HAMILTON-JACOBI SCALING LIMITS OF PARETO PEELING IN 2D

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Abstract. Pareto hull peeling is a discrete algorithm, generalizing convex hull peeling, for sorting points in Euclidean space. We prove that Pareto peeling of a random point set in two dimensions has a scaling limit described by a first-order Hamilton-Jacobi equation and give an explicit formula for the limiting Hamiltonian, which is both non-coercive and non-convex. This contrasts with convex peeling, which converges to curvature flow. The proof involves direct geometric manipulations in the same spirit as Calder (2016).

1. Introduction

1.1. **Overview.** Consider $\mathbb{R}^2$ equipped with a norm $\varphi(\cdot)$ and let $A \subset \mathbb{R}^2$. A point $x \in \mathbb{R}^2$ is in the *Pareto hull* of A if, for every $y \in \mathbb{R}^2$, there exists $a \in A$ such that $\varphi(a-x) < \varphi(a-y)$. The Pareto hull peeling process proceeds by repeatedly taking the Pareto hull, $\mathcal{P}(A)$, and removing points on its boundary:

$$E_1(A) = \mathcal{P}(A) \quad \text{and} \quad E_{k+1}(A) = \mathcal{P}(A \cap \mathbf{int}(E_k(A))). \tag{1}$$

When the unit ball of $\varphi(\cdot)$ has no flat spots, the Pareto hull coincides with the convex hull [TWW84]. Calder-Smart showed that convex hull peeling of random points converges to curvature flow as the number of points goes to infinity[CS20]. Here we consider the more general case and find that whenever the unit ball of $\varphi(\cdot)$ has a facet, the scaling limit of Pareto hull peeling solves a first-order Hamilton-Jacobi equation. Hence the facets lead to faster, 'ballistic' motion contrasting with the strictly convex case which has a slower, 'diffusive' limit. Higher dimensional analogues are discussed in Section 8.

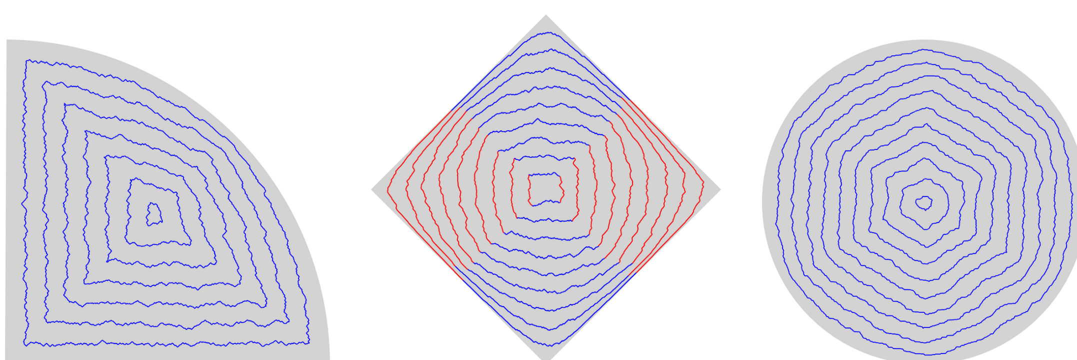

Figure 1. Peels, E_k, for k a multiple of 30, of Pareto peeling of homogeneous Poisson clouds in the shaded domains, with respect to various φ displayed in Figure 2. Peels are colored blue if 'constrained' by a facet and red otherwise — this is made precise in Section 2.

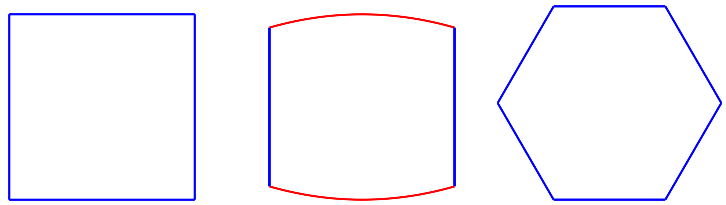

FIGURE 2. Unit balls, $\{\varphi \leq 1\}$, with flat edges outlined in blue and 'round' edges in red.

1.2. **Background.** The term Pareto hull or Pareto envelope originates from computer science [CKO00, Nou05]; however, these hulls were studied much earlier under the name *sets of strictly efficient points* in (what is now known as) the field of location analysis [LNSdG19, SLH09]. Briefly, location analysis is a specialized branch of combinatorial optimization which studies the 'best location' for a set of 'facilities' under various constraints.

The so-called *Fermat-Weber* or *point objective* problem aims to determine the location of a single facility which minimizes the distance to a finite number of demand points, *e.g.*, deciding where to build a factory serving multiple customers. Since this is a multi-objective problem, there are several ways to define optimal — one way to do so is with the Pareto hull. (Other definitions, in the context of this paper, are discussed in Section 8.)

Under this notion of optimality, Kuhn showed that when the chosen distance metric is Euclidean, the set of optimal solutions to the Fermat-Weber problem lie in the convex hull of the set of demand points [Kuh67, Kuh73]. In particular, the Pareto hull coincides with the convex hull in this case. Following earlier work of Ward-Wendell, Thisse-Ward-Wendell extended this characterization in two dimensions to any distance induced by a norm with strictly convex unit ball [WW85, TWW84]. Geometric properties of the Pareto hull and general algorithms to compute them appear in papers by Ndiaye-Michelot [NM97, NM98], Durier [Dur87, Dur90] Durier-Michelot [DM85, DM94] and Pelegrin-Fernandez [PF88, PF89].

Notably, Durier-Michelot [DM86] present a beautiful and deep characterization of the Pareto hull in terms of supporting cones — this generalizes the halfspace description of convex hulls. We use this to extend the dynamic programming principle for convex hull peeling [CS20] to Pareto peeling. In fact, the scaling limits in this paper may be thought of as continuum versions of this dynamic programming principle.

The limiting equations we derive are closely related to the continuum limit of nondominated sorting, proving a conjecture of Calder [Cal]. Briefly, nondominated sorting is an algorithm for sorting points in Euclidean space according to the coordinatewise partial order. Calder-Esedoglu-Hero showed that nondominated sorting of random points has a scaling limit described by an explicit Hamilton-Jacobi equation [CEH14a, CEH14b, CEH15, Cal17]. Recently Calder-Cook established a rate of convergence to this continuum limit [CC20] — it would be interesting to adapt those ideas to Pareto peeling.

1.3. **Main result.** Our convergence result is captured via the *height function* of Pareto hull peeling,

$$u_A = \sum_{k \geq 1} 1_{\mathbf{int}(E_k(A))}. \tag{2}$$

Let U be a bounded, open convex subset of $\mathbb{R}^2$ which is 'compatible' with φ. The definition of compatibility is somewhat technical and will be made precise in Section 4. Importantly, we indicate some necessity of this condition via an explicit counterexample below. For now, we note that when the unit ball of φ is a polygon, any convex set is compatible.

For simplicity, we model our random data via a Poisson process [Kin92], X_{nf}, of intensity nf in U, where f is a bounded, strictly positive, continuous function in U. Write $u_n := u_{X_{nf}}$, and rescale by $\bar{u}_n(x) = n^{-1/2}u_n(x)$.

In our main result, we assume that φ is a norm in $\mathbb{R}^2$ satisfying:

(3) The unit ball $\{\varphi \leq 1\}$ has at least one boundary facet,

(4) The unit ball $\{\varphi \leq 1\}$ has at most finitely many boundary facets.

A canonical example is when $\{\varphi \leq 1\}$ is a polygon.

Theorem 1.1. *If φ satisfies* (3) *and* (4) *and* U *is a bounded, open convex set in* $\mathbb{R}^2$ *that is compatible with φ, then, on an event of probability 1, the sequence of rescaled height functions converges uniformly in* $\bar{\mathrm{U}}$ *to the unique viscosity solution of the PDE:*

$$\begin{cases} \bar{H}_\varphi(Du) = f & \text{in } \mathrm{U} \\ \bar{u} = 0 & \text{on } \partial\mathrm{U} \end{cases} \tag{5}$$

where $\bar{H}_\varphi(\cdot)$ is a non-negative, continuous Hamiltonian that only depends on $\{\varphi \leq 1\}$ and is neither convex nor coercive.

Assumption (4) is only used to overcome a technical difficulty at one stage of the proof. We expect the result remains true without it and the majority of the arguments do not use it at all.

Assumption (3), on the other hand, is necessary. When (3) fails, $\{\varphi \leq 1\}$ is strictly convex and classical results in location analysis imply that the Pareto hull is nothing but the convex hull. In this case, Calder-Smart [CS20] have already shown that convex hull peeling converges, but with a larger $n^{\frac{2}{3}}$ rescaling. As a natural byproduct of our arguments, we give a self-contained proof that the $n^{\frac{1}{2}}$ scaling of convex hull peeling is trivial.

Corollary 1.1. *If $\{\varphi \leq 1\}$ is strictly convex, i.e., if* (3) *does not hold, then $\bar{u}_n \to \infty$ locally uniformly in* U.

Finally, where the compatibility assumption is concerned, we prove that convergence may fail if it does not hold.

Corollary 1.2. *There is a norm φ satisfying* (3) *and* (4) *and an open, bounded convex set* U *in* $\mathbb{R}^2$ *that is* not *compatible with φ for which the rescaled height functions do not converge uniformly to a continuous function in* $\bar{\mathrm{U}}$.

Our proof explicitly identifies the effective Hamiltonian $\bar{H}_\varphi(\cdot)$ in Theorem 1.1. The full description of $\bar{H}_\varphi$ will be postponed till Section 2. For now, we give an example that already demonstrate the main qualitative features of these functions. When φ is the ℓ^1 norm, this confirms a conjecture of Calder [Cal] and reflects the fact that nondominated sorting describes the local behavior of Pareto peeling.

Example. *For $p' \in \mathbb{R}^2$ when $\varphi(x) = |x_1| + |x_2|$, the effective Hamiltonian is*

$$\bar{H}_\varphi(p') = |p'_1 p'_2|.$$

If instead $\varphi(x) = \max(x_1 - x_2, x_2 - x_1, \|x\|_2)$, then we have

$$\bar{H}_\varphi(p') = \max(p'_1 p'_2, 0).$$

The unit balls for the prior two norms are displayed in Figure 3.

Interestingly, every effective Hamiltonian, including the above, is non-convex and non-coercive. This contrasts with some of the usual assumptions made in stochastic homogenization of Hamilton-Jacobi equations, see *e.g.*, [ACS14].

1.4. **Level Set Formulation.** The scaling limit of the height functions can be rephrased in terms of the Pareto hull peeling process itself. Notice that if the limit function u solves (5), then the function v defined in $\mathrm{U} \times (0, \infty)$ by

$$v(x,t) = u(x) - t \tag{6}$$

is a solution of the parabolic PDE

$$-\sqrt{f} v_t + \sqrt{\bar{H}_\varphi(Dv)} = 0 \quad \text{in } \mathrm{U} \times (0, \infty). \tag{7}$$

This can be understood as the level set formulation of a geometric flow.

More precisely, if we define $(E_t)_{t \geq 0}$ by

$$E_t = \{x \in \mathrm{U} \mid v(x,t) > 0\} = \{x \in \mathrm{U} \mid u(x) > t\}, \tag{8}$$

then these sets form a generalized level set evolution with noraml velocity

$$V_{\partial E_t} = \sqrt{f^{-1} \bar{H}_\varphi(n_{\partial E_t})}. \tag{9}$$

The correspondence between level set PDE such as (7) and generalized level set evolutions is explained in [BS98].

Stated in these terms, our result reads as follows:

Corollary 1.3. *Given $n \in \mathbb{N}$, let $\{E_k^{(n)}\}_{k \in \mathbb{N}}$ be the Pareto hull peeling process associated with X_{nf}. If φ satisfies* (3) *and* (4) *and* U *is a bounded, open convex set compatible with φ, then*

$$\bar{E}^{(n)}_{\lfloor n^{\frac{1}{2}} t \rfloor} \to \bar{E}_t \quad \textit{for each } t > 0,$$

where $(E_t)_{t \geq 0}$ is the generalized level set evolution with velocity (9) *and initial datum $E_0 = \mathrm{U}$ and the convergence is in the Hausdorff metric.*

We reiterate that when (3) fails, the norm ball $\{\varphi \leq 1\}$ is strictly convex, Pareto hull peeling coincides with convex hull peeling, and the scaling is different. In [CS20], it is shown that, in this case,

$$\partial E^{(n)}_{\lfloor n^{\frac{2}{3}} t \rfloor} \to \partial E_t$$

where $(E_t)_{t \geq 0}$ shrinks according to affine curve shortening.

1.5. **Paper and method outline.** The paper proceeds by exploiting three ideas. First, as in previous work on scaling limits of nondominated sorting and convex hull peeling [Cal16, CS20], we begin by translating a geometric characterization of Pareto hulls into a dynamic programming principle satisfied by the height functions. Next, we show how nondominated sorting can be used as a cell problem for Pareto hull peeling. Finally, following the intuition that the height functions necessarily grow like $|A|^{\frac{1}{2}} n^{\frac{1}{2}}$ in any set A, we use the dynamic programming principle to show that the rescaled height functions do not have asymptotic gradients in certain directions — in other words, for large n, the Pareto peels start developing corners.

The outline of the paper is as follows. In Section 2, the effective Hamiltonian is defined, its properties are discussed, and the necessary geometric preliminaries are reviewed. This is also where the dynamic programming principle for the height functions is stated and proved. In Section 3, we discuss a generalization of nondominated sorting and the basic results needed in what follows. The proof of Theorem 1.1 is outlined in Section 4, which also includes preliminaries on viscosity solutions, preliminary estimates on the asymptotic behavior of the height functions, and proofs of the corollaries. Sections 5, 6, 7 comprise the main technical contributions of the paper and are devoted to proving that the limiting height functions solve (5). Finally, Section 8 is a discussion of open questions for future work.

1.6. **Notation and conventions.**

- Unless made explicit, C, c are positive constants which may change from line to line.
- For a subset A of $\mathbb{R}^2$ write $|A|$ for its Lebesgue measure, $\bar{A}$ for closure, $\mathbf{int}(A)$ for interior, and ∂A for boundary.
- For $x, y \in \mathbb{R}^2$,
$$[x, y] = [x_1, y_1] \times [x_2, y_2]$$
and for $a, b \in \bar{\mathbb{R}}$
$$[a, b]^2 = [a, b] \times [a, b]$$
and for $x \in \mathbb{R}^2$ and $b \in \bar{\mathbb{R}}$,
$$[x, b]^2 = [x_1, b] \times [x_2, b]$$
and vice-versa.
- $\langle x, y \rangle$ denotes the Euclidean inner product of $x, y \in \mathbb{R}^2$.
- $v \times w$ is the cross product of two vectors $v, w \in \mathbb{R}^2$. Recall this can be computed via the determinant
$$v \times w = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$
Alternatively, using wedge products, $v \times w$ is the real number such that $v \wedge w = (v \times w)(1, 0) \wedge (0, 1)$.
- Given a vector $q \in \mathbb{R}^2 \setminus \{0\}$, we denote by H_q the halfspace determined by q by
$$H_q = \{x \in \mathbb{R}^2 \mid \langle q, x \rangle \geq 0\}.$$
- $\|x\|_\infty$ denotes the ℓ^∞ norm and $\|x\| = \|x\|_2$ denotes the Euclidean or ℓ^2 norm.
- $B(x_0, r) = \{x \in \mathbb{R}^2 \mid \|x - x_0\| \leq r\}$ denotes the ball of radius r centered around x_0.
- If $p \in \mathbb{R}^2$, write
$$p^\perp = (-p_2, p_1) \quad \text{if } p = (p_1, p_2).$$
- $\mathbf{cone}(\mathrm{C}) = \{av \in \mathbb{R}^2 \mid a \geq 0 \text{ and } v \in \mathrm{C}\}$ and $\mathbf{conv}(C)$ is the convex hull of C.

