origin with toppling rule
$\Delta v(x,y) = -7v(x,y) + v(x+1,y) + v(x,y-1) + v(x-1,y+1) +$
$v(x+1,y) + v(x-1,y) + v(x,y+1) + v(x,y-1)$

Single-source sandpile on random background - $\mathbb{Z}^2$

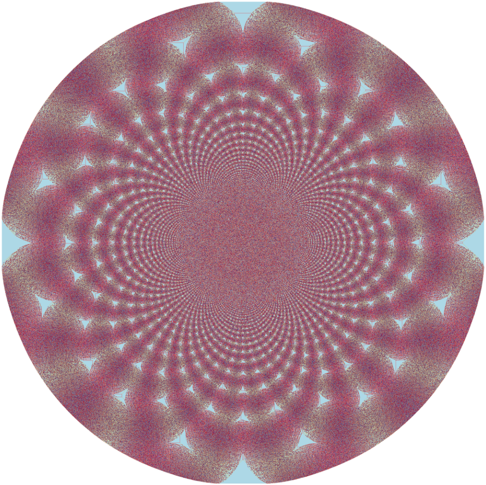


Figure 13: 2^{25} chips at origin on $\mathbb{Z}^2$ with an iid uniform $\{-1, 0\}$ background

Single-source sandpile on random background - $\mathbb{Z}^2$

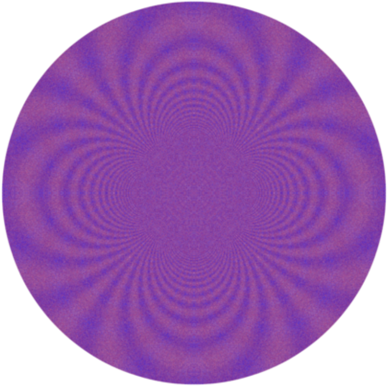


Figure 14: 2^{25} chips at origin on $\mathbb{Z}^2$ with an iid uniform $\{-2, -1, 0\}$ background

Single-source sandpile - $\mathbb{Z}^3$

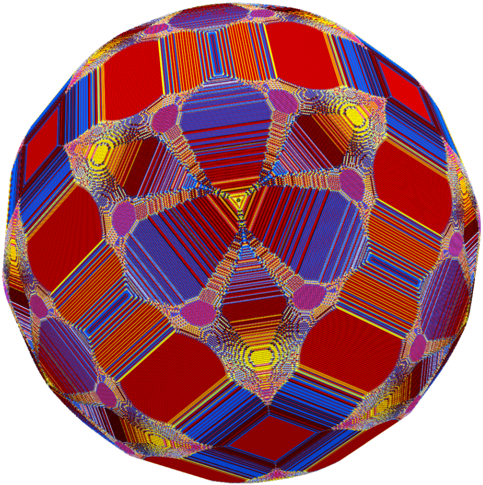


Figure 15: 2^{30} chips at origin on $\mathbb{Z}^3$

Single-source sandpile on random background - $\mathbb{Z}^3$

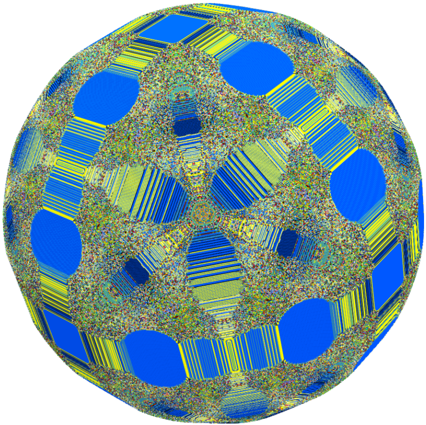


Figure 16: 2^{30} chips at origin on $\mathbb{Z}^3$ with an iid uniform $\{-1, 0\}$ background

Single-source sandpile on random background - $\mathbb{Z}^3$

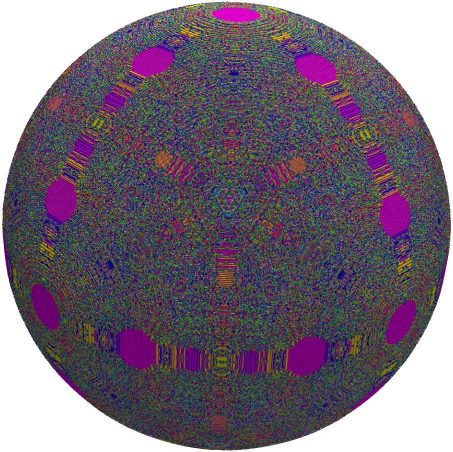


Figure 17: 2^{27} chips at origin on $\mathbb{Z}^3$ with an iid uniform $\{-2, -1, 0\}$ background

Single-source sandpile on random background - $\mathbb{Z}^3$

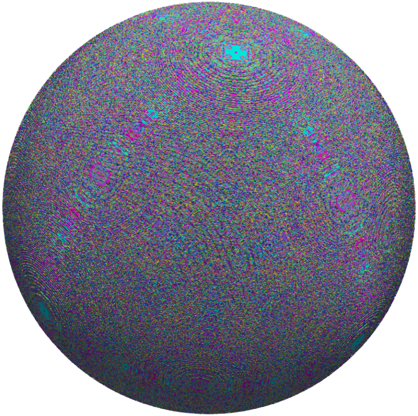


Figure 18: 2^{27} chips at origin on $\mathbb{Z}^3$ with an iid uniform $\{-3, -2, -1, 0\}$ background

