CONVERGENCE OF THE RANDOM ABELIAN SANDPILE

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Abstract. We prove that Abelian sandpiles with random initial states converge almost surely to unique scaling limits. The proof follows the Armstrong-Smart program for stochastic homogenization of uniformly elliptic equations. Using a different approach, we prove an analogous result for the divisible sandpile and identify its scaling limit as exactly that of the averaged divisible sandpile. As a corollary, this gives a new quantitative proof of known results on the stabilizability of random divisible and Abelian sandpiles.

1. Introduction

The Abelian sandpile is a simple combinatorial model which produces striking, fractal-like patterns [BTW87, HLM+08]. C. Smart and W. Pegden began the rigorous understanding of these patterns by showing that scaling limits of sandpiles exist and are Laplacians of solutions to elliptic obstacle problems [PS13]. Their proof technique is flexible: it was first applied to the single-source sandpile and it works for any sandpile with a periodic initial configuration. However, their proof does not extend to random initial configurations. In this paper, as a first step towards understanding random sandpiles, we show that sandpiles with random initial states also have scaling limits.

As a simple example, consider the following random sandpile on the two-dimensional square lattice. For each site x in a ball of radius n in $\mathbb{Z}^2$, flip a fair coin. If the coin lands heads, put 3 grains of sand at x; otherwise put 5 grains of sand at x. Then, let the sandpile stabilize. If you repeat this procedure for large n and rescale, a non-random pattern emerges. The pattern looks remarkably similar to the scaling limit of the single-source sandpile - compare Figure 1 with Figure 2. Our main result explains this similarity by proving that the scaling limit of the random sandpile is the Laplacian of the solution to an elliptic obstacle problem with two operators. One operator depends on the distribution of the randomness. The other operator is the exact same one appearing in the scaling limit of the single source sandpile.

More generally let $\eta : \mathbb{Z}^d \to \mathbb{Z}$ be stationary, ergodic, bounded, and satisfy $\mathbf{E}(\eta(0)) > 2d-1$. Let $W \subset \mathbb{R}^d$ be a bounded Lipschitz subset. For each $n \in \mathbb{N}$, let $W_n = \mathbb{Z}^d \cap nW$ denote the finite difference approximation of W. Initialize the sandpile according to η in W_n and set it to be 0 elsewhere. Then, stabilize the sandpile, counting how many times each square topples through the *odometer function* $v_n : \mathbb{Z}^d \to \mathbb{N}$. Denote the stable sandpile by $s_n : \mathbb{Z}^d \to \mathbb{Z}$. Our main result is the following.

Theorem 1.1. *Almost surely, as $n \to \infty$, the rescaled functions $\bar{v}_n := n^{-2} v_n([nx])$ converge uniformly to the unique solution of the elliptic obstacle problem*

$$\min\{v \in C(\mathbb{R}^d) : v \geq 0 \text{ and } \bar{F}_\eta(D^2 v) \leq 0 \text{ in } W \text{ and } \bar{F}_0(D^2 v) \leq 0 \text{ in } \mathbb{R}^d\},$$

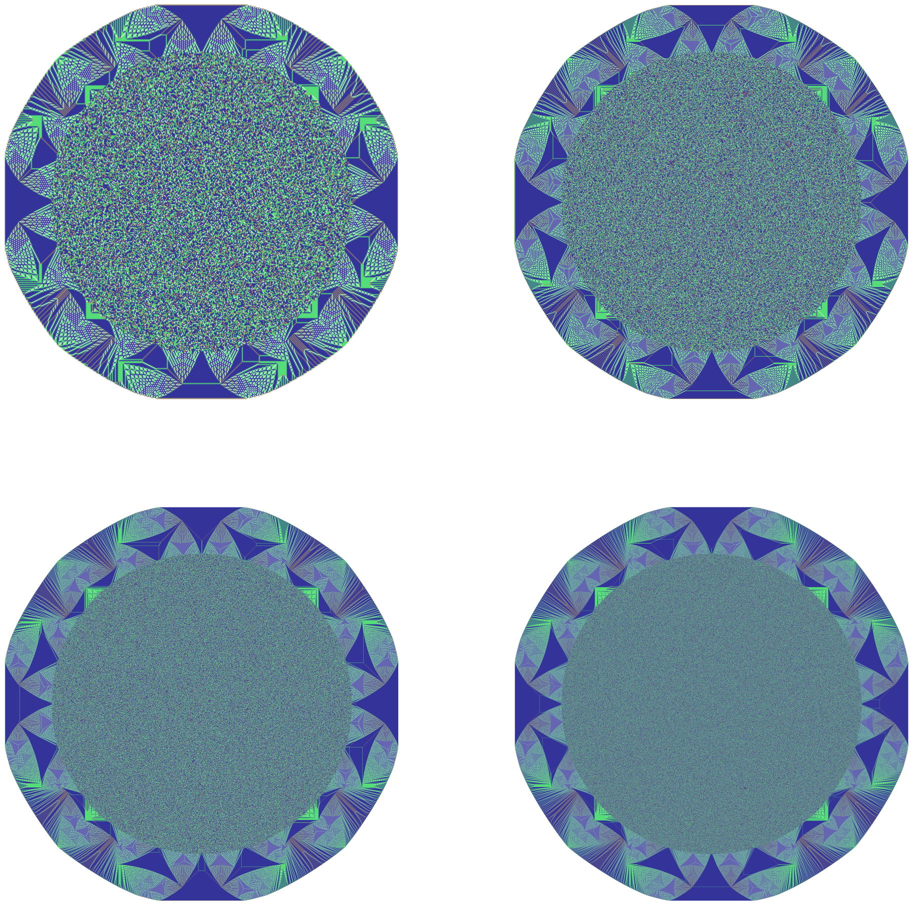

FIGURE 1. For each x in a ball of radius $n = 500, 1000, 2000$, and 3000, flip a fair coin. If it lands heads, put 3 grains of sand at x, otherwise put 5 grains of sand. Then, stabilize. Sites with 0,1,2, and 3 grains of sand are represented by white, brown, green, and blue respectively.