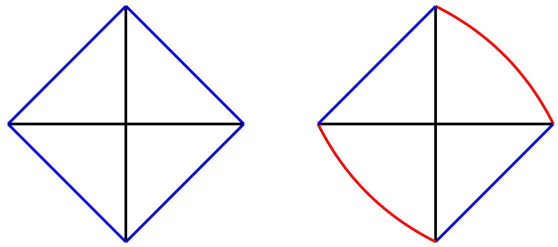

FIGURE 3. Unit balls of the two norms described in Example 1.3 partitioned into cones.

- Differential inequalities are interpreted in the viscosity sense.

1.7. **Code.** Programs used to generate the figures are included in the arXiv submission.

1.8. **Acknowledgments.** We thank Jeff Calder for helpful suggestions and encouragement. A.B. thanks Charles K. Smart for many inspiring discussions. P.S.M. gratefully acknowledges his thesis advisor, P.E. Souganidis, for introducing him to viscosity solutions and homogenization and for unwavering support these past few years.

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2. Pareto Hulls and the Effective Hamiltonian

In this section, we give an explicit formula for the effective Hamiltonian $\bar{H}_\varphi$ in terms of certain geometric objects associated to the norm φ. We then review the definition and main properties of the the Pareto hull and show that the height function associated with Pareto hull peeling satisfies a dynamic programming principle. The section concludes with the derivation of some properties of $\bar{H}_\varphi$, including continuity and noncoercivity.

2.1. **Partitioning the norm.** We present a cone partition of the unit ball and some auxiliary definitions which will be used in the sequel.

Recall φ denotes a norm in $\mathbb{R}^2$. In this section, we impose no assumptions on φ other than being a norm.

Since φ is a norm, it has a dual norm φ^* given by

$$\varphi^*(p) = \max \left\{ \frac{\langle p, q \rangle}{\varphi(q)} \mid q \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

It is well known that φ^* is also a norm.

Let $\mathcal{N}^*$ be the set of corner points of $\{\varphi^* = 1\}$, that is,

$$\mathcal{N}^* = \{p \in \{\varphi^* = 1\} \mid \#\partial\varphi^*(p) > 1\}. \tag{10}$$

Recall that $\mathcal{N}^*$ is at most countable. We will be interested in the collection of cones $\{\mathcal{Q}_p\}_{p \in \mathcal{N}^*}$ given by

$$\mathcal{Q}_p = \{q \in \mathbb{R}^2 \mid \langle q, p \rangle = \varphi(q)\}.$$

These are precisely the cones determined by the facets of $\{\varphi \leq 1\}$. Since we are working in dimension $d = 2$, for each $p \in \mathcal{N}^*$, we can fix a basis $\{w_p, v_p\} \subseteq \mathbb{R}^2$ such that

$$\mathcal{Q}_p = \{cv_p - dw_p \mid c, d \geq 0\}. \tag{11}$$

We can and will assume that $\|w_p\| = \|v_p\| = 1$ and $w_p \times v_p > 0$.

In what follows, we denote by Q_p the convex cone obtained from $\mathcal{Q}_p$ by

$$Q_p = \{aw_p + bv_p \mid a, b \geq 0\}. \tag{12}$$

We will refer to the cones $\{Q_p\}_{p \in \mathcal{N}^*}$ as *flat cones.* Finally, let $\mathcal{E}$ be the set of extreme points of $\{\varphi \leq 1\}$.

When $\{\mathcal{Q}_p \mid p \in \mathcal{N}^*\}$ is a finite partition of $\mathbb{R}^2$, we say φ is *polyhedral.* This is equivalent to saying $\{\varphi \leq 1\}$ is a polygon.

2.2. **Pareto Hulls and Dynamic Programming.** With the notation of the previous section, we can now define the effective Hamiltonian:

$$\bar{H}_\varphi(\xi) = \max\left\{\sup_{p \in \mathcal{N}^*} \frac{\langle \xi, v_p\rangle\langle \xi, w_p\rangle}{|v_p \times w_p|}, 0\right\}. \tag{13}$$

In addition to making $\bar{H}_\varphi$ non-negative, which is convenient for the level-set description (9), the zero in the definition can be understood as the contribution from the 'round parts' $\mathcal{E}$ of $\{\varphi = 1\}$.

The derivation of the Hamilton-Jacobi equation (5) and the formula (13) uses the fact that, in a certain sense, the height functions of Pareto peeling themselves satisfy a discrete PDE. We make this precise via a dynamic programming formulation.

Before doing so, let us recall the definition of the Pareto hull. Let $A \subseteq \mathbb{R}^2$ be a given finite set of points and recall that the *Pareto hull* with respect to A, $\mathcal{P}(A)$, is

$$\mathcal{P}(A) := \{x \in \mathbb{R}^2 : \forall y \neq x \text{ there exists } a \in A \text{ with } \varphi(a - x) < \varphi(a - y)\}. \tag{14}$$

It is worth emphasizing at this point that the Pareto hull is monotone, that is,

$$\mathcal{P}(A) \subseteq \mathcal{P}(B) \quad \text{if} \quad A \subseteq B. \tag{15}$$

It will be convenient to abuse notation by defining $\mathcal{P}(\emptyset) = \emptyset$.

Pareto hull peeling with respect to A is a collection of sets, $\{E_k(A)\}_{k \in \mathbb{N}}$, defined recursively via

$$E_1(A) = \mathcal{P}(A) \quad \text{and} \quad E_{k+1}(A) = \mathcal{P}(A \cap \mathbf{int}(E_k(A))).$$

Associated to this process is a height function u_A defined in $\mathbb{R}^2$ by

$$u_A(x) = \sum_{k=1}^{\infty} 1_{\mathbf{int}(E_k(A))}(x).$$

The next result, which identifies the dynamic programming principle satisfied by u_A, plays a fundamental role in what follows.

Proposition 2.1. *Given $A \subseteq \mathbb{R}^2$ finite, the height function u_A satisfies a dynamic programming principle:*

$$u_A(x) = \inf\left\{\inf_{p \in \mathcal{N}^*} \sup_{y \in x + Q_p} u(y) + 1_A(y), \inf_{q \in \mathcal{E}} \sup_{y \in x + H_{q^\perp}} u(y) + 1_A(y)\right\} \quad \text{for } x \in \mathbb{R}^2. \tag{16}$$

2.3. **Cone characterization of Pareto hulls.** The dynamic programming principle follows from a characterization of Pareto hulls using the cones $\{Q_p\}$ and $\{H_{q^\perp}\}$ defined above.

Theorem 2.1 ([DM86, Dur87]). *Let A be a finite set of points in $\mathbb{R}^2$, then*

$$x \in \mathbf{int}(\mathcal{P}(A)) \iff \text{for every } p \in \mathcal{N}^*,\ A \cap (x + \mathbf{int}(Q_p)) \neq \emptyset$$
$$\text{and for every } q \in \mathcal{E},\ A \cap (x + \mathbf{int}(H_{q^\perp})) \neq \emptyset.$$

If φ is polyhedral, the halfspace constraint is unnecessary,

$$x \in \mathbf{int}(\mathcal{P}(A)) \iff \text{for every } p \in \mathcal{N}^*,\ A \cap (x + \mathbf{int}(Q_p)) \neq \emptyset.$$

Proof. The proof combines Remark 2.1 in [DM86] and Theorem 4.1 in [DM86] together with Proposition 2.5 in [Dur87]. For a different proof (and algorithms) see [PF88, PF89]. $\square$

The previous representation is reminiscent of the halfspace separation characterization of convex hulls. In fact, the following reformulation, which will be useful in what follows, shows that the Pareto hull can be thought of as a constrained convex hull.

Corollary 2.1. *Given $A \subseteq \mathbb{R}^2$ finite and $x \in \mathbb{R}^2$, the inclusion $x \in \mathbf{int}(\mathcal{P}(A))$ holds if and only if the following two conditions are satisfied:*

(i) $A \cap (x + \mathbf{int}(Q_p)) \neq \emptyset$ *for each* $p \in \mathcal{N}^*$,

(ii) $x \in \mathbf{int}(\mathbf{conv}((A))$.

As an immediate consequence of the previous corollary, we recover the classical result that in dimension 2, the Pareto hull coincides with the convex hull whenever $\{\varphi \leq 1\}$ is strictly convex [TWW84].

Corollary 2.2. *If $\{\varphi \leq 1\}$ is strictly convex or, equivalently, $\varphi^* \in C^1(\mathbb{R}^2 \setminus \{0\})$, then $\mathcal{P}(A) = \mathbf{conv}(A)$ for any finite $A \subseteq \mathbb{R}^2$.*

Proof. If $\varphi^* \in C^1(\mathbb{R}^2 \setminus \{0\})$, then $\mathcal{N}^* = \emptyset$. Hence condition (i) in the previous corollary is always vacuous in this setting. Accordingly, that result reduces to the simple identity $\mathcal{P}(A) = \mathbf{conv}(A)$. $\square$

2.4. **Proof of the dynamic programming principle.** We next use Theorem 2.1 to prove the dynamic programming principle. In the proof, we let $\mathcal{L} = \{Q_p : p \in \mathcal{N}^*\} \cup \{H_{q^\perp} : q \in \mathcal{E}\}$. Observe that all cones in $\mathcal{L}$ are convex.

Proof of Proposition 2.1. Let $A_k = A \cap \mathrm{int}(E_k(A))$ and set $A_0 = A$. By monotonicity (15), $E_k(A) \supset E_{k+1}(A)$ so $A_k \supset A_{k+1}$ and hence $h_A(x) = k$ if $x \in A_k \setminus A_{k+1}$ for all $k \geq 0$.

Take $x \in \mathbb{R}^2$ and set $j = h_A(x)$. Thus, $x \notin \mathbf{int}(E_{j+1}(A))$ and, hence by Theorem 2.1, there is $Q \in \mathcal{L}$ so that

$$A_j \cap (x + \mathbf{int}(Q)) = \emptyset. \tag{17}$$

This implies, together with Q being a convex cone, that $h_A(z) \leq (j-1)$ for all $z \in (x + \mathbf{int}(Q))$. Indeed, if $h_A(z) \geq j$, then there is $z' \in A_j \cap (z + \mathbf{int}(Q))$. However, by convexity of Q, $\mathbf{int}(Q) + \mathbf{int}(Q) \subseteq \mathbf{int}(Q)$ hence $z' \in A_j \cap (x + \mathbf{int}(Q))$, contradicting (17). Thus,

$$\inf_{Q \in \mathcal{L}} \sup_{z \in A \cap (x + \mathbf{int}(Q))} (1_A(z) + h_A(z)) \leq 1 + (j-1) = h_A(x).$$

For the other direction, assume $j \geq 1$ and let $Q \in \mathcal{L}$ be given. Since $x \in \mathbf{int}(E_j(A))$, by Theorem 2.1, there is $z \in A_{j-1} \cap (x + \mathbf{int}(Q))$ and so

$$\inf_{Q \in \mathcal{L}} \sup_{z \in A \cap (x + \mathbf{int}(Q))} (1_A(z) + h_A(z)) \geq 1 + (j-1) = h_A(x).$$

□

2.5. **Duality.** In the sequel, convex duality will play a recurring role. First, the dual cones $\{Q_p^* \mid p \in \mathcal{N}^*\}$ determine the directions in which the Hamiltonian is nonzero. These are defined by

$$Q_p^* = \bigcap_{v \in Q_p} \{\xi \in \mathbb{R}^2 \mid \langle \xi, v \rangle \leq 0\}. \tag{18}$$

Next, in the analysis of the Hamiltonian $\bar{H}_\varphi$, it will be convenient to define dual bases $\{(v_p^*, w_p^*) \mid p \in \mathcal{N}^*\}$ by the rule

$$\langle v_p^*, v_p \rangle = \langle w_p^*, w_p \rangle = 1, \quad \langle v_p^*, w_p \rangle = \langle w_p^*, v_p \rangle = 0.$$

This is well-defined since $\{(w_p, v_p) \mid p \in \mathcal{N}^*\}$ are themselves bases.

Note that these provide coordinates for $\mathbb{R}^2$ in the sense that, given any $p \in \mathcal{N}^*$,

$$\zeta = \langle \zeta, v_p^* \rangle v_p + \langle \zeta, w_p^* \rangle w_p \quad \text{for each } \zeta \in \mathbb{R}^2. \tag{19}$$

Using these coordinates, we obtain an alternative formula for the expressions appearing in the definition of $\bar{H}_\varphi$.

Proposition 2.2. *For each $\xi \in \mathbb{R}^2$ and $p \in \mathcal{N}^*$,*

$$\frac{\langle \xi, v_p \rangle \langle \xi, w_p \rangle}{|v_p \times w_p|} = -|v_p \times w_p| \langle \xi^\perp, v_p^* \rangle \langle \xi^\perp, w_p^* \rangle. \tag{20}$$

Proof. Observe that we can write

$$\langle \xi, v_p \rangle = \langle -(\xi^\perp)^\perp, v_p \rangle = -\langle \langle \xi^\perp, v_p^* \rangle v_p^\perp + \langle \xi^\perp, w_p^* \rangle w_p^\perp, v_p \rangle = -\langle \xi^\perp, w_p^* \rangle \langle w_p^\perp, v_p \rangle.$$

A similar computation shows that $\langle \xi, w_p \rangle = -\langle \xi^\perp, v_p^* \rangle \langle v_p^\perp, w_p \rangle$. At the same time,

$$\langle w_p^\perp, v_p \rangle = -\langle v_p^\perp, w_p \rangle = w_p \times v_p.$$

Combining these gives the desired result. □

The same basic idea driving the previous proof enables us to derive an important bijective correspondence between Q_p^* and $\mathcal{Q}_p$.

Proposition 2.3. *If $p \in \mathcal{N}^*$ and $\xi \in \mathbb{R}^2$, then $-\xi \in Q_p^*$ (resp. $-\xi \in \mathbf{int}(Q_p^*)$) if and only if $\xi^\perp \in \mathcal{Q}_p$ (resp. $\xi^\perp \in \mathbf{int}(\mathcal{Q}_p)$).*

Proof. $-\xi \in Q_p^*$ if and only if $\langle \xi, v_p \rangle \geq 0$ and $\langle \xi, w_p \rangle \geq 0$. Since $w_p \times v_p > 0$ by the choice of $\{w_p, v_p\}$, the computations in the previous proposition show that this occurs if and only if $\langle \xi^\perp, v_p^* \rangle \geq 0$ and $\langle \xi^\perp, w_p^* \rangle \leq 0$. That is, by definition of $\mathcal{Q}_p$, $-\xi \in Q_p^*$ if and only if $\xi^\perp \in \mathcal{Q}_p$.

The previous argument works just as well if the interiors are considered instead. □

The last proposition helps us to unpack the formula (13). Among the consequences, it shows that $\bar{H}_\varphi$ is never coercive.

