DYNAMIC DIMENSIONAL REDUCTION IN THE ABELIAN SANDPILE

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Abstract. We prove the dimensional reduction conjecture of Fey, Levine, and Peres (2010) on the hypercube. The proof shows that dimensional reduction, symmetry, and regularity of the Abelian sandpile persist during the parallel toppling process. This stronger result verifies empirical observations first documented by Liu, Kaplan, and Gray (1990).

1. Introduction

1.1. **Overview.** Let $\mathcal{C}_N^{(d)}$ be a hypercube of side length N in $\mathbb{Z}^d$. A *sandpile* is a function $s : \mathcal{C}_N^{(d)} \to \mathbb{Z}$. Start with $2d$ chips in the hypercube, $s_0 = s = 2d$, then iterate the following rule: every site with at least $2d$ chips on it becomes *unstable* and *topples* in *parallel*, simultaneously giving one chip to each of its $2d$ neighbors. If a boundary site topples, it loses chips over the edge. Eventually every site is stable and the process stops [HLM+08, LP10, LP17, J+18, Kli18].

Let $\{s_t^{(d)}\}_{t \geq 1}$ denote the evolution of this process over time. When N is large, growing, self-similar patterns appear in $s_t^{(2)}$. Moreover, central cross-sectional slices of $s_t^{(3)}$ coincide almost exactly with $s_t^{(2)}$ for all t. See Figures 1 and 2.

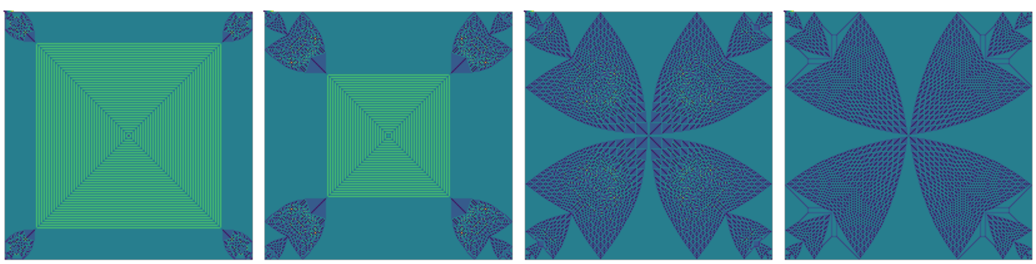

Figure 1. $s_t^{(2)}$ for $t = (25)^2, (50)^2, (100)^2, \infty$, $N = 200$, and $s_0 = 4$. Sites with $0, \ldots, 7$ chips are represented by different colors.

These observations were first made in the literature by Liu, Kaplan, and Gray in 1990 [LKG90]. In particular, they noted that this only occurred in domains bounded by right angles. Several years later, in 2010, Fey, Levine, and Peres highlighted an approximate dimensional reduction conjecture in the context of the single-source sandpile [FLP10].

While dimensional reduction is exact in our setting, it can also be understood through the scaling limit of the sandpile. The large-scale patterns which appear in sandpiles have been studied in a series of papers by Levine, Pegden, Smart, and the author [PS+13, LPS16, LPS17, PS20, BR19]. The foundation of these works is the link between the sandpile and a limit *sandpile PDE*. The structure of this PDE reveals that it is possible to construct solutions in $\mathbb{Z}^d$ from solutions in $\mathbb{Z}^{d-1}$.

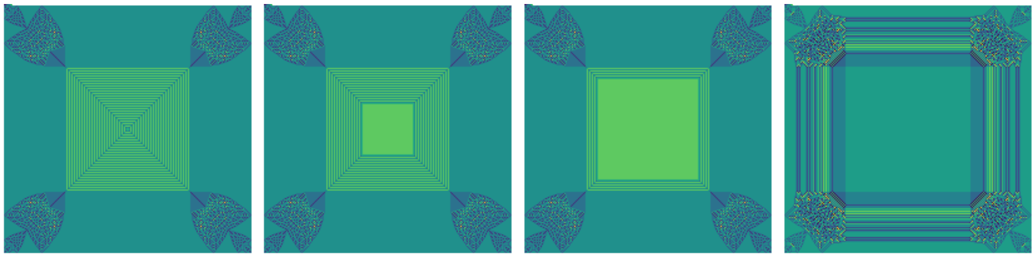

FIGURE 2. $s_t^{(3)}(\cdot, \cdot, \text{offset})$ for $t = (50)^2$, offset $= 0, 20, 40, 60$, $N = 200$, and $s_0 = 6$. Sites with $0, \ldots, 11$ chips are represented by different colors.

However, scaling limits for sandpiles are universal, but dimensional reduction is not. While scaling limits can be used to understand rough, macroscopic properties of sandpiles, they are too coarse to uncover *exact*, microscopic behavior.

In this paper, we leverage discrete techniques to prove exact dimensional reduction and self-similarity of the Abelian sandpile on the hypercube when $s_0 = 2d$. Our main insight is recognizing the parallel toppling process together with strong induction can be used to control the sandpile as it stabilizes. In fact, we do not know how to prove dimensional reduction for only the terminal sandpile. Our proof involves no technology from viscosity solutions or knowledge of the continuum sandpile PDE.

We observe this result as a special case of a more general phenomena which we cannot yet prove: for integer $k \geq 0$, when $s_0 = 2d + k$, above a critical dimension, $d > d_0 := k + 1$, exact dimensional reduction, modulo $(d_0 - 1)$ dimensional defects, appears to persist throughout the parallel toppling process. We show that when $d = d_0$, dimensional reduction fails to occur, providing one explanation for why $(d_0 - 1)$ dimensional defects appear in higher dimensions. See Table 1.

1.2. **Main Result.** Our proof begins with the *odometer*, v_t, which encodes the number of topples per site over time. Let $v_0 : \mathcal{C}_N^{(d)} \to \mathbb{N}$ be the initial odometer $v_0 = 0$; then, recursively,

$$v_{t+1} = v_t + 1\{s_t \geq 2d\}, \tag{1}$$

$$s_{t+1} = s_t + \Delta(v_{t+1} - v_t), \tag{2}$$

where Δ is the graph Laplacian on $\mathcal{C}_N^{(d)}$ with dissipating boundary conditions. We indicate dependence on d and N by writing $v_t^{(d,N)}$ and $s_t^{(d,N)}$.

For a more succinct presentation of the result, we first make a symmetry reduction. Let $\mathrm{Aut}_{\mathcal{C}_d}$ denote the group of symmetries of the d-dimensional hypercube and let it act on $\mathbb{Z}^d$ by matrix-vector multiplication. In Section 2.4 below, we show $v_t^{(d,N)}(\mathbf{x}) = v_t^{(d,N)}(\sigma_i \mathbf{x})$ for all $t \geq 1$, $\mathbf{x} \in \mathcal{C}_N^{(d)}$, and $\sigma_i \in \mathrm{Aut}_{\mathcal{C}_d}$. Hence the odometer and sandpile are fully determined by their restrictions to a fundamental domain of the hypercube.

Theorem 1.1. *Let $N \geq 1$, $d \geq 2$, and $M = \lceil N/2 \rceil$. Denote the fundamental domain of $C_N^{(d)}$ consisting of sorted coordinates in decreasing order by $\mathcal{S}_M^{(d)} := \{(x_1, \ldots, x_d) : M \geq x_1 \geq \cdots \geq x_d \geq 1\}$.*

(1) For all $t \geq 1$ and $\mathbf{x}_{d-1} \in \mathcal{S}_M^{(d-1)}$

$$v_t^{(d,M)}(\mathbf{x}_{d-1}, 1) = v_t^{(d-1,M)}(\mathbf{x}_{d-1})$$

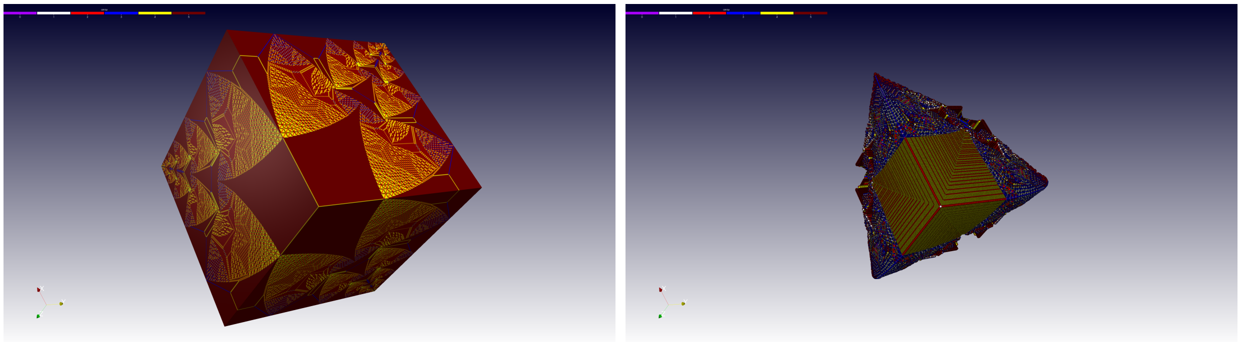

FIGURE 3. On the left, a cut-out corner cube of $s_\infty^{(N,3)}$ and the right the same cube with all layers which match $s_\infty^{(N,2)}$ removed. $N = 2^{11}$ and $s_0 = 7$.

and for $\mathbf{x}_{d-1} \geq 2$
$$v_\infty^{(d,M)}(\mathbf{x}_{d-1}, 2) = v_\infty^{(d-1,M)}(\mathbf{x}_{d-1}).$$

(2) *For all* $j \leq M$*,* $t \leq \tau_j$*, and* $\mathbf{x} \in \mathcal{S}_M^{(d)}$ *with* $\mathbf{x} > M - j$*,*
$$v_t^{(d,M)}(\mathbf{x}) = v_t^{(d,j)}(\mathbf{x} - j)$$
where
$$\tau_j = \min\{t \geq 1 : v_t^{(1,j)}(x) \geq j \text{ for } x \in \partial\mathcal{C}_j\}.$$

These results also translate to the sandpile.

Corollary 1.1.1.

