Scaling limits of Abelian sandpiles

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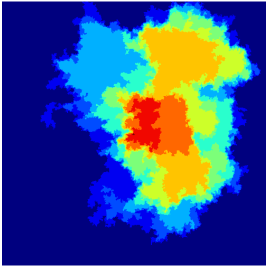
Self-organized criticality

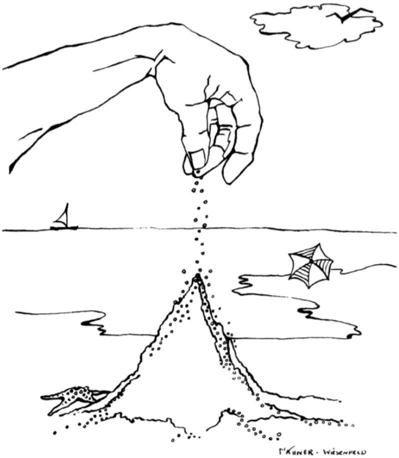
The laws of physics can explain how an apple falls but not why Newton, a part of a complex world, was watching the apple

—Per Bak

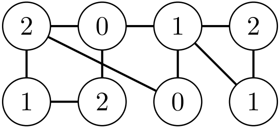
- how can complex behavior arise from simple rules?
  - **self-organized criticality:** nature perpetually self-organizes itself into a critical state, in which microscopic fluctuations can lead to macroscopic, complex changes
- in 1987, physicists Bak, Tang, Wiesenfeld, invented the **Abelian sandpile** as a simple prototype of self-organized criticality



What is a sandpile?

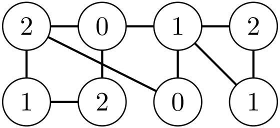


What is an Abelian sandpile?



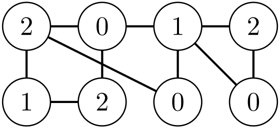
- collection of indistinguishable grains distributed among the vertices of a graph

Sandpile dynamics

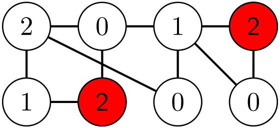


- one rule
  - a vertex is *unstable* if it has at least as many grains as its degree
  - an unstable vertex can *topple* sending one grain to each neighboring vertex

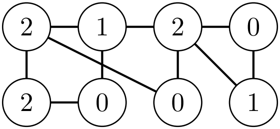
Sandpile dynamics



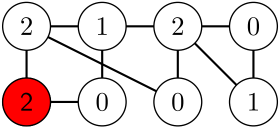
Sandpile dynamics



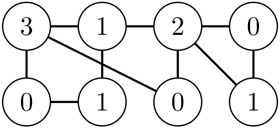
Sandpile dynamics



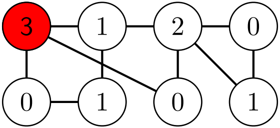
Sandpile dynamics



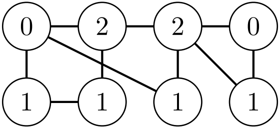
Sandpile dynamics



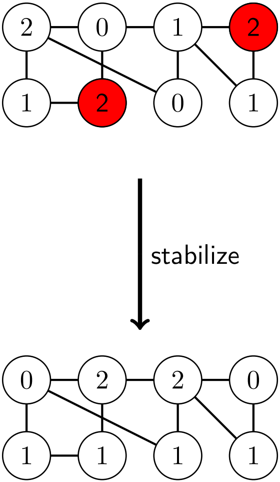
Sandpile dynamics



Sandpile dynamics



Stabilizing sandpiles



order of topples doesn't change final sandpile - model is Abelian!

Do Abelian sandpiles always stabilize?

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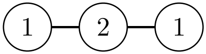
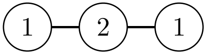
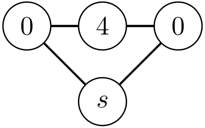


Do Abelian sandpiles always stabilize?

Do Abelian sandpiles always stabilize – no!

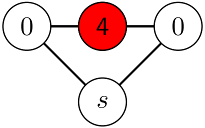


One way to ensure stabilization



add a 'sink' vertex which absorbs grains

One way to ensure stabilization



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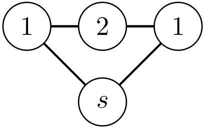

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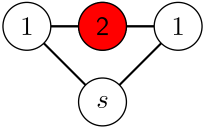
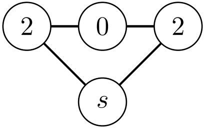
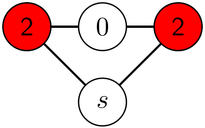


One way to ensure stabilization



One way to ensure stabilization



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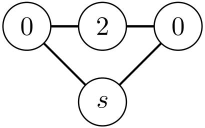

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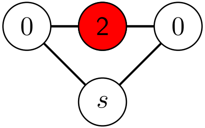
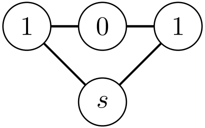
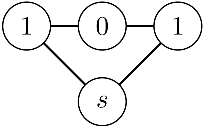


One way to ensure stabilization

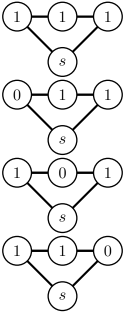


One way to ensure stabilization



Abelian sandpiles on finite, connected graphs with a sink vertex always stabilize.