Proposition 2.4. *(i) Given $\xi \in \mathbb{R}^2$, there are at most two $p \in \mathcal{N}^*$ such that*

$$\frac{\langle \xi, v_p\rangle\langle \xi, w_p\rangle}{|v_p \times w_p|} > 0.$$

Furthermore, such p necessarily satisfy $\xi^\perp \in \mathbf{int}(\mathcal{Q}_p) \cap (-\mathbf{int}(\mathcal{Q}_p))$.

(ii) Given $\xi \in \mathbb{R}^2$,

$$\bar{H}_\varphi(\xi) = 0 \quad \iff \quad \xi^\perp \in \mathbf{cone}(\mathcal{E}).$$

Proof. (i) If $\langle \xi, v_p\rangle\langle \xi, v_p\rangle > 0$, then $\xi \in \mathbf{int}(Q_p^*) \cup (-\mathbf{int}(Q_p^*))$. Hence the previous result implies $\xi^\perp \in \mathbf{int}(\mathcal{Q}_p) \cup (-\mathbf{int}(\mathcal{Q}_p))$. Since the sets $\{\mathbf{int}(\mathcal{Q}_p) \mid p \in \mathcal{N}^*\}$ are disjoint and $\mathcal{Q}_{-p} = -\mathcal{Q}_p$, this determines p up to negation.

(ii) Notice that if $\zeta \in \mathbf{int}(\mathcal{Q}_p)$ for some $p \in \mathcal{N}^*$, then $\varphi^*(\zeta)^{-1}\zeta \notin \mathcal{E}$ since $\{\varphi^* = 1\}$ is flat in $\mathbf{int}(\mathcal{Q}_p)$. Accordingly, $\mathbf{cone}(\mathcal{E}) \cap \mathbf{int}(\mathcal{Q}_p) = \emptyset$ for each $p \in \mathcal{N}^*$. We conclude by combining this last observation with the previous result. $\square$

Combining everything we have done in this section, we obtain the following alternative formula for $\bar{H}_\varphi$:

$$\bar{H}_\varphi(\xi) = \begin{cases} -|v_p \times w_p|\langle \xi^\perp, v_p^*\rangle\langle \xi^\perp, w_p^*\rangle, & \text{if } \xi^\perp \in \mathcal{Q}_p \text{ for some } p \in \mathcal{N}^*, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

This formula suggests that $\bar{H}_\varphi$ is, in some sense, a function of the tangent vector $n_{\partial E_t}^\perp$ rather than the normal vector $n_{\partial E_t}$ in (9). Note that this explains the otherwise counter-intuitive 90° discrepancy between the middle images in Figures 1 and 2.

2.6. **Continuity of the Hamiltonian.** In this section, we show that the effective Hamiltonian $\bar{H}_\varphi$ given by (13) is a continuous function for an arbitrary norm φ. Note, in particular, that assumption (4) is not used here.

We start by proving that $\bar{H}_\varphi$ is locally bounded.

Lemma 2.2. *For each $p \in \mathcal{N}^*$, if $\theta(p) = (\pi - \arcsin(w_p \times v_p))/2$, then*

$$\frac{\langle \xi, v_p\rangle\langle \xi, w_p\rangle}{|v_p \times w_p|} \le \frac{\|\xi\|^2 \tan(\theta(p))}{2} \quad \textit{for each } \xi \in Q_p^*.$$

As we will see below, the angle $\theta(p)$ is small for all but finitely many $p \in \mathcal{N}^*$. This will be used to prove local boundedness of $\bar{H}_\varphi$.

Proof. For convenience, we write $v = v_p$, $w = w_p$, and $\theta = \theta(p)$. Since rotations don't change inner or cross products, we can rotate the plane so that

$$w = (\cos(\theta), \sin(\theta)) \quad \text{and} \quad v = (-\cos(\theta), \sin(\theta)).$$

Note that, after this rotation, we have $\xi \in Q_p^*$ only if $\xi = \|\xi\|(\cos(\psi), \sin(\psi))$ for some $\psi \in [\theta, \pi - \theta]$. Henceforth, we will restrict attention to such angles ψ.

Since $\|v\| = \|w\| = 1$, our assumptions imply that

$$\frac{\langle \xi, v\rangle\langle \xi, w\rangle}{|v \times w|} = \frac{\|\xi\|^2 \cos(\pi - \psi - \theta)\cos(\psi - \theta)}{\sin(\pi - 2\theta)}.$$

Rewriting the numerator, we see that

$$\cos(\pi - \psi - \theta)\cos(\psi - \theta) = 1/2(-\cos(2\theta) - \cos(2\psi)),$$

which is maximized at $\psi = \frac{\pi}{2}$. Thus,

$$\frac{\langle \xi, v\rangle\langle \xi, w\rangle}{|v \times w|} \leq \frac{\|\xi\|^2 \cos\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)}{\sin(\pi - 2\theta)} = \frac{\|\xi\|^2 \tan(\theta)}{2}.$$

$\square$

Proposition 2.5. *For each $R, M > 0$, let $\mathcal{N}^*_{R,M} \subseteq \mathcal{N}^*$ denote the subset*

$$\mathcal{N}^*_{R,M} = \{p \in \mathcal{N}^* \mid |v_p \times w_p|^{-1}\langle \xi, v_p\rangle\langle \xi, w_p\rangle \geq M \quad \textit{for some } \xi \in B(0,R)\}.$$

Then $\mathcal{N}^*_R$ is a finite set.

Proof. Since $\{\varphi^* = 1\}$ has finite perimeter, the disjoint segments $\{\mathcal{Q}_p \cap \{\varphi^* = 1\} \mid p \in \mathcal{N}^*\}$ have summable lengths. From this, it follows that, for each $\delta > 0$, the sets $\mathcal{N}^*(\delta)$ given by

$$\mathcal{N}^*(\delta) = \{p \in \mathcal{N}^* \mid |v_p \times w_p| > \delta\}$$

are finite.

In view of the previous claim, it only remains to show that, for each $R, M > 0$, there is a $\delta > 0$ such that $\mathcal{N}^*_{R,M} \subseteq \mathcal{N}^*(\delta)$. However, this follows from the previous result. Indeed, if $\xi \in B(0,R)$ and $|v_p \times w_p|^{-1}\langle \xi, v_p\rangle\langle \xi, w_p\rangle \geq M$, then

$$\frac{2M}{\tan(\theta(p))} \leq \|\xi\|^2 \leq R^2$$

where $\tan(\theta(p)) = (\pi - \arcsin(w_p \times v_p))/2$. From this, we deduce that $\tan(\theta(p)) \geq \frac{2M}{R^2}$ and this readily implies $p \in \mathcal{N}^*(\delta)$ for some $\delta > 0$ depending on M and R. $\square$

Proposition 2.6. *$\bar{H}_\varphi$ is continuous.*

Proof. Note that $\bar{H}_\varphi$ is a lower semicontinuous function, being the supremum of a family of continuous functions.

To show that it is continuous, it suffices to prove that $\bar{H}_\varphi$ restricts to a continuous function in $B(0,R)$ for each $R > 0$. Given such an R and a sequence $(\xi_n)_{n\in\mathbb{N}} \subsetneq B(0,R)$ converging to some $\xi \in \mathbb{R}^2$, there are two possibilities: either $\bar{H}_\varphi(\xi_n) \to 0$ or else $\bar{H}_\varphi(\xi_n) \geq M$ for some $M > 0$ and sufficiently large n.

In the second case, the previous result implies that $(\xi_n)_{n\geq N} \subseteq \cup_{p\in\mathcal{N}^*_{R,M}} Q^*_p$. At the same time, since $\mathcal{N}^*_{R,M}$ is finite, $\bar{H}_\varphi$ restricts to a continuous function in that set by Proposition 2.3 and (21). Therefore, $\bar{H}_\varphi(\xi) = \lim_{n\to\infty} \bar{H}_\varphi(\xi_n)$.

In the other case, we know that $\lim_{n\to\infty} \bar{H}_\varphi(\xi_n) = 0$. Therefore, by lower semicontinuity, $\bar{H}_\varphi(\xi) \leq 0$. At the same time, $\bar{H}_\varphi$ is a non-negative function so this implies $0 = \bar{H}_\varphi(\xi)$, and hence $\bar{H}_\varphi(\xi) = \lim_{n\to\infty} \bar{H}_\varphi(\xi_n)$. $\square$

2.7. **Affine invariance.** We note that Pareto peeling has a certain invariance with respect to linear transforms of the plane.

Lemma 2.3. *Let A be a finite set of points in $\mathbb{R}^2$, then for any bijective linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$*

$$x \in \mathbf{int}(\mathcal{P}(A)) \iff L(x) \in \mathbf{int}(\mathcal{P}_L(L(A))),$$

where $\mathcal{P}_L$ denotes the Pareto hull with respect to the norm, $\varphi \circ L$.

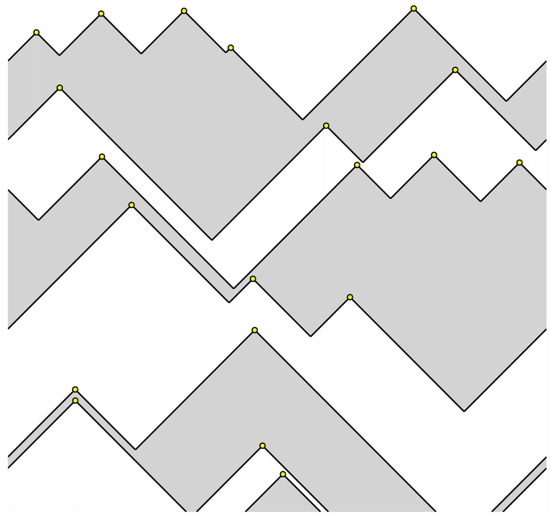

FIGURE 4. Q-nondominated sorting of a Poisson point cloud where $Q = \{x \in \mathbb{R}^2 : \|x\|_\infty = x_2\}$. The shading indicates alternating layers.

Proof. As the linear maps are bijective, it suffices to prove one direction. If $x \in \mathcal{P}(A)$, then by Theorem 2.1, for every $Q \in \{Q_p, H_{q^\perp}\}$,

$$A \cap (x + \mathbf{int}(Q)) \neq \emptyset$$

Let such Q be given and let $x \in \mathcal{P}(A)$ so that there is

$$y = x + q_i$$

for $y \in A$ and $q_i \in \mathbf{int}(Q)$. Since L is linear,

$$L(y) = L(x) + L(q_i)$$

and $L(y) \in L(A)$, $L(q_i) \in \mathbf{int}(L(Q))$, meaning

$$L(A) \cap (L(x) + \mathbf{int}(L(Q))) \neq \emptyset,$$

and we conclude by Theorem 2.1. $\square$

Corollary 2.3. *For any finite set $A \subseteq \mathbb{R}^2$ and any bijective linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$, the height functions are related by $h_A(x) = h_{L(A)}(L(x))$.*

3. A generalization of nondominated sorting

In this section we present a generalization of nondominated sorting and some convergence results. Roughly speaking, the generalization sorts points with respect to arbitrary partial orders induced by cones. In two dimensions, the cones are all bijections of the quadrant, leading to a simple relationship between the generalization and 'vanilla' nondominated sorting.

3.1. **Definitions.** We recall the definition of nondominated sorting as well as a useful generalization. This generalization describes the local behavior of Pareto peeling — it is the 'cell problem'.

Given a finite set of distinct points, $A \subseteq \mathbb{R}^2$, nondominated sorting arranges the set of points into layers by repeatedly removing or peeling the set of minimal elements. Specifically, let

$$x \leq y \iff x_1 \leq y_1 \quad \text{and} \quad x_2 \leq y_2 \tag{22}$$

denote the component-wise partial order. Given $q \in \mathbb{R}^2$ and $q' \in A$, we say q' *dominates* q if $q' \leq q$. If there is no such q', say that q is a *nondominated* point for A. Write $\mathcal{N}(A)$ for the set of nondominated points. If A is empty, designate $\mathcal{N}(A) = \emptyset$. *Nondominated sorting* of A is then

$$S_1(A) = \mathcal{N}(A) \quad \text{and} \quad S_{n+1}(A) = \mathcal{N}(A \cap \mathbf{int}(S_n(A))). \tag{23}$$

The nondominated sorting *depth function* of A is defined to be

$$s_A(x) = \sum_{m=1}^{\infty} 1_{\mathbf{int}(S_n(A))}. \tag{24}$$

Calder-Esedoglu-Hero showed that if A consists of randomly scattered points, then, in the large sample limit, h_A converges to the solution of a Hamilton-Jacobi equation [CEH14a, CEH14b, CEH15, Cal17]. A rate of convergence was recently establish by Cook-Calder [CC20]. These analyses relied on an equivalence between nondominated sorting of a set A and the *longest chain* in A, a longest, totally ordered subset of A, $l_1 \leq \cdots \leq l_n$, $\{l_i\} \in A$. Denote the length of the longest chain in A by $\ell(A)$ and observe that

$$s_A(x) = \ell((-\infty, x]^2 \cap A).$$

The definition of nondominated sorting is easily extended beyond (22) to any partial order on $\mathbb{R}^2$. Recall that any proper cone $Q \subseteq \mathbb{R}^2$ induces a partial order [BV04]. A cone Q is *proper* if it is convex, closed, has nonempty interior, and is pointed: $Q \cap -Q \subset \{0\}$. In particular, given any proper cone Q, let

$$x \leq_Q y \iff (y - x) \in Q \tag{25}$$

denote the associated partial order. Observe that when $Q = [0, \infty)^2$, (25) coincides with (22).

By replacing (22) in nondominated sorting by (25), we get *Q-nondominated sorting*, see Figure 4. We distinguish between the two by a superscript, *e.g.*, $S_n^Q(A)$, u_A^Q, and $\ell^Q(A)$. We also note the following *dynamic programming principle* for Q-nondominated sorting,

$$s_A(x) = \sup_{y \in (x + \mathbf{int}(Q))} (s(y) + 1_A(y)) \tag{26}$$

Remark 1. *Topological properties of Q-nondominated sets have been analyzed previously in an optimization context* [Luc85, DT05].

3.2. **Convergence of the longest chain.** In this section we record known results on the longest chain of a Poisson points in rectangles and simplicial domains. Let $\mathcal{B}$ denote the set of bounded rectangles in $\mathbb{R}^2$ and let X_{nf} denote a Poisson process of intensity nf in $\mathbb{R}^2$.