(1) *Dimensional reduction: for all* $\mathbf{x}_{d-1} \geq 2$
$$s_\infty^{(d,M)}(\mathbf{x}_{d-1}, 1) = s_\infty^{(d-1,M)}(\mathbf{x}_{d-1}) + 2,$$

(2) *Self-similarity: for each* $t \leq \tau_j$*,* $j \leq M$*, and* $\mathbf{x} > M - j + 1$
$$s_t^{(d,M)}(\mathbf{x}) = s_t^{(d,j)}(\mathbf{x} - j)$$

In Section 3 we show that dimensional reduction does not occur when $s_0 = 2d + d$ and $N = 2$. We also provide an explicit solution to the odometer v_t when $s_0 = 2d + (d-1)$ and $N = 4$ for all $d \geq 1$. The solution suggests that the proof template in this paper may help with the following.

Problem 1.2. *Show that Theorem 1.1 holds when* $s_0 = 2d + k$ *for all* $N \geq 2$ *and* $d > d_0 := (k+1)$*.*

We expect an even stronger result to be true, although it is likely the proof will require techniques beyond those presented here. In simulations, exact dimensional reduction appears to occur away from the central slice. For example, when $s_0 = 4$, $d = 2$, and N is large, the center of the sandpile contains large curved triangles of 3s. In fact, for every dimension and size we could simulate, whenever dimensional reduction occurs along the axes of the hypercube, it extends; see Figure 3 for an example in three dimensions.

Problem 1.3. *Extend dimensional reduction on the hypercube to a domain of codimension zero.*

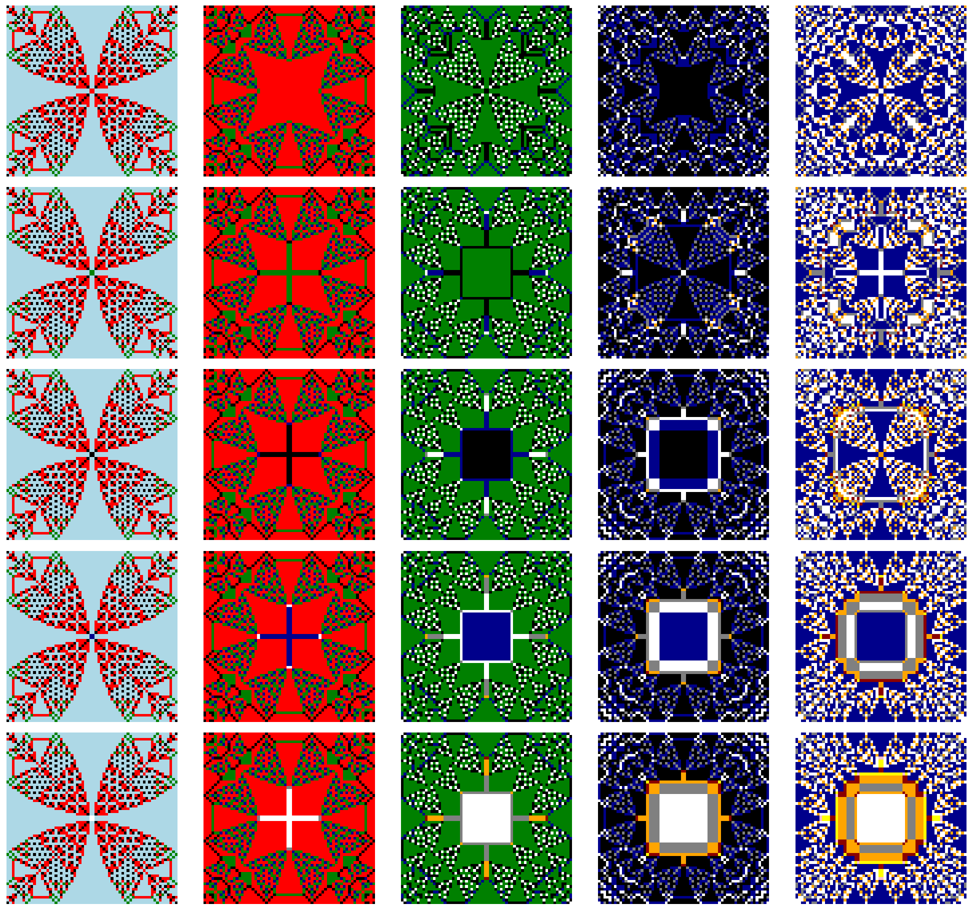

TABLE 1. Terminal sandpile configurations for $s_0 = 2d + k$ and $N = 64$. Site colors are normalized by column so that in dimension d a site with z chips has the same color as a site with $(z-2)$ chips in dimension $(d-1)$.

The following is closely related.

Problem 1.4. *Show that the odometer for any bounded initial sandpile on the hypercube has bounded second differences. In particular, show for all $t \geq 1$ and $i = 1, 2$ that*

$$-3 \leq -2v_t(\mathbf{x}) + v_t(\mathbf{x} + e_i) + v_t(\mathbf{x} - e_i) \leq 2$$

when $d = 2$ and $s_0 = 4$.

Numerical evidence indicates the hypercube is a necessary hypothesis in Problem 1.4. In fact, in most other domains, including the circle, the odometer does not appear to have bounded second differences. On the hypercube, our proof of dimensional reduction shows that the odometer has bounded second difference along the central axes; however, a method to propagate those bounds to the interior remains out of reach.

1.3. **Outline of the proof.** Our main technical tool is a technique introduced by Babai and Gorodezy to prove discrete quasiconcavity of the single-source sandpile odometer in $\mathbb{Z}^2$

[BG07]. By an iteration of their technique, we gain symmetry of the odometer, a derivative comparison result, and a parabolic least action principle. These results, which appear in Section 2, extend beyond the hypercube and so may be of independent interest.

The proof of dimensional reduction is a careful induction on hypercube dimension, side length, and time. Some parts of the argument can be simplified, but we present it in this fashion to suggest a template for proving dimensional reduction with more general initial data. In Section 3 we explicitly determine v_t when $N = 4$ for all $s_0 = 2d + d - 1$. This is done by mapping the hypercube to a line graph via a radial decomposition. The explicit form of v_t provides both a base case for our proof and progress towards Problem 1.2. We also show that when $N = 2$ dimensional reduction does not occur at the critical dimension d_0.

The explicit solution when $N = 4$ establishes the base case for an odometer regularity result which is then proved in Section 4. Finally, in Section 5, we use the established regularity of the odometer in dimension $(d-1)$ to prove dimension reduction in dimension d

An efficient algorithm for computing high-dimensional sandpiles. In Section 2.4 we show that $v_t^{(d,N)}$ can be computed via the parallel toppling procedure restricted to the simplex. In fact, the argument shows that any sandpile with a symmetric initial condition on $\mathbb{Z}^d$, including the single-source sandpile, retains symmetry throughout the parallel toppling process and can be computed in this way.

For d large, computing sandpiles on the simplex improves space complexity by a factor of d^d. Moreover, the reduction in size also leads to a faster algorithm when using parallelization. We wrote a program in Julia [BEKS17, BFDS18] for computing arbitrary dimensional sandpiles which implements these improvements. The program, which may be freely used and modified, is included in the arXiv post.

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2. Preliminaries

2.1. **Notation and conventions.** When we need to distinguish between vectors and scalars, we reserve $\mathbf{x}$ for vectors and x for scalars. The i-th element of $\mathbf{x}$ is x_i and $\mathbf{x}_i = (x_1, \ldots, x_i)$. We refer to coordinate basis vectors of $\mathbb{Z}^d$ as e_i and the ones vector of length i by $\mathbf{1}_i = (1, \ldots, 1)$. Equalities, inequalities, addition, and multiplication between vectors and scalars are to be understood pointwise.

We embed $\mathcal{C}_N^{(d)}$ into $\mathbb{Z}^d$ in two different ways depending on whether or not N is even or odd. If $N = 2M+1$, $\mathcal{C}_N^{(d)} = \{\mathbf{x}+1 : |x_i| \leq M\}$ otherwise $\mathcal{C}_N^{(d)} = \{\mathbf{x} - M + 1 : 1 \leq x_i \leq 2M\}$. The graph Laplacian on $\mathcal{C}_N^{(d)}$ operates on functions $f : \mathcal{C}_N^{(d)} \to \mathbb{R}$ as

$$\Delta^{(d)} f(x) = -2df(x) + \sum_{y \sim x} f(y), \tag{3}$$

where we pad $f(x) := 0$ for $x \notin \mathcal{C}_N^{(d)}$ and the sum $y \sim x$ is over the 2d nearest neighbors of x, $|y - x| = 1$. When the hypercube size or dimension is not used, we omit distinguishing sub/superscripts.

2.2. **Babai-Gorodezky technique.** In this section and the next, let s_0 be an arbitrary initial sandpile on $\mathcal{C}$ and v_t its odometer. A straightforward induction argument and the definition of the graph Laplacian yields the following lemma.

Lemma 2.1. *For each $x \in \mathcal{C}$ and all $t \geq 0$,*

$$v_{t+1}(x) = \lfloor \frac{s_0(x) + \sum_{y \sim x} v_t(y)}{2d} \rfloor. \tag{4}$$

Babai and Gorodezky used this simple lemma to prove a nontrivial discrete quasiconvexity property of the single-source sandpile in $\mathbb{Z}^d$ [BG07]. A more general version of their argument appears below in Lemma 2.4. Roughly, their technique recognizes that if a property of the odometer holds at $t = 1$, is consistent across the symmetry axes, and can be verified on the boundaries of the domain, it must hold for all $t \geq 1$.

Lemma 2.1 is used many other times throughout this paper; notably we use it to prove a parabolic least action principle and symmetry of the odometer on the hypercube.

2.3. **Parabolic least action principle.** The least action principle [FLP10] shows that v_∞ is minimal among all $w : \mathcal{C} \to \mathbb{N}$ which stabilize s_0: $\Delta w + s_0 \leq 2d - 1$. We upgrade this to a parabolic least action principle by observing the parallel toppling procedure as a directed sandpile on $\mathcal{G} = \mathbb{N} \times \mathcal{C}$. The initial sandpile and odometer over time are stacked, $s(t, x) := s_0(x)$ and $v(t, x) := v_t(x)$ for all $t \geq 1$ and $x \in \mathcal{C}$. The graph Laplacian operates on functions $f : \mathcal{G} \to \mathbb{R}$ as

$$\Delta^{\mathcal{G}} f(t, x) = -2df(t, x) + \sum_{y \sim x} f(t - 1, y), \tag{5}$$

for $t \geq 1$ and $x \in \mathcal{C}$, where the sum $y \sim x$ is over the nearest neighbors of x in $\mathcal{C}$.