Recurrent sandpiles

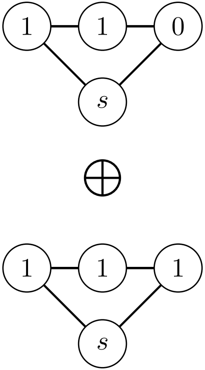


- by adding grains to the sandpile and stabilizing, you eventually will enter the set of *recurrent* sandpiles.

The sandpile group

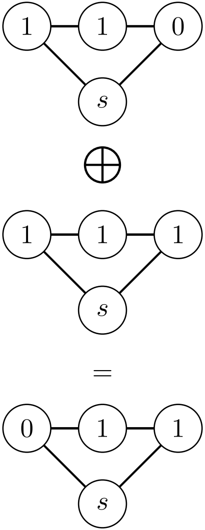
- to any finite connected graph with a sink, we can associate an Abelian group called *the sandpile group* which consists of recurrent sandpiles.

The sandpile group



- the set of stable sandpile configurations forms a commutative monoid under the operation $\bigoplus$ of adding pointwise and then stabilizing

The sandpile group

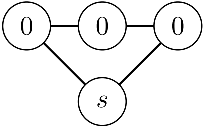


addition in the sandpile group is not linear!

The sandpile group

- the set of stable sandpile configurations forms a commutative monoid under the operation of adding pointwise and then stabilizing
- the minimal ideal of this commutative monoid is an Abelian group which we call *the sandpile group*
  - recall that an *ideal* of a monoid $(M, \bigoplus)$ is a subset $J \subset M$ satisfying $\sigma \bigoplus J \subset J$ for all $\sigma \in M$
  - the *minimal ideal* is the intersection over all nonempty ideals
  - it is a general fact that the minimal ideal of a finite commutative monoid is an Abelian group

What is the identity of the sandpile group?



?

What is the identity of the sandpile group?

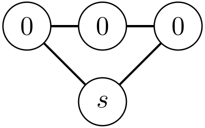


Figure: not the identity

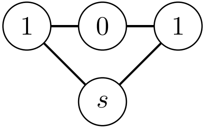


Figure: identity sandpile

- the identity element of a group constructed in this way is not easy to guess
- for the sandpile group, it is generally *not* the all 0 configuration

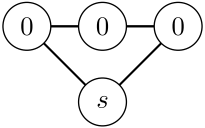
How do you find the identity sandpile?

- the group is finite: can enumerate all stable sandpile configurations and check
  - this is unfeasible for large graphs
  - also, the cardinality of the sandpile group is exponential in the size of the graph

Dhar's burning algorithm

- an algorithm introduced by statistical physicist and sandpile pioneer Deepak Dhar in 1987
- can be used to find the identity of the sandpile group
- roughly: push in sand through the sink until every vertex topples

Example of burning algorithm



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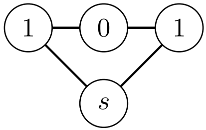

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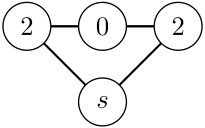

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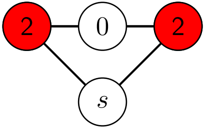

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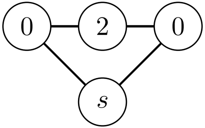

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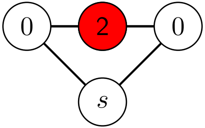
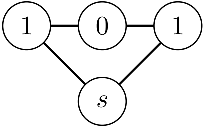
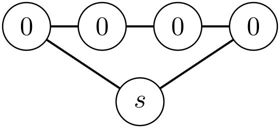


Example of burning algorithm



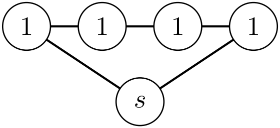
identity - line graph with 3 vertices

Identity on a line graph



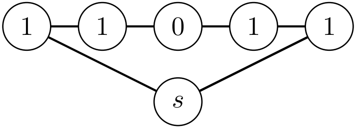
what about a longer line graph?

Identity on a line graph



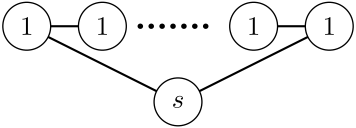
identity - line graph with 4 vertices

Identity on a line graph



identity - line graph with 5 vertices

Identity on a line graph



can show by induction that the identity for a line graph with n vertices is 1 everywhere, except for a 0 at the center if n is odd.