Proposition 3.1 ([Ham72, CEH15])**.** *There exists a constant $c_2 = 2$, so that on an event of probability 1, for all $B \in \mathcal{B}$ and $f \in C(\mathbb{R}^2)$*

$$\limsup_{n\to\infty} n^{-1/2}\ell(X_{nf} \cap B) \leq c_2 \left((\sup_{x\in B} f(x))|B| \right)^{1/2} \tag{27}$$

and

$$\liminf_{n\to\infty} n^{-1/2}\ell(X_{nf} \cap B) \geq c_2 \left((\inf_{x\in B} f(x))|B| \right)^{1/2}. \tag{28}$$

Proof. For completeness, we sketch the proof. By the subadditive ergodic theorem [Ham72], for each $q \in \mathbb{Q}^2$, on an event of probability 1, $\lim n^{-1/2}\ell(X_n \cap (q+[0,1]^2)) = c_2$. In particular, by scaling and taking the intersection over a countable number of probability 1 events, for each $q, q' \in \mathbb{Q}^2$

$$\lim n^{-1/2}\ell(X_n \cap (q + R_{q'})) = c_2|R_{q'}|^{1/2},$$

where $R_{q'} = [0, q']$, on an event of probability 1. By approximation, this implies

$$\lim n^{-1/2}\ell(X_n \cap R) = c_2|B|^{1/2},$$

for any $B \in \mathcal{B}$ on an event of probability 1. The extension to arbitrary f uses the standard coupling of Poisson processes [Kin92]. $\square$

We next record a consistency result for nondominated sorting which is used in Section 5 below. For $v \in (0,\infty)^2$, denote the simplex

$$S_v := \{x \in (-\infty, 0]^2 \mid 1 + \langle x, v\rangle \geq 0\}. \tag{29}$$

Proposition 3.2 ([CEH15, Cal])**.** *Let $f \in C^0(\mathbb{R}^2)$. Then with probability one, for any $p \in (0,\infty)^2$ and $x \in \mathbb{R}^2$*

$$\limsup_{n\to\infty} \frac{\ell((x + S_p) \cap X_{nf})}{n^{1/2}} \leq \left(\frac{\sup_{x+S_p} f}{p_1 p_2} \right)^{1/2}. \tag{30}$$

3.3. **Isomorphism between versions of nondominated sorting.** We note a useful change of variables which will allow us to translate between Q-nondominated sorting and standard nondominated sorting. This change of variables immediately leads to versions of Propositions 3.1 and 3.2 for Q-nondominated sorting.

Let $Q \subseteq \mathbb{R}^2$ be a proper cone and let a, b denote its extremal directions, *i.e.*,

$$Q = \{xa + yb \mid x, y \geq 0\}.$$

As Q is proper, $\{a, b\}$ form a basis of $\mathbb{R}^2$ and we may define a linear bijection $L_Q : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$L_Q(xa + yb) := (x, y). \tag{31}$$

Note that L_Q preserves the order:

$$L_Q(x) \leq L_Q(y) \iff x \leq_Q y$$

for $x, y \in \mathbb{R}^2$. Therefore, given a set of points $A \subseteq \mathbb{R}^2$,

$$q \in \mathcal{N}^Q(A) \iff L_Q(q) \in \mathcal{N}(L(A)) \tag{32}$$

The above observations imply the following.

Lemma 3.1. *If X_g is a Poisson point process of intensity $g \in C(\mathbb{R}^2)$ then*

$$\ell^Q(X_g \cap A) = \ell(L_Q(X_g \cap A)) = \ell(X_{g'} \cap L_Q(A))$$

for all finite subsets A of $\mathbb{R}^2$ where $g' = |a \times b| g$.

4. Viscosity solutions and basic estimates

In this section, we set-up the proof of our main result, Theorem 1.1. We begin by stating the intermediate results that will be used in the proof and showing how they imply the theorem. The remainder of the section establishes L^∞ and boundary Hölder estimates for the rescaled height functions.

The key compatibility assumption is defined in Section 4.3.

4.1. Proof of Theorem 1.1.

We follow what is now a classical approach in the viscosity solutions literature. To start with, define so-called upper and lower half-relaxed limits u^* and u_* in $\bar{\mathrm{U}}$ by

$$u^*(x) = \lim_{\delta \to 0^+} \sup \left\{ n^{-\frac{1}{2}} u_n(y) \mid \|y - x\| + n^{-1} \leq \delta \right\}, \tag{33}$$

$$u_*(x) = \lim_{\delta \to 0^+} \inf \left\{ n^{-\frac{1}{2}} u_n(y) \mid \|y - x\| + n^{-1} \leq \delta \right\}. \tag{34}$$

To prove our main result, we will argue that u^* and u_* are, respectively, viscosity sub- and supersolutions of the Hamilton-Jacobi equation (5) and apply the comparison principle to conclude that $u^* = u_*$.

First, let us recall the relevant definitions from viscosity solutions. In the next definition, $\mathrm{USC}(\mathcal{O})$ (resp. $\mathrm{LSC}(\mathcal{O})$) denotes the set of functions that are upper (resp. lower) semicontinuous at all points in $\mathcal{O}$.

Definition 4.1. *(i) We say that $w \in \mathrm{USC}(\mathrm{U})$ is a* viscosity subsolution *of the equation $\bar{H}_\varphi(Du) = f$ in U if for each $x_0 \in \mathrm{U}$ and each smooth function ψ defined in a neighborhood of x_0, the following statement holds: if there is an $r > 0$ such that $\psi \geq w$ in $B(x_0, r)$ and $\psi(x_0) = w(x_0)$, then*

$$\bar{H}_\varphi(D\varphi(x_0)) \leq f(x_0).$$

We abbreviate this by writing $\bar{H}_\varphi(Dw) \leq f$ in U.

(ii) We say that $v \in \mathrm{LSC}(\mathrm{U})$ is a viscosity supersolution *of the equation $\bar{H}_\varphi(Du) = f$ in U if, for each $x_0 \in \mathrm{U}$ and each smooth function ψ defined in a neighborhood of x_0, the following statement holds: if there is an $r > 0$ such that $\psi \leq v$ in $B(x_0, r)$ and $\psi(x_0) = v(x_0)$, then*

$$\bar{H}_\varphi(D\varphi(x_0)) \geq f(x_0).$$

We abbreviate this by writing $\bar{H}_\varphi(Dv) \geq f$ in U.

(iii) A function $u \in C(\bar{\mathrm{U}})$ is a viscosity solution *of $\bar{H}_\varphi(Du) = f$ in U if it is both a viscosity sub- and supersolution.*

As is customary in the viscosity solutions literature, we will abbreviate the condition in (i) and (ii) by saying "ψ touches w from above at x_0" and "ψ touches v from below at x_0", respectively.

To prove that u^* and u_* are viscosity sub- and supersolutions, we start with an L^∞ estimate.

Lemma 4.1. *If φ satisfies* (3)*, then there is a constant $0 < C := C_{\varphi,f,\mathrm{U}} < \infty$ so that on an event of probability 1*

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}^2} \bar{u}_n(x) \leq C.$$

This will be proved below by comparing to nondominated sorting.

Next, we show that the boundary behavior is controlled provided U is compatible with φ.

Lemma 4.2. *If* U *is a bounded, open convex set compatible with φ, then there exists a constant $C := C_{\mathrm{U},f,\varphi}$, so that on an event of probability 1, for each $\delta > 0$ and $x_0 \in \partial\mathrm{U}$,*

$$\sup_{x\in B(x_0,\delta)} u_n(x) \leq C\sqrt{\delta n}. \tag{35}$$

for all n sufficiently large.

Appealing again to nondominated sorting, we show that u^* is a subsolution.

Proposition 4.1. *On an event of probability 1, the upper half-relaxed limit u^* defined by* (33) *satisfies $\bar{H}_\varphi(Du^*) \leq f$ in* U.

Finally, in the core of the paper, we show that u_* is a supersolution.

Proposition 4.2. *If φ satisfies* (3) *and* (4)*, then, on an event of probability 1, the lower half-relaxed limit u_* defined by* (33) *satisfies $\bar{H}_\varphi(Du_*) \geq f$ in* U.

The remainder of the paper is devoted to the proof of these results. For the sake of completeness, we show how they imply Theorem 1.1. The main step involves an invocation of the comparison principle. Let us recall that result and briefly sketch its proof.

Lemma 4.3. *Suppose $f > 0$ in* U*, an open, bounded set in $\mathbb{R}^2$. If $w \in \mathrm{USC}(\bar{\mathrm{U}})$ satisfies $\bar{H}_\varphi(Dw) \leq f$ in* U *and if $v \in \mathrm{LSC}(\bar{\mathrm{U}})$ satisfies $\bar{H}_\varphi(Dv) \geq f$ in* U *and $v \geq w$ on $\partial\mathrm{U}$, then $v \geq w$ in* U

Proof. For each $\epsilon \in (0,1)$, if we define $w^\epsilon = (1-\epsilon)w$, then $\bar{H}_\varphi(Dw^\epsilon) = (1-\epsilon)^2\bar{H}_\varphi(Dw) \leq (1-\epsilon)^2 f$. Hence w^ϵ is a strict subsolution and, thus, by strict comparison [CIL92], $\sup_{\bar{\mathrm{U}}}(w^\epsilon - v) \leq \sup_{\partial U}(w^\epsilon - v)$. We recover the result upon sending $\epsilon \to 0^+$. ☐

Proof of Theorem 1.1. A direct argument shows that u^* is upper semi-continuous and u_*, lower semi-continuous (see, for instance, [BCD08, Chapter 5]). Lemma 4.1 shows they are, in fact, bounded, hence not identically infinity.

By Lemma 4.2, $u_* = u^* = 0$ on $\partial\mathrm{U}$. Together with Propositions 4.1 and 4.2, this implies u^* and u_* satisfy the hypotheses of the comparison principle. In particular, $u^* \leq u_*$ in U. At the same time, the definitions (33) and (34) directly give $u^* \geq u_*$. Therefore, $u^* = u_*$ in $\bar{\mathrm{U}}$, this function is the viscosity solution of (5) to which $(\bar{u}_n)_{n\in\mathbb{N}}$ converges uniformly. ☐

4.2. **Global upper bound.** As an immediate application of the dynamic programming principle, Proposition 2.1, we prove an L^∞ estimate.

Proof of Lemma 4.1. By Proposition 2.1 and our assumption that $\mathcal{N}^*$ is nonempty, there is a Q_p so that for all $x \in \mathbb{R}^2$,

$$u_n(x) \leq \sup_{y \in x + Q_p} (u_n(y) + 1_{X_n}(y)) \leq \ell^{Q_p}(X_{nf} \cap \mathrm{U}).$$

Conclude by Proposition 3.1 and Lemma 3.1. ☐

4.3. **Consistency and boundary Hölder estimate.** In this section, we prove a boundary Hölder estimate on open, bounded convex domains that are compatible with the given norm.

Let U be a convex set. Recall that for $x_0 \in \partial \mathrm{U}$, the vector $q \in \mathbb{R}^2 \setminus \{0\}$ supports U at x_0 if $\langle q, x\rangle \leq \langle q, x_0\rangle$ for all $x \in \mathrm{U}$. The set of all vectors supporting U at x_0 will be denoted by $\mathcal{H}(\mathrm{U}, x_0)$. Note that this set is always nonempty.

Definition 4.2. *We say a convex set* $\mathrm{U} \subseteq \mathbb{R}^2$ *is compatible with* φ *if for each* $x_0 \in \partial U$*, there is a* $q \in \mathcal{H}(\mathrm{U}, x_0)$ *and a* $p \in \mathcal{N}^*$ *such that* $q^\perp \in \mathcal{Q}_p$*.*

Note that if φ is polyhedral, then $\mathbb{R}^2 = \cup_{p \in \mathcal{N}^*} \mathcal{Q}_p$ and, thus, any convex set is compatible.

Proof. Observe that since U is convex, by Corollary (2.1),

$$u_n(x) = 0 \text{ for all } x \notin \mathrm{U}. \tag{36}$$

Let $x_0 \in \partial \mathrm{U}$ be given. Translate so that $x_0 = 0$ and take $q \in \mathcal{H}(\mathrm{U}, 0)$ with $q^\perp \in Q_p$ and $H_q \cap \mathrm{U} = x_0$. Rescaling q if necessary, we can assume that $\|q\| = 1$. Write $Q := Q_p$.

Let $M = C\mathrm{diam}(\mathrm{U})$ for some $C > 0$ to be determined and for $\delta > 0$, consider the parallelogram P_δ with vertices

$$y_1^\pm = \delta q^\perp \pm Mq \quad \text{and} \quad y_2^\pm = -\delta q^\perp \pm Mq,$$

i.e., $P_\delta = \mathbf{conv}(y_1^\pm, y_2^\pm)$. Note that by possibly making C larger, independent of δ,

$$(H_q - \delta q^\perp) \cap \mathrm{U} \subseteq P_\delta, \tag{37}$$

$$u_n = 0 \quad \text{on } \mathbf{conv}(y_1^+, y_2^+) \cup \mathbf{conv}(y_1^-, y_2^-) \cup \mathbf{conv}(y_1^+, y_1^-)). \tag{38}$$

and $B(0, \delta) \subseteq P_\delta$ (since $\|q^\perp\| = 1$).

We claim that

$$\sup_{x \in P_\delta} u_n(x) \leq \ell^Q(P_\delta) \leq C\sqrt{M\delta n}, \tag{39}$$

where $\ell^Q(P_\delta)$ denotes the Q-longest chain in $X_{nf} \cap P_\delta$. By asymptotics of the longest chain, Proposition 3.1 and Lemma 3.1, it suffices to show the first inequality. By the dynamic programming principle for both Q-nondominated sorting (26) and Pareto peeling, Proposition 2.1, it suffices to show that

$$\sup_{x \in P_\delta} s_{X_n}(x) = \sup_{y \in P_\delta} s_{X_n \cap P_\delta}(y). \tag{40}$$

Let $\ell_1, \ldots, \ell_n$ be a Q-longest chain which ends in P_δ but possibly has points outside of P_δ. Let ℓ_{i-1} be the last point (maximal i) not in P_δ. By definition, $\ell_{i-1} \in \ell_i + Q$, but $Q \subseteq H_q$; therefore, by (37), $\ell_{i-1} \in P_\delta$, showing (40) and completing the proof. ☐

The boundary estimate implies that $u^* = u_* = 0$ on $\partial \mathrm{U}$ almost surely.

Proposition 4.3. *If* U *is compatible with* φ*, then* $u^* = u_* = 0$ *on* ∂U*.*

Proof. Since $u_n = 0$ on $\mathbb{R}^2 \backslash \mathrm{U}$, $u_* = 0$ on $\partial\mathrm{U}$. Lemma 4.2 shows that $u^* = 0$ on $\partial\mathrm{U}$. $\square$

4.4. **Convex hull peeling.** We now show that when $\{\varphi \leq 1\}$ is strictly convex, or, equivalently, $\mathcal{N}^* = \emptyset$, the $n^{\frac{1}{2}}$ results in a trivial limit.

Proof of Corollary 1.1. In this case, Proposition 4.2 implies $\bar{H}_\varphi(Du_*) \geq f$ in U, where $\bar{H}_\varphi \equiv 0$. Hence no smooth function can touch u_* from below. We claim this implies $u_* \equiv \infty$.

Indeed, suppose there were an $x_0 \in \mathrm{U}$ for which $u_*(x_0) < \infty$. Given $\zeta > 0$, the function $x \mapsto u_*(x) + \frac{\|x-x_0\|^2}{2\zeta}$ achieves its minimum at some point $x_\zeta \in \bar{\mathrm{U}}$ by lower semi-continuity. Since $u_*(x_0) \geq u_*(x_\zeta) + \frac{\|x_\zeta - x_0\|^2}{2\zeta}$, we know that $x_\zeta \to x_0$. Hence, for small enough ζ, u_* is touched from below by the smooth function $u_*(x_\zeta) - \frac{\|x-x_0\|^2}{2\zeta}$ at x_ζ, contradicting our previous deduction.

From the identity $u_* \equiv \infty$, one readily deduces that $\bar{u}_n \to \infty$ locally uniformly in U. $\square$

4.5. **Counterexample when compatibility fails.** In the next result, we prove that the Hamilton-Jacobi equation (5) in a convex domain does not always have a classical viscosity solution. As we will show, this implies that the height functions do not converge uniformly in $\bar{\mathrm{U}}$ in general without the compatibility assumption. This does not rule out the possibility that they converge locally uniformly in U to the minimal supersolution, for example.