Lemma 2.2 (Parabolic least action principle)**.**

$$v(t, x) = \min\{u : \mathcal{G} \to \mathbb{N} : \Delta^{\mathcal{G}} u(t, x) + s(t, x) \leq 2d - 1 \textit{ for all } x \in \mathcal{C} \textit{ and } t \geq 1\} \tag{6}$$

Proof. Let $w(t, x)$ denote the right-hand side of (6). We show using Lemma 2.1 and induction that $v(t, x) = w(t, x)$. Indeed, by the directed structure of $\mathcal{G}$, it suffices to show this equality one time slice at a time. Equality holds at $t = 0$ as $v(0, x) = w(0, x) = 0$. Assume that $v(t', \cdot) = w(t', \cdot)$ for $t' \leq t$ and let $x \in \mathcal{C}$ be given. The monotonicity of the graph Laplacian implies $\Delta^{\mathcal{G}} w + s \leq 2d - 1$, hence,

$$\Delta^{\mathcal{G}} w(t + 1, x) + s(t + 1, x) = -2dw(t + 1, x) + s_0(x) + \sum_{y \sim x} w(t, y) < 2d,$$

and a rearrangement yields,

$$w(t + 1, x) \geq \lfloor \frac{s_0(x) + \sum_{y \sim x} w(t, y)}{2d} \rfloor.$$

By Lemma 2.1 and the inductive hypothesis, the right-hand side of the above is exactly $v(t + 1, x)$. Similarly, for the other direction,

$$\begin{aligned} \Delta^{\mathcal{G}} v(t + 1, x) + s(t + 1, x) &= 2d \left(\frac{s_0(x) + \sum_{y \sim x} w(t, y)}{2d} - \lfloor \frac{s_0(x) + \sum_{y \sim x} w(t, y)}{2d} \rfloor \right) \\ &< 2d, \end{aligned}$$

which concludes the proof by minimality of w. $\square$

Our usage of the parabolic least action principle in the main argument is minimal and can be avoided. And, in some sense, it is a restatement of Lemma 2.1. We included it as it may be of independent interest.

2.4. **Symmetry and fundamental domains.** In this section we show that sandpile dynamics on $\mathcal{C}_N^{(d)}$ preserve the symmetry structure of the d-dimensional hypercube. This is then used to reduce to the sandpile on a *fundamental domain* of the hypercube with reflecting boundary conditions.

We briefly provide a presentation of the group of automorphisms of the hypercube and its action on $\mathbb{Z}^d$; for more details see, for example, [GR13]. Let $\mathrm{Aut}_{\mathcal{C}_d}$ be the group of $(d \times d)$ matrices with exactly one ± 1 in each row and in each column and 0s elsewhere. Let $\sigma_i \in \mathrm{Aut}_{\mathcal{C}_d}$ act on $x \in \mathbb{Z}^d$ by matrix-vector multiplication followed by a translation and let it act on $f : \mathbb{Z}^d \to \mathbb{R}$ by $\sigma_i f(x) := f(\sigma_i x)$. The translation is chosen to preserve $\mathcal{C}_N^{(d)}$ when N is even or odd in our choice of coordinates.

Each σ_i is an isometry and hence preserves nearest neighbors and $\mathcal{C}_N^{(d)}$. That is, if $y \notin \mathcal{C}_N^{(d)}$, then $\sigma_i y \notin \mathcal{C}_N^{(d)}$. And, if $|y - x| = 1$, then $|\sigma_i y - \sigma_i x| = 1$, so

$$\sum_{y' \sim \sigma_i x} f(y') = \sum_{y \sim x} f(\sigma_i y). \tag{7}$$

We say $\Omega \subseteq \mathcal{C}_N^{(d)}$ is a fundamental domain if there exists $\{\sigma_j\} \in \mathrm{Aut}_{\mathcal{C}_d}$ so that $\cup_j \sigma_j \Omega = \mathcal{C}_N^{(d)}$. For example, a fundamental domain of an interval is half of it, while a fundamental domain of a square is a right triangle with one side along the central axes of the square. The fundamental domains which we consider have coordinate consistency across dimensions. Let $M = \lceil N/2 \rceil$ and

$$\mathcal{S}_M^{(d)} := \{(x_1, \dots, x_d) : M \geq x_1 \geq \cdots \geq x_d \geq 1\}. \tag{8}$$

Observe that $\{\mathbf{x}_{d-1} : (\mathbf{x}_{d-1}, 1) \in \mathcal{S}_M^{(d)}\} = \mathcal{S}_M^{(d-1)}$; this is the first step towards proving dimensional reduction on the hypercube.

Let v_t be the odometer function for an initial sandpile, s_0, on $\mathcal{C}_N^{(d)}$ which is *symmetric*, $\sigma_i s_0 = s_0$ for all $\sigma_i \in \mathrm{Aut}_{\mathcal{C}_d}$. We show that the parallel toppling odometer coincides with the *symmetrized* odometer $v_t^{\mathcal{S}}$ on $\mathcal{S}_M^{(d)}$ with appropriate reflecting boundary conditions. That is, for each $x \in \mathcal{S}_M^{(d)}$ and $y \sim x$ there exists a unique rotation or reflection, $\sigma_i \in \mathrm{Aut}_{\mathcal{C}_d}$ so that $\sigma_i y \in \mathcal{S}_M^{(d)}$. Denote the iteration over rotated and reflected neighbors by $y \overset{\mathcal{S}}{\sim} x$. For all $t \geq 0$ and $x \in \mathcal{S}_M^{(d)}$, let

$$v_{t+1}^{\mathcal{S}}(x) = \lfloor \frac{s_0(x) + \sum_{y \overset{\mathcal{S}}{\sim} x} v_t^{\mathcal{S}}(y)}{2d} \rfloor, \tag{9}$$

where $v_0^{\mathcal{S}} := 0$.

We provide an algorithmic construction of this which we use to prove Lemma 2.4 below. Let $x \in \mathcal{S}_M^{(d)}$ be given and define $x_0 = 1$ and $x_{d+1} = M$. The follow iteration produces a sequence of indices describing the symmetrized nearest neighbors of x. Start with $l_0 = 0$ and pick the largest $(d+1) \geq r_0 \geq l_0$ with $x_{l_0} = x_{r_0}$. If $r_0 = (d+1)$, stop, otherwise, set $l_1 = (r_0 + 1)$ and repeat, generating

$$\mathbf{I}^{(M,d)}(x) := \{(l_0, r_0), \dots, (l_n, r_n)\}, \tag{10}$$

where $n \leq (d+1)$. Then,

$$\sum_{y \overset{\mathcal{S}}{\sim} x} v_t(y) = \sum_{k=1}^{n-1} (1 + r_k - l_k) \left(v_t(x + e_{r_k}) + v_t(x - e_{l_k}) \right) \tag{11}$$
$$+ (r_0 - l_0) \left(v_t(x + e_{r_0}) + v_t(x - e_{l_0}) \right)$$
$$+ (r_n - l_n) v_t(x - e_{l_n})$$

where $v_t(x - e_{d+1}) := 0$ and

$$v_t(x - e_{l_0}) = \begin{cases} v_t(x) & \text{if } N \text{ is even} \\ v_t(x + e_{r_0}) & \text{if } N \text{ is odd} \end{cases} \tag{12}$$

Lemma 2.3 (Symmetry). *For each $t \geq 0$ and each $\sigma_i \in \mathrm{Aut}_{\mathcal{C}_d}$, $\sigma_i v_t = v_t$. Hence, $v_t = v_t^{\mathcal{S}}$ on $\mathcal{S}_M^{(d)}$.*

Proof. We prove symmetry of v_t by induction and Lemma 2.1. At $t = 0$, $v_t = 0$, so suppose symmetry holds at time t. Let $\sigma_i \in \mathrm{Aut}_{\mathcal{C}_d}$, $x \in \mathcal{C}_N^{(d)}$ be given. By Lemma 2.1, (7), and the inductive hypothesis,

$$\begin{aligned} v_{t+1}(\sigma_i x) &= \lfloor \frac{s_0(\sigma_i x) + \sum_{y' \sim \sigma_i x} v_t(y')}{2d} \rfloor \\ &= \lfloor \frac{s_0(\sigma_i x) + \sum_{y \sim x} v_t(\sigma_i y)}{2d} \rfloor \\ &= \lfloor \frac{s_0(x) + \sum_{y \sim x} v_t(y)}{2d} \rfloor \\ &= v_{t+1}(x). \end{aligned}$$

□

Note that the proof indicates that Lemma 2.3 can be extended in a natural way to other graphs and domains which are preserved under the automorphism group of the graph.

Henceforth, we consider $v_t^{\mathcal{S}}$ in $\mathcal{S}_M^{(d)}$ and drop superscripts. To reduce the number of cases with similar arguments, we only consider $N = 2M$. Indeed, when N is odd, the proofs are identical except for slight changes to the boundary arguments. Also, we will use $\mathcal{S}_M^{(d)}$ to refer exclusively to the sorted fundamental domain of $\mathcal{C}_{2M}^{(d)}$.

