Superdiffusion for Brownian motion with random drift

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Joint work with Scott Armstrong (Courant Institute) and Tuomo Kuusi (University of Helsinki)

Probabilistic Field Theories, Aalto University, Finland

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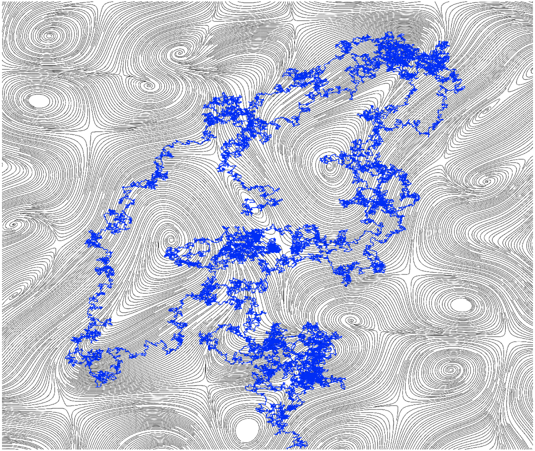
Long time behavior of Brownian motion with isotropic random drift



Brownian motion X_t with an isotropic "random" drift $\mathrm{f} : \mathbb{R}^d \to \mathbb{R}^d$

$$\boxed{dX_t = \mathrm{f}(X_t)\, dt + dW_t}$$

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    - solenoidal: $\operatorname{div} \mathrm{f} = 0$, *superdiffusive*

      $$\mathsf{E}^0[|X_t|^2] \sim t\sqrt{\log t}$$

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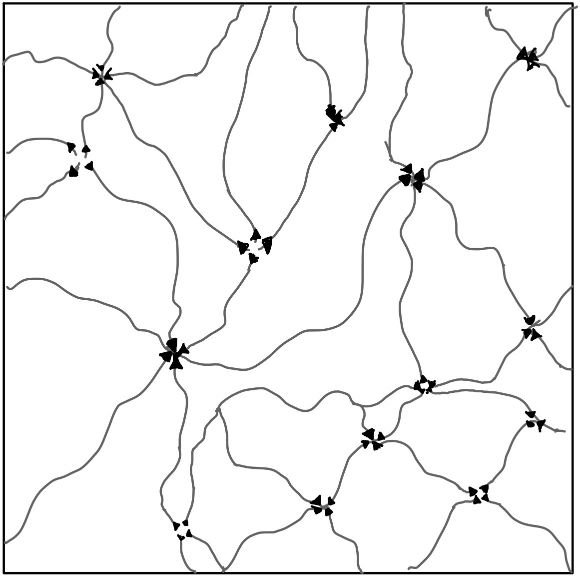
Potential



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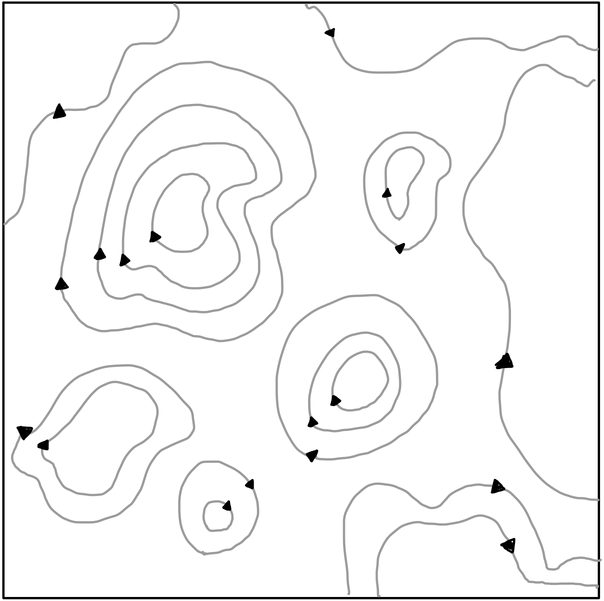

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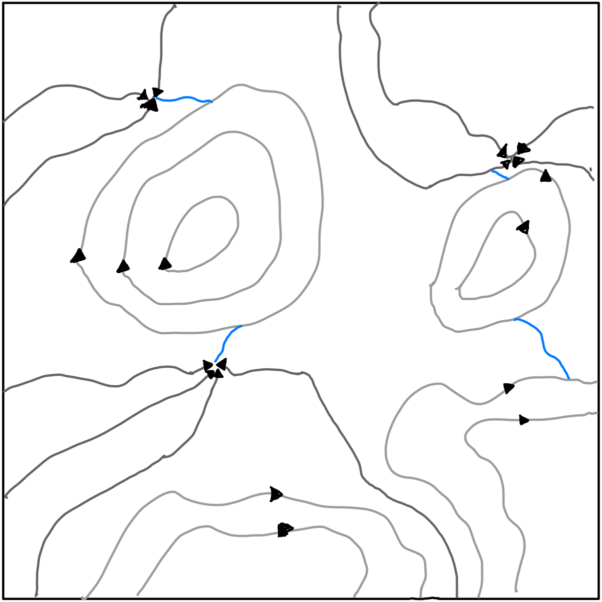