Proposition 4.4. *If* $\mathrm{U} = (-1,1)^2$*,* $\varphi(x) = \max(x_1, -x_1, \|x\|_2)$*, and* $f \equiv 1$*, then* U *is not compatible with* φ *and* (5) *has no viscosity solution.*

Proof. First of all, U is not compatible with φ since $\mathcal{H}(U, (-1,0)) = \{(-1,0)\}$.

Next, notice that if we define $\{\tilde{H}_N\}_{N \in \mathbb{N}}$ by

$$\tilde{H}_N(p') = \max\left(\frac{2}{N}|p_1'|, 2|p_2'|\right),$$

then

$$\bigcup_{N=1}^{\infty} \{\tilde{H}_N = 1\} \subseteq \{\bar{H}_\varphi < 1\}. \tag{41}$$

See Figure 4.5 for a depiction of this; the key point is the rectangle can grow arbitrarily long. We will use $\{\tilde{H}_N\}$ to build subsolutions that are nonzero near $(-1,0)$.

Let us argue by contradiction. Suppose $u \in C(\bar{\mathrm{U}})$ is a viscosity solution of (5). By (41), if we let $(u_N)_{N \in \mathbb{N}} \subseteq C(\bar{\mathrm{U}})$ denote the solutions of the Eikonal equations

$$\begin{cases} \tilde{H}_N(Du_N) = 1 & \text{in } (-1,1)^2, \\ u_N = 0 & \text{on } \partial[-1,1]^2, \end{cases}$$

then $\bar{H}_\varphi(Du_N) \leq 1$ in $(-1,1)^2$ independent of N. Thus, $\sup_N u_N \leq u$ by Lemma 4.3.

At the same time, notice that if we rescale by setting $v_N(y) = u_N(N^{-1}y_1, y_2)$, then $(v_N)_{N \in \mathbb{N}}$ are viscosity solutions of the problems

$$\begin{cases} \tilde{H}_1(Dv_N) = 1 & \text{in } (-N,N) \times (-1,1), \\ v_N = 0 & \text{on } \partial[-N,N] \times [-1,1]. \end{cases}$$

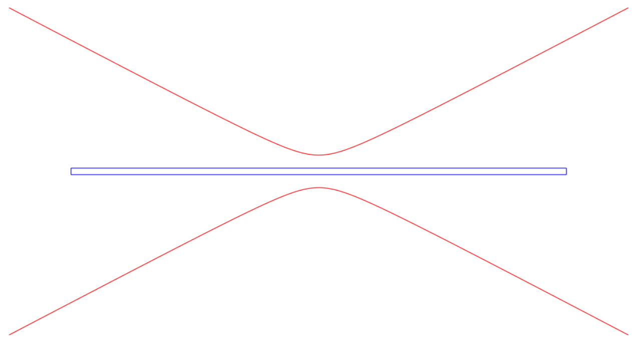

FIGURE 5. The red lines are the curves $\{\bar{H}_\varphi = 1\}$, while the rectangle enclosed in blue is $\{\tilde{H}_N \leq 1\}$.

These have explicit representations as the distance functions to the boundary with respect to the dual norm $\tilde{H}_1^*$, that is,

$$v_N(y) = \inf\left\{\tilde{H}_1^*(y-z) \;\mid\; z \in \partial[-N,N]\times[-1,1]\right\}.$$

A well known computation shows that $\tilde{H}_1^*(q') = \frac{1}{2}(|q_1'| + |q_2'|)$ and, thus,

$$v_N(-(N-1),0) = \tilde{H}_1^*(1,0) = \frac{1}{2}.$$

Scaling back, we have $u_N(-1+N^{-1},0) = \frac{1}{2}$ and, thus,

$$u(-1+N^{-1},0) \geq u_N(-1+N^{-1},0) = \frac{1}{2}.$$

Since $u(-1,0) = 0$, this contradicts the continuity of u. $\square$

For the sake of completeness, we show how the nonexistence of viscosity solutions implies the failure of uniform convergence of the height functions.

Proof of Corollary 1.2. Let U, φ, and f be as in the previous proposition. If the rescaled height functions converged uniformly to some continuous function, then this would be a solution of (5) by Proposition 4.1 and 4.2, which was just shown to be impossible. $\square$

4.6. **Level Set PDE.** We conclude the section by proving that the Pareto peels themselves converge to surfaces moving with normal velocity given by (9).

Proof of Corollary 1.3. To start with, we need to justify the claim that the sets $(E_t)_{t\geq 0}$ given by (8) are, in fact, the generalized level set evolution associated with (9). That they are a generalized level set evolution in the sense of [BS98] readily follows from the fact that the function v defined by (6) is a viscosity solution of (7). By Proposition 2.2 in [BS98], it suffices to prove that there is no fattening, or, in other words, ∂E_t has empty interior for all $t \geq 0$.

This part follows readily from the equation solved by u. Recall that $\partial E_t = \{u = t\}$. If $B(x_0, r) \subseteq \{u = t\}$ for some $x_0 \in$ U and $r > 0$, then the constant function t touches u below at x_0. This implies $\bar{H}_\varphi(0) \geq f(x_0) > 0$, which is absurd. Hence ∂E_t has empty interior as claimed.

Finally, the convergence $\bar{E}^{(n)}_{\lfloor n^{\frac{1}{2}} t\rfloor} \to \bar{E}_t$ in the Hausdorff distance for any fixed $t > 0$ follows readily from the fact that ∂E_t has empty interior and $\bar{u}_n \to u$ uniformly in $\bar{\mathrm{U}}$. $\square$

5. Subsolution proof

In this section we show that the upper half-relaxed limits are subsolutions. The proof applies the subsolution argument for nondominated sorting [CEH15] to Q-nondominated sorting.

Proof of Proposition 4.1. Suppose that ψ is a smooth function in U, and $u^* - \psi$ has a strict local maximum at 0 and $u^*(0) = \psi(0)$. Assume that $\langle D\psi(0), v_p\rangle\langle D\psi(0), w_p\rangle > 0$ for some $p \in \mathcal{N}^*$, otherwise the claim is immediate as $f \geq 0$. By possibly reflecting (using symmetry of the norm), we may further suppose $\min(\langle D\psi(0), v_p\rangle, \langle D\psi(0), w_p\rangle) > 0$.

After this, the proof follows the nondominated sorting subsolution direction of Calder [Cal16, Cal] closely — the only change is that nondominated sorting is replaced by Q_p-nondominated sorting. We reproduce the argument for the reader's convenience.

By varying ψ away from 0, assume ψ is strictly increasing with respect to the partial order induced by Q_p and $u^*(z) \leq \psi(z)$ for all $z \in -Q_p$. Let $\epsilon > 0$, $v \in Q_p$ and set

$$A = \{x \in -Q_p \mid \psi(x) \geq \psi(-\epsilon v) - \epsilon^2\} \tag{42}$$

and

$$A_n = \{x \in -Q_p \mid u_n(x) \geq n^{1/2}\psi(-\epsilon v)\}. \tag{43}$$

Note that since ψ is smooth and Q_p-strictly increasing, $A \subseteq B(0, C\epsilon)$. Also, $A_n \subseteq A$ for all n sufficiently large.

By the dynamic programming principle, Proposition 2.1,

$$u_n(0) \leq n^{1/2}\psi(-\epsilon v) + \ell^{Q_p}(X_{nf} \cap A_n). \tag{44}$$

The construction of ψ together with (44) then implies

$$\langle \epsilon D\psi(0), v\rangle - C\epsilon^2 \leq \limsup_{n\to\infty} n^{-1/2}\ell^{Q_p}(X_{nf} \cap A). \tag{45}$$

Also, by Taylor's formula in A, we have for any $x \in A$ and $y \in Q_p$ with $|y| \leq \epsilon^2$ that

$$1 + \langle (x - y), q\rangle \geq 0$$

where

$$q = \frac{D\psi(0)}{\langle \epsilon D\psi(0), v\rangle + C\epsilon^2}.$$

This shows $A \subseteq S_q := \{x \in -Q_p \mid 1 + \langle x, q\rangle \geq 0\}$, the simplex with sides parallel to v_p and w_p. By (45) and Proposition 3.2 together with Lemma 3.1,

$$\langle \epsilon D\psi(0), v\rangle - C\epsilon^2 \leq |w_p \times v_p|^{1/2} \left(\frac{\sup_{y+S_q} f}{\langle q, v_p\rangle\langle q, w_p\rangle}\right)^{1/2} \tag{46}$$

which implies

$$\frac{\langle \epsilon D\psi(0), v\rangle - C\epsilon^2}{\langle \epsilon D\psi(0), v\rangle + C\epsilon^2} \leq |w_p \times v_p|^{1/2} \left(\frac{\sup_{y+S_q} f}{\langle D\psi(0), v_p\rangle\langle D\psi(0), w_p\rangle}\right)^{1/2}$$

and by sending $\epsilon \to 0^+$, using the continuity of f,

$$\frac{\langle D\psi(0), v_p\rangle\langle D\psi(0), w_p\rangle}{|w_p \times v_p|} \le f(0),$$

concluding the proof. $\square$

6. Conditional supersolution proof

In this section we reduce the proof of Proposition 4.2 to checking a certain gradient condition at contact points. In the subsequent section, we verify this gradient condition.

To simplify the proofs, we will frequently change variables so that $Q_p = [0,\infty)^2$. This can be done using the change of coordinates $T(aw_p + bv_p) = (a,b)$. By affine invariance (Section 2.7), this is no loss of generality so long as the proper accounting is carried out.

6.1. **Legendre transform.**

Once again, convex duality will play a role. In particular, we will use the following bit of elementary convex analysis, the proof of which is left to the reader.

Lemma 6.1. *The function $p : \mathbb{R}^2 \to (-\infty, 0] \cup \{\infty\}$ given by*

$$p(a,b) = \begin{cases} -\sqrt{ab}, & \text{if } a,b \in [0,\infty)^2, \\ +\infty, & \text{otherwise} \end{cases}$$

is convex. Its Legendre transform $p^* : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ is given by

$$p^*(c,d) = \begin{cases} 0, & \text{if } c,d \in (-\infty,0]^2,\ |cd| \ge \frac{1}{4}, \\ +\infty, & \text{otherwise} \end{cases}$$

Recall that the Legendre transform is defined by

$$\begin{aligned} p^*(c,d) &= \sup\left\{ca + db - p(a,b) \mid (a,b) \in \mathbb{R}^2\right\} \\ &= \sup\left\{ca + db + \sqrt{ab} \mid a,b \ge 0\right\}. \end{aligned} \tag{47}$$

6.2. **Forcing a peeling direction.**

We introduce a certain 'flattened' norm which we use to control the growth of general Pareto peeling from below.

For $p \in \mathcal{N}^*$, denote the *Q_p-flattened norm* φ_{Q_p} of φ to be the norm with unit ball the parallelogram, $\mathbf{conv}(v_p, w_p, -w_p, -v_p)$. The Q_p-flattened Pareto hull is

$$\mathcal{P}_{Q_p}(A) = \{x \in \mathbb{R}^2 \mid \forall y \neq x \text{ there exists } a \in A \text{ with } \varphi_{Q_p}(a-x) < \varphi_{Q_p}(a-y)\}. \tag{48}$$

We use the dynamic programming principle to compare Pareto hulls with their flattened hulls.

Lemma 6.2. *Let A be a finite set of points in $\mathbb{R}^2$, then for all $p \in \mathcal{N}^*$,*

$$x \in \mathbf{int}(\mathcal{P}_{Q_p}(A)) \implies x \in \mathbf{int}(\mathcal{P}(A)).$$

In particular, if $x \in \mathbb{R}^2$ and for some $p \in \mathcal{N}^*$,

$$y_1^{\pm} \in (x \pm \mathbf{int}(Q_p)) \cap A \quad \text{and} \quad y_2^{\pm} \in (x \pm \mathbf{int}(\mathcal{Q}_p)) \cap A,$$

then $x \in \mathbf{int}(\mathcal{P}(A))$.

Proof. Let $p \in \mathcal{N}^*$ be given and let $x \in \mathbf{int}(\mathcal{P}_{Q_p}(A))$. By Corollary 2.1, $x \in \mathbf{int}(\mathbf{conv}(A))$ follows from $x \in \mathbf{int}(\mathcal{P}_{Q_p}(A))$. It remains to check the cone condition.

By construction of φ_{Q_p} and Corollary 2.1, there is $y_1^{\pm} \in (x \pm \mathbf{int}(Q_p))$ and $y_2^{\pm} \in (x \pm \mathbf{int}(\mathcal{Q}_p))$. Now, let $p' \in \mathcal{N}^*$. If $Q_p = \pm Q_{p'}$, then $y_1^{\pm} \in (x + \mathbf{int}(Q_{p'}))$, otherwise, by definition, $\mathbf{int}(\pm\mathcal{Q}_p)$ and $\mathbf{int}(\mathcal{Q}_{p'})$ are disjoint. Since $(\pm Q_p) \cup (\pm \mathcal{Q}_p) = \mathbb{R}^2$, we have $\mathcal{Q}_{p'} \subseteq \pm Q_p$. This implies $\pm\mathcal{Q}_p \subseteq Q_p'$ so $y_2^{\pm} \in (x + \mathbf{int}(Q_p'))$. □

We next transfer this lower bound to the height function. To avoid introducing additional notation, we suppose without loss of generality the given cone is a quadrant.

Lemma 6.3. *Let $A \subseteq \mathbb{R}^2$ finite, $x_0 \in \mathbb{R}^2$ and $p \in \mathcal{N}^*$. Suppose $Q_p = [0,\infty)^2$. Let $R_z = [z, x_0]$ for $z \leq x_0$. If z is such that there are points*

$$y_1 \in \bigcap_{z' \in R_z} (z' + \mathbf{int}(Q_p)) \quad \textit{and} \quad y_2^{\pm} \in \bigcap_{z' \in R_z} (z' \pm \mathbf{int}(\mathcal{Q}_p))$$

where

$$\min(u_A(y_1), u_A(y_2^{\pm}) \geq u_A(x_0) + 1, \tag{49}$$

then

$$u_A(z) + \ell_A(R_z \cap A) \leq u_A(x_0).$$

Proof. Let $l_1, \ldots, l_n$ denote a (possibly empty) longest chain in $A \cap R_z$, $l_1 \leq \cdots \leq l_n$ and set $l_{n+1} = x_0$ and $l_0 = z$. By (49), Lemma 6.2, and the dynamic programming principle, for all $1 \leq i \leq n$

$$u_A(l_{i+1}) \geq \min(u_A(l_i) + 1, u_A(x_0) + 1)$$

and $u_A(l_1) \geq \min(u_A(z), u_A(x_0) + 1)$. Thus, by summing

$$u_A(l_{i+1}) \geq \min(u_A(z) + i, u_A(x_0) + 1),$$

and so $u_A(x_0) \geq \min(u_A(z) + n, u_A(x_0) + 1)$. Since the other inequality is impossible, $u_A(x_0) \geq u_A(z) + n = u_A(z) + \ell_A(R_z)$. □

6.3. **Conditional supersolution proof.** We finally give a proof of the supersolution property under a strictness assumption on the touching point.