Single-source sandpile - $\mathbb{Z}^4$

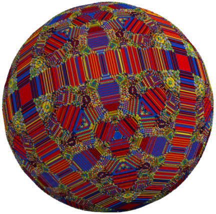



Figure 19: center and final 3D slice of 2^{30} chips at origin on $\mathbb{Z}^4$

Dihedral symmetry of the PDE

Proposition

If η is iid, for all $O \in Aut_{\mathcal{C}_d}$, if $\bar{F}_\eta(M) = 0$ then $\bar{F}_\eta(M^*) = 0$ for $M^* := O^T M O$.

Proof.

- by construction $\bar{F}_\eta(M) \leq 0$ iff there exists $u_N \in \mathcal{C}_N$ with $\Delta u_N(x) + \eta \leq 2d - 1$ and $u_N(x) \geq q_M(x) + o(N^2)$
- since η is iid $\eta^*(x) := \eta(Ox)$ has the same distribution of η and $\Delta u_N^* + \eta^* \leq 2d - 1$ on $\mathcal{C}_N$
- also $u_N^*(x) = u_N(Ox) \geq q_{M^*}(x) + o(N^2)$ and we conclude

$\square$

Dihedral symmetry

Proposition

$\bar{v}(x) = \bar{v}^*(x) := \bar{v}(Ox)$ *for all* $O \in Aut_{\mathcal{C}_d}$

Proof.

- by radial symmetry of G, $G^* = G$ and hence $\bar{w}^* \geq -G$
- by passing the orthogonal transformation to the (quadratic) test function and using the previous slide, $\bar{w}^*$ and $\bar{w}$ are both solutions and hence by comparison equal

$\square$

Axis-monotonicity

Proposition

For any $x, y \in \mathbb{R}^d \backslash \{0\}$ *such that* $(x - y)$ *is orthogonal to a mirror symmetry hyperplane of the unit cube*

$$\bar{v}(x) \geq \bar{v}(y) \qquad \textit{if } |x| \leq |y|$$

Proof.

- suppose $|x| \leq |y|$ and translate symmetry plane T so that it is equal distance from x and y
- let $\bar{v}^*(z) = \bar{v}(z^*)$ where v^* is the reflection of z across hyperplane T, note $\bar{v}^*(x) = \bar{v}(y)$ and $\bar{v}^*(y) = \bar{v}(x)$
- choose rectangle Ω so that $v^* \geq 0 = v$ on outward face, $v = v^*$ on inner face and $v = v^* = 0$ on parallel faces
- by dihedral symmetry of the PDE, $\bar{v}^*$ solves $\bar{F}_\eta(D^2 v) = 0$ on Ω, hence by comparison $\bar{v}^* \geq \bar{v}$

$\square$

Radial symmetry

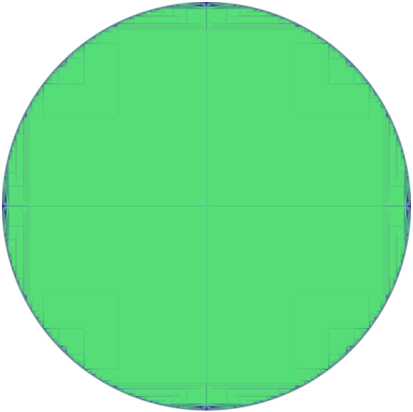


Figure 20: stabilization of 4 chips at every site in a ball of radius 2^{12} on $\mathbb{Z}^2$. the dominating color corresponds to 2 chips.

Radially symmetric solutions on the ball

Proposition

If η is stationary ergodic and $0 \leq \eta \leq d-1$, the scaling limit of the recurrent equivalent to η in B_1 on $\mathbb{Z}^d$ is radial.

Proof.

- candidate $\bar{o}(x) = \frac{|x|^2-1}{2}$ on B_1 satisfies desired boundary conditions
- the function $o(x) = \sum_{i=1}^d \frac{x_i(x_i+1)}{2}$ shows $\bar{F}(D^2\bar{o}) = 0$ as

$$d \leq \Delta o + \eta \leq 2d - 1$$

  is stable and recurrent

□

- also, same argument shows stabilization of $(2d) + \eta$ is radially symmetric

Other η?

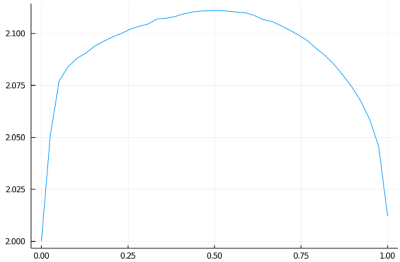


Figure 21: terminal density of Bernoulli$(2, 4, p)$ on a circle of radius 2000

- beginning part of curve looks like hockey-stick!