where $\bar{F}_\eta$ is a nonrandom, degenerate elliptic operator,

$$\bar{F}_0(M) := \inf\{s \in \mathbb{R} | \exists u : \mathbb{Z}^d \to \mathbb{Z} \text{ such that } \Delta_{\mathbb{Z}^d} u \leq 2d - 1 \text{ and } u(y) \geq \tfrac{1}{2} y^T (M - sI) y\},$$

and the differential inequalities are interpreted in the viscosity sense. In turn, the rescaled sandpiles, $\bar{s}_n(x) := s_n([nx])$ converge weakly-$*$ to a deterministic function $s \in L^\infty(\mathbb{R}^d)$ as $n \to \infty$. Moreover, the limit satisfies $\int_{\mathbb{R}^d} s dx = |W| \, \mathbf{E}(\eta(0))$, $s \leq 2d-1$, $s = 0$ in $\mathbb{R}^d \backslash B_R(W)$ for some constant $R > 0$ depending on W and $\mathbf{E}(\eta(0))$.

The main challenge in proving the above theorem is that there is no inherent linear or subadditive quantity governing the behavior of the sandpile. The Abelian sandpile is nonlocal: one unstable pile can cause a far-reaching avalanche of topplings. This difficulty is the same one faced by those studying stochastic homogenization of fully nonlinear elliptic

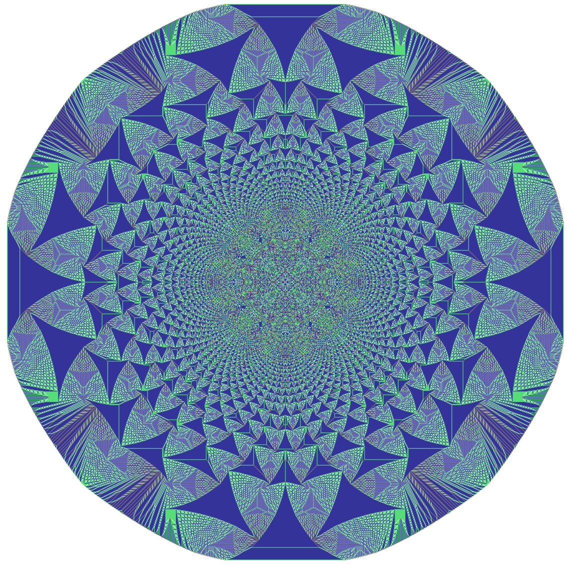

Figure 2. The single source sandpile: start with 10^6 grains of sand at the origin and stabilize. Sites with 0,1,2, and 3 grains of sand are represented by white, brown, green, and blue respectively.

PDEs. Fortunately, since the sandpile can be expressed as the solution to a nonlinear discrete PDE, we can use those same methods here. To be specific, we import the stochastic homogenization tools introduced by S. Armstrong and C. Smart in [AS14a]. Most of the work will be in identifying the appropriate sandpile ingredients. Some of the tools also need to be rebuilt due to the lack of uniform ellipticity.

1.1. **Outline of the paper.** In Section 2, we precisely state the assumptions of our result. Then, in Section 3, we recall some necessary properties of the Abelian sandpile. Next, in Section 4, we define a subadditive quantity, μ, which will implicitly control the random sandpile. Through an appropriate perturbation of μ, we identify $\bar{F}_\eta$ in Section 5. In Section 6 we prove the main result. It is here that the alternative proof of stabilizability of random sandpiles appears. Then, in Section 7 we show a simpler proof of convergence of a related model, the random *divisible* sandpile, invented by L. Levine and Y. Peres [LP09, LP10]. In stark contrast to the Abelian sandpile, the limit for the random divisible sandpile is exactly the limit of the averaged divisible sandpile. We end with some generalizations and open questions in Section 8.

1.2. **Acknowledgments.** I am grateful to Charles K. Smart for suggesting the program in [AS14a], patiently providing essential advice throughout this project, and carefully reviewing a previous draft of this paper. I am also grateful to Steven P. Lalley for useful conversations, encouragement, and first introducing me to this problem. I also acknowledge Khalid Bou-Rabee, Nawaf Bou-Rabee, Gregory Lawler, Lionel Levine, and Micol Tresoldi for inspiring conversations.

2. Preliminaries

2.1. **Notation and Conventions.** The constant $d \in \mathbb{N}$ will refer to dimension. We denote $\mathbf{S}^d$ as the set of symmetric $d \times d$ matrices with real entries. If $M \in \mathbf{S}^d$, we write $M \geq 0$, if M has nonnegative eigenvalues. $|M|_2$ will also refer to the largest absolute eigenvalue of M. For a vector $x \in \mathbb{R}^d$, $|x|_\infty$ denotes the maximum norm and $|x|_2$ the 2-norm. We sometimes omit the subscript, in which case $|x|$ refers to the 2-norm. We also write $q_M(x) := \frac{1}{2}x^T Mx$ and $q_l(x) := \frac{1}{2}|x|^2$. For both functions and vectors, inequalities between them are to be interpreted as point-wise. We write $y \sim x$ when $y - x \in \mathbb{Z}$ and $|y - x| = 1$. For a subset $A \subset \mathbb{Z}^d$, $d \geq 1$, denote its discrete boundary by

$$\partial A = \{y \in \mathbb{Z}^d \backslash A : \exists x \in A : y \sim x\}.$$

and its discrete closure by

$$\bar{A} = A \cup \partial A.$$

The diameter of A is

$$\mathrm{diam}(A) = \max\{|x - y|_2 : x, y \in A\}$$

For $x \in \mathbb{Z}^d$,

$$Q_n(x) = \{y \in \mathbb{Z}^d : |x - y|_\infty < n\},$$

is the cube of radius n centered at x: and

$$B_n(x) = \{y \in \mathbb{Z}^d : |x - y|_2 < n\},$$

is the ball of radius n centered at x. For short, $B_n := B_n(0)$, $Q_n := Q_n(0)$. We will also use Q_n and B_n to refer to cubes and balls on $\mathbb{R}^d$.