2.5. **Derivative comparison.** In this section we provide a general parabolic comparison result for first order differences of v_t on $\mathcal{S}_M$ when the initial sandpile, s_0 is constant. For $w \in \mathbb{Z}^d$, let D_w denote a first order difference operator of the form $D_w v_t(\cdot) = v_t(\cdot) - v_t(\cdot + w)$. Pad v_t by $v_t(x) := 0$ for all $x \notin \mathcal{C}_M$. Denote the interior with respect to w as

$$\mathrm{Int}_w(\mathcal{S}_M) = \{x \in \mathcal{S}_M : y' \in \mathcal{S} \text{ for all } |y' - (x + w)| = 1\}. \tag{13}$$

Observe that every $y \sim x \in \mathcal{S}_M$ is of the form $y = (x \pm e_i) + d_i$, where d_i is either a reflection, $d_i = e_i$ or a rotation $d_i = e_i - e_j$. We will show that if one can control $D_w v_t$ over the reflecting, rotating, and dissipating boundaries of $\mathcal{S}_M$, then that control persists over time. The dissipating boundary on $\mathcal{S}_M$ is

$$\mathcal{B}_w^{(disp)} \mathcal{S}_M = \{x \in \mathcal{S}_M : (x + w)_i \geq M \text{ for some } 1 \leq i \leq d \ \} \tag{14}$$

while the reflecting and rotating boundaries are

$$\mathcal{B}_w^{(ref)}\mathcal{S}_M = \{x \in \mathcal{S}_M : (x+w)_i \leq 1 \text{ for some } 1 \leq i \leq d\ \} \tag{15}$$

and

$$\mathcal{B}_w^{(rot)}\mathcal{S}_M = \{x \in \mathcal{S}_M : (x+w)_i \leq (x+w)_j + 1 \text{ for some } 1 \leq i < j \leq d\}. \tag{16}$$

for notational convenience write,

$$\mathcal{B}_w = \{\mathcal{B}_w^{(disp)} \cup \mathcal{B}_w^{(ref)} \cup \mathcal{B}_w^{(rot)}\}. \tag{17}$$

Lemma 2.4. *Let $\mathbf{w} := \{w_1, \dots, w_n\}$ be a set of points in $\mathbb{Z}^d$ each equipped with a function $g_j : \mathcal{S} \to \mathbb{Z}$ which is superharmonic in the interior of $\mathcal{S}$. If,*

$$\sup_j \big(D_{w_j} v_{t_0}(x) - g_j(x)\big) \leq 0 \qquad \text{for all } x \in \mathcal{S} \tag{18}$$

and for all $t \geq t_0$ and $x \in \{\mathcal{B}_0 \cup \mathcal{B}_{\mathbf{w}}\}\mathcal{S}$,

$$\begin{gathered} \sup_j\Big(\sum_{y\sim x} v_t(y) - \sum_{y'\sim(x+w_j)} v_t(y') - 2dg_j(x)\Big) \leq 0 \\ \text{or} \\ \sup_j(D_{w_j} v_{t+1}(x) - g_j(x)) \leq 0 \end{gathered} \tag{19}$$

then

$$\sup_j \big(D_{w_j} v_{t+1}(x) - g(x)\big) \leq 0 \tag{20}$$

for all $t \geq t_0$ and $x \in \mathcal{S}$.

Proof. We prove this by induction on t, starting at t_0, the base case guaranteed by (18). Suppose (20) holds at t and let w_j, $x \in \mathcal{S}$ be given. First suppose $x \in \{\mathrm{Int}_{w_j} \cap \mathrm{Int}_0\}(\mathcal{S})$. By Lemma 2.1

$$\begin{aligned} D_{w_j} v_{t+1}(x) - g_j(x) &\leq \lfloor \frac{(2d-1) + \sum_{y\sim x} v_t(y) - \sum_{y'\sim(x+w_j)} v_t(y')}{2d} \rfloor - g_j(x) \\ &= \lfloor \frac{(2d-1) + \sum_{y\sim x} D_{w_j} v_t(y)}{2d} \rfloor - g_j(x) \\ &= \lfloor \frac{(2d-1) + \sum_{y\sim x}\big(D_{w_j} v_t(y) - g_j(y)\big) + \sum_{y\sim x}(g_j(y) - g_j(x))}{2d} \rfloor \\ &\leq 0 \end{aligned}$$

as g_j is superharmonic and integer-valued. If $x \in \{\mathcal{B}_0 \cup \mathcal{B}_w\}$, then we either use the same argument or conclude depending on the case in (19).

□

As a corollary, we deduce the following discrete quasiconcavity property of v_t on a hypercube, which was proved in [BG07] for axis monotonic initial sandpiles on $\mathbb{Z}^2$.

Corollary 2.4.1 (Axis monotonicity [BG07])**.** *For all $t \geq 1$, $x \in \mathcal{S}$ and all sets of indices*

$$I = \{i_1, \dots, i_n | 1 \leq i_1 < \dots < i_n \leq d\}$$

and

$$J = \{j_1, \dots, j_m | i_n < j_1 < \dots < j_m \le d\}$$

where $n \ge 1$ *and* $m \ge 0$ *we have*

$$v_t(x) \ge v_t(x + e_I - e_J),$$

We also have control on the derivative given an odometer upper bound on the dissipating boundary.

Corollary 2.4.2 (Derivative bound)**.** *Suppose* $v_\infty(M, \mathbf{1}_{d-1}) \le kM$ *for integer* $k \ge 1$*. Then, for all* $1 \le j \le d$ *and* $t \ge 0$

$$v_t(x) - v_t(x + e_j) \le kx_j \tag{21}$$

Proof. Let e_j and x be given and let

$$\mathbf{I}^{(M,d)}(x) := \{(l_0, r_0), \dots, (l_n, r_n)\}, \tag{22}$$

be the indices describing the nearest neighbors of x as defined in Section 2.4 above. Pick the largest J so that

$$l_J \le j \le r_J,$$

$v_t(x + e_j) = v_t(x + e_{r_J})$, and

$$\mathbf{I}^{(M,d)}(x + e_{r_J}) = \begin{cases} \{\dots, (l_J, -1 + r_J), (r_J, r_{J+1}), \dots\} & \text{if } (x_{r_J} + 1) = x_{l_{J+1}} \\ \{\dots, (l_J, -1 + r_J), (r_J, r_J), (l_{J+1}, r_{J+1}), \dots\} & \text{if } (x_{r_J} + 1) < x_{l_{J+1}} \end{cases} \tag{23}$$

As $g_j(x) := kx_j$ is harmonic in the interior of $\mathcal{S}$, it remains to check (19) in Lemma 2.4. For later reference, we label the expression we bound,

$$\sum_{y \sim x} v_t(x) - \sum_{y' \sim x + e_j} v_t(y'). \tag{24}$$

The computations are similar in other cases, so we assume $x_{l_{J+1}} = x_{r_J} + 1$ and $l_J \le -1 + r_J$.

Case 1: $J = n$

As we are on the dissipating boundary, $v_{t+1}(x + e_j) = 0$ and $x_j = \dots = x_{r_J} = M$, hence

$$v_{t+1}(x) - v_{t+1}(x + e_{r_J}) = v_{t+1}(x) \le v_\infty(M, \mathbf{1}_{d-1}) \le kM$$

by axis monotonicity and our assumption on the odometer.

Case 2: $0 < J < n - 2$

We compute (24), observing that all differences except for those near r_J are unaffected by the symmetrization;

$$\begin{aligned}
(24) = &\sum_{k=1, k \notin [J, J+1]}^{n-1} (1 + r_k - l_k)\left(v_t(x + e_{r_k}) - v_t(x + e_{r_k} + e_{r_J}) + v_t(x - e_{l_k}) - v_t(x - e_{l_k} + e_{r_J})\right) \\
&+ (r_0 - l_0)\left(v_t(x + e_{r_0}) - v_t(x + e_{r_0} + e_{r_J}) + (v_t(x) - v_t(x + e_{r_J})\right) \\
&+ (r_n - l_n)\left(v_t(x - e_{l_n}) - v_t(x - e_{l_n} + e_{r_J})\right) \\
&+ \star_{\{J, J+1\}} \\
&\le (2d - 2(r_{J+1} - l_J + 1))kx_{r_J} + \star_{\{J, J+1\}},
\end{aligned}$$

where $\star_{\{J,J+1\}}$ is defined as sum of terms in the difference with indices $\{J, J+1\}$. This can then be computed,

$$\begin{aligned}
\star_{\{J,J+1\}} &= (r_J - l_J + 1)\,(v_t(x + e_{r_J}) + v_t(x - e_{l_J})) \\
&- (r_J - l_J)\,(v_t(x + e_{r_J - 1} + e_{r_J}) + v_t(x + e_{r_J} - e_{l_J})) \\
&+ (r_{J+1} - l_{J+1} + 1)\,\big(v_t(x + e_{r_{J+1}}) + v_t(x - e_{l_{J+1}})\big) \\
&- (r_{J+1} - l_{J+1} + 2)\,(v_t(x + e_{r_{J+1}} + e_{r_J}) + v_t(x)) \\
&\leq (r_J - l_J)\,(kx_{r_J - 1} + kx_{r_J}) + (r_{J+1} - l_{J+1} + 1)(kx_{r_J} + k(x_{l_J+1} - 1)) \\
&+ \big(v_t(x - e_{l_j}) - v_t(x)\big) + (v_t(x + e_{r_J}) - v_t(x + e_{r_J} + e_{r_J+1})) \\
&\leq (r_J - l_J)2kx_{r_J} + (r_{J+1} - r_J)2kx_{r_J} \\
&+ k(x_{r_J} - 1) + k(x_{r_J} + 1) \\
&= 2(r_{J+1} - l_J + 1)kx_{r_J}.
\end{aligned}$$

Case 3: $J = (n-1) > 0$

We bound differences with indices $\{n-1, n\}$ in (24),

$$\begin{aligned}
\star_{\{n-1,n\}} &\leq (r_{n-1} - l_{n-1})2kx_{r_{n-1}} \\
&+ (r_n - l_n)k(x_{r_{n-1}}) \\
&+ (v_t(x + e_{r_{n-1}}) - 0) + (v_t(x - e_{l_{n-1}}) - v_t(x)) \\
&\leq (r_{n-1} - l_{n-1})2kx_{r_{n-1}} + (r_n - l_n)kx_{r_{n-1}} + k(x_{r_{n-1}} + 1) + k(x_{l_{n-1}} - 1) \\
&\leq 2(r_n - l_{n-1})kx_{r_{n-1}}.
\end{aligned}$$

Case 4: $J = 0 < n - 1$

We bound differences with indices $\{0, 1\}$ in (24),

$$\begin{aligned}
\star_{\{0,1\}} &\leq (r_0 - l_0 - 1)\,(kx_{r_0 - 1} + kx_{r_0}) \\
&+ (r_1 - l_1 + 2)\,(kx_{r_0} + k(x_{l_1} - 1)) \\
&+ v_t(x + e_{r_0}) - v_t(x + e_{r_0} + e_{r_1}) + v_t(x) - v_t(x) \\
&\leq (r_0 - l_0 - 1)2kx_{r_0} + (r_1 - l_1 + 2)2kx_{r_0} \\
&+ k(x_{r_1}) \\
&\leq 2(r_1 - l_0 + 1)kx_{r_0}.
\end{aligned}$$

In the last step we used $x_{r_1} = x_{r_0} + 1 = 2x_{r_0}$.