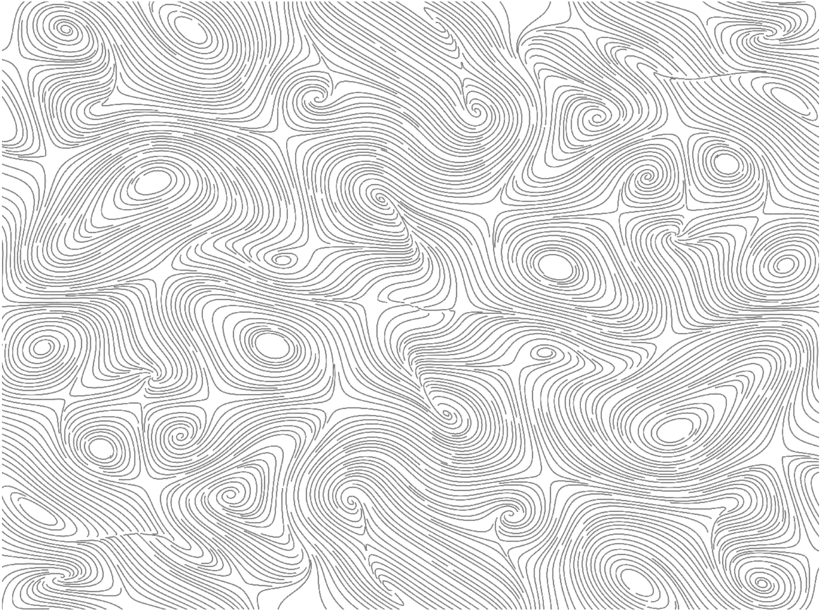

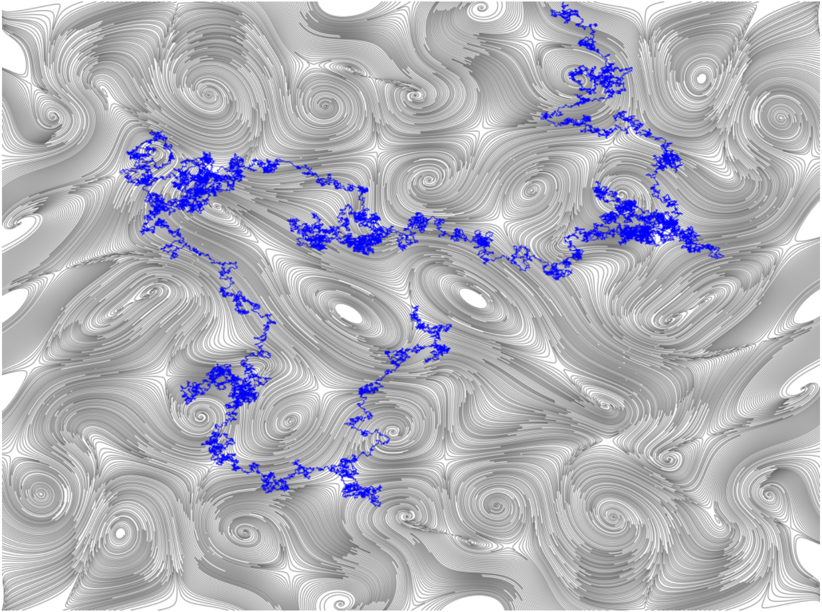

Divergence-drift — passive tracer

$$dX_t = \mathsf{f}(X_t)dt + \sqrt{2\nu}dW_t$$

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  - critical — barely superdiffusive, $\mathrm{E}^0[|X_t|^2] \sim t\sqrt{\log t}$

    $$\mathrm{cov}[f(x), f(y)] \simeq |x-y|^{-2}$$

    this talk

Critical divergence-free drift in $d = 2$ — past work

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- above results are annealed: $\mathbb{E}[\mathrm{E}^0[\cdot]]$

Quenched superdiffusive invariance principle in two dimensions

Theorem (Armstrong, B., Kuusi 2024)

$\mathbb{P}$*-almost surely,*

$$|\log \varepsilon^2|^{-1/4} \varepsilon X_{t/\varepsilon^2} \Rightarrow (2\pi)^{-1/2} W_t \quad \textit{as } \varepsilon \to 0\,,$$

where $\{W_t\}$ *is a standard, two-dimensional Brownian motion and*

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- special case of a much more general result in $\mathbb{R}^d$

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- thus,

$$\mathfrak{L}u = \nabla\cdot(\nu\mathrm{I}_d + \mathsf{k})\nabla u$$

  is a pure divergence form operator, and, in our case k is the two-dimensional GFF mollified at the unit scale (curl of the GFF)

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- in our case:

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is equivalent to

$$\text{``}\mathfrak{L}^\varepsilon \to \Delta''$$

where

$$\mathfrak{L}^\varepsilon = \left((4\sqrt{2}\pi)^{-2}|\log \varepsilon|\right)^{-1/2}\nabla \cdot \left(\nu \mathrm{I}_d + \mathsf{k}^\varepsilon\right)\nabla\,, \quad \text{for} \quad \mathsf{k}^\varepsilon(x) := \mathsf{k}\left(x/\varepsilon\right)$$

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- goal: show convergence of the *effective diffusivities*, $\bar{\mathsf{s}}_n$