We similarly overload notation so that for $A \subset \mathbb{Z}^d$, $|A|$ refers to the number of elements in A and for measurable $L \subset \mathbb{R}^d$, $|L|$ is the Lebesgue measure of L. For $f : \mathbb{Z}^d \to \mathbb{R}$, we denote its discrete Laplacian by

$$\Delta_{\mathbb{Z}^d} f = \sum_{y \sim x} (f(y) - f(x)),$$

and its discrete second-differences by

$$\Delta_i f = f(x + e_i) + f(x - e_i) - 2f(x),$$

where $\{\pm e_i\}$ are the $2d$ coordinate directions in $\mathbb{Z}^d$. Δ will refer to the Laplace operator on the continuum. The convex hull of a set of points A will be denoted

$$\mathbf{conv}(A) = \{y \in \mathbb{R}^d : y = \sum_{i=1}^{d+1} \lambda_i x_i \text{ for } x_i \in A, \textstyle\sum_{i=1}^{d+1} \lambda_i = 1, \lambda_i \in [0, 1]\}.$$

Throughout the proofs the constant C will denote a positive constant which may change from line to line. When needed, explicit dependence of C on other constants will be denoted by, for example, C_d.

2.2. **Assumptions.** We consider the sandpile on the integer lattice $\mathbb{Z}^d$ for $d \geq 2$, (although this assumption is not an essential one). Let Ω denote the set of all bounded functions $\eta : \mathbb{Z}^d \to \mathbb{Z}$. Endow Ω with the σ-algebra $\mathcal{F}$ generated by $\{\eta \to \eta(x) : x \in \mathbb{Z}^d\}$. We model the randomness by a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ with the following properties. First, there exists $\eta_{\min}, \eta_{\max} \in \mathbb{Z}$ so that for every $x \in \mathbb{Z}^d$,

$$\text{Uniform Boundedness: } \mathbf{P}\left[\eta_{\min} \leq \eta(x) \leq \eta_{\max}\right] = 1. \tag{1}$$

Note this may be replaced by an appropriate concentration assumption. We further assume that $\mathbf{P}$ is stationary and ergodic. Denote the action of translation by $T : \mathbb{Z}^d \times \Omega \to \Omega$,

$$T(y, \eta)(z) = (T_y \eta)(z) := \eta(y + z),$$

and extend this to $\mathcal{F}$ by defining $T_y E := \{T_y \eta : \eta \in E\}$. Stationarity and ergodicity is then

$$\text{Stationarity: for all } E \in \mathcal{F}, y \in \mathbb{Z}^d\text{: } \mathbf{P}(T_y E) = \mathbf{P}(E), \tag{2}$$

$$\text{Ergodic: } E = \cap_{y \in \mathbb{Z}^d} T_y E \text{ implies that } \mathbf{P}(E) \in \{0, 1\} \ . \tag{3}$$

Lastly, we assume that the density of sand in the initial sandpile is high:

$$\text{High density: } \mathbf{E}(\eta(0)) > 2d - 1. \tag{4}$$

High density is a natural, weak assumption which forces interesting behavior to occur. See Section 8 for further discussion of this.

3. Sandpiles

The results in this section are reformulations of fundamental facts about the Abelian sandpile. See, for example, [CP18, Red05, Jár18]. Fix a bounded, connected $A \subset \mathbb{Z}^d$ and a starting sandpile $\eta : \bar{A} \to \mathbb{Z}$. We call positive integer valued functions on $\bar{A}$, *toppling functions*. Recall that a toppling function u is *legal* for η if it can be decomposed into a sequence of topplings so that only sites x where $\eta(x) \geq 2d$ are toppled. More precisely, u is legal for η if we can express

$$u = u_0 + u_1 + \cdots + u_n,$$

for some $n \geq 0$ where $u_0 = 0$ and for $i \geq 1$, $u_i(x) = 0$ for all $x \in \mathbb{Z}^d$ except for one $\hat{x}_i \in A$ for which $u_i(\hat{x}_i) = 1$ and

$$\left(\Delta_{\mathbb{Z}^d}(u_1 + \cdots + u_{i-1}) + \eta\right)(\hat{x}_i) \geq 2d.$$

When $\eta \leq 2d - 1$, the only legal toppling function is $u = u_0 = 0$. An important observation is that any legal toppling function satisfies $\Delta_{\mathbb{Z}^d} u + \eta \geq \min(0, \eta)$ but this inequality does not imply u is legal. A toppling function v is *stabilizing* for η in a set A if $\Delta_{\mathbb{Z}^d} v + \eta \leq 2d - 1$ in A.