Identity on a square graph

0	0	0
0	0	0
0	0	0

each square is a vertex which has nearest neighbor edges; all boundary squares have edges to an invisible sink.

Identity on a square graph

2	1	2
1	0	1
2	1	2

push in sand through the boundary

Identity on a square graph

4	2	4
2	0	2
4	2	4

keep pushing in sand through the boundary and stabilizing until every vertex topples

Identity on a square graph

4	2	4
2	0	2
4	2	4

Identity on a square graph

0	4	0
4	0	4
0	4	0

Identity on a square graph

0	4	0
4	0	4
0	4	0

Identity on a square graph

2	0	2
0	4	0
2	0	2

Identity on a square graph

2	0	2
0	4	0
2	0	2

Identity on a square graph

2	1	2
1	0	1
2	1	2

every vertex has now toppled, this is the identity for the 3×3 square graph

Identity on a square graph

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

5×5 square graph

Identity on a square graph

2	3	2	3	2
3	2	1	2	3
2	1	0	1	2
3	2	1	2	3
2	3	2	3	2

identity - 5×5 square graph

Identity on a square graph

2	3	3	0	3	3	0	3	3	2
3	2	2	1	2	2	1	2	2	3
3	2	2	3	3	3	3	2	2	3
0	1	3	2	2	2	2	3	1	0
3	2	3	2	2	2	2	3	2	3
3	2	3	2	2	2	2	3	2	3
0	1	3	2	2	2	2	3	1	0
3	2	2	3	3	3	3	2	2	3
3	2	2	1	2	2	1	2	2	3
2	3	3	0	3	3	0	3	3	2

identity - 10×10 square graph

Identity on a square graph

2	3	3	0	3	3	0	3	3	2
3	2	2	1	2	2	1	2	2	3
3	2	2	3	3	3	3	2	2	3
0	1	3	2	2	2	2	3	1	0
3	2	3	2	2	2	2	3	2	3
3	2	3	2	2	2	2	3	2	3
0	1	3	2	2	2	2	3	1	0
3	2	2	3	3	3	3	2	2	3
3	2	2	1	2	2	1	2	2	3
2	3	3	0	3	3	0	3	3	2

identity - 10×10 square graph

Identity on a square graph

2	3	3	0	3	3	0	3	0	3	3	0	3	0	3	3	0	3	3	2
3	2	3	2	3	3	2	2	2	3	3	2	2	2	3	3	2	3	2	3
3	3	0	3	3	3	3	0	3	3	3	3	0	3	3	3	3	0	3	3
0	2	3	0	2	2	3	2	3	3	3	3	2	3	2	2	0	3	2	0
3	3	3	2	0	3	0	2	2	2	2	2	2	0	3	0	2	3	3	3
3	3	3	2	3	2	3	3	3	3	3	3	3	3	2	3	2	3	3	3
0	2	3	3	0	3	2	2	2	2	2	2	2	2	3	0	3	3	2	0
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3	3	3	3	2	3	2	2	2	2	2	2	2	2	3	2	3	3	3	3
0	2	3	3	2	3	2	2	2	2	2	2	2	2	3	2	3	3	2	0
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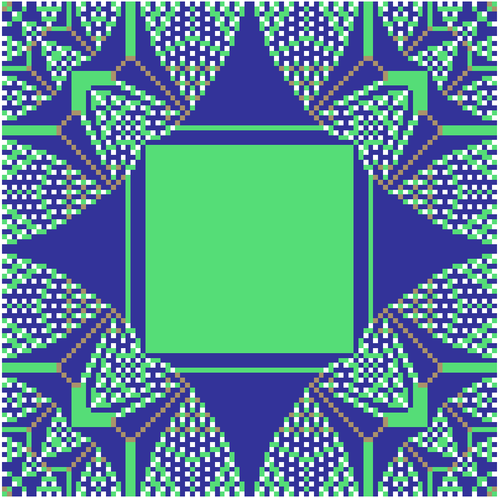
3	3	3	2	0	3	0	2	2	2	2	2	2	0	3	0	2	3	3	3
0	2	3	0	2	2	3	2	3	3	3	3	2	3	2	2	0	3	2	0
3	3	0	3	3	3	3	0	3	3	3	3	0	3	3	3	3	0	3	3
3	2	3	2	3	3	2	2	2	3	3	2	2	2	3	3	2	3	2	3
2	3	3	0	3	3	0	3	0	3	3	0	3	0	3	3	0	3	3	2