A tractable example - $d = 1$

- let η denote a stationary ergodic initial sandpile on $\mathbb{Z}$
- stabilize η in intervals $[1, n]$, letting sand fall off edges, denote stable sandpile by s_n
- scaling limit theorem tells us that there exists, $c_\eta \in \mathbb{R}$ so that, almost surely

$$\frac{1}{n}\sum_{x=1}^{n} s_n(x) \to c_\eta$$

- stabilizability result of Fey, Meester, Redig implies $c_\eta = \min(\mathrm{E}(\eta), 1)$
  - this is because the 'only' recurrent configuration on $\mathbb{Z}$ is the all 1s configuration

More general one-dimensional examples

- let $d = |\mathcal{K}|$, $\mathcal{K} \subset \mathbf{N}$,

$$\Delta_{\mathcal{K}} u(x) = \sum_{k \in \mathcal{K}} (-2u(x) + u(x+k) + u(x-k))$$

- stabilize η on the interval with toppling rule $\Delta_{\mathcal{K}}$, define limit density by

$$c_{\mathcal{K},\eta} := \lim_{n \to \infty} \frac{1}{n} \sum_{x=1}^{n} s_n(x)$$

- when $\mathcal{K} = \{1\}^2$, $c_{\mathcal{K},4} = 2$, but every sandpile s with $2 \leq s \leq 3$ is stable and recurrent
  - in this case there is extra room for the initial condition to have some influence

A parabola - $\mathcal{K} = \{1, 1\}$

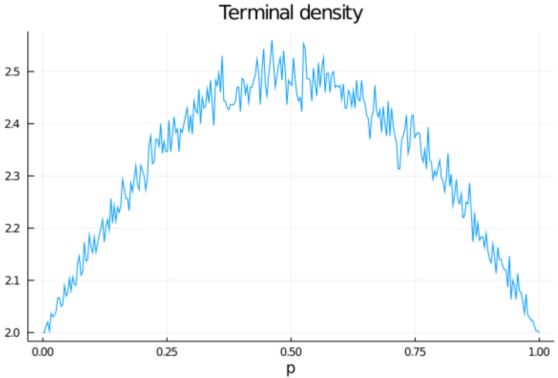


Figure 22: Terminal density for starting density iid η_p

$$\eta_p(x) = \begin{cases} 5 & \text{with probability } p \cdot (1-p) \\ 4 & \text{with probability } 1 - 2p \cdot (1-p) \\ 3 & \text{with probability } p \cdot (1-p) \end{cases}$$

One proof!

Proposition (Fey, Levine, Wilson '09)

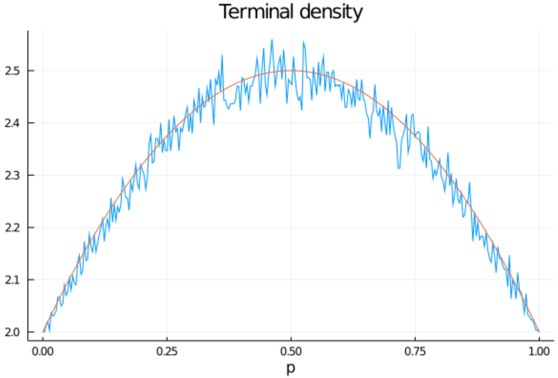
When $\mathcal{K} = \{k\}^d$, $c_\eta = d + \mathsf{E}(\eta \mod d)$

Proof.

- let $s_0 = \eta$
- key: toppling preserves parity: $s_t = s_{t+1} \mod d$
- decompose $s_0 = d_0 + f_0$ where $d_0 \mod d = 0$
- stabilizing / finding recurrent equivalent to d_0 equivalent to same operation for d_0/d on $\mathbb{Z}^1$

$\square$

A parabola - $\mathcal{K} = \{1, 1\}$



- above proposition shows

$$c(p) = 2 + 2p \cdot (1 - p)$$

A quartic (?) - $\mathcal{K} = \{1, 2\}$

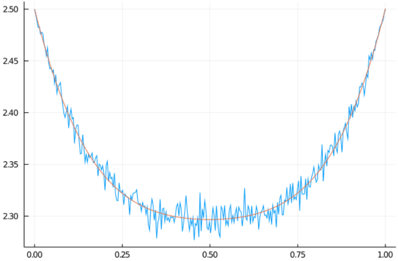


Figure 23: Terminal density iid η_p

- least squares suggests

$$c(p) \approx 5/2 - 5/9 p_1 - 4/3 p_2$$

where

$$p_1 = (p \cdot (1-p))^2$$
$$p_2 = (p \cdot (1-p))\,(1 - 2p \cdot (1-p))$$

but no proof

A foothold

Proposition

Let $q_c(x) = \frac{cx^2}{2\sum_{k\in\mathcal{K}} k^2}$*. For all* $c < c_{\mathcal{K},\eta} - \mathrm{E}\,\eta$*,*

$$s := \Delta_{\mathcal{K}}\lceil q_c \rceil + \eta$$

is stabilizable

- proof of this uses $d = 1$ in an essential way,
- believe (?) but still cannot prove result for $d \geq 2$

Thank you!