Proposition 6.1. *On an event of probability 1, if $x_0 \in \mathrm{U}$, ψ is a smooth function touching u_* from below at x_0, and*

$$\bar{H}_\varphi(D\psi(x_0)) > 0, \tag{50}$$

then

$$\bar{H}_\varphi(D\psi(x_0)) \geq f(x_0).$$

Proof. Suppose that ψ is a smooth function in U, $x_0 \in \mathrm{U}$, and $u_* - \psi$ has a strict local minimum at x_0 and $u_*(x_0) = \psi(x_0)$. By Corollary 2.3 and (50), we may take a change of variables so that there is $p \in \mathcal{N}^*$ with $Q_p = (0,\infty)^2$ and $\min(\psi_{x_1}(x_0), \psi_{x_2}(x_0)) > 0$.

Note this transformation turns the Poisson process of intensity nf to a Poisson process of intensity $nf|v_p \times w_p|$. Also note that since $Q_p = [0,\infty)^2$, by our convention $w_p \times v_p > 0$, $\pm\mathcal{Q}_p = \pm((-\infty, 0] \times [0,\infty)])$.

Since ψ touches u_* from below at x_0, we know that there is a $r > 0$ such that $\{\psi > u_*(x_0)\} \cap B(x_0, r) \subseteq \{u_* > u_*(x_0)\} \cap B(x_0, r)$. Accordingly, since $\min(\psi_{x_1}(x_0), \psi_{x_2}(x_0)) > 0$, we can find $\delta > 0$ and points $y_1, y_2^{\pm} \in B(x_0, r)$ so that

$$\begin{gathered}\min(u_*(y_1), u_*(y_2^{\pm})) \geq \min(\psi(y_1), \psi(y_2^{\pm})) \geq u_*(x_0) + \delta,\\ y_1 \in x_0 + \mathbf{int}(Q_p) \quad \text{and} \quad y_2^{\pm} \in x_0 \pm \mathbf{int}(\mathcal{Q}_p).\end{gathered} \tag{51}$$

In fact, (51) can be quantified. For a set $\mathrm{C} \subseteq \mathbb{R}^2$ and $\delta' > 0$, let

$$\mathrm{C}^{\delta'} = \{x \in \mathrm{C} \mid x + B_{\delta'} \subseteq \mathrm{C}\}$$

be a strict subset of $\mathbf{int}(\mathrm{C})$. By the definition of u_*, we can fix $(x_0^{(n)})_{n\in\mathbb{N}} \subseteq \mathrm{U}$ such that

$$\lim_{n\to\infty} x_0^{(n)} = x_0, \quad \lim_{n\to\infty} n^{-\frac{1}{2}} u_n(x_0^{(n)}) = u_*(x_0).$$

Thus, there is $\delta' > 0$ and $N \in \mathbb{N}$ such that, for each $n \geq N$,

$$\min(u_n(y_1), u_n(y_2^{\pm})) \geq u_n(x_0^{(n)}) + 1$$

and

$$y_1 \in x_0 + (Q_p)^{\delta'} \quad \text{and} \quad y_2^{\pm} \in x_0 \pm (\mathcal{Q}_p)^{\delta'}. \tag{52}$$

Let $z = -(a, b) \leq 0$. We claim that (52) implies there is a $\zeta_0^z > 0$ small such that if $n \geq N$ and $\zeta \in (0, \zeta_0^z)$, then the point $x_*^{(n)} = x_0^{(n)} - \zeta z$ satisfies the hypotheses of Lemma 6.3. Indeed, for $Q \in \{Q_p, \pm\mathcal{Q}_p\}$, since $y_1 + B_{\delta'} \subseteq x_0 + Q$, we have $y_1 \in (x_0 - B_{\delta'}) + Q$. Thus, $[\zeta z, 0] \subseteq B_{\delta'}$ for small ζ_0.

Hence, by Lemma 6.3, if we set $R_\zeta^{(n)} = x_0^{(n)} + [\zeta z, 0]$, then

$$n^{-\frac{1}{2}} u^{(n)}(x_0^{(n)} + \zeta z) + n^{-\frac{1}{2}} \ell_n(R_\zeta^{(n)}) \leq n^{-\frac{1}{2}} u^{(n)}(x_0^{(n)}),$$

where $\ell_n(\mathrm{B})$ denotes the length of the longest chain in $\mathrm{B} \cap X_{|w_p \times v_p| nf}$. Sending $n \to \infty$ (and recalling $z = -(a, b)$) we find

$$\begin{aligned}\lim_{n\to\infty} n^{-\frac{1}{2}} \ell_n(R_\zeta^{(n)}) &\geq c_2 \zeta \left(\left(\inf_{x \in R_\zeta^{(n)}} f(x) \right) (ab) |v_p \times w_p| \right)^{1/2} \\ &:= c_{f,p,\zeta} \zeta \sqrt{ab},\end{aligned}$$

thus,

$$u_*(x_0 - \zeta z) + c_{f,p,\zeta} \zeta \sqrt{ab} \leq u_*(x_0). \tag{53}$$

Since ψ touches u_* from below at x_0, we can transfer (53) to ψ:

$$\psi(x_0 - \zeta z) + c_{f,p,\zeta} \zeta \sqrt{ab} \leq \psi(x_0). \tag{54}$$

Dividing by ζ and taking the limit $\zeta \to 0^+$ in (54), we conclude, by continuity of f, that

$$\psi_{x_1}(x_0) a + \psi_{x_2}(x_0) b \geq \bar{c}_{f,p} \sqrt{ab} \quad \text{if } (a, b) \in (0, \infty)^2, \tag{55}$$

where $\bar{c}_{f,p} = c_2 \sqrt{f(x_0) |v_p \times w_p|}$.

Recalling the Legendre transformation from (47), (55) implies,

$$p^*(-\psi_{x_1}(x_0), -\psi_{x_2}(x_0)) = \sup \left\{ \left(-\frac{\psi_{x_1}(x_0)}{\bar{c}_{f,p}} \right) a + \left(-\frac{\psi_{x_2}(x_0)}{\bar{c}_{f,p}} \right) b + \sqrt{ab} \;\middle|\; a, b \geq 0 \right\} \leq 0.$$

Therefore, by the explicit representation of p^* in Lemma 6.1,

$$\sqrt{\psi_{x_1}(x_0)\psi_{x_2}(x_0)} \geq \frac{\bar{c}_{f,p}}{2} = \sqrt{f(x_0)|v_p \times w_p|},$$

as $c_2/2 = 1$. $\square$

7. Gradient control

We now rule out the degenerate directions. This together with Proposition 6.1 will complete the proof of Proposition 4.2. Specifically, in this section we prove the following:

Proposition 7.1. *On an event of probability 1, if $x_0 \in \mathrm{U}$, ψ is a smooth function touching u_* from below at x_0, then*

$$\bar{H}_\varphi(D\psi(x_0)) > 0.$$

We break the proof of Proposition 7.1 into several pieces. First, we show that $D\psi(x_0)$ is strictly in the dual cone if it is in one of the the dual cones $\{Q_p^*\}$ to begin with.

Lemma 7.1. *On an event of probability 1, if $x_0 \in \mathrm{U}$, $p \in \mathcal{N}^*$, ψ is a smooth function touching u_* from below at x_0, and $D\psi(x_0) \neq 0$, then*

$$\langle D\psi(x_0), v_p\rangle\langle D\psi(x_0), w_p\rangle > 0 \quad \text{if} \quad \langle D\psi(x_0), v_p\rangle\langle D\psi(x_0), w_p\rangle \geq 0.$$

Next, we show that if $D\psi(x_0)$ is non-zero, then it is certainly dual to one of the flat cones.

Lemma 7.2. *On an event of probability 1, if $x_0 \in U$, ψ is a smooth function touching u_* from below at x_0, and $D\psi(x_0) \neq 0$, then*

$$\sup_{p\in\mathcal{N}^*} \langle D\psi(x_0), v_p\rangle\langle D\psi(x_0), w_p\rangle \geq 0.$$

Finally, we show that $D\psi(x_0)$ is always non-zero.

Lemma 7.3. *On an event of probability 1, if $x_0 \in U$ and ψ is a smooth function touching u_* from below at x_0, then $D\psi(x_0) \neq 0$.*

The proof of Lemmas 7.1, 7.2, and 7.3 will be based on certain growth lemmas, which are stated and proved in the next two sections. As in Section 2.7, we will change variables so that $Q_p = [0,\infty)^2$ where it simplifies the exposition.

7.1. **Box growth.**

We start with a fundamental growth estimate for the height function in a square. In addition to proving that the height function of n random points in a square is at least order $\sqrt{n}$ at the center, the estimate implies that the limiting height function has no local minima — or, more precisely, it cannot be touched from below by a test function that is flat at the touching point.

Lemma 7.4. *Suppose $Q_p = [0,\infty)^2$ for $p \in \mathcal{N}^*$. There is a function $\rho : (0,\infty) \to (0,\infty)$ so that on an event of probability 1, if $x_0 \in \mathbb{Q}^2$ and $a \in \mathbb{Q} \cap (0,\infty)$ is chosen so that*

$$x_0 + [-a,a]^2 \subseteq \{f > f(x_0)/2\},$$

and

$$y_1^\pm \in (x_0 \pm (a,a)) \pm \mathbf{int}(Q_p) \quad \text{and} \quad y_2^\pm \in (x_0 \pm (-a,a)) \pm \mathbf{int}(\mathcal{Q}_p)$$

then for all n sufficiently large,

$$u_n(x_0) \geq \min(u_n(y_1^\pm), u_n(y_2^\pm)) + \rho(f(x_0))a\sqrt{n}.$$

Proof. We split the proof into steps. We first identify an event of full probability which we then show leads to the desired lower bound.

(Recall that since $Q_p = [0,\infty)^2$, by our convention $w_p \times v_p > 0$, $\pm\mathcal{Q}_p = \pm((-\infty,0] \times [0,\infty)])$.)

Step 1. Translate so that $x_0 = 0$ and let $\gamma = f(0)/2$. For $n \geq 1$, cover $[-a,a]^2$ by a disjoint grid of identical cubes of side length $n^{-1/2}$ where each such cube, R_z, is centered at a point $n^{-1/2}z$ for $z \in \{\mathbb{Z}^2 \cap [-m,m]^2\}$ where $m = \lceil a\sqrt{n} \rceil$.

For each $\ell \in \{1,\ldots,m\}$, let

$$A_\ell = \prod 1\{X_{nf} \cap R_{(\pm\ell,\mp\ell)} \neq \emptyset\},$$

denote the indicator of the event that four 'corner' cubes contain a point from the Poisson process. Observe that A_ℓ dominate independent Bernoulli(p) random variables where p is independent of n:

$$p^{1/4} = P(\text{Poisson}(\gamma) \geq 1) = 1 - \exp(-\gamma).$$

Therefore,

$$\Gamma_n = \sum_{\ell=1}^{m} A_\ell \tag{56}$$

dominates a Binomial with mean mp. Thus, by, say, the strong law of large numbers, on an event, Ω_{x_0}, of probability one, for all n sufficiently large $\Gamma_n \geq mp/2$.

Step 2. We next argue as in Lemma 6.3 to transfer the lower bound on Γ_n to the height functions. Let $\ell_1 \geq \cdots \geq \ell_k$ for $k = \Gamma_n$ be a sequence of indices with $A_{\ell_i} = 1$. Observe that by construction

$$R_{(\pm\ell_i,\pm\ell_i)} \subseteq x \mp \mathbf{int}(Q_p) \quad \text{and} \quad R_{(\mp\ell_i,\pm\ell_i)} \subseteq x \mp \mathbf{int}(\mathcal{Q}_p)$$

for all $x \in R_{(\pm\ell_i',\pm\ell_i')}$ with $\ell_i' > \ell_i$. Hence, since $A_{\ell_i} = 1$ for all i, there exists a list of quadruples of (random) points, $x_{\ell_i}^{\pm,\mp} \in R_{(\pm\ell_i,\mp\ell_i)} \cap X_n$ which, in view of Lemma 6.2 and dynamic programming, satisfy

$$\min_{\pm,\mp}(u_n(x_{\ell_i}^{\pm,\mp})) \geq \min_{\pm,\mp}(u_n(x_{\ell_{i+1}}^{\pm,\mp})) + 1.$$

Therefore, by induction,

$$u_n(0) \geq \min(u_n(y_1^{\pm}), u_n(y_2^{\pm})) + k,$$

and

$$k = mp/2 \geq ca\sqrt{n}(1 - \exp(-Cf(0))^4.$$

Step 3. Conclude by observing $\Omega = \bigcap_{x_0 \in \mathbb{Q}^2} \Omega_{x_0}$ has full probability. $\square$

7.2. **Planar growth.** Next, we prove a planar growth lemma that gives a lower bound on the size of the height function near points where its level set looks straight. As we will show below, this lower bound is useful in proving that, roughly speaking, the gradient of the limiting height function always misses certain directions, *i.e.*, Lemma 7.1.

Lemma 7.5. *Suppose $Q_p = [0,\infty)^2$ for $p \in \mathcal{N}^*$. There is a function $\rho : (0,\infty) \to (0,\infty)$ so that on an event of probability 1, if $x_0 \in \mathbb{Q}^2$ and $a,b \in \mathbb{Q} \cap (0,\infty)$ are chosen so that*

$$x_0 + [-a,0] \times [-b,b] \subseteq \{f > f(x_0)/2\},$$

and there are

$$y_1^+ \in (x_0 + (0,b) + \mathbf{int}(Q_p)) \quad \textit{and} \quad y_2^+ \in (x_0 + (0,-b) - \mathbf{int}(\mathcal{Q}_p))$$

and

$$s := \min\{u_n(z) \mid z \in x_0 + [-a,0] \times [-b,b]\},$$

then for all n sufficiently large,

$$u_n(x_0) \geq \min\left\{u_n(y_1^+), u_n(y_2^+), s + \rho(f(x_0))\sqrt{abn}\right\}. \tag{57}$$

Proof. Define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\zeta) = \sqrt{\frac{b}{a}}\zeta_1 + \sqrt{\frac{a}{b}}\zeta_2$$

and observe that T maps $[-a,0] \times [b,b]$ to $[-\sqrt{ab},0] \times [-\sqrt{ab},\sqrt{ab}]$ and $\det(T) = 1$. After making this transformation, the rest of the proof is almost identical to that of Lemma 7.4, the only change is that the growth bound is in one direction. Thus, we only sketch it.

Step 1. Set $d = \sqrt{ab}$, translate so that $x_0 = 0$, and let $\gamma = f(0)/2$. For $n \geq 1$, cover $[-d,0] \times [-d,d]$ by a disjoint grid of identical cubes of side length $n^{-1/2}$ where each, R_z, is centered at a point $n^{-1/2}z$ for $z \in \{\mathbb{Z}^2 \cap [-m,0] \times [-m,m]\}$ where $m = \lceil d\sqrt{n} \rceil$.

For each $\ell \in \{1,\ldots,m\}$, let

$$A_\ell = \prod 1\{X_{nf} \cap R_{(-\ell,\pm\ell)} \neq \emptyset\},$$

denote the indicator of the event that two corners contain a point from the Poisson process. By the domination argument in Lemma 7.4, on an event Ω_{x_0}, of probability one, for all n sufficiently large, $\Gamma_n \geq \rho(f(x_0))\sqrt{abn}$.