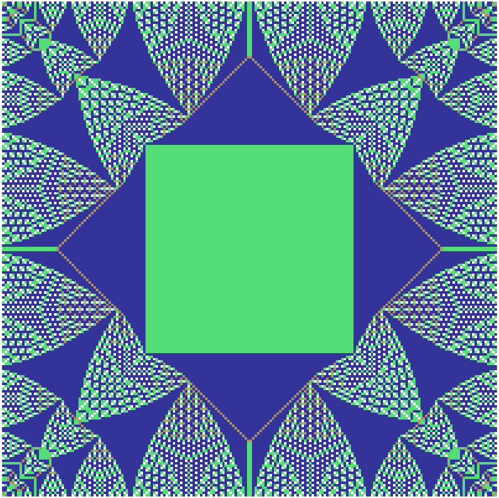
identity - 20×20 square graph

Identity on a square graph



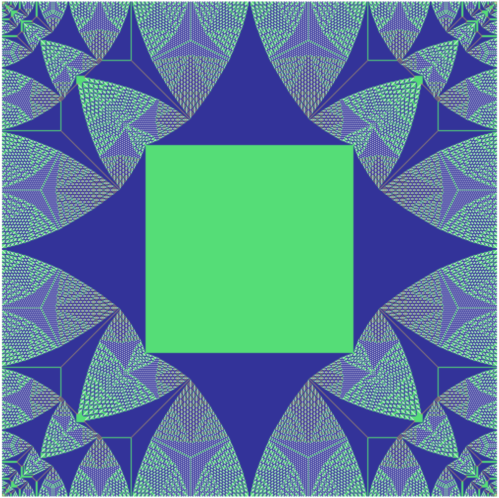
identity - 100×100 square graph

Identity on a square graph



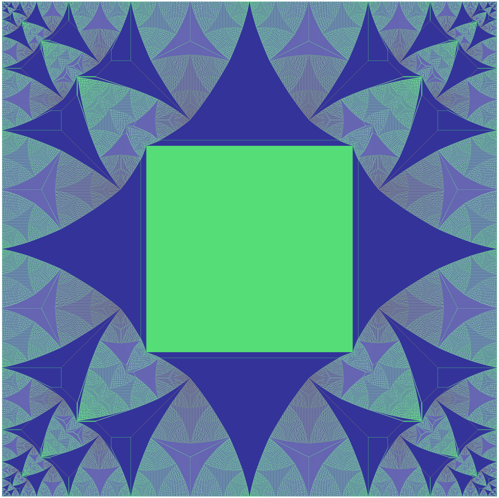
identity - 200×200 square graph

Identity on a square graph



identity - 500×500 square graph

Identity on a square graph

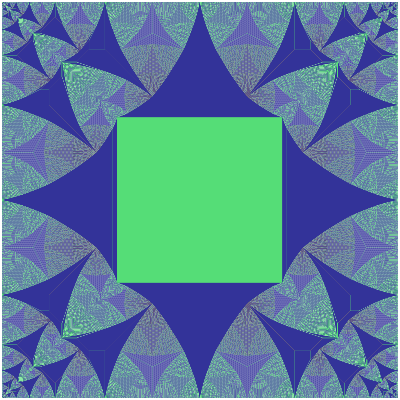


identity - 2000×2000 square graph

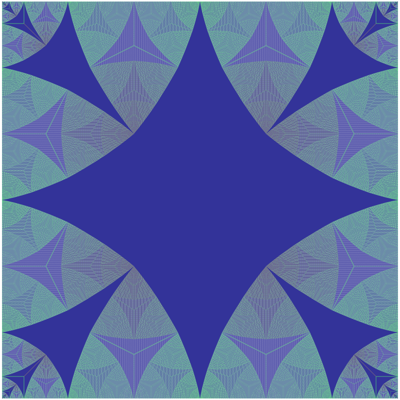
Convergence of the Abelian sandpile

Theorem (Pegden-Smart 2011, Duke J. Math)

The scaling limit of the sandpile identity on a square exists and is the Laplacian of the solution to an elliptic obstacle problem.



Other scaling limits in the sandpile group.



start with any *periodic* initial state of sand and push in sand through the boundary until every vertex topples; the sandpile that remains has a scaling limit by the Pegden-Smart theorem.

Random initial states?