Denote the set of legal topplings for η in A as

$$\begin{aligned}\mathcal{L}(\eta, A) = \{&u : \mathbb{Z}^d \to \mathbb{N} : \text{ there exists } w : \mathbb{Z}^d \to \mathbb{N} \text{ and } \hat{u} : \mathbb{Z}^d \to \mathbb{N} \text{ with } u = w + \hat{u} \\ &\text{so that } w(x) = 0 \text{ for } x \in A,\ \hat{u}(x) = 0 \text{ for } x \in \mathbb{Z}^d \backslash A \text{ and } \hat{u} \text{ is legal for } \Delta_{\mathbb{Z}^d} w + \eta\}\end{aligned}$$

and the set of stabilizing topplings for η in A as

$$\mathcal{S}(\eta, A) = \{v : \mathbb{Z}^d \to \mathbb{N} : \Delta_{\mathbb{Z}^d} v(x) + \eta(x) \leq 2d - 1 \text{ for } x \in A\}.$$

It is important to note that these sets only enforce their constraints in A, they may include arbitrary topplings outside A.

The *odometer* function, $v : \mathbb{Z}^d \to \mathbb{N}$, counts the number of times each square in η topples when stabilizing. Here we distinguish between two common scenarios. In the first scenario, once a grain of sand leaves A, it falls off and disappears. We call this the *closed boundary* condition. In this case $v = 0$ on $\mathbb{Z}^d \backslash \bar{A}$. In the second scenario, the grain does not disappear and topplings will occur outside of A. This is the *open boundary* condition. The sandpile we consider in our main theorem has the open boundary condition. However, as we will discuss in Section 8, our methods also apply to other sandpiles including those with closed boundaries. In this section, we state results for sandpiles with the closed boundary condition.

First, we recall the least-action principle for sandpiles and rephrase it in a way amenable to the methods of this paper. We will refer to this as the *discrete sandpile PDE*.

Proposition 3.1. *The odometer function v uniquely solves each of the following problems.*

1. *Longest legal toppling:* $v = \sup\{u : \mathbb{Z}^d \to \mathbb{N} : u \in \mathcal{L}(\eta, A),\ v = 0 \text{ on } \mathbb{Z}^d \backslash A\}$
2. *Shortest stabilizing toppling:* $v = \inf\{u : \mathbb{Z}^d \to \mathbb{N} : u \in \mathcal{S}(\eta, A),\ u = 0 \text{ on } \mathbb{Z}^d \backslash A\}$
3. *Stabilizing, legal toppling:* $v \in \mathcal{L}(\eta, A) \cap \mathcal{S}(\eta, A)$ *and* $v = 0$ *on* $\mathbb{Z}^d \backslash A$

A certain class of sandpiles, known as *recurrent* sandpiles will help in the sequel. We say η is recurrent if we can find $s : A \to \mathbb{N}$ and $u \in \mathcal{L}(s + 2d - 1, A)$ with $u = 0$ on $\mathbb{Z}^d \backslash A$ so that $\eta = 2d - 1 + s + \Delta_{\mathbb{Z}^d} u$. In other words, η is recurrent if we can reach η by starting with $2d - 1$ grains at every site in A, adding grains of sand at some sites in A and then toppling some sites legally. Also we call η *stable* in A if $\eta \leq 2d - 1$ in A.

A useful consequence of Dhar's burning algorithm will aid in controlling topplings on stable, recurrent sandpiles. Recall that the burning algorithm provides a recipe for checking if a stable sandpile is recurrent: topple the boundary of a sandpile once, if the sandpile is recurrent, each inner square will topple exactly once when stabilizing. More generally, topple squares along ∂A and then legally stabilize s in A. If s is a stable sandpile, no square in A will topple more times than a boundary square has toppled. And, if s is a recurrent sandpile, some square in A will topple at least as many times as some boundary square. This leads to both a maximum principle and a comparison principle for the sandpile.

Proposition 3.2. *For $f : \partial A \to \mathbb{Z}$, a sandpile $s : A \to \mathbb{Z}$, let v solve*

$$\begin{cases} v \in \mathcal{L}(s) \cap \mathcal{S}(s) & \text{on } A \\ v = f & \text{on } \partial A \end{cases}$$

If s is stable, then

$$\sup_{x \in A} v(x) \leq \sup_{x \in \partial A} v(x).$$

If s is recurrent, then

$$\inf_{x \in A} v(x) \geq \inf_{x \in \partial A} v(x).$$

In particular, we have the following comparison principle for stable, recurrent sandpiles. Suppose s is stable, recurrent and let u solve

$$\begin{cases} u \in \mathcal{L}(s) \cap \mathcal{S}(s) & \text{on } A \\ u = f' & \text{on } \partial A. \end{cases}$$

Then,

$$\inf_{x \in A} (u - v)(x) \geq \inf_{y \in \partial A} (u - v)(y).$$

Furthermore, for any integer valued functions $g, h : \bar{A} \to \mathbb{Z}$ with $\Delta_{\mathbb{Z}^d} g = \Delta_{\mathbb{Z}^d} h = 0$ in A,

$$\inf_{x \in A} ((u+g) - (v+h))(x) \geq \inf_{y \in \partial A} ((u+g) - (v+h))(y).$$

We will also use the following consequence of the Abelian property: any legal stabilizing toppling function can be decomposed into the usual odometer function for η and an odometer function which keeps track of topplings originating from the boundary.

Proposition 3.3. *If $v \in \mathcal{L}(\eta, A) \cap \mathcal{S}(\eta, A)$ and $v = f$ on ∂A, then v can be decomposed into*

$$v = v_1 + v_2,$$

where

$$\begin{cases} v_1 \in \mathcal{L}(\eta) \cap \mathcal{S}(\eta) & \text{on } A \\ v_1 = 0 & \text{on } \partial A \end{cases}$$

and

$$\begin{cases} v_2 \in \mathcal{L}(\eta + \Delta_{\mathbb{Z}^d} v_1) \cap \mathcal{S}(\eta + \Delta_{\mathbb{Z}^d} v_1) & \text{on } A \\ v_2 = f & \text{on } \partial A \end{cases}$$

We conclude the section by noting a useful alternative characterization of recurrent sandpiles. If each site in η has toppled at least once, then what remains is recurrent.