A recurrence for the effective diffusivities

- assuming no interaction between the waves,

$$\begin{aligned}\mathfrak{L}_{n+k} &\approx \nabla \cdot \Big(\bar{\mathsf{s}}_n \mathrm{I}_d + (\mathsf{j}_{n+1} + \cdots + \mathsf{j}_{n+k})\Big)\nabla \\ &= \bar{\mathsf{s}}_n \nabla \cdot \Big(\mathrm{I}_d + \bar{\mathsf{s}}_n^{-1}(\mathsf{j}_{n+1} + \cdots + \mathsf{j}_{n+k})\Big)\nabla\end{aligned}$$

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  $\nabla \cdot \big(\mathrm{I}_d + \boldsymbol{\delta}\big) \nabla \approx \overline{\mathsf{s}} \Delta$, where $\overline{\mathsf{s}} = 1 + \underbrace{\mathbb{E}\big[|\nabla \Delta^{-1} \nabla \cdot \boldsymbol{\delta} e_1|^2(0) \big]}_{kc_* \log 3} + O\big(\|\boldsymbol{\delta}\|^4 \big)$

- we deduce that

$$\overline{\mathsf{s}}_{n+k} = \overline{\mathsf{s}}_n \big(1 + (kc_* \log 3) \overline{\mathsf{s}}_n^{-2} \big) + O(\overline{\mathsf{s}}_n^{-1}),$$

  and so $\frac{\overline{\mathsf{s}}_n}{\sqrt{2c_* \log 3^n}} \to 1$

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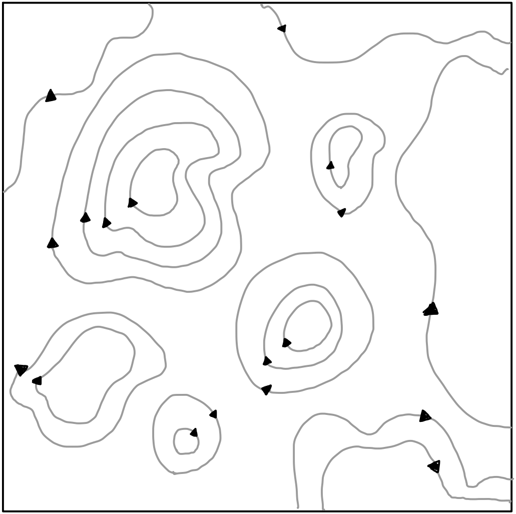

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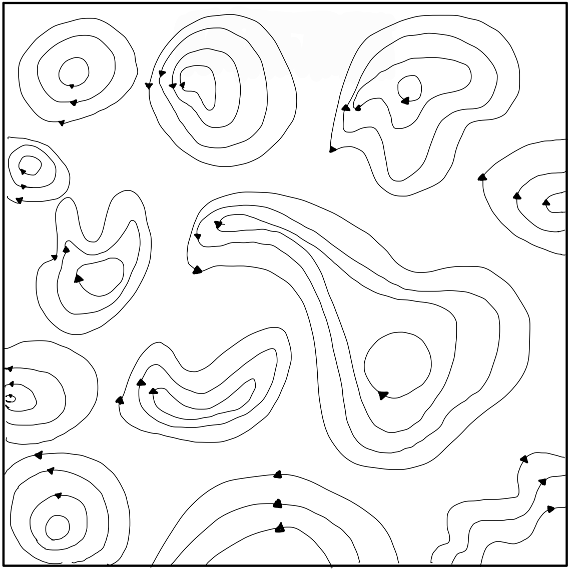

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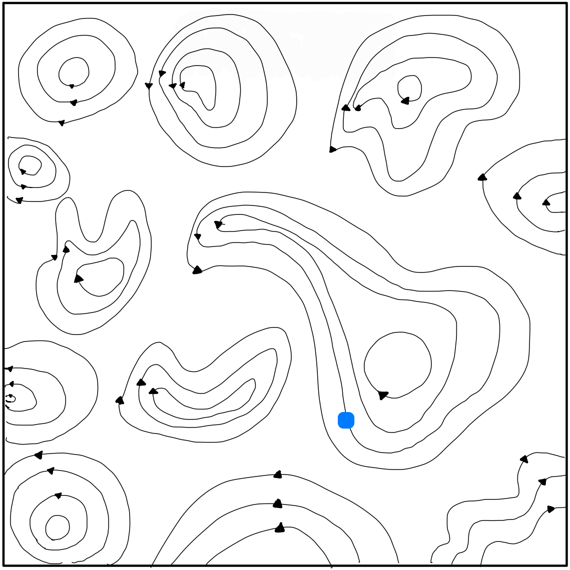

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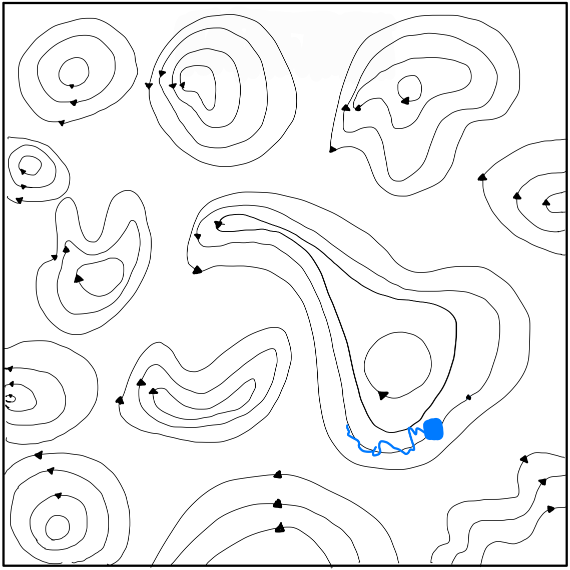