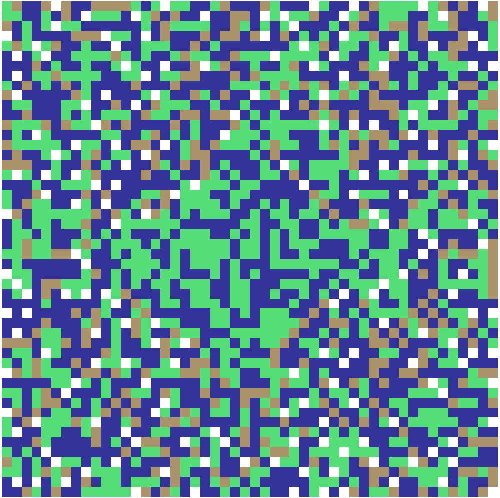
0	1	1
0	1	0
1	1	1

Random initial state

2	2	3
1	1	1
3	2	3

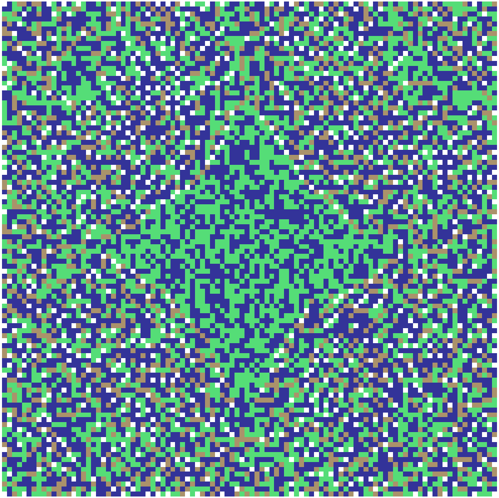
3×3 square

Random initial state



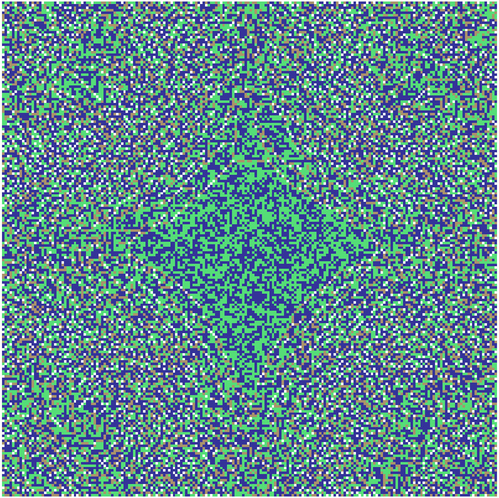
50×50 square

Random initial state



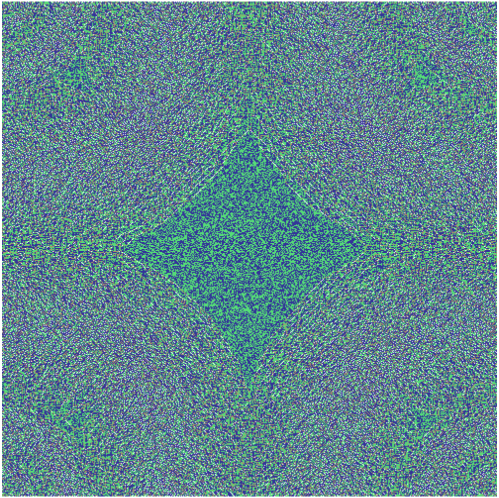
100×100 square

Random initial state



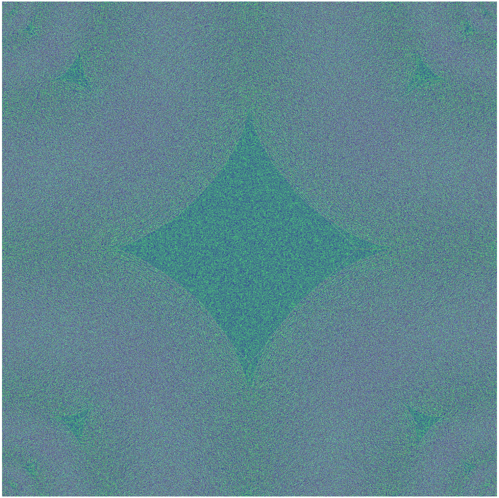
200×200 square

Random initial state



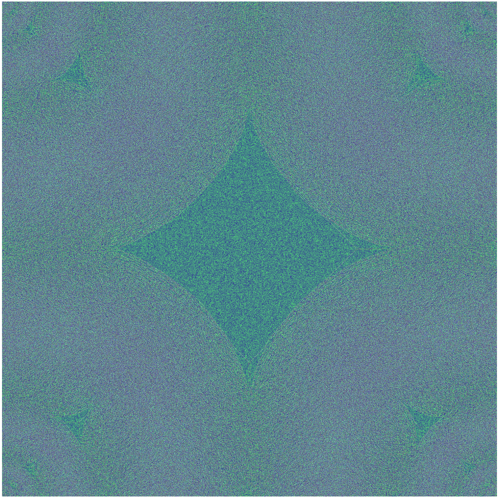
500×500 square

Random initial state



1000×1000 square

Random initial state

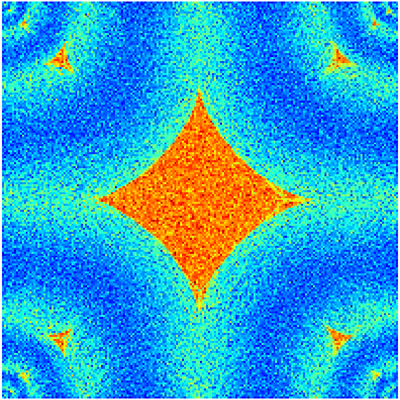


2000×2000 square

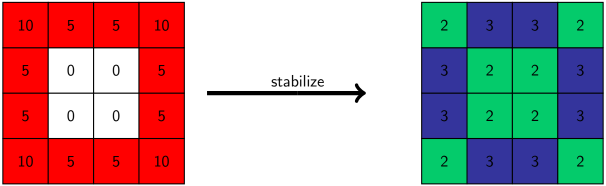
Random initial state

Theorem (B. 2019)

The scaling limit of the random sandpile exists and is the Laplacian of the solution to an elliptic obstacle problem.