Case 5: $J = (n-1) = 0$

We bound differences with indices $\{0, n\}$ in (24),

$$\begin{aligned}
\star_{\{0,n\}} &\leq (r_0 - l_0 - 1)\,(kx_{r_0 - 1} + kx_{r_0}) \\
&+ (r_n - l_n)\,(k(x_{l_n} - 1)) \\
&+ (v_t(x + e_{r_0}) - 0) + (v_t(x) - v_t(x)) \\
&\leq (r_0 - l_0 - 1)2kx_{r_0} + (r_n - l_n)2kx_{r_0} \\
&+ k(x_{r_0} + 1) \\
&\leq 2(r_n - l_0 - 1)kx_{r_0}.
\end{aligned}$$

□

2.6. **Weak topple control.** We provide a difference-in-time analogue of Lemma 2.4

Lemma 2.5. *For all $t \geq t_0$ and $j \geq 0$,*

$$\max_{z \in \mathcal{S}} \left(v_{t+j}(z) - v_t(z)\right) \leq \max_{z \in \mathcal{S}} \left(v_{t_0+j}(z) - v_{t_0}(z)\right)$$

Proof. We induct on t starting at t_0. Suppose the result holds for $(t-1)$ and let $x \in \mathcal{S}$ be given. Lemma 2.1 implies

$$v_{t+j}(x) - v_t(x) \leq \lfloor \frac{(2d-1) + \sum_{y \sim x} \left(v_{t+j-1}(y) - v_{t-1}(y)\right)}{2d} \rfloor,$$

hence, by induction

$$v_{t+j}(x) - v_t(x) \leq \lfloor \frac{(2d-1) + 2d \cdot \left(\max_{z \in \mathcal{S}} \left(v_{t_0+j}(z) - v_{t_0}(z)\right)\right)}{2d} \rfloor = \max_{z \in \mathcal{S}} \left(v_{t_0+j}(z) - v_{t_0}(z)\right).$$

$\square$

3. Explicit solutions when $M \leq 2$

In this section, we compute v_t when $s_0 = 2d + (d-1)$ for all $d \geq 1$ when $M = 2$. We also show that dimensional reduction does not occur between dimensions (d_0) and $(d_0 - 1)$ when $s_0 = 2d + (d_0 - 1)$ and $M = 1$.

3.1. $M = 1$**.** As we do not know how to define a 0-dimensional sandpile, suppose $d_0 \geq 2$.

Proposition 3.1. *When $s_0 = 2d + (d_0 - 1)$, $v_\infty^{(d_0)} = 1$ but $v_\infty^{(d_0-1)} = 2$*

Proof. In dimension d, a corner site of the hypercube has d internal neighbors so $\Delta^{(d)}(\mathbf{1}) = -2d + d = -d$. Hence,

$$s_1^{(d_0)}(\mathbf{1}) = (2d_0 + (d_0 - 1)) - d_0 = 2d_0 - 1$$

however,

$$s_1^{(d_0-1)}(\mathbf{1}) = (2(d_0 - 1) + (d_0 - 1)) - (d_0 - 1) = 2(d_0 - 1).$$

$\square$

3.2. $M = 2$**.** After a radial reparameterization of $\mathcal{S}_2$, arbitrary dimensional sandpiles become one-dimensional with a simple nearest-neighbor toppling rule. Indeed, every $\mathbf{x} \in \mathcal{S}_2$ is of the form $\mathbf{x} = (\mathbf{2}_x, \mathbf{1}_{d-x})$, for $x = 0, \ldots, d$. Overload notation and consider s_t and v_t as functions on $\{0, \ldots, d\}$. The Laplacian on the line graph can then be computed using symmetry.

Lemma 3.1. *If we define $0 = v_t(d+1) = v_t(-1)$, then*

$$\Delta v_t(x) = (-d - x)v_t(x) + (d - x)v_t(x+1) + x v_t(x-1).$$

Proof. Let $\mathbf{x} = (\mathbf{2}_x, \mathbf{1}_{d-x})$ so that $\mathbf{I}^{(2,d)}(\mathbf{x}) = \{(0, d-x), (d-x+1, d+1)\}$. Hence, by definition of the symmetric Laplacian,

$$\begin{aligned}\Delta v_t(\mathbf{x}) &= -2d v_t(\mathbf{x}) + (d-x)\left(v_t(\mathbf{x}) + v_t(\mathbf{x} + e_{d-x})\right) + x v_t(\mathbf{x} - e_{d-x+1}) \\ &= -2d v_t(x) + (d-x)v_t(x) + (d-x)v_t(x+1) + x v_t(x-1)\end{aligned}$$

$\square$

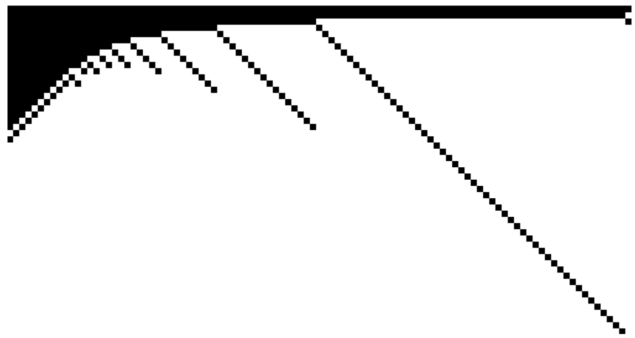

FIGURE 4. The parallel toppling odometer for $s_0 = 2d + (d-1)$ when $d = 100$ and $M = 2$. A black pixel in row t and column x indicates that $v_t(x) = v_{t-1}(x) + 1$.

See Figure 4 for a display of the odometer throughout the parallel toppling process when $s_0 = 2d + (d-1)$. Visually, a contiguous block of decreasing size fires at each step, followed by a ripple of outwards firings. For $t > t_d := (\lceil\sqrt{d-1}\rceil + 1)$, the firing block appears to decreases by one every step. In particular, if a_t indexes the right edge of the block at time t, then $a_1 = d$ and

$$a_t = \begin{cases} \lfloor \frac{d-1}{t-1} \rfloor & \text{for } t \le t_d \\ a_{t-1} - 1 & \text{for } t > t_d. \end{cases}$$

This leads to a simple formula for v_t.

Proposition 3.2. *For all $t \ge 1$,*

$$v_t(x) = \begin{cases} v_{t-1}(x) + 1 & \textit{for } x \le a_t \\ v_{t-1}(x) & \textit{for } a_t < x \le a_{t-1} \end{cases}. \tag{25}$$

And for each $t' < t$ and $a_{t'-1} \ge x > a_{t'}$

$$v_t(x) = v_{t-1}(x-1). \tag{26}$$

Proof. We induct on t. Since $s_0 \ge 2d$, $v_1 = 1$. Suppose (25) and (26) hold for all $t' \le t$.

Step 1: (25)

By strong induction for $t' \le t$, (25) implies $v_t(x) = t$ for $x \le a_t$. Thus,

$$s_t(x) = \begin{cases} 2d + (d-1) - tx & \text{for } x < a_t \\ 2d + (d-1) - tx - (d-x) & \text{for } x = a_t. \end{cases}$$

Let $g(x) := (d-1) - tx$. If $g(x) \ge 0$, $v_{t+1}(x) = v_t(x) + 1$, otherwise $v_{t+1}(x) = v_t(x)$. When $a_{t+1} < x \le a_t$, $g(x) < 0$. Indeed, $g(a_{t+1}) \le (t-1)$ and $g(x+1) - g(x) = -t$.

As g is increasing in x, it remains to check $g(a_{t+1}) \ge 0$ for all t. If $x \le \frac{d-1}{t}$ then $g(x) \ge 0$. If $(t+1) > t_d$, then

$$\frac{d-1}{t-1} - \frac{d-1}{t} \le \frac{d-1}{\sqrt{d-1}\left(\sqrt{d-1}+1\right)} \le 1,$$

thus

$$\begin{aligned} a_{t+1} &= a_t - 1 \\ &\le \frac{d-1}{t-1} - 1 \\ &\le \frac{d-1}{t}. \end{aligned}$$

Step 2: (26)

Now, take $a_{t'-1} \ge x > a_{t'}$ for $1 \le t' \le (t-1)$. If $v_t(x-1) = v_t(x) + 1$, then by strong induction for $t' \le t$, (26) and (25) imply that $v_t(x-2) = v_t(x-1) = t'$ and $v_t(x) = t' - 1$ and $v_t(x+1) \le t' - 1$. Thus,

$$\begin{aligned} s_t(x) &\ge 2d + (d-1) - (t'-2)x \\ &\ge 2d + (d-1) - (t'-2)a_{t'-1} \\ &\ge 2d + (d-1) - (d-1) \\ &= 2d. \end{aligned}$$

However,

$$\begin{aligned} s_t(x-1) &= 2d + (d-1) - t'x - d + x \\ &= 2d - 1 - (t'-1)x \\ &\le 2d - 1, \end{aligned}$$

as $t' \ge 1$. If $v_t(x-1) = v_t(x) = v_t(x+1) \ge (t'-1)$, then

$$\begin{aligned} s_t(x) &\le 2d + (d-1) - (t'-1)x \\ &< 2d + (d-1) - (t'-1)(a_{t'}) \\ &\le 2d. \end{aligned}$$

$\square$

4. Odometer regularity when $d = 1$

From here onward, suppose $s_0 = 2d$. We start the inductive proof of Theorem 1.1, by establishing some regularity of the odometer in the critical dimension $d = 1$. In the next section, we inductively use dimensional reduction to show that $d \ge 2$ sandpiles inherit this regularity. This regularity ensures that the dynamics of lower-dimensional sandpiles agree with their higher-dimensional embeddings.

When reading Section 5 below, the reader should observe that whenever Proposition 4.1 (or something close to it) holds, dimensional reduction follows. For example, if a version of this result is established in every critical dimension $d_0 \ge 1$, then dimensional reduction follows for all sandpiles of the form $2d + (d_0 - 1)$ in dimensions $d > d_0$. Proposition 3.2 should be understood as a step in this direction.