Step 2. Following the argument in Lemma 7.4, there exists a sequence of indices $\ell_1 \geq \cdots \geq \ell_{\Gamma_n}$ and $x_{\ell_i}^{-,\pm} \in R_{(-\ell_i,\pm\ell_i)} \cap X_n$ which satisfy, by dynamic programming,

$$\min_{\pm}(u_n(x_{\ell_i}^{-,\pm})) \geq \min(u_n(y_1^+), u_n(y_2^+), \min_{\pm}(u_n(x_{\ell_{i+1}}^{-,\pm})) + 1).$$

We iterate to conclude. □

7.3. **Proof of Lemma 7.1.** By an affine transformation, using Corollary 2.3, we may assume $Q_p = [0,\infty)^2$ so that $\mathcal{Q}_p = (-\infty,0] \times [0,\infty)$ and $\min(\psi_{x_1}(x_0), \psi_{x_2}(x_0)) \geq 0$. Suppose, for sake of contradiction, that $\psi_{x_2}(x_0) = 0$. Let $t = u_*(x_0) = \psi(x_0)$.

Since $D\psi(x_0) \neq 0$, we must have $\psi_{x_1}(x_0) > 0$. This means, by Taylor approximation, there are positive constants ϵ_0 and c such that

$$\psi(x) \geq t - c\epsilon^2 \quad \text{for } x \in \{x_0 \pm 2R_\epsilon\}$$

and $\epsilon \in (0,\epsilon_0)$ where $R_\epsilon = [-\epsilon^2,0] \times [-\epsilon,\epsilon]$. Given such ϵ, pick rational points $\{x_i^\pm\} \in (x_0 + \{R_\epsilon \cup -R_\epsilon\})$ such that

$$x_1^\pm \in \bigcap_{x_0' \in B(x_0,\delta)} (x_0' \pm \mathbf{int}(Q_p)) \quad \text{and} \quad x_2^\pm \in \bigcap_{x_0' \in B(x_0,\delta)} (x_0' \pm \mathbf{int}(\mathcal{Q}_p)),$$

where $\delta = \min(\epsilon^2/4, \epsilon/4)$. Also, let

$$y_1^+ = x_0 + (2\epsilon^2, 0) + (0, 4\epsilon) \quad \text{and} \quad y_2^+ = x_0 + (2\epsilon^2, 0) - (0, 4\epsilon)$$

and observe that $y_1^+ \in (x_i^\pm + (0,\epsilon) + \mathbf{int}(Q_p))$ and $y_2^+ \in (x_i^\pm + (0,-\epsilon) - \mathbf{int}(\mathcal{Q}_p))$ for $i = 1, 2$. Furthermore, making ϵ_0 smaller if necessary, we have

$$\min(\psi(y_1^+), \psi(y_2^+)) \geq t + c\epsilon^2.$$

Since ψ touches u_* below, we deduce that for all n sufficiently large

$$\begin{gathered}\min(u_n(y_1^+), u_n(y_2^+)) \geq (t + c\epsilon^2)n^{1/2}, \\ u_n(x) \geq (t - c\epsilon^2)n^{1/2} \quad \text{for } x \in \{x_0 \pm 2R_\epsilon\}.\end{gathered}$$

Therefore, by Lemma 7.5 and Lemma 6.2, again taking ϵ_0 smaller if necessary so that $x_0 \pm 2R_\epsilon \subseteq \{f > f(x_0)/2\}$,

$$\min_{x_0' \in B(x_0,\delta)} u_n(x_0') \geq \min\left[(t - c\epsilon^2)n^{1/2} + (C\epsilon^{3/2}\rho(f(x_0)))n^{1/2}, (t + c\epsilon^2)n^{1/2}\right].$$

Sending $n \to \infty$, this implies,

$$t = u_*(x_0) \geq \min(t - c\epsilon^2 + C\epsilon^{3/2}\rho(f(x_0)), t + c\epsilon^2),$$

a contradiction for ϵ sufficiently small as $\rho(f(x_0)) > 0$.

7.4. **Proof of Lemma 7.3.** The argument is similar to the proof of Lemma 7.1 so we only sketch it.

We argue by contradiction. Again, by an affine transformation, using Corollary 2.3, we may assume $Q_p = [0,\infty)^2$ so that $\mathcal{Q}_p = [0,\infty) \times (-\infty, 0]$ and $\psi_{x_1}(x_0) = \psi_{x_2}(x_0) = 0$. Let $t = u_*(x_0) = \psi(x_0)$. By Taylor approximation, there are positive constants ϵ_0 and c such that

$$\psi(x) \geq t - c\epsilon^2 \quad \text{for } x \in \{x_0 + R_\epsilon\}$$

and $\epsilon \in (0, \epsilon_0)$ where $R_\epsilon = [-\epsilon, \epsilon]^2$. Given such ϵ, let

$$y_1^\pm = x_0 \pm (2\epsilon, 2\epsilon) \quad \text{and} \quad y_2^\pm = x_0 \pm (2\epsilon, -2\epsilon)$$

and observe that

$$y_1^\pm \in ((x_0 \pm (\epsilon, \epsilon)) \pm \mathbf{int}(Q_p)) \quad \text{and} \quad y_2^\pm \in ((x_0 \pm (-\epsilon, \epsilon)) \pm \mathbf{int}(\mathcal{Q}_p)).$$

Making ϵ sufficiently small, exactly as in the proof of Lemma 7.1, we deduce that for all n large

$$u_n(x) \geq (t - c\epsilon^2)n^{1/2} \quad \text{for } x \in \{x_0 + R_\epsilon\},$$

and hence by Lemma 7.4,

$$\min_{x_0' \in B(x_0,c\epsilon)} u_n(x_0') \geq (t - c\epsilon^2 + C\epsilon\rho(f(x_0)))n^{1/2},$$

which is a contradiction.

7.5. **Proof of Lemma 7.2.** This is the only part of the paper where we use the finiteness of $\mathcal{N}^*$. The approach is similar to Lemma 7.1, except that more work is needed to satisfy the flat cone constraints.

To start with, it will be helpful to introduce some notation. First, let $\tilde{\mathcal{E}}$ be the set $\tilde{\mathcal{E}} = \mathcal{E} \setminus \cup_{p \in \mathcal{N}^*} \mathcal{Q}_p$. Since $\mathcal{N}^*$ is finite, $\mathbf{cone}(\tilde{\mathcal{E}})$ is open in $\mathbb{R}^2$.

It will be convenient to define $\tilde{\mathcal{E}}^\perp$ by

$$\tilde{\mathcal{E}}^\perp = \{v^\perp \mid v \in \mathbf{cone}(\tilde{\mathcal{E}})\}.$$

Notice that $\tilde{\mathcal{E}}^\perp$ is open in $\mathbb{R}^2$. With this notation, Lemma 7.2 boils down to proving that $D\psi(x_0) \notin \tilde{\mathcal{E}}^\perp$.

We will frequently use certain cones associated to the basis $(e, e^\perp)$. Given $e \in S^1$ and $\mu_1, \mu_2 \in \mathbb{R}$, these are defined as follows:

$$Q^e_{NE} = \{x \in \mathbb{R}^2 \mid \langle x, e\rangle \geq 0,\ \langle x, e^\perp\rangle \leq 0\}, \quad Q^e_{NW} = \{x \in \mathbb{R}^2 \mid \langle x, e\rangle \geq 0,\ \langle x, e^\perp\rangle \geq 0\},$$
$$Q^e_{SE} = \{x \in \mathbb{R}^2 \mid \langle x, e\rangle \leq 0,\ \langle x, e^\perp\rangle \leq 0\}, \quad Q^e_{SW} = \{x \in \mathbb{R}^2 \mid \langle x, e\rangle \leq 0,\ \langle x, e^\perp\rangle \geq 0\},$$
$$D^e(\mu_1, \mu_2) = \{\cos(\theta)e + \sin(\theta)e^\perp \mid \theta \in [\mu_1, \mu_2]\}.$$

When $e \in S^1 \cap \tilde{\mathcal{E}}^\perp$, it is convenient to introduce the set $\mathcal{N}^*_e \subseteq \mathcal{N}^*$ determined by

$$\mathcal{N}^*_e = \{p \in \mathcal{N}^* \mid \mathcal{Q}_p \subseteq \mathbf{int}(H_e)\}.$$

The reason this is useful is, by the definition of $\tilde{\mathcal{E}}^\perp$, it gives a natural partition of $\mathcal{N}^*$:

$$\mathcal{N}^* = \mathcal{N}^*_e \cup (-\mathcal{N}^*_e). \tag{58}$$

We will use the following lemma, which follows from the assumption that $\mathcal{N}^*$ is finite.

Lemma 7.6. *There is a continuous function $\mu : S^1 \cap \tilde{\mathcal{E}}^\perp \to (0, \pi/2)$ such that if $e \in S^1 \cap \tilde{\mathcal{E}}^\perp$, then*

$$D^e(\mu(e), \pi/2) \subseteq -\mathbf{int}(Q_p) \quad \text{and} \quad D^e(-\pi/2, -\mu(e)) \subseteq \mathbf{int}(Q_p) \quad \text{for each } p \in \mathcal{N}^*_e.$$

Proof. Given $p \in \mathcal{N}^*_e$, we know that $\mathcal{Q}_p \subseteq \mathbf{int}(H_e)$. Hence there is a minimal angle $0 < \mu_e(p) < \pi/2$ such that

$$\mathcal{Q}_p \subseteq D^e(-\mu_e(p), \mu)_e(p)).$$

From this, taking complements, we deduce that

$$D^e(\mu_e(p), \pi/2) \subseteq -Q_p \quad \text{and} \quad D^e(-\pi/2, -\mu_e(p)) \subseteq Q_p.$$

Since $\mathcal{N}^*$ is finite, we conclude upon setting $\mu(e) = \max\{\mu_e(p) \mid p \in \mathcal{N}^*_e\}$. This function is continuous since $e \mapsto \mathcal{N}^*_e$ is locally constant in $S^1 \cap \tilde{\mathcal{E}}^\perp$ and the functions $e \mapsto \mu_e(p)$ are all continuous. □

The next elementary geometric lemma is fundamental to the proofs that follow.

Lemma 7.7. *If $e \in S^1$ and there are points $y_{NE}, y_{NW}, y_{SE}, y_{SW} \in \mathbb{R}^2$ satisfying*

$$y_\bullet \in \mathbf{int}(Q^e_\bullet) \quad \text{for each } \bullet \in \{NE, NW, SE, SW\},$$

then

$$0 \in \mathbf{int}(\mathbf{conv}(y_{NE}, y_{NW}, y_{SE}, y_{SW})).$$

Proof. This is clear pictorially and can be proved using halfplane separation theorems for convex sets. Alternatively, one can argue as in Lemma 6.2 using the norm

$$\varphi_e(q) = \max(|\langle q, e\rangle|, |\langle q, e^\perp\rangle|)$$

and invoke Corollary 2.1. $\square$

Finally, define rectangles $Q^e_-(a,b)$ by

$$Q^e_-(a,b) = \{te + se^\perp \mid -a \leq t \leq 0,\ |s| \leq b\}.$$

In order to avoid measurability issues, henceforth fix a countable dense subset $\mathcal{D} \subseteq S^1 \cap \tilde{\mathcal{E}}^\perp$.

We will use the following planar growth lemma, which is similar to Lemma 7.5.

Lemma 7.8. *Let $\mu : S^1 \cap \tilde{\mathcal{E}}^\perp \to (0, \pi/2)$ be the function defined in Lemma 7.6. There is a function $\rho : (0,\infty) \to (0,\infty)$ so that on an event of probability 1, if $e \in \mathcal{D}$, $x_0 \in \mathbb{Q}^2$, $a, b \in \mathbb{Q} \cap (0,\infty)$ are chosen so that*

$$x_0 + Q^e_-(a,b) \subseteq \{f > f(x_0)/2\},$$

$y_{NE}, y_{NW} \in \mathbb{R}^2$ are points satisfying

$$x_0 + Q^e_-(a,b) \subseteq (y_{NW} - D^e(\mu(e), \pi/2)) \cap (y_{NE} - D^e(-\pi/2, -\mu(e)))$$

and

$$s := \min\left\{u_n(z) \mid z \in x_0 + Q^e_-(a,b)\right\},$$

then for all n sufficiently large,

$$u_n(x_0) \geq \min\left\{u_n(y_{NE}), u_n(y_{NW}), s + \rho(f(x_0))\sqrt{abn}\right\}. \tag{59}$$

Proof. The argument is similar in spirit to that of Lemma 7.5 so we only sketch the proof. As in that lemma, for all n large enough, there is a random integer $L(n) \geq \rho(f(x_0))\sqrt{2abn}$ and a random subset $\{x_1, \ldots, x_{2L}\} \subseteq X_n \cap (x_0 + Q^e_-(a,b))$ such that

$$x_0 \in (x_1 - \mathbf{int}(Q^e_{SW})) \cap (x_2 - \mathbf{int}(Q^e_{SE})),$$
$$\{x_{2i-1}, x_{2i}\} \subseteq (x_{2i+1} - \mathbf{int}(Q^e_{SW})) \cap (x_{2(i+1)} - \mathbf{int}(Q^e_{SE})) \quad \text{for each } i \in \{1, 2, \ldots, L(n) - 1\}.$$

(Compared to Lemma 7.5, all that is different is $[0,\infty)^2$ is replaced by Q^e_{NE}.)

We now invoke the dynamic programming principle. The key point is, for each $i \in \{1, 2, \ldots, L(n)\}$,

$$y_{NW} \in (x_{2i-1} + D^e(\mu(e), \pi/2)) \cap (x_{2i} + D^e(\mu(e), \pi/2)),$$
$$y_{NE} \in (x_{2i-1} + D^e(-\pi/2, -\mu(e)) \cap (x_{2i} + D^e(-\pi/2, -\mu(e))).$$

Hence, by (58) and Lemma 7.6,

$$\{y_{NW}, y_{NE}\} \cap (x_j + \mathbf{int}(Q_p)) \neq \emptyset \quad \text{for each } j \in \{1, \ldots, 2L\},\ p \in \mathcal{N}^*.$$

At the same time, by Lemma 7.7,

$$x_{2i-1}, x_{2i} \in \mathbf{int}(\mathbf{conv}(y_{NE}, y_{NW}, x_{2i+1}, x_{2(i+1)})) \quad \text{for each } i \in \{1, 2, \ldots, L(n) - 1\}.$$

Therefore, Corollary 2.1 implies that

$$x_{2i-1}, x_{2i} \in \mathcal{P}(y_{NE}, y_{NW}, x_{2i+1}, x_{2(i+1)}),$$

which, in terms of the height function, gives

$$u_n(x_{2i-1}), u_n(x_{2i}) \geq \min(u_n(y_{NE}), u_n(y_{NW}), u_n(x_{2i+1}) + 1, u_n(x_{2(i+1)}) + 1). \tag{60}$$

We conclude by iterating the bounds in (60). $\square$

Finally, we are in a position to prove that $D\psi(x_0) \notin \tilde{\mathcal{E}}^{\perp}$.

Proof of Lemma 7.2. We assume that

$$\sup_{p \in \mathcal{N}^*} \langle D\psi(x_0), v_p \rangle \langle D\psi(x_0), w_p \rangle < 0$$

and derive a contradiction. Note that this assumption implies $D\psi(x_0)^{\perp} \notin \cup_{p \in \mathcal{N}^*} \mathcal{Q}_p$ and hence, by definition, $D\psi(x_0) \in \tilde{\mathcal{E}}^{\perp}$.

Define $e = \|D\psi(x_0)\|^{-1} D\psi(x_0)$, $t = u_*(x_0) = \psi(x_0)$, and fix a (random) sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x_0$ and $u_n(x_n) \to t$. Note that $e \in S^1 \cap \tilde{\mathcal{E}}^{\perp}$ and, thus, can be approximated by points in $\mathcal{D}$.