How do you prove anything about the Abelian Sandpile?



Do not study the patterns

10	5	5	10
5	0	0	5
5	0	0	5
10	5	5	10

$\xrightarrow{\text{stabilize}}$

2	3	3	2
3	2	2	3
3	2	2	3
2	3	3	2

2	3	3	2
3	2	2	3
3	2	2	3
2	3	3	2

Study the odometer function

10	5	5	10
5	0	0	5
5	0	0	5
10	5	5	10

2	3	3	2
3	2	2	3
3	2	2	3
2	3	3	2

3	2	2	3
2	1	1	2
2	1	1	2
3	2	2	3

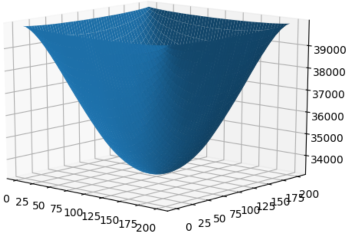
$$\eta_n + \Delta v_n = s_n$$

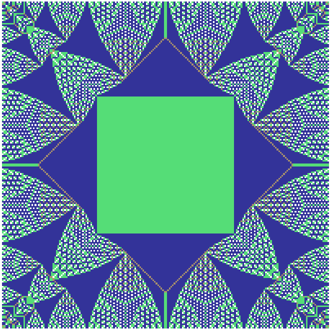
Odometer Function

$$\eta_n + \Delta v_n = s_n$$

- $\eta_n : \square_n \to \mathbf{Z}$ - initial configuration of pushed in sand
- $s_n : \square_n \to \mathbf{Z}$, $s_n \leq 3$ - stable configuration
- $v_n : \square_n \to \mathbf{N}$ - number of topples per site when stabilizing
- $\Delta v_n(x) = \sum_{y \sim x} (v_n(y) - v_n(x))$ - graph Laplacian

Odometer Function





(More) General Framework

- sample a random background $\eta : \mathbf{Z}^2 \to \mathbf{Z}$ from a distribution which is uniformly bounded and stationary, ergodic under spatial translations
- first example: $\eta \sim \text{Bernoulli}(0, 1)$
- $\eta_n(x) = \eta(x)$ for $x \in \square_n$

Convergence of the Random Abelian Sandpile

Theorem (B. 2019)

- *There exists a unique* $\bar{s} : \Box_1 \to [0, 3]$ *and* $\bar{v} : \Box_1 \to \mathbf{R}^+$ *so that almost surely*

$$n^{-2} v_n([nx]) \to \bar{v} \text{ uniformly}$$

$$s_n([nx]) \to \bar{s} \text{ weakly-*}$$

and

$$\bar{s}(x) = \Delta \bar{v}(x) + \mathbf{E}(\eta(0)).$$

- $\bar{v}$ *is the unique viscosity solution to the elliptic obstacle problem*

$$\min\{v \in C(\Box_1) : v \geq 0, D^2 v \in \bar{\Gamma}_\eta \text{ in } \Box_1\},$$

where $\bar{\Gamma}_\eta$ *is a unique downwards closed set with Lipschitz boundary.*

Convergence of the (Identity) Abelian Sandpile

Theorem (Pegden-Smart 2011, Duke J. Math)

- *There exists a unique* $\bar{s} : \Box_1 \to [0,3]$ *and* $\bar{v} : \Box_1 \to \mathbf{R}^+$ *so that almost surely*

$$n^{-2} v_n([nx]) \to \bar{v} \text{ uniformly}$$

$$s_n([nx]) \to \bar{s} \text{ weakly-*}$$

and

$$\bar{s}(x) = \Delta \bar{v}(x).$$

- $\bar{v}$ *is the unique viscosity solution to the elliptic obstacle problem*

$$\min\{v \in C(\Box_1) : v \geq 0, D^2 v \in \bar{\Gamma}_0\},$$

where

$$\bar{\Gamma}_0 = \{M \in S^2 \text{ so that there exists } u : \mathbf{Z}^2 \to \mathbf{Z}$$
$$\Delta u \leq 3 \text{ and } u(x) \geq \tfrac{1}{2} x^T M x + o(|x|^2))\}.$$

Proof outline

discrete adaption of the program of Armstrong-Smart (2014) for stochastic homogenization