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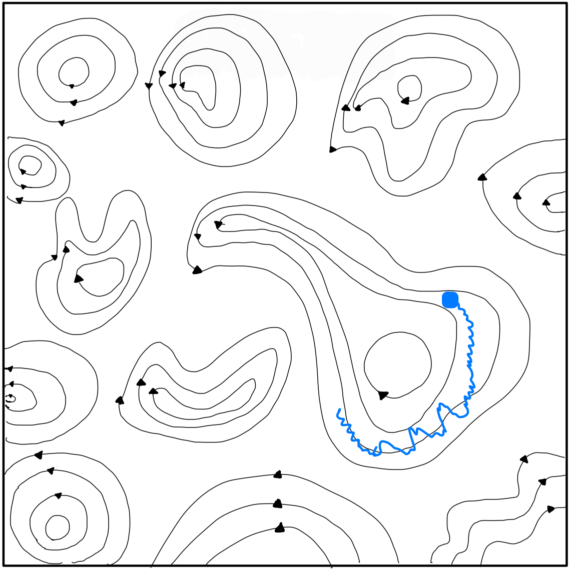

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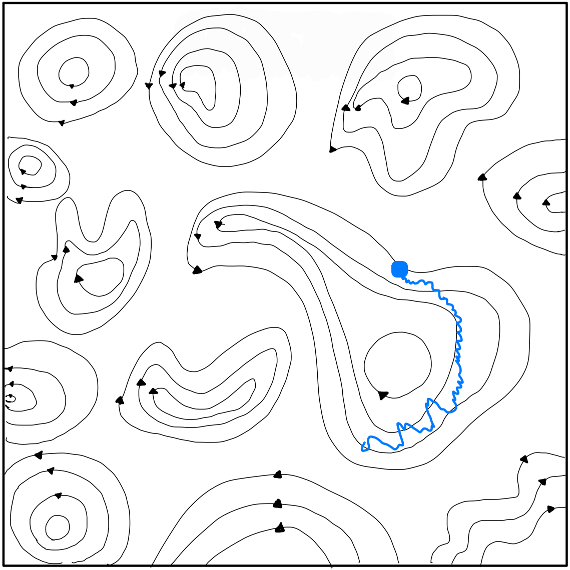

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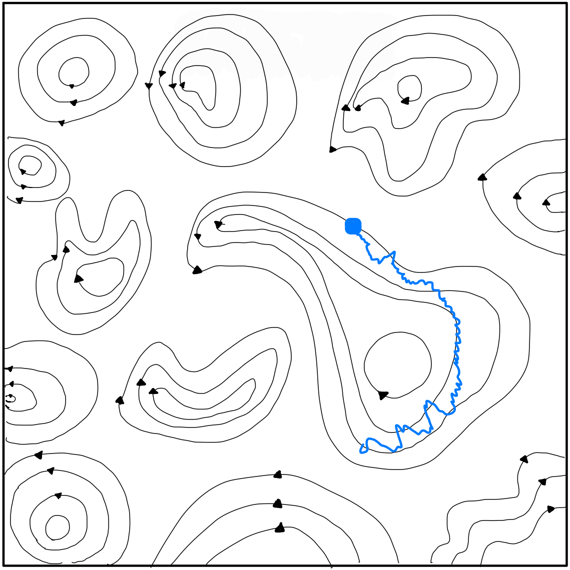

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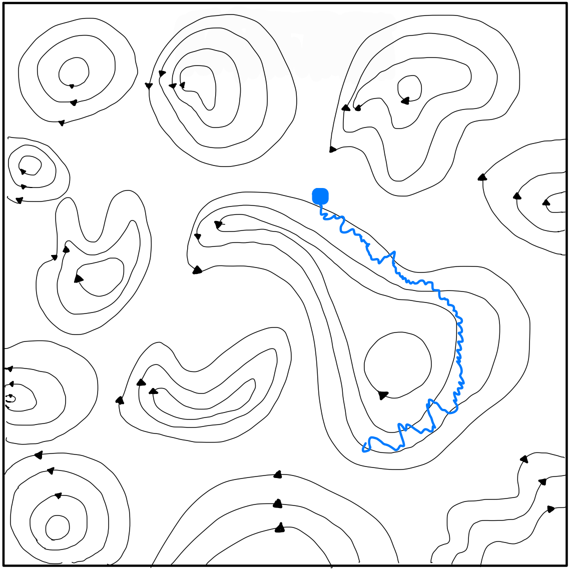