Let $\bar{\mu} = \frac{1}{2}(\frac{\pi}{2} + \mu(e))$, where μ is the function from Lemma 7.6. In what follows, for $\epsilon > 0$, let $x^{\epsilon}_{NE}, x^{\epsilon}_{NW} \in \mathbb{R}^2$ be the points defined by

$$x^{\epsilon}_{NE} = x_0 + \epsilon \cos(\bar{\mu}) e - \epsilon \sin(\bar{\mu}) e^{\perp}, \quad x^{\epsilon}_{NW} = x_0 + \epsilon \cos(\bar{\mu}) e + \epsilon \sin(\bar{\mu}) e^{\perp}.$$

Note that $x^{\epsilon}_{NW} \in x_0 + \mathbf{int}(D^e(\mu(e), \pi/2)$ and $x^{\epsilon}_{NE} \in x_0 + \mathbf{int}(D^e(-\pi/2, -\mu(e))$.

A contradiction argument shows that are constants $\nu, \epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$,

$$x_0 + Q^e_-(\epsilon^2, \nu\epsilon) \subseteq (x^{\epsilon}_{NW} - \mathbf{int}(D^e(\mu(e), \pi/2))) \cap (x^{\epsilon}_{NE} - \mathbf{int}(D^e(-\pi/2, -\mu(e)))).$$

Note, furthermore, that there is a constant $c > 0$ such that

$$\psi(x^{\epsilon}_{NE}), \psi(x^{\epsilon}_{NW}) \geq t + c\epsilon,$$
$$\min\left\{\psi(y) \;\middle|\; y \in Q^e_-(\epsilon^2, \nu\epsilon)\right\} \geq t - c\epsilon^2,$$

and, thus, for small enough ϵ,

$$u_*(x^{\epsilon}_{NE}), u_*(x^{\epsilon}_{NW}) \geq t + c\epsilon,$$
$$\min\left\{u_*(y) \;\middle|\; y \in Q^e_-(\epsilon^2, \nu\epsilon)\right\} \geq t - c\epsilon^2.$$

Given ϵ, we can choose $e_\epsilon \in \mathcal{D}$ close to e and points $x^{\epsilon}_{SE}, x^{\epsilon}_{SW} \in \mathbb{Q}^2$ such that

$$x^{\epsilon}_{SE} \in x_0 + \mathbf{int}(Q^e_{SE}), \quad x^{\epsilon}_{SW} \in x_0 + \mathbf{int}(Q^e_{SW}),$$
$$\{x^{\epsilon}_{SE}, x^{\epsilon}_{SW}\} + Q^{e_\epsilon}_-(\epsilon^2/4, \nu\epsilon/2) \subseteq x_0 + Q^e_-(\epsilon^2, \nu\epsilon).$$

Furthermore, since μ is continuous and $\mu(e) < \bar{\mu} < \pi/2$, e_ϵ can be chosen in such a way that

$$x_0 + Q^e_-(\epsilon^2, \nu\epsilon) \subseteq (x^{\epsilon}_{NW} - \mathbf{int}(D^{e_\epsilon}(\mu(e_\epsilon), \pi/2))) \cap (x^{\epsilon}_{NE} - \mathbf{int}(D^{e_\epsilon}(-\pi/2, -\mu(e_\epsilon)))).$$

Finally, invoking Lemma 7.8 for a fixed ϵ, we deduce that

$$u_n(x^{\epsilon}_{SE}), u_n(x^{\epsilon}_{SW}) \geq \min\left(t + c\epsilon, t - c\epsilon^2 + \rho(f(x_0))\sqrt{\nu\epsilon^3/8}\right) n^{\frac{1}{2}}.$$

If n is large enough, then, since $x_n \to x_0$, we have

$$x^{\epsilon}_{\bullet} \in x_n + Q^e_{\bullet} \quad \text{for } \bullet \in \{SE, SW\}.$$

Therefore, by the same argument as in Lemma 7.8, we deduce that

$$x_n \in \mathcal{P}(x^{\epsilon}_{NE}, x^{\epsilon}_{NW}, x^{\epsilon}_{SE}, x^{\epsilon}_{SW})$$

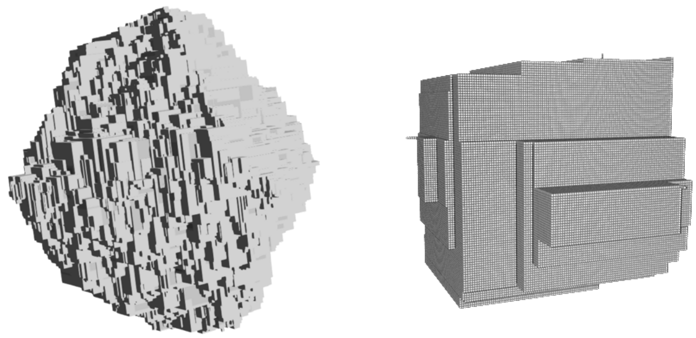

FIGURE 6. Snapshots of ℓ^∞ (left) and ℓ^1 (right) Pareto peeling of Poisson points in a cube in 3D.

and

$$n^{-\frac{1}{2}}u_n(x_n) \geq \min\left(t + c\epsilon, t - c\epsilon^2 + \rho(f(x_0))\sqrt{\nu\epsilon^3/8}\right) = t - c\epsilon^2 + \rho(f(x_0))\sqrt{\nu\epsilon^3/8}.$$

In the limit $n \to \infty$, we find $t \geq t - c\epsilon^2 + \rho(f(x_0))\sqrt{\nu\epsilon^3/8}$ for all $\epsilon \in (0, \epsilon_0)$, which is absurd. $\square$

8. Further remarks and some open problems

8.1. **Higher dimensions.** A natural followup is to analyze the behavior of Pareto peeling in $\mathbb{R}^d$ for $d > 2$. In higher dimensions, only much weaker versions of Theorem 2.1 are available [DM86] and we expect this reflects new phenomena that occur in higher dimensions. On the one hand, when $\varphi(\cdot) = \|\cdot\|_\infty$, the situation is similar to the two dimensional case. The family of cones that describe the Pareto hull are rotated quadrants,

$$Q_k^\pm = \{x \in \mathbb{R}^d \mid \pm x_k = \|x\|_\infty\}, \tag{61}$$

for $k = 1, \ldots, d$. Equivalently, Q_k are cones generated by 0 and facets of the cube $[-1, 1]^d$. In this case, it is straightforward to extend the above arguments to prove the following.

Theorem 8.1. *If X_n are Poisson point processes in* U*, convex, bounded, and open, with intensities n and $\varphi(x) = \|x\|_\infty$ then, on an event of probability 1, the sequence of rescaled height functions $n^{-1/d}u_n := \bar{u}_n \to \bar{u}$, where $\bar{u}$ is the unique viscosity solution to the PDE*

$$\begin{cases} \max_k \left(\prod_j \langle D\bar{u}, v_{k,j}\rangle\right) = c_d & \text{in } \mathrm{U} \\ \bar{u} = 0 & \text{on } \partial\mathrm{U}, \end{cases}$$

$\{v_{k,j}\}$ range over the extremal directions of $Q_k^\pm$, and $c_d > 0$ is a finite constant.

On the other hand, cones with more complex geometries are also possible in higher dimensions. For example, when $\varphi(\cdot) = \|\cdot\|_1$ in three dimensions, the dynamic programming principle becomes

$$u_A(x) = \inf_{C_\delta} \sup_{z \in x - C_\delta} (u_A(z) + 1_A(z)) \tag{62}$$

where δ ranges over $\{(\pm 1, \mp 1, 0), (\pm 1, 0, \mp 1), (0, \pm 1, \mp 1)\}$ and if, say, $\delta = (1, 1, 0)$ then $C_\delta = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Importantly, C_δ are convex, but not pointed. However, they are pointed in one dimension lower which leads to the inequality

$$u_A(x_1,x_2,x_3) \le \min(h_A^1(x_1,x_2), h_A^2(x_1,x_3), h_A^3(x_2,x_3))$$

where h_A^i is ℓ^1-Pareto peeling in two dimensions and A is projected from $\mathbb{R}^3$ to $\mathbb{R}^2$ in the indicated way. These considerations suggest the following.

Conjecture 8.1. *Under the same assumptions as Theorem 8.1, in dimension 3, when $\varphi = \|\cdot\|_1$, $n^{-1/2}u_n$ almost surely converges to $\bar{h}$ locally uniformly where $\bar{h}$ is the unique viscosity solution to*

$$\begin{cases} \max(|\bar{h}_{x_1}\bar{h}_{x_2}|, |\bar{h}_{x_1}\bar{h}_{x_3}|, |\bar{h}_{x_2}\bar{h}_{x_3}|) = 1 & \text{on } \mathrm{U} \\ \bar{h} = 0 & \text{on } \partial\mathrm{U} \end{cases}$$

Of note, this conjecture suggests different scalings for ℓ^1 and ℓ^∞-Pareto peeling in dimensions higher than 2.

8.2. **Other versions of peeling.** In this article, we have only considered one particularly convenient notion of Pareto efficiency. Our definition of Pareto hull corresponds to what is known in location analysis as a *strictly efficient set* but there are also efficient and weakly efficient sets, which we now discuss.

Consider $\mathbb{R}^d$ equipped with a norm $\varphi(\cdot)$ and let $A \subset \mathbb{R}^d$ and denote by $B_{a_i}(x)$ the closed ball of radius $\varphi(x - a_i)$ centered at a_i.

(1) The set of *efficient* points with respect to A is

$$\begin{aligned} E(A) = &\{x \in X \mid \forall y \neq x, (\exists a \in A, \varphi(a-x) < \varphi(a-y)) \text{ or} \\ &(\forall a \in A, \varphi(a-x) \le \varphi(a-y)). \end{aligned}$$

(2) The set of *strictly efficient* points with respect to A is

$$\begin{aligned} c(A) = \{&x \in \mathbb{R}^d \mid \forall y \neq x \\ &\text{there exists } a \in A \text{ with } \varphi(a-x) < \varphi(a-y)\} \end{aligned}$$

or equivalently $x \in c(A)$ if and only if $\bigcap_{i=1}^n B_{a_i}(x) = \{x\}$.

(3) The set of *weakly efficient* points with respect to A is

$$\begin{aligned} C(A) = \{&x \in \mathbb{R}^d \mid \forall y \neq x, \\ &\text{there exists } a \in A \text{ with } \varphi(a-x) \le \varphi(a-y)\} \end{aligned}$$

or equivalently $x \in C(A)$ if and only if $\bigcap_{i=1}^n \operatorname{int} B_{a_i}(x) = \emptyset$.

The definitions imply $A \subseteq c(A) \subseteq E(A) \subseteq C(A)$. Moreover, one may check that if $A \subseteq B$ then, $c(A) \subseteq c(B)$ and $C(A) \subseteq C(B)$. However, counterexamples demonstrate $A \subseteq B$ with $E(A) \not\subset E(B)$ — see [DM86]. The monotonicity of strictly and weakly efficient sets suggest both enjoy scaling limits in general; however, it is not clear how to use these to tightly bound $E(A)$. Indeed, weakly and strictly efficient peeling may have different scalings as indicated in Section 8.3.

On the other hand, if $\varphi(\cdot)$ is the norm induced by an inner product or its unit ball is strictly convex and $d = 2$, then $c(A) = C(A) = E(A) = \mathbf{conv}(A)$. Interestingly, Durier-Michelot have an example of a strictly convex norm ball in $d = 3$ where $\mathbf{conv}(A) \not\subset c(A)$ — see Section 4.3 of [DM86].

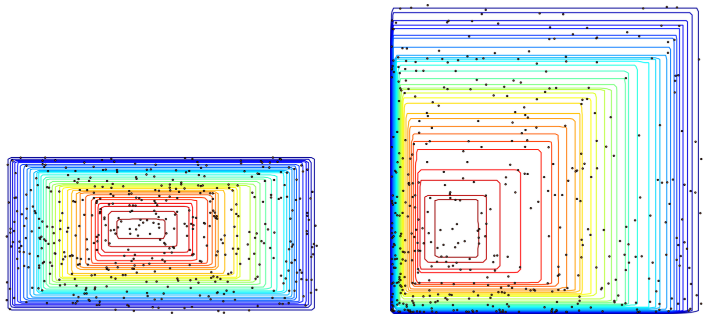

FIGURE 7. Level sets of weakly-efficient ℓ^1-peeling for Poisson points in rectangular domains

8.3. **Weakly efficient peeling in two dimensions and higher.** In two dimensions, weakly efficient Pareto hulls are simpler to analyze than strictly efficient sets but the analogous height functions appear to have a different scaling. Specifically, for a finite set of points $X \subseteq \mathbb{R}^2$ and a norm $\varphi(\cdot)$, denote the *weak Pareto hull* by

$$W_1(A) = C(A) \quad \text{and} \quad W_{n+1}(A) = C(A \cap \mathbf{int}(W_n(A))) \tag{63}$$

and the height function by

$$h_X = \sum_{n\geq 1} 1_{\mathbf{int}(W_n(A))}.$$

Weak Pareto hulls have a simpler inclusion constraint than strictly efficient sets — see Theorem 4.3 of [DM86] and Theorem 3 of [PF89]. For example, when $\varphi(\cdot) = |\cdot\|_1$ the height function satisfies the dynamic programming principle,

$$h_X(x) = \inf_{C_i} \sup_{z\in x-C_i} (h_X(z) + 1_X(z))$$

where

$$C_i = \{x \in \mathbb{R}^2 \mid \langle x, \zeta_i\rangle \geq 0\}$$

for

$$\zeta_1 = e_1 \quad \zeta_2 = -e_1 \quad \zeta_3 = e_2 \quad \zeta_4 = -e_2$$

where e_1, e_2 is the standard basis of $\mathbb{R}^2$. Equivalently $C(X)$ is the bounding rectangle of X,

$$\mathbf{br}(X) := \{z \in \mathbb{R}^2 \mid \ m \leq z \leq M\}, \tag{64}$$

where $m_i := \min_{z\in X} z_i$ and $M_i := \max_{z\in X} z_i$ for $i = 1, 2$ and the vector inequalities are pointwise. A straightforward analysis, essentially counting Poisson points, leads to the following.

Example. *Let $\varphi(\cdot) = \|\cdot\|_1$. If X_n are Poisson point processes in $[-1, 1]^2$ with intensities n, then, almost surely, the sequence of rescaled height functions $n^{-1}h_n$ converges to $\bar{h}$ where*

$$\bar{h}(x) = \min(1 - x_1^2, 1 - x_2^2). \tag{65}$$

One can also check that if $f_i : \mathbb{R}^2 \to \mathbb{R}$ are continuous and invertible, the map

$$F(x_1, x_2) = (f_1(x_1), f_2(x_2)) \tag{66}$$

'preserves bounding rectangles'. That is if $(x, y) \in \mathbf{br}(A)$ then $F(x, y) \in \mathbf{br}(F(A))$. This can be used to extend Example 8.3 to rectangular domains — see Figure 7. However, it is not clear if there is a simple description of the limit when the Poisson intensity is not strictly positive in $\mathbf{br}(A)$.

The situation in higher dimensions again appears to be even more difficult — the cones describing weakly efficient sets may not always be convex — see Example 2 in Section 4.2 of [DM86].

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