Proposition 4.1. *For all $M \ge 2$ and $d \ge 1$, the odometer maintains the following properties throughout the parallel toppling process.*

Self-similarity: *For each $1 \le j \le M$ and $t \le \tau_j$*

(27) $$v_t^{(M)}(\mathbf{x}) = v_t^{(j)}(\mathbf{x} - (M - j)) \text{ for } \mathbf{x} > M - j,$$

Weak facet compatibility: *For all* $\mathbf{x}_i \geq 2$, $t \geq 1$, $i \geq 0$, $j \geq 0$

$$\begin{aligned} v_t^{(M)}(\mathbf{x}_i, 1, \mathbf{1}_j) &= v_t^{(M)}(\mathbf{x}_i, 2, \mathbf{1}_j) + 1 \\ &\Longrightarrow \\ v_{t+1}^{(M)}(\mathbf{x}_i, 1, \mathbf{1}_j) &= v_{t+1}^{(M)}(\mathbf{x}_i, 1, \mathbf{1}_j) \end{aligned} \tag{28}$$

Strong facet compatibility: *For all* $\mathbf{x}_i \geq 2$, $j \geq 0$, *(*$t < \tau_M$ *and* $i \geq 0$*) or (*$i \geq 1$ *and* $t \geq \tau_M$*)*

$$v_t^{(M)}(\mathbf{x}) - v_t^{(M)}(\mathbf{x} + 2e_{i+1}) \leq 2 \tag{29}$$

and

$$\begin{aligned} v_t^{(M)}(\mathbf{x}_i, 1, \mathbf{1}_j) &= v_t^{(M)}(\mathbf{x}_i, 2, \mathbf{1}) + 1 \\ &\Longrightarrow \\ v_{t+1}^{(M)}(\mathbf{x}_i, 1, \mathbf{1}_j) &= v_t^{(M)}(\mathbf{x}_i, 1, \mathbf{1}_j) \\ v_{t+1}^{(M)}(\mathbf{x}_i, 2, \mathbf{1}_j) &= v_t^{(M)}(\mathbf{x}_i, 2, \mathbf{1}_j) + 1 \end{aligned} \tag{30}$$

Strong topple control: *For all* $t \geq \tau_{M-1}$,

$$\sup_{x \in \mathcal{S}_M} (v_{t+2}(x) - v_t(x)) \leq 1 \tag{31}$$

Proof of Proposition 4.1 for $d = d_0 = 1$. The proof proceeds by induction on M and then on t. When $M = 2$,

	$v_t^{(M)}(1)$	$v_t^{(M)}(2)$
$t=1$	1	1
$t=2$	2	1
$t=3$	2	2
$t=4$	3	2

and $v_1^{(M-1)}(1) = v_2^{(M-1)}(1) = 1$ which verifies the base case. Now, let M be given and note that $v_1^{(M)}(x) = 1$ for all $1 \leq x \leq M$ and $v_2^{(M)}(x) = 2$ for $1 \leq x < M$ and $v_2^{(M)}(M) = 1$. Hence, suppose (27),(28),(29),(30), hold for $(M-1)$ for all $t \geq 1$ and suppose they hold for M for all $t' \leq (t-1)$. We verify each inductive step.

Self-similarity: (27). By strong induction, it suffices to show $v_t^{(M)}(x) = v_t^{(M-1)}(x-1)$ for $x \geq 2$. By Lemma 2.1,

$$v_t^{(M)}(x) = 1 + \lfloor \frac{v_{t-1}^{(M)}(x+1) + v_{t-1}^{(M)}(x-1)}{2} \rfloor$$

for $x > 1$. Hence, by (27) at $(t-1)$, for $x > 2$,

$$v_t^{(M)}(x) = 1 + \lfloor \frac{v_{t-1}^{(M-1)}(x) + v_{t-1}^{(M-1)}(x-2)}{2} \rfloor = v_t^{(M-1)}(x-1)$$

For $x = 1$, we have reflection at the origin,

$$v_t^{(M-1)}(1) = 1 + \lfloor \frac{v_{t-1}^{(M-1)}(2) + v_{t-1}^{(M-1)}(1)}{2} \rfloor = 1 + \lfloor \frac{v_{t-1}^{(M)}(3) + v_{t-1}^{(M)}(2)}{2} \rfloor.$$

Hence, if $v_{t-1}^{(M)}(1) = v_{t-1}^{(M)}(2)$, then $v_t^{(M-1)}(1) = v_t^{(M)}(2)$.

When $v_{t-1}^{(M)}(1) = v_{t-1}^{(M)}(2) + 1$, we instead use strong facet compatibility in both layers. If $v_{t-1}^{(M-1)}(1) = v_{t-1}^{(M-1)}(2)$, then $v_t^{(M-1)}(1) = v_{t-1}^{(M-1)}(1) + 1$ and we are done, so suppose not. Since sites topple at most once per time step by Lemma 2.5, the odometer must then be, for some $v \geq 2$:

	$v_\cdot^{(M)}(1)$	$v_\cdot^{(M)}(2)$	$v_\cdot^{(M)}(3)$
$t-2$	v	v	$v-1$
$t-1$	$v+1$	v	$v-1$
t	$v+1$	$v+1$	$\geq (v-1)$

This contradicts strong facet compatibility for $v_t^{(M-1)}(1)$ from $(t-2) \to (t-1)$, which we can use as $t \leq \tau_{M-1}$ and hence $(t-1) < \tau_{M-1}$.

Weak facet compatibility: (28). If $v_t(1) = v_t(2)+1$, then $\Delta v_t(1) = -1$ and so $v_{t+1}(1) = v_t(1)$.

Strong facet compatibility: (29) *and* (30). We use Lemma 2.4 to show (29). The function $g(x) = 2x$ is harmonic in the interior of the interval so it suffices to check the dissipating and reflecting boundaries. We control the dissipating boundary using $t < \tau_M$ and the reflecting boundary with (28).

As $t < \tau_M$, $v_t(M) \leq (M-1)$ and hence by Corollary 2.4.2, $v_t(M-1) \leq v_t(M) + (M-1) \leq 2(M-1)$. For the reflecting boundary, i.e, $x = 1$, we check

$$\sum_{y \sim x} v_t(y) - \sum_{y' \sim (x+2)} v_t(y) = (v_t(1) - v_t(2)) + (v_t(2) - v_t(4)) \leq (v_t(1) - v_t(2)) + 4.$$

If $v_t(1) - v_t(2) = 1$, then $v_{t+1}(1) = v_t(1)$ by weak facet compatibility. Otherwise, $\sum_{y \sim x} v_t(y) - \sum_{y' \sim (x+2)} v_t(y) \leq 4$ and we conclude that

$$v_t(x) - v_t(x+2) \leq 2x.$$

Taking $x = 1$, this implies

$$\Delta v_t(2) \geq -2v_t(2) + v_t(1) + v_t(1) - 2 \geq 0,$$

which shows (30).

Strong topple control: (31). By Lemma 2.5, it suffices to show

$$\sup_{x \in \mathcal{S}_M} \left(v_{\tau_{M-1}+2}(x) - v_{\tau_{M-1}}(x) \right) \leq 1$$

First observe that (27) for $v_t^{(M)}$ and (31) for $v_t^{(M-1)}$ imply that

$$\tau_{M-1} \geq \tau_{M-2} + 2. \tag{32}$$

Suppose for sake of contradiction that

$$\left(v_{\tau_{M-1}+2}(x) - v_{\tau_{M-1}}(x) \right) = 2$$

for some $1 \leq x \leq M$. Lemma 2.5 then implies that some neighbor $y \sim x$ must have toppled twice previously. Pick the maximal such y. We consider three cases for y.

Case 1: $y \geq 3$
We first note that $v^{(M)}_{\tau_{M-1}+1}(y) = v^{(M-1)}_{\tau_{M-1}+1}(y-1)$. Indeed, by (27), as $(y-1) \geq 2$,

$$\begin{aligned} v^{(M)}_{\tau_{M-1}+1}(y) &= 1 + \lfloor \frac{v^{(M)}_{\tau_{M-1}}(y+1) + v^{(M)}_{\tau_{M-1}}(y-1)}{2} \rfloor \\ &= 1 + \lfloor \frac{v^{(M-1)}_{\tau_{M-1}}(y) + v^{(M-1)}_{\tau_{M-1}}(y-2)}{2} \rfloor \\ &= v^{(M-1)}_{\tau_{M-1}+1}(y-1) \end{aligned}$$

Hence,

$$2 = v^{(M)}_{\tau_{M-1}+1}(y) - v^{(M)}_{\tau_{M-1}-1}(y) = v^{(M-1)}_{\tau_{M-1}+1}(y-1) - v^{(M-1)}_{\tau_{M-1}-1}(y-1),$$

which contradicts (31) for $v^{(M-1)}_t$.
Case 2: $y = 2$
We claim that $v^{(M)}_{\tau_{M-1}+1}(2) = v^{(M-1)}_{\tau_{M-1}+1}(1)$, in which case we can use the argument of Case 1. If not, then $v^{(M)}_{\tau_{M-1}}(2) = v^{(M)}_{\tau_{M-1}}(3) + 1$ but $v^{(M)}_{\tau_{M-1}}(1) = v^{(M)}_{\tau_{M-1}}(2) + 1$. This implies that either

$$v^{(M)}_{\tau_{M-1}-1}(1) = v^{(M)}_{\tau_{M-1}-1}(2) + 2$$

or

$$v^{(M)}_{\tau_{M-1}-1}(1) = v^{(M)}_{\tau_{M-1}-1}(2) + 1$$
and
$$v^{(M)}_{\tau_{M-1}}(1) = v^{(M)}_{\tau_{M-1}-1}(1) + 1$$

both which contradict weak facet compatibility.
Case 3: $y = 1$
In this case, the odometer near the center must be, for some $v \geq 1$,

	$v^{(M)}_{\cdot}(1)$	$v^{(M)}_{\cdot}(2)$	$v^{(M)}_{\cdot}(3)$
$\tau_{M-1} - 1$	v	v	v
τ_{M-1}	$v+1$	$v+1$	v
$\tau_{M-1} + 1$	$v+2$	$v+1$	$\geq (v)$

This shows $v^{(M)}_{\tau_{M-1}-2}(2) = v - 1$. Indeed, if $v^{(M)}_{\tau_{M-1}-2}(1) = v - 1$, then as $\Delta v^{(M)}_{\tau_{M-1}-2}(1) \geq 0$ $v^{(M)}_{\tau_{M-1}-2}(2) = v-1$. If $v^{(M)}_{\tau_{M-1}-2}(1) = v$, then $\Delta v^{(M)}_{\tau_{M-1}-2}(1) \leq -1$ and $v^{(M)}_{\tau_{M-1}-2}(2) = v - 1$. Hence,

$$v^{(M-1)}_{\tau_{M-1}}(1) = v^{(M)}_{\tau_{M-1}}(2) = v^{(M)}_{\tau_{M-1}-2}(2) + 2 = v^{(M-1)}_{\tau_{M-1}-2}(2) + 2,$$

which contradicts (31) for $v^{(M-1)}_t$ using (32).