1. show convergence of $\bar{v}_n$ along subsequences using PDE regularity theory
2. find a subadditive quantity μ
   - show it controls the sandpile
   - show that it is nice
   - implicitly define $\bar{\Gamma}_\eta$ with μ and the subadditive ergodic theorem
3. conclude that every subsequential limit solves PDE defined by $\bar{\Gamma}_\eta$

Start of the proof

- the odometer function solves a discrete obstacle problem

$$v = \inf\{w : \square_1 \to \mathbf{N} : \Delta w + \eta_n \leq 3\},$$

  where $\eta_n : \mathbf{Z}^2 \to \mathbf{N}$ is the initial configuration at step n
- called the *least action* principle: sandpiles are lazy
- equivalent to the *Abelian* property, the order of topplings doesn't change the final, stable configuration

What is $\bar{\Gamma}_0$?

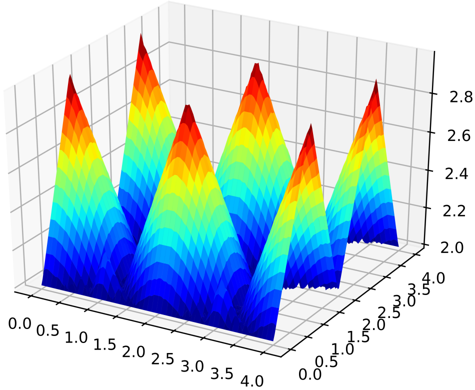
$$\bar{\Gamma}_0 = \{M \in S^2 \text{ so that there exists } u : \mathbf{Z}^2 \to \mathbf{Z}$$
$$\Delta u \leq 3 \text{ and } u(x) \geq q_M(x) + o(|x|^2))\}$$

- can look at the boundary $\partial\bar{\Gamma}_0$ with a computer algorithm
- parameterize $M \in \mathbf{S}^2$ by

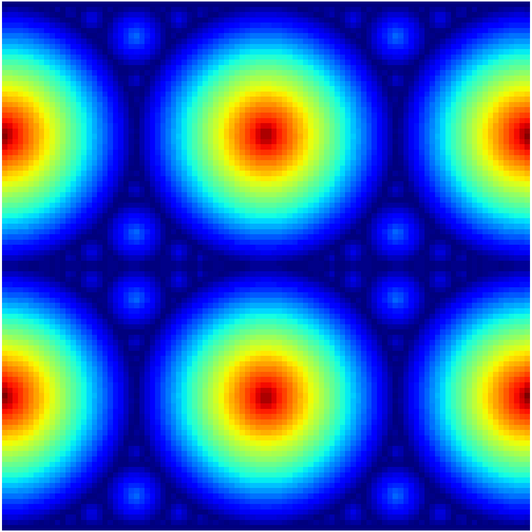
$$M(a,b,c) = \frac{1}{2}\begin{bmatrix} c-a & b \\ b & c+a \end{bmatrix}$$

and view $\partial\bar{\Gamma}_0$ as a surface in $\mathbf{R}^3$

What is $\bar{\Gamma}_0$?



What is $\bar{\Gamma}_0$?

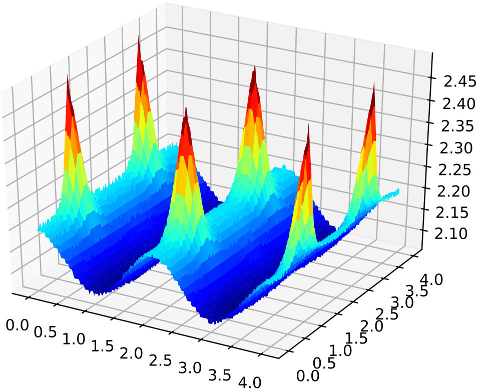


$\partial\bar{\Gamma}_0$ is an Appolonian circle packing (Levine, Pegden, Smart, Ann. Math 2017)

What is $\bar{\Gamma}_\eta$?

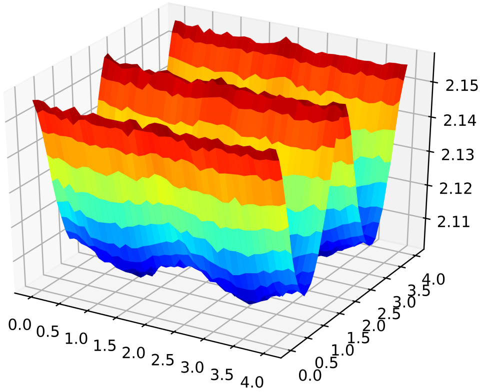
- can also look at the boundary $\partial\bar{\Gamma}_\eta$ with a computer
- will depend on the distribution of η

What is $\bar{\Gamma}_\eta$?