Proposition 3.4. *If W is a connected subset of $\mathbb{Z}^d$ and $A \subset W$ is also connected if $w \in \mathcal{L}(\eta, W)$, and $w \geq 1$ in A, $w \geq 0$ on ∂A, then $\Delta_{\mathbb{Z}^d} w + \eta$ is recurrent in A.*

4. A monotone quantity

4.1. **The definition of μ.** In this section we introduce μ, a monotone quantity which will control solutions to the discrete sandpile PDE. For a function $v : \mathbb{Z}^d \to \mathbb{Z}$ and $x \in A \subset \mathbb{Z}^d$ let

$$\partial^+(v, x, A) = \{p \in \mathbb{R}^d : v(x) + p \cdot (y - x) \geq v(y) : \text{ for all } y \in \bar{A}\}$$

denote the supergradient set of v at x in A. Similarly,

$$\partial^-(v, x, A) = \{p \in \mathbb{R}^d : v(x) + p \cdot (y - x) \leq v(y) : \text{ for all } y \in \bar{A}\}$$

is the subgradient set at x. For short-hand, we omit the set A when it is clear and write

$$\partial^+(v, A) = \cup_{x \in A} \partial^+(v, x).$$

To completely identify a fully nonlinear PDE, it suffices to recognize when a parabola is a supersolution or a subsolution. This fundamental observation is due to Caffarelli [Caf99] and was employed by Caffarelli, Souganidis, and Wang in their obstacle problem argument for homogenization of fully nonlinear equations [CSW05, AS14c].

Our method is similar: we will perturb solutions by a parabola and then define the effective equation, $\bar{F}_\eta$, through these perturbed limits. For $l \in \mathbb{R}$ and $M \in \mathbf{S}^d$, denote the set of perturbed subsolutions as

$$S(A, \omega, l, M) = \{u : \mathbb{Z}^d \to \mathbb{Z} : u \in \mathcal{L}(\eta, A)\} - (q_l + q_M)$$

and the set of perturbed supersolutions as

$$S^*(A, \omega, l, M) = \{v : \mathbb{Z}^d \to \mathbb{Z} : v \in \mathcal{S}(\eta, A)\} - (q_l + q_M).$$

The monotone quantity controlling subsolutions is then

$$\mu(A, \omega, l, M) = \sup\{|\partial^+(w, A)| : w \in S(A, \omega, l, M)\},$$

while the monotone quantity which controls supersolutions is

$$\mu^*(A, \omega, l, M) = \sup\{|\partial^-(w, A)| : w \in S^*(A, \omega, l, M)\}.$$

4.2. **Comparing subsolutions and supersolutions.** We will need to compare legal and stabilizing toppling functions throughout this paper. However, the discrete sandpile PDE is nonlinear: if v is a stabilizing toppling function for η, then $-v$ is not a legal toppling function for $-\eta$ (unless $v = 0$). This makes it difficult to compare legal toppling functions and stabilizing toppling functions. However, through μ, we can compare the two using the following lemma.

Lemma 4.1. *If $u \in \mathcal{L}(\eta, A)$, the solution of*

$$h \in \mathcal{L}(\eta, A) \cap \mathcal{S}(\eta, A) \text{ and } h = u \text{ on } \partial A,$$

satisfies $\partial^+(u, A) \subseteq \partial^+(h, A)$. Similarly, if $v \in \mathcal{S}(\eta, A)$, then the solution of

$$h^* \in \mathcal{L}(\eta, A) \cap \mathcal{S}(\eta, A) \text{ and } h^* = v \text{ on } \partial A,$$

satisfies $\partial^-(v, A) \subseteq \partial^-(h^*, A)$.

Proof. By the least action principle, we know that for all $x \in A$, $h(x) \geq u(x)$. Also, by assumption $u(x) = h(x)$ on ∂A. Hence, for each $p \in \partial^+(u, x, A)$ if we let

$$t = \inf\{c \in \mathbb{R} : u(x) + p \cdot (y - x) + c \geq h(y) \text{ for all } y \in \bar{A}\}$$

we see that $t \geq 0$. And since A is finite, t is bounded. Further since $h = u$ on ∂A, we must have $y \in A$ for which

$$u(x) + p \cdot (y - x) + t = h(y),$$

which shows $p \in \partial^+(h, y, A)$. The proof for subgradients is similar. □

4.3. **Basic properties of μ.** We now establish control on solutions from above and below which will follow from the proof of the Alexandroff-Bakelman-Pucci (ABP) inequality [RCC95, GB01].

Lemma 4.2. *There exists $C_d > 0$ so that for all $w \in S(B_n, \omega, l, M)$,*

$$\max_{x \in B_n} w(x) \leq \max_{x \in \partial B_n} w(x) + C_d n \mu(B_n, \omega, l, M)^{1/d} \tag{5}$$

and for all $w \in S^*(B_n, \omega, l, M)$,

$$\inf_{x \in \partial B_n} w(x) \leq \inf_{x \in B_n} w(x) + C_d n \mu^*(B_n, \omega, l, M)^{1/d}. \tag{6}$$

Proof. Let $a = \max_{x \in B_n} w(x) - \max_{x \in \partial B_n} w(x)$. Assume $a > 0$, otherwise the claim is immediate. Choose x_0 so that $\max_{x \in B_n} w(x) = w(x_0)$. Let $p \in \mathbb{R}^d$ satisfy $|p| \leq a \mathrm{diam}(B_n)^{-1} = C_d a / n$. Then, for each $x \in B_n$,

$$w(x_0) + p \cdot (x - x_0) \geq w(x_0) - |p||x - x_0| > w(x_0) - w(x_0) + \max_{x \in \partial B_n} w(x) = \max_{x \in \partial B_n} w(x). \tag{7}$$