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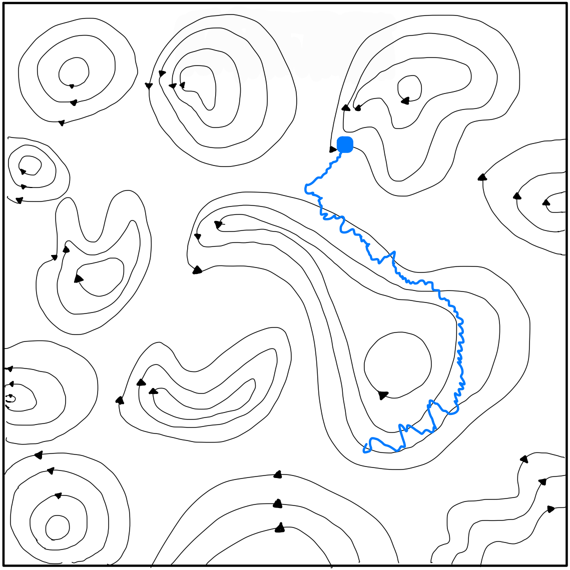

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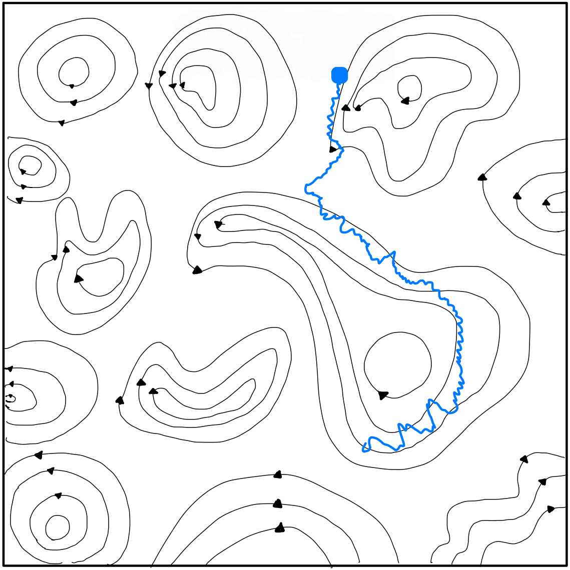

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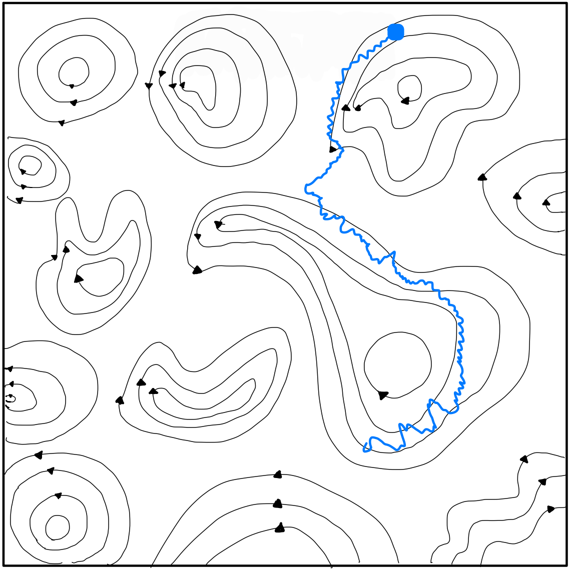

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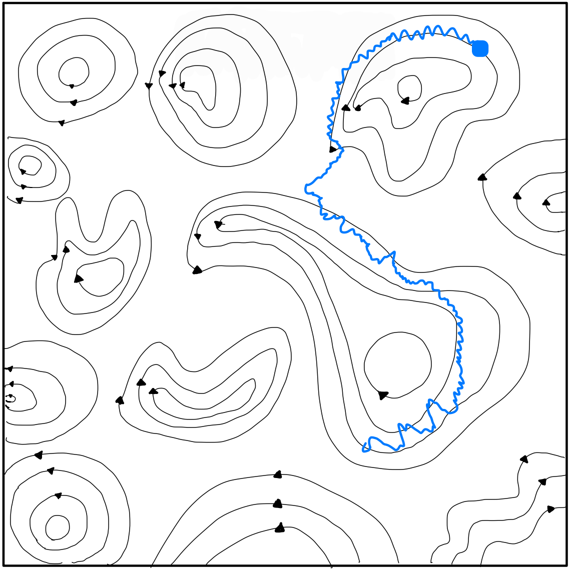

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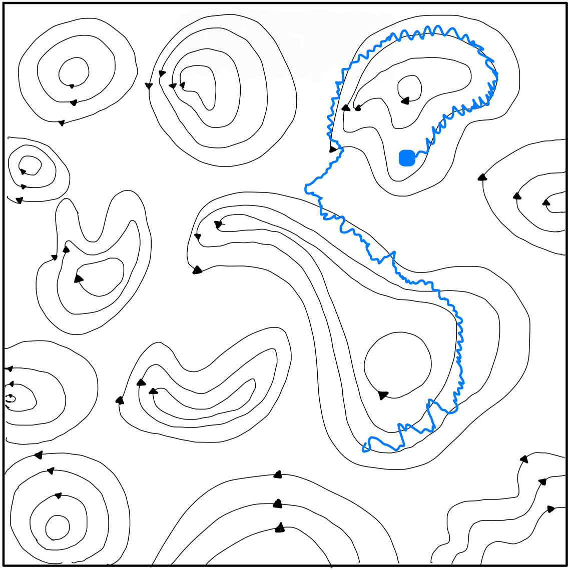