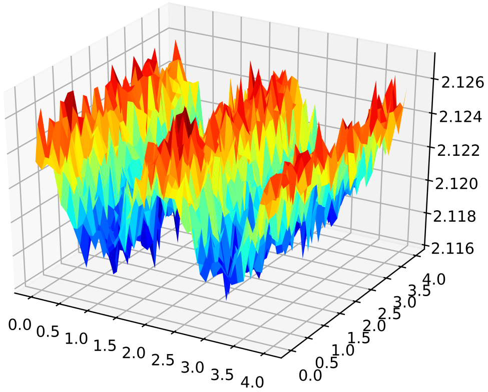
$\eta \sim$ Bernoulli$(3,4)$

What is $\bar{\Gamma}_\eta$?



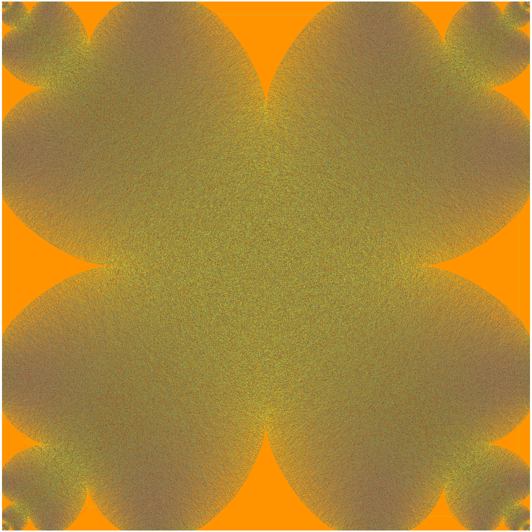
$\eta \sim \text{Bernoulli}(3, 5)$

What is $\bar{\Gamma}_\eta$?



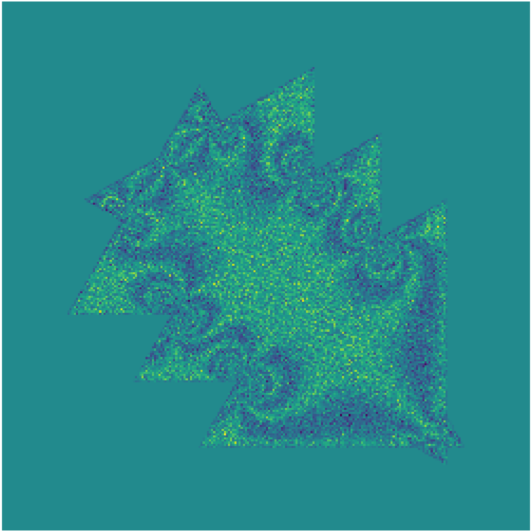
$\eta \sim$ Bernoulli$(2, 6)$

Convergence of the Random Abelian Sandpile



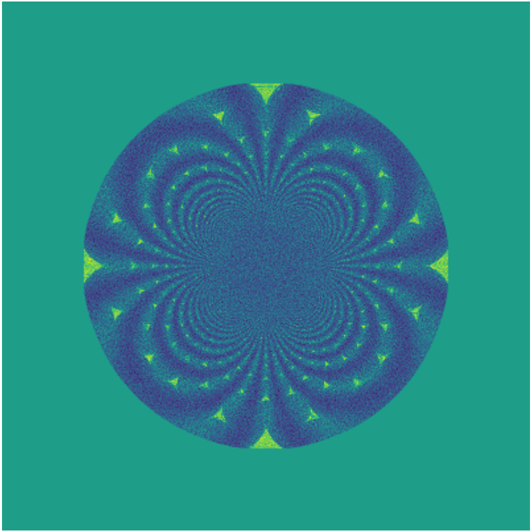
Dirichlet problem on square domain $\eta \sim$ Bernoulli$(3, 4)$

Convergence of the Random Abelian Sandpile



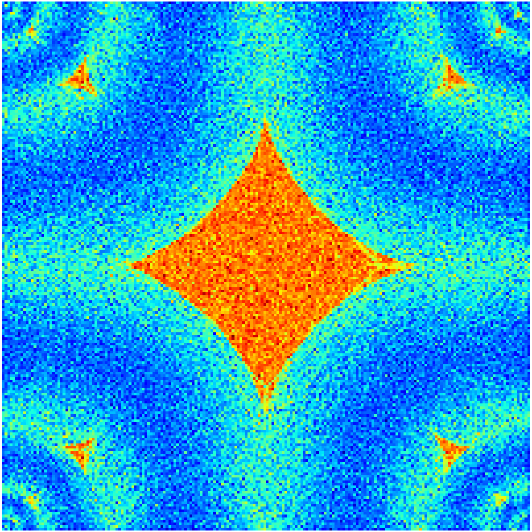
Dirichlet problem on stingray domain $\eta \sim \mathrm{Bernoulli}(3, 5)$

Convergence of the Random Abelian Sandpile



free boundary problem with random background $\eta \sim \text{Bernoulli}(0, -1)$

Thank you for listening!



arXiv:1909.07849