Now, we shift up the hyperplane just enough so that it lies above w in $\bar{B}_n$: let

$$t = \inf\{c \in \mathbb{R} : w(x_0) + p \cdot (x - x_0) + c \geq w(x) \text{ for all } x \in \bar{B}_n\}$$

and note that $t \geq 0$ and that there exists $y \in \bar{B}_n$ with

$$w(y) = w(x_0) + p \cdot (y - x_0) + t.$$

If $t > 0$, then (7) shows that $y \in B_n$. If $t = 0$, we can choose $y = x_0$. Hence, there is a $y \in B_n$ with $p \in \partial^+(w, y, B_n)$. Since this holds for every $|p| < a/|\mathrm{diam}(B_n)|$, this implies that

$$|\partial^+(w, B_n)| > C_d \frac{a^d}{\mathrm{diam}(B_n)^d}.$$

And so rearranging, we get

$$a \leq |\partial^+(w, B_n)|^{1/d} C_d \mathrm{diam}(B_n) \leq C_d n \mu(B_n, \omega, l, M)^{1/d}$$

The proof for μ^* is identical.

□

Next we introduce the concave envelope of a subsolution. First, we extend the discrete domain Q_n and its closure to their convex hulls: $\mathcal{Q}_n := \mathbf{conv}\, Q_n$ and $\bar{\mathcal{Q}}_n := \mathbf{conv}\, \bar{Q}_n$. Then, define the concave envelope of w by, $\Gamma_w : \bar{\mathcal{Q}}_n \to \mathbb{R}$,

$$\Gamma_w(x) = \inf_{p \in \mathbb{R}^d} \max_{y \in \bar{Q}_n} \left(w(y) + p \cdot (x - y) \right),$$

noting that Γ_w is the point-wise least concave function so that on $\bar{Q}_n$, $\Gamma_w \geq w$. We recall a useful representation of the concave envelope.

Proposition 4.1. *We can alternatively represent*

$$\Gamma_w(x) = \sup\{\sum_{i=1}^{d+1} \lambda_i w(x_i) : x_i \in \bar{Q}_n, \sum_{i=1}^{d+1} \lambda_i x_i = x, \lambda_i \in [0,1], \sum_{i=1}^{d+1} \lambda_i = 1\},$$

and if

$$\Gamma_w(x) = \sum_{i=1}^{d+1} \lambda_i w(x_i),$$

then for each x_i, $\Gamma_w(x_i) = w(x_i)$ and Γ_w is linear in $\mathbf{conv}(x_1, \ldots, x_{d+1})$.

The next statement uses this representation to show that the measure of the supergradient set is preserved under the operation of taking the concave envelope.

Lemma 4.3. *If we define*

$$\partial^+(\Gamma_w, \mathcal{Q}_n) = \{p \in \mathbb{R}^d : \exists x \in \mathcal{Q}_n : \Gamma_w(x) + p \cdot (y - x) \geq \Gamma_w(y) : \text{ for all } y \in \bar{\mathcal{Q}}_n\},$$

then

$$\sum_{x \in Q_n} |\partial^+(w, x, Q_n)| = |\partial^+(w, Q_n)| = |\partial^+(\Gamma_w, \mathcal{Q}_n)| = \sum_{\{x : \Gamma_w(x) = w(x)\}} |\partial^+(\Gamma_w, x, Q_n)|.$$

Proof. We split the proof into two steps.

Step 1. We first show that

$$|\partial^+(w, Q_n)| = \sum_{x\in Q_n} |\partial^+(w, x, Q_n)|,$$

which follows from the proof in the continuous setting: since

$$|\partial^+(w, Q_n)| = | \cup_{x\in Q_n} \partial^+(w, x)|,$$

it suffices to show that

$$S = \{p \in \mathbb{R}^d : \text{ there exists } x, y \in Q_n,\ x \neq y \text{ and } p \in \partial^+(u_l, x) \cap \partial^+(u_l, y)\}$$

has measure zero. Denote the discrete Legendre transform $w^* : \mathbb{R}^d \to \mathbb{R}$ by $w^*(p) := \min_{x\in\bar{Q}_n}(x \cdot p - w(x))$. This is a concave, finite function as Q_n is bounded and it is a minimum of affine functions. Further, if $p \in \partial^+(w, x)$, then $w^*(p) = x\cdot p - w(x)$. And hence, if $p \in S$, then $w^*(p) = x_1 \cdot p - w(x_1) = x_2 \cdot p - w(x_2)$ for $x_1 \neq x_2$. This implies that $w^*(p)$ is not differentiable at p. But, since w^* is concave it is differentiable almost everywhere, which implies S has measure zero since it is a subset of a measure zero set. This completes the proof of Step 1.

Step 2. We now show that

$$|\partial^+(w, Q_n)| = |\partial^+(\Gamma_w, \mathcal{Q}_n)| = \sum_{\{x:\Gamma_w(x)=w(x)\}} |\partial^+(\Gamma_w, x, Q_n)|.$$

First consider $p \in \partial^+(w, x)$ and the affine function $L(y) = w(x) + p\cdot(y - x)$. By definition of the concave envelope, $\Gamma = \min(\Gamma, L)$, and so $p \in \partial^+(\Gamma_w, \mathcal{Q}_n)$. Next, suppose $p \in \partial^+(\Gamma_w, x)$ for $x \in \mathcal{Q}_n$ and

$$\Gamma_w(x) = \sum_{i=1}^{k} \lambda_i w(x_i),$$

for $\lambda_i > 0$, $x_i \in \bar{Q}_n$, and some $k \geq 1$. This implies that $p \in \partial^+(\Gamma_w, x_i)$ for some x_i, if $k = 1$ and $x_i = x \in Q_n$, we are done, so suppose not. Then, we have some $x_i \neq x$ and $p \in \partial^+(\Gamma_w, x) \cap \partial^+(\Gamma_w, x_i)$. However, the argument in Step 1 implies that such p have measure zero. This also implies the third equality.