□

Note that the comparison principle for sandpiles shows

$$v_\infty(x) = \frac{1}{2}\left(M(M+1) - x(x-1)\right),$$

and so $v_\infty(x) - v_\infty(x+1) = x$. Hence we must use an assumption like $t < \tau_M$ for strong facet compatibility.

5. Odometer regularity and dimensional reduction

We now prove Proposition 4.1 for $d \geq 2$ together with dimensional reduction,

$$v_t^{(d,M)}(\mathbf{x}_{d-1}, 1) = v_t^{(d-1,M)}(\mathbf{x}_{d-1}), \tag{33}$$

by strong induction on M, d, and t. Specifically, given M, d, and t, suppose (27),(28),(29),(30), (31), hold for $v_{t'}^{(M',d')}$ for all $M' \geq 1$, $t' \geq 1$, $d' < d$, for $v_{t'}^{(d,M')}$ for all $M' < M$, $t' \geq 1$, and for $v_{t'}^{(M,d)}$ for all $t' < t$. We also suppose (33) holds for $v_{t-1}^{(d',M')}$ for all $d' \geq 2$ and $M' \leq M$. Indeed, $v_1^{(d,M)} = \mathbf{1}$ for all $d \geq 1$ and $M \geq 1$.

5.1. **Dimensional reduction.** We start the induction in time by proving dimensional reduction given odometer regularity at $(t-1)$. Let $\mathbf{x}_{d-1}$ be given and pick the smallest $d > i \geq 0$ so that $(\mathbf{x}_i, \mathbf{1}_{d-i}) = (\mathbf{x}_{d-1}, 1)$. By symmetry,

$$\begin{aligned}\Delta^{(d)} v_{t-1}^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) = &-2d v_{t-1}^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) \\ &+ \sum_{\mathbf{y}_i \sim \mathbf{x}_i} v_{t-1}^{(d)}(\mathbf{y}_i, \mathbf{1}_{d-i}) \\ &+ (d-i)\left(v_{t-1}^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) + v_{t-1}^{(d)}(\mathbf{x}_i, 2, \mathbf{1}_{d-i-1})\right)\end{aligned}$$

We consider two cases at time $(t-1)$.

Case 1: $v_{t-1}^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) = v_{t-1}^{(d)}(\mathbf{x}_i, 2, \mathbf{1}_{d-i-1})$

By (33) at $(t-1)$, $v_{t-1}^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) = v_{t-1}^{(i+1)}(\mathbf{x}_i, 1)$. Thus,

$$\begin{aligned}\Delta^{(d)} v_{t-1}^{(d,M)}(\mathbf{x}_i, \mathbf{1}_{d-i}) &= -2i v_{t-1}^{(i+1)}(\mathbf{x}_i, 1) + \sum_{\mathbf{y}_i \sim \mathbf{x}_i} v_{t-1}^{(i+1)}(\mathbf{y}_i, 1) \\ &= \Delta^{(i+1)} v_{t-1}^{(i+1,M)}(\mathbf{x}_i, 1)\end{aligned}$$

which concludes this case as $v_t^{(d)} = v_{t-1}^{(d)} + 1(\Delta^{(d)} v_{t-1}^{(d)} \geq 0)$.

Case 2: $v_{t-1}^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) = v_{t-1}^{(d)}(\mathbf{x}_i, 2, \mathbf{1}_{d-i-1}) + 1$

If $i \leq (d-2)$, then (28) for $(t-1) \to t$ for both $v_{t-1}^{(i+1)}$ and $v_{t-1}^{(d)}$ imply that

$$v_{t-1}^{(i+1)}(\mathbf{x}_i, 1) = v_{t-1}^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) = v_t^{(d)}(\mathbf{x}_i, \mathbf{1}_{d-i}) = v_t^{(i+1)}(\mathbf{x}_i, 1)$$

If $i = (d-1)$, then (30) and (33) at $(t-1)$ and $(t-2)$ imply that

$$v_{t-1}^{(d-1)}(\mathbf{x}_{d-1}) = v_{t-1}^{(d)}(\mathbf{x}_{d-1}, 1) = v_{t-2}^{(d)}(\mathbf{x}_{d-1}, 1) + 1 = v_{t-2}^{(d-1)}(\mathbf{x}_{d-1}) + 1.$$

If $(t-2) \geq \tau_{M-1}$, (31) for $v_t^{(i)}$ and $v_t^{(d)}$ imply that

$$v_t^{(d-1)}(\mathbf{x}_{d-1}) = v_{t-1}^{(d-1)}(\mathbf{x}_{d-1}) = v_{t-1}^{(d)}(\mathbf{x}_{d-1}, 1) = v_t^{(d)}(\mathbf{x}_{d-1}, 1).$$

If $(t-2) < \tau_{M-1}$, then (27) and (33) for $v_{t-1}^{(M-1)}$ show

$$v_{t-1}^{(d,M)}(\mathbf{x}_{d-1}, 2) = v_{t-1}^{(d,M-1)}(\mathbf{x}_{d-1} - 1, 1) = v_{t-1}^{(d-1,M-1)}(\mathbf{x}_{d-1} - 1).$$

Similarly,

$$v_{t-1}^{(d,M)}(\mathbf{x}_{d-1}, 1) = v_{t-1}^{(d-1,M)}(\mathbf{x}_{d-1}) = v_{t-1}^{(d-1,M-1)}(\mathbf{x}_{d-1} - 1)$$

Therefore, for all $t' \leq \tau_{M-1}$,

$$v_{t'}^{(d,M)}(\mathbf{x}_{d-1}, 2) = v_{t'}^{(d,M)}(\mathbf{x}_{d-1}, 1), \tag{34}$$

this however contradicts the case we are in.

5.2. **Odometer regularity for $d \geq 2$.** We verify each inductive step.

Self-similarity: (27). As (27) holds for $(M-1)$ at t, it suffices to show that if $t \leq \tau_{M-1}$ and $\mathbf{x} > 1$,

$$v_t^{(M)}(\mathbf{x}) = v_t^{(M-1)}(\mathbf{x}-1). \tag{35}$$

We split verification of this into cases.

Case 1: $\mathbf{x} > 2$

As (35) holds for $(t-1)$, by Lemma 2.1,

$$\begin{aligned} v_t^{(M)}(\mathbf{x}) &= \lfloor \frac{s_0(\mathbf{x}) + \sum_{\mathbf{y}\sim\mathbf{x}} v_{t-1}^{(M)}(\mathbf{y})}{2d} \rfloor \\ &= \lfloor \frac{s_0(\mathbf{x}-1) + \sum_{\mathbf{y}\sim(\mathbf{x}-1)} v_{t-1}^{(M-1)}(\mathbf{y})}{2d} \rfloor \\ &= v_t^{(M-1)}(\mathbf{x}-1). \end{aligned}$$

Case 2: $\mathbf{x} = (\mathbf{x}_j, \mathbf{2})$ for $\mathbf{x}_j > 2$ and $0 \leq j < d$

We show that if $\Delta v_{t-1}^{(M)}(\mathbf{x}) \geq 0$, then $\Delta v_{t-1}^{(M-1)}(\mathbf{x}-1) \geq 0$. First, decompose the Laplacian into a sum of discrete second differences,

$$\Delta v_{t-1}^{(M)}(\mathbf{x}) = \sum_{i=1}^{d} \Delta_{(i)} v_{t-1}^{(M)}(\mathbf{x}),$$

where

$$\Delta_{(i)} v_{t-1}^{(M)}(\mathbf{x}) = -2v_{t-1}^{(M)}(\mathbf{x}) + v_{t-1}^{(M)}(\mathbf{x}+e_i) + v_{t-1}^{(M)}(\mathbf{x}-e_i).$$

Observe that (27) at $(t-1)$ implies, $\Delta_{(i)} v_{t-1}^{(M)}(\mathbf{x}) = \Delta_{(i)} v_{t-1}^{(M-1)}(\mathbf{x}-1)$ for all $i \leq j$ and for $i > j$,

$$\Delta_{(i)}(v_{t-1}^{(M-1)}(\mathbf{x}-1) - v_{t-1}^{(M)}(\mathbf{x})) = v_{t-1}^{(M-1)}(\mathbf{x}-e_i-1) - v_{t-1}^{(M)}(\mathbf{x}-e_i).$$

By reflectional symmetry, for each $i > j$,

$$v_{t-1}^{(M-1)}(\mathbf{x}-e_i-1) = v_{t-1}^{(M-1)}(\mathbf{x}-1) = v_{t-1}^{(M)}(\mathbf{x}),$$

thus

$$\Delta_{(i)}(v_{t-1}^{(M-1)}(\mathbf{x}-1) - v_{t-1}^{(M)}(\mathbf{x})) = v_{t-1}^{(M)}(\mathbf{x}) - v_{t-1}^{(M)}(\mathbf{x}-e_i). \tag{36}$$

If $v_{t-1}^{(M)}(\mathbf{x}) = v_{t-1}^{(M)}(\mathbf{x}-e_i)$ for all $i > j$, we are done, so suppose otherwise.

Take $i > j$ where (36) $\neq 0$. By (30),

$$v_{t-2}^{(M)}(\mathbf{x}) = v_{t-1}^{(M)}(\mathbf{x}) = v_{t-1}^{(M)}(\mathbf{x}-e_i) - 1 = v_{t-2}^{(M)}(\mathbf{x}-e_i). \tag{37}$$

By (27) and (30) for $v_{t-2}^{(M-1)}$, if

$$v_{t-1}^{(M-1)}(\mathbf{x}-1) = v_{t-1}^{(M-1)}(\mathbf{x}-1+e_i) + 1$$

then

$$v_{t-1}^{(M)}(\mathbf{x}) = v_{t-1}^{(M-1)}(\mathbf{x}-1) = v_{t-2}^{(M-1)}(\mathbf{x}-1) + 1 = v_{t-2}^{(M)}(\mathbf{x}) + 1$$

which contradicts (37). Moreover, by (30) we must have for each neighbor $(\mathbf{y}-e_i) \sim (\mathbf{x}-e_i)$, $v_{t-1}^{(M)}(\mathbf{y}) \geq v_{t-2}^{(M)}(\mathbf{y}-e_i)$. Thus,

$$\Delta v_{t-1}^{(M-1)}(\mathbf{x}-1) \geq \Delta v_{t-2}^{(M)}(\mathbf{x}-e_i) \geq 0 \tag{38}$$

Strong facet compatibility: (29) *and* (30). We first use

$$v_t(\mathbf{x}_{i-1},1,\mathbf{1}_j) - v_t(\mathbf{x}_{i-1},3,\mathbf{1}_j) \leq 2 \tag{39}$$

together with the inductive hypotheses to show (30), then we verify (29) below.