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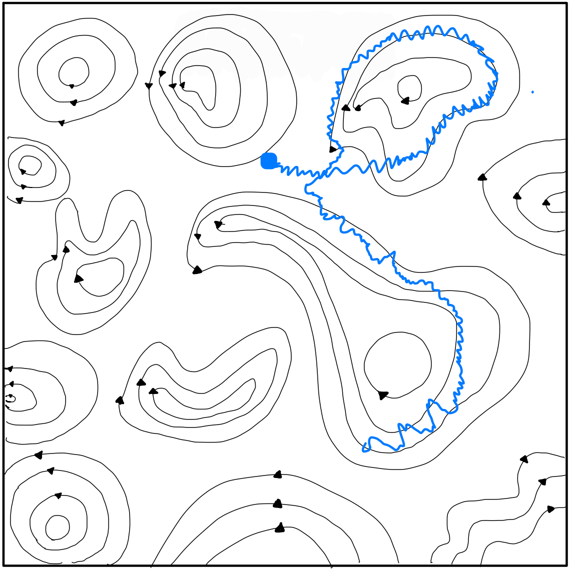

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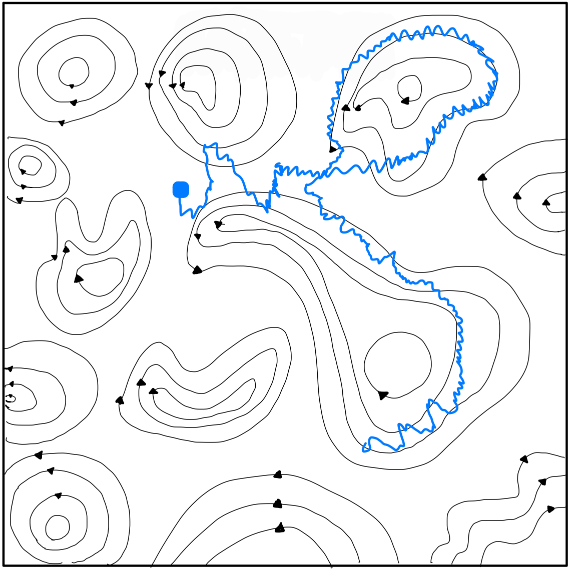

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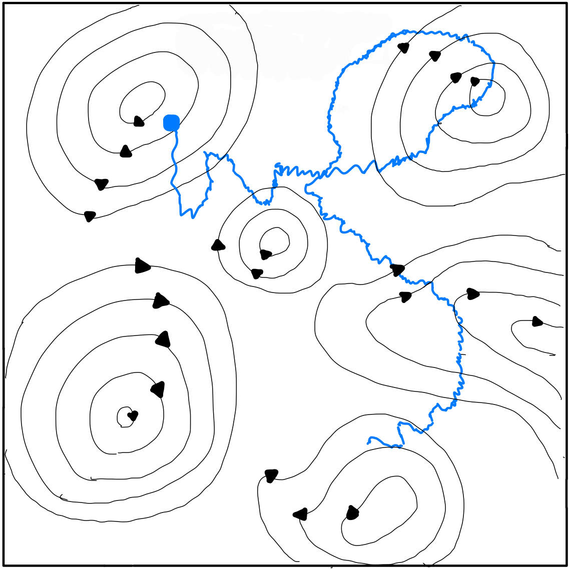

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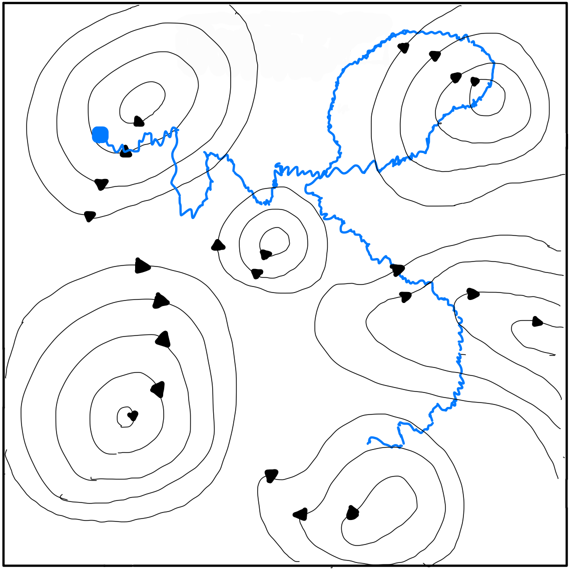

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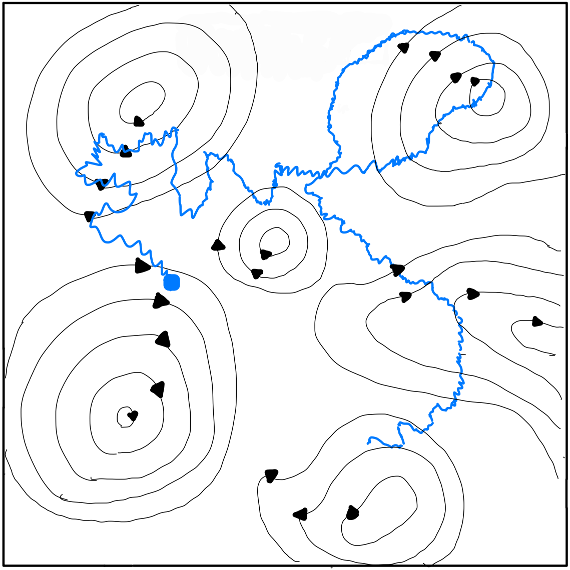