□

The arithmetic geometric-mean inequality and the lower bound on the Laplacian of subsolutions imply an upper bound on μ.

Lemma 4.4. *There is $C := C_{\eta_{\max},l,M,d}$ and $C^* := C^*_{\eta_{\min},l,M,d}$ for which*

$$\mu(Q_n, \omega, l, M) < C|Q_n|,$$

$$\mu^*(Q_n, \omega, l, M) < C^*|Q_n|.$$

For $l \geq \eta_{\max} - Tr(M)$

$$\mu(Q_n, \omega, l, M) = 0$$

and for $l \leq \eta_{\min} - (2d - 1) - Tr(M)$

$$\mu^*(Q_n, \omega, l, M) = 0.$$

Proof. Let $w := u - q_l - q_M \in S(A, \omega, l, M)$. Since u is legal, $\Delta_{\mathbb{Z}^d} u \geq \min(-\eta_{\max}, 0)$ in Q_n. Using $\Delta_{\mathbb{Z}^d} q_M = \mathrm{Tr}(M)$, we get $\Delta_{\mathbb{Z}^d} w \geq l + \mathrm{Tr}(M) - \eta_{\max}$. Choose, $x \in Q_n$ so that $|\partial^+(w, x)| > 0$. For $p \in \partial^+(w, x)$, by definition

$$w(x) + p \cdot (x + e_i - x) \geq w(x + e_i),$$

and

$$w(x) + p \cdot (x - e_i - x) \geq w(x - e_i).$$

Putting these two inequalities together, we get for each direction $i = 1, \ldots, d$,

$$w(x + e_i) - w(x) \leq p_i \leq w(x) - w(x - e_i). \tag{8}$$

And so,

$$|\partial^+(w, x)| \leq \prod_{i=1}^{d} (2w(x) - w(x - e_i) - w(x + e_i)) = \prod_{i=1}^{d} (-\Delta_i w).$$

The inequality (8) implies $\Delta_i w \geq 0$, and so an application of the arithmetic geometric mean inequality yields

$$-\Delta_{\mathbb{Z}^d} w = \sum_{i=1}^{d} (-\Delta_i w) \geq d \left(\prod_{i=1}^{d} (-\Delta_i w) \right)^{1/d}.$$

And so

$$|\partial^+(w, x)| \leq d^{-d} (-\Delta_{\mathbb{Z}^d} w)^d \leq d^{-d} (\eta_{\max} - \mathrm{Tr}(M) - l)^d, \tag{9}$$

which implies the claim by Lemma 4.3. The other direction is similar. □

Our next lemma uses the bound on the discrete Laplacian of subsolutions together with the representation of Γ_l to establish a bound on $\mathrm{diam}(\partial^+(\Gamma_w, x))$ when $\Gamma_w(x) = w(x)$.

Lemma 4.5. *Let $w \in S(Q_n, \omega, l, M)$. There exists a constant $C := C_{l,M,\eta_{\min}}$ so that for every $x_0 \in \{x \in Q_n : \Gamma_w(x) = w(x)\}$*

$$\mathit{diam}(\partial^+(\Gamma_w, x_0)) \leq C.$$

Proof. Suppose $\Gamma_w(x) = w(x)$ and note that since $\Gamma_w \geq w$ on $\bar{Q}_n$,

$$\begin{aligned} -\Delta_i \Gamma_w(x) &= 2\Gamma_w(x) - \Gamma_w(x + e_i) - \Gamma_w(x - e_i) \\ &= 2w(x) - \Gamma_w(x + e_i) - \Gamma_w(x - e_i) \\ &\leq 2w(x) - \Gamma_w(x + e_i) - \Gamma_w(x - e_i) \\ &= -\Delta_i w(x), \end{aligned}$$

which shows $-\Delta_i w(x) \geq 0$. And so,

$$-\Delta_i \Gamma_w(x) \leq -\Delta_i w(x) \leq -\Delta w(x) \leq C_{l,M,\eta_{\min}}$$

This fact, together with the linearity of Γ_w between contact points shows the claim. □

Next, we use the following consequence of the discrete Harnack inequality [LL10] to regulate the growth of the concave envelope in balls around contact points.

Proposition 4.2 (Lemma 2.17 in [LP10])**.** *Fix $0 < \beta < 1$. For any $f : \mathbb{Z}^d \to \mathbb{R}$ nonnegative, with $f(0) = 0$ and $|\Delta_{\mathbb{Z}^d} f| \leq \lambda$ in $B(0, R)$ there is a constant $C_{\beta,\lambda}$ so that*

$$f(x) \leq C_{\beta,\lambda}|x|^2 \tag{10}$$

for $x \in B(0, \beta R)$.

Using this, we show the following.