Suppose $v_t(\mathbf{x}_{i-1},1,\mathbf{1}_j) = v_t(\mathbf{x}_{i-1},2,\mathbf{1}_j)+1$. By (30) at $(\mathbf{x}_i,1,\mathbf{1}_j)$ from $(t-1)\to t$, $\Delta v_{t-1}(\mathbf{x}_i,1,\mathbf{1}_j)\geq 0$. Hence, it suffices to show

$$\Delta v_t(\mathbf{x}_i,2,\mathbf{1}_j) \geq \Delta v_{t-1}(\mathbf{x}_i,1,\mathbf{1}_j).$$

We use symmetry to decompose each Laplacian;

$$\begin{aligned}
\Delta v_t(\mathbf{x}_{i-1},2,\mathbf{1}_j) &= -2d v_t(\mathbf{x}_{i-1},2,\mathbf{1}_j) \\
&\quad + \sum_{j'=1}^{(i-1)} \left(v_t(\mathbf{x}_i+e_{j'},2,\mathbf{1}_j) + v_t(\mathbf{x}_i-e_{j'},2,\mathbf{1}_j)\right) && (40)\\
&\quad + v_t(\mathbf{x}_i,1,\mathbf{1}_j) + v_t(\mathbf{x}_i,3,\mathbf{1}_j) && (41)\\
&\quad + j\left(v_t(\mathbf{x}_i,2,\mathbf{1}_j) + v_t(\mathbf{x}_i,2,2,\mathbf{1}_{j-1})\right) && (42)
\end{aligned}$$

while

$$\begin{aligned}
\Delta v_{t-1}(\mathbf{x}_{i-1},1,\mathbf{1}_j) &= -2d v_{t-1}(\mathbf{x}_{i-1},1,\mathbf{1}_j) \\
&\quad + \sum_{j'=1}^{(i-1)} \left(v_{t-1}(\mathbf{x}_i+e_{j'},1,\mathbf{1}_j) + v_{t-1}(\mathbf{x}_i-e_{j'},1,\mathbf{1}_j)\right) && (43)\\
&\quad + v_{t-1}(\mathbf{x}_i,1,\mathbf{1}_j) + v_{t-1}(\mathbf{x}_i,2,\mathbf{1}_j) && (44)\\
&\quad + j\left(v_{t-1}(\mathbf{x}_i,1,\mathbf{1}_j) + v_{t-1}(\mathbf{x}_i,2,\mathbf{1}_j)\right) && (45)
\end{aligned}$$

By (30) from $(t-1)\to t$, $v_{t-1}(\mathbf{x}_{i-1},1,\mathbf{1}_j) = v_t(\mathbf{x}_{i-1},2,\mathbf{1}_j)$. Also (30) shows that each $\mathbf{y}_i \sim \mathbf{x}_i$ with $\mathbf{y}_i \geq 2$, $v_t(\mathbf{y}_i,2,\mathbf{1}_j) \geq v_{t-1}(\mathbf{y}_i,1,\mathbf{1}_j)$. If $\mathbf{x}_i - e_{j'} \not\geq 2$, $v_t(\mathbf{x}_i-e_{j'},2,\mathbf{1}_j) = v_t(\mathbf{x}_i,1,\mathbf{1}_j) \geq v_{t-1}(\mathbf{x}_i,1,\mathbf{1}_j)$. This shows that (40) $\geq$ (43). Next, (39) implies

$$(41) \geq 2v_t(x_i,1,\mathbf{1}_j)-2,$$

while (30) from $(t-1)\to t$ implies

$$(44) \leq 2v_{t-1}(x_i,1,\mathbf{1}_j),$$

hence (41) $\geq$ (44). Finally, by (30) from $(t-1)\to t$,

$$v_t(x_i,2,2,\mathbf{1}_{j-1}) \geq v_{t-1}(x_i,2,\mathbf{1}_j)$$

which implies (42) $\geq$ (45).

We now verify (29) for different choices of t.

Case 1: $t < \tau_M$**,** $i \geq 1$

We use Lemma 2.4 as in the proof for $d=1$ to show that

$$v_t(\mathbf{x}) - v_t(\mathbf{x}+2e_i) \leq 2x_i$$

for all $d \leq i \leq 1$ and $\mathbf{x} \in \mathcal{S}_M$. Indeed as $t < \tau_M$

$$v_t(\mathbf{x}) - v_t(\mathbf{x} + e_j) \leq x_j \tag{46}$$

for all $\mathbf{x} \in \mathcal{S}_M$ and $1 \leq j \leq d$. Hence, $v_t(\mathbf{x}) - v_t(\mathbf{x} + 2e_j) \leq 2(M-1)$ on $\partial_{2e_j}\mathcal{S}_M$ for all $1 \leq j \leq d$. The reflecting boundary is checked in the same way as $d = 1$, using weak facet compatibility in higher dimensions.

Case 2: $t \geq \tau_M$**,** $i \geq 2$
Here we show that

$$v_t(\mathbf{x}) - v_t(\mathbf{x} + 2e_i) \leq 2x_i$$

for all $d \geq i \geq 2$ and $x_1 \geq 2$. We again use Lemma 2.4 except the region in which we have the derivative bound shrinks and therefore our boundaries change. The dissipating boundary gets smaller, $\mathcal{B}^{(disp)} := \{x \in \mathcal{S}_M | x_j = M-1 \text{ for some } 2 \leq j \leq d\}$ and the reflecting boundary remains the same except for the removal of a single point, $\mathbf{1}$. By axis monotonicity, $\sup_{x \in \mathcal{B}^{(disp)}} v_t(x) \leq v_t(M, \mathbf{1}) \leq M$. Checking the reflective boundary is as in $d = 1$ except for the point $(2, \mathbf{1}_{d-1})$. We show directly that

$$v_t(2, \mathbf{1}_{d-1}) \leq v_t(3, 2, \mathbf{1}_{d-2}) + 2. \tag{47}$$

Suppose for sake of contradiction that $v_t(2, \mathbf{1}_{d-1}) = v_t(3, 2, \mathbf{1}_{d-2}) + 2$ and $\Delta v_t(2, \mathbf{1}_{d-1}) \geq 0$ but $\Delta v_t(3, 2, \mathbf{1}_{d-1}) < 0$. As (29) has been verified for all $x \in \mathcal{S}_M$ other than $(2, \mathbf{1}_{d-1})$, (30) holds for $(3, 2, \mathbf{1}_{d-1})$ and so $v_t(3, 2, \mathbf{1}_{d-2}) = v_t(3, \mathbf{1}_{d-1})$. Then, by definition of the symmetric Laplacian, weak facet compatibility, and axis monotonicity,

$$\begin{aligned}
\Delta v_t(2, \mathbf{1}_{d-1}) &= -2d v_t(2, \mathbf{1}_{d-1}) \\
&\quad + v_t(3, \mathbf{1}_{d-1}) + v_t(1, \mathbf{1}_{d-1}) \\
&\quad + (d-1)\left(v_t(2, \mathbf{1}_{d-1}) + v_t(2, 2, \mathbf{1}_{d-2})\right) \\
&\leq -2v_t(2, \mathbf{1}_{d-1}) + v_t(3, \mathbf{1}_{d-1}) + v_t(1, \mathbf{1}_{d-1}) \\
&\leq -1,
\end{aligned}$$

which is a contradiction.

Weak facet compatibility: (28). The only remaining case is

$$v_t^{(d)}(1, \mathbf{1}_j) = v_t^{(d)}(2, \mathbf{1}_j) + 1$$

By symmetry,

$$\Delta v_t^{(d)}(1, \mathbf{1}_j) = d \cdot (v_t^{(d)}(2, \mathbf{1}_j) - v_t^{(d)}(1, \mathbf{1}_j)) = -d$$

Strong topple control: (31). We use strong topple control established in dimension $(d-1)$. Suppose for sake of contradiction there exists $\mathbf{x} \in \mathcal{S}_M^{(d)}$ with $v_{\tau_{M-1}+2}(\mathbf{x}) - v_{\tau_{M-1}}(\mathbf{x}) = 2$. Pick $\mathbf{x} = (\mathbf{x}_{d-1}, x)$ so that $x \geq 1$ is minimal.

Case 1: $x = 1$

By dimensional reduction at time τ_{M-1}, $v_{\tau_{M-1}}^{(M,d)}(\mathbf{x}_{d-1}, 1) = v_{\tau_{M-1}}^{(M,d-1)}(\mathbf{x}_{d-1})$. By the parabolic least action principle,

$$v_{\tau_{M-1}+2}^{(M,d-1)}(\mathbf{x}_{d-1}) \geq v_{\tau_{M-1}+2}^{(M,d)}(\mathbf{x}_{d-1}, 1),$$

which contradicts (31) for $v_t^{(M,d-1)}$.

Case 2: $x = 2$
By (34),

$$v^{(M,d)}_{\tau_{M-1}}(\mathbf{x}_{d-1}, 1) = v^{(M,d)}_{\tau_{M-1}}(\mathbf{x}_{d-1}, 2),$$

which in turn, by axis monotonicity, implies $v^{(M,d)}_{\tau_{M-1}+2}(\mathbf{x}_{d-1}, 1) = v^{(M,d)}_{\tau_{M-1}}(\mathbf{x}_{d-1}, 1) + 2$, which contradicts the minimality of x.

Case 3: $x \geq 3$
Some neighbor $\mathbf{y} \sim \mathbf{x}$ must have toppled twice previously. As $x \geq 3$, $\mathbf{y} = (\mathbf{y}_{d-1}, y)$ for $y \geq 2$. The same argument for $d = 1$ when $y \geq 2$ then implies, $v^{(M,d)}_{\tau_{M-1}+1}(\mathbf{y}) = v^{(M-1,d)}_{\tau_{M-1}+1}(\mathbf{y})$ which contradicts (31) for $v^{(M-1,d)}_t$.

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