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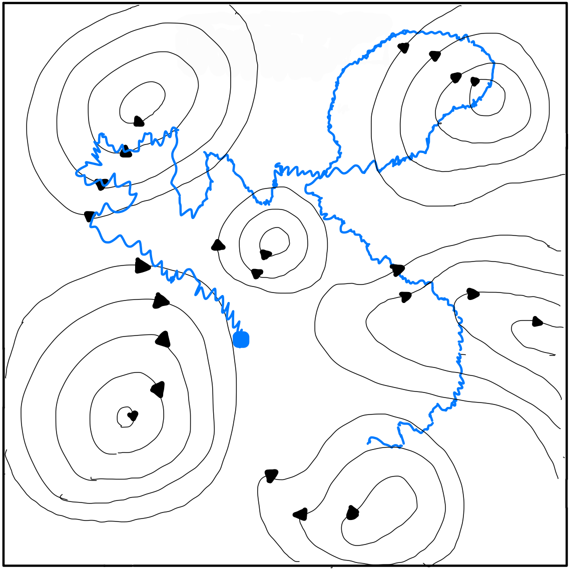

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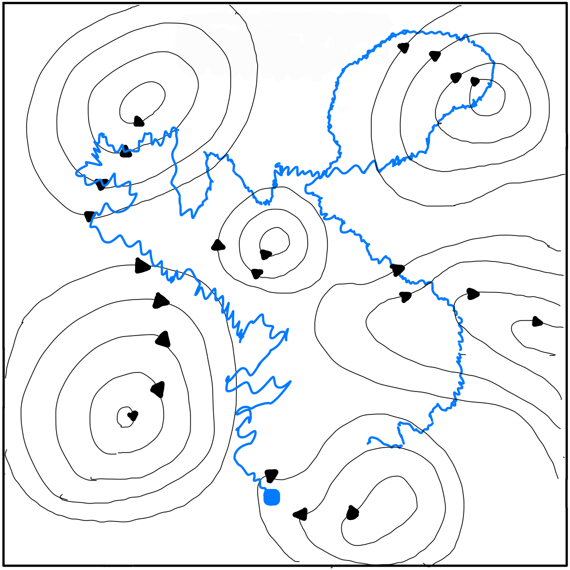

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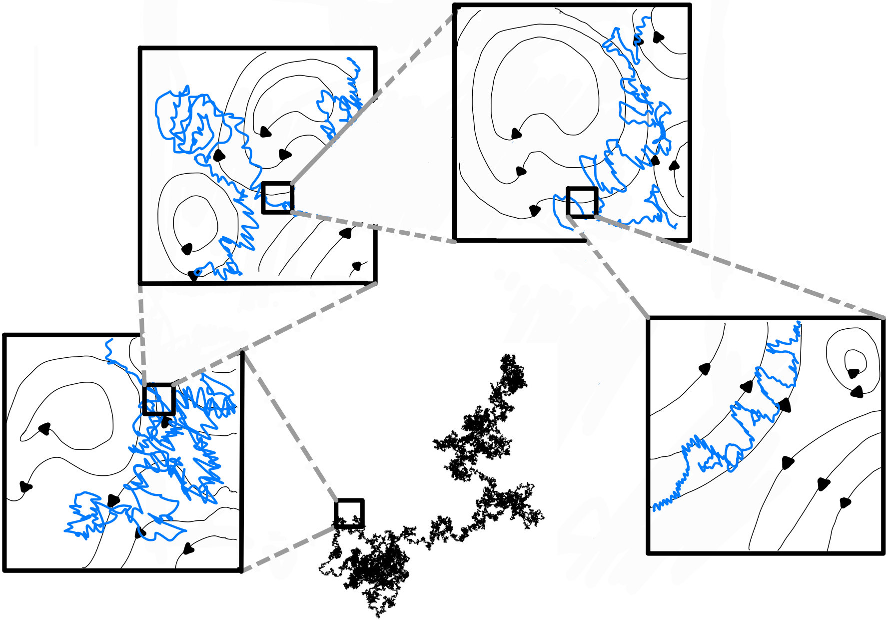

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    - also, need to argue that the nearby waves do not affect the homogenization

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assume the stream matrix is given by the formal sum

$$\mathsf{k}(x) = \sum_{n=0}^{\infty} \mathsf{j}_n(x)\,,$$

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- canonical example: entries given by independent copies of mollified log-correlated Gaussian field

Quenched superinvariance principle

Theorem (Armstrong, B., Kuusi 2024)

$\mathbb{P}$-almost surely,

$$|\log \varepsilon^2|^{-1/4} \varepsilon X_{t/\varepsilon^2} \Rightarrow \sqrt{2c_*} W_t \quad \textit{as } \varepsilon \to 0\,,$$

where $\{W_t\}$ is a standard Brownian motion in $\mathbb{R}^d$ and

$$\lim_{t\to\infty} \left| \frac{1}{2c_* t(\log t)^{1/2}} \mathrm{E}^0\left[|X_t|^2\right] - d \right| = 0\,,$$

and, consequently, for every $p \in [1,\infty)$

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- corollary of analogous result for the generators (homogenization)
- actual result is quantitative — has explicit rates

Qualitative homogenization

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For every Lipschitz domain $U \subset \mathbb{R}^d$, $f \in L^\infty(U)$ and $g \in Lip(U)$ if we let $u^\varepsilon, u_{\mathrm{hom}} \in H^1(U)$ denote the solutions of the boundary value problems

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- qualitative version of our result

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Theorem (Armstrong, B., Kuusi 2024)

For every Lipschitz domain $U \subset \mathbb{R}^d$ and $\alpha \in (0, 1/4)$, there exists a finite random variable $\mathcal{Z}$ such that for every $f \in L^\infty(U)$ and $g \in Lip(U)$ and $\varepsilon^{-1} \geq \mathcal{Z}$

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- actual statement is slightly more quantitative

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  - a special case is classicial and goes back to (De Giorgi (1972))

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- technical heart: *coarse-grained elliptic regularity estimates* and high contrast homogenization estimates established in the companion work (Armstrong, Kuusi 2024)

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- superdiffusive Caccioppoli

$$\|\nabla u\|_{\underline{L}^2(B_r)} \lesssim \frac{(\log r)^{1/4}}{r}\|u-(u)_{B_{2r}}\|_{\underline{L}^2(B_{2r})}$$

  and superdiffusive Poincaré

$$\frac{(\log r)^{1/4}}{r}\|u-(u)_{B_r}\|_{\underline{L}^2(B_r)} \lesssim \|\nabla u\|_{\underline{L}^2(B_r)}$$

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For each $\alpha \in (0,1)$*, there exists* $C < \infty$ *such that for every* $R \geq \mathcal{X}$, $f \in L^\infty(B_R)$ *and solution* $u \in H^1(B_R)$ *of the equation*

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- any solution growing slower than $x^{1-\alpha}$ is constant
- states that if you mollify at scale $R(\log R)^{-1000}$, the solution is Hölder

Subquadratic solutions

- recall the classical result

$$\{u : \mathbb{R}^d \to \mathbb{R} : \Delta u = 0 \text{ and u grows slower than a quadratic}\}$$
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  - a is random, but bounded uniformly (Armstrong, Smart (2014)) (Gloria, Neukamm, Otto (2014))
- false for us, since a $:= \nu Id + \mathrm{k}$ is *not uniformly bounded*

Quantitative first-order Liouville theorem

$$\mathcal{A}^1(\mathbb{R}^d) = \{u : \mathbb{R}^d \to \mathbb{R} : \nabla \cdot \mathrm{a}\nabla u = 0 \text{ and } u = O(|x|^{2-\delta})\}$$

Theorem (Armstrong, B., Kuusi 2024)

1. *Finite dimensionality. We have that* $\mathbb{P}\big[\dim \mathcal{A}^1(\mathbb{R}^d) = d+1\big] = 1$.
2. *Flatness at every scale. There exists* $C < \infty$ *such that, for every* $\phi \in \mathcal{A}^1(\mathbb{R}^d)$ *and* $r \geq \mathcal{X}$, *we have*

$$\inf_{e \in \mathbb{R}^d} \|\phi - \ell_e\|_{\underline{L}^2(B_r)} \leq C(\log r)^{-1/2+\delta} \|\phi\|_{\underline{L}^2(B_r)}\,.$$

3. *Large-scale* $C^{1,\gamma}$ *estimate. For each* $R \in [\mathcal{X}, \infty)$ *and* u *solving* $\nabla \cdot \mathrm{a}\nabla u = 0$ *in* B_R *there exists* $\phi \in \mathcal{A}^1(\mathbb{R}^d)$ *such that*

$$\|\nabla u - \nabla \phi\|_{\underline{L}^2(B_r)} \leq C\left(\frac{r}{R}\right)^{\gamma} \|\nabla u\|_{\underline{L}^2(B_R)}\,, \quad \forall r \in [\mathcal{X}, R)\,.$$

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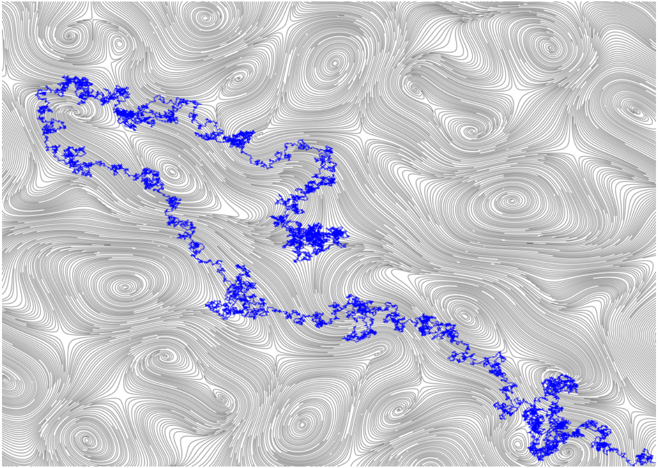
- approach seems flexible and could work for other types of drift (and other problems which are renormalizable)



"Superdiffusive central limit theorem for a Brownian particle in a critically-correlated incompressible random drift"

(Scott Armstrong, Ahmed Bou-Rabee, and Tuomo Kuusi.)
[arXiv:2404.01115]