Lemma 4.6. *Suppose $w \in S(Q_n, \omega, l, M) \cap S^*(Q_n, \omega, l, M)$. There exists $C := C_{\eta_{\min},\eta_{\max},l,M,d}$ so that for every $x_0 \in \{x \in Q_n : \Gamma_w(x) = w(x)\}$, $p \in \partial^+(\Gamma_w, x_0)$, and every $12 < 2r < R$ with $B_R \subset Q_n$,*

$$\partial^+(\Gamma_w, B_r(x_0)) \subset B_{C(r+2)^2}(p). \tag{11}$$

Proof. By an affine transformation and translation, we can assume $x_0 = 0$, $p = 0$, and $w(0) = \Gamma_w(0) = 0$. Suppose $q \in \partial^+(\Gamma_w, B_r(0))$ and that $|q|_\infty = |q_i|$ for some direction i. As Γ_w is concave, by moving in the e_i direction, we can find $y_0 \in B_{r+2} \cap \mathbb{Z}^d$ and $q^{(2)} \in \partial^+(\Gamma_w, y_0)$ so that $|q_i^{(2)}|_\infty \geq |q_i|_\infty$, so it suffices to bound $|q_i^{(2)}|_\infty$. Write $q = q^{(2)}$.

As $w \in S(Q_n, \omega, l, M) \cap S^*(Q_n, \omega, l, M)$, we have $|\Delta_{\mathbb{Z}^d} w| \leq C$. Hence, by Proposition 4.2 and the definition of Γ_w, for every $y \in B_{4r/3}$,

$$\Gamma_w(y) \geq w(y) \geq -C|y|^2.$$

Then,

$$\Gamma_w(y_0) + q \cdot (y_0 - e_i - y_0) \geq \Gamma_w(y_0 - e_1)$$

becomes

$$\Gamma_w(y_0) - q_i \geq \Gamma_w(y_0 - e_i)$$

and so since $\Gamma_w \leq 0$,

$$q_i \leq \Gamma_w(y_0) - \Gamma_w(y_0 - e_i) \leq C|y_0|^2 \leq C(r + 2)^2,$$

Repeating this for $-q_i$ completes the proof. ∎

4.4. **Convergence of μ.** We next use the multi-parameter subadditive ergodic theorem to show almost sure convergence of μ.

Lemma 4.7. *For each M and l, there exists constants $0 \leq \mu(l, M) \leq C_{M,l,\eta_{\min}}$, $0 \leq \mu(l, M)^* \leq C^*_{M,l,\eta_{\max}}$ and an event $\Omega_{l,M}$ of full probability so that for each $\omega \in \Omega_{l,m}$ and bounded Lipschitz set W,*

$$\mu(l, M) := \lim_{n\to\infty} \frac{\mu(nW \cap \mathbb{Z}^d, \omega, l, M)}{|nW \cap \mathbb{Z}^d|}$$

and

$$\mu^*(l, M) := \lim_{n\to\infty} \frac{\mu^*(nW \cap \mathbb{Z}^d, \omega, l, M)}{|nW \cap \mathbb{Z}^d|}.$$

Proof. Fix M and l and let $W_n = nW \cap \mathbb{Z}^d$. We apply the multiparameter subadditive ergodic theorem as stated in [AS14b] to

$$f(W_n, \omega) = \sup\{|\partial^+(w, W_n)| : w \in S(W_n, \omega, l, M)\}.$$

By Lemma 4.4, $0 \le f(W_n,\omega) \le C|W|n^d$ for all ω. Also stationarity and ergodicity of f follow from the corresponding assumptions on the random background η and the probability space. It remains to check subadditivity for connected subsets of $\mathbb{Z}^d$. Let A be a connected subset of $\mathbb{Z}^d$ and let $A_1,\dots,A_k$ be pairwise disjoint connected subsets of $\mathbb{Z}^d$ which satisfy $\cup_{j=1}^k A_j = A$.

Let u be a legal toppling function in A. For each A_i, we can decompose u into illegal topplings on ∂A_i followed by legal topplings inside A. Hence $u - q_l - q_M$ is a subsolution for each A_i. Suppose $p \in \partial^+(u - q_l - q_M, x, A)$. Then since $x \in A_i$ for some i, we have $p \in \partial^+(u - q_l - q_M, x, A_i)$. And so for each $x \in A$

$$\partial^+(u - q_l - q_M, x, A) \subset \partial^+(u - q_l - q_M, x, A_i),$$

hence by Lemma 4.3 and disjointness of the A_i, this implies

$$|\partial^+(u - q_l - q_M, A)| = \sum_{x\in A} |\partial^+(u - q_l - q_M, x, A)| \le \sum_{j=1}^k \sum_{x\in A_j} |\partial^+(u - q_l - q_M, A_j, x)| = \sum_{j=1}^k |\partial^+(u - q_l - q_M, A_j)|.$$

Since this holds for any legal toppling u of A, taking the supremum of both sides implies that

$$f(A,\omega) \le \sum_{j=1}^k f(A_j,\omega),$$

which completes the proof. The exact same argument, using the fact that any stabilizing toppling for A also stabilizes each A_i shows convergence of μ^*. □

As in the continuous case, if both $\mu(l, M)$ and $\mu^*(l, M)$ are 0, we have a comparison principle in the limit. This will allow us to identify the effective equation; and hence is what we carry out in the next section.

5. The effective equation

5.1. **Finding the effective equation.** We will identify, for each parabola M, the largest real number l_M, so that in the limit $\mu(l_M, M) = \mu^*(l_M, M) = 0$. This then defines the effective equation $\bar{F}_\eta$. To show that such a number exists, since μ is bounded, it suffices to show that μ is continuous in the limit. In the continuum, this is done with an argument that utilizes a certain regularity of concave envelopes of subsolutions which we do not have. This difficulty is circumvented by a consequence of the stationarity of η, Lemma 5.2. We first prove the easier direction of continuity, monotonicity of the curvature.

Lemma 5.1. *For $s \ge 0$,*

$$\mu(Q_n, \omega, l + s, M) \ge \mu(Q_n, \omega, l, M).$$

and

$$\mu^*(Q_n, \omega, l - s, M) \ge \mu^*(Q_n, \omega, l, M).$$

Proof. Let $w \in S(Q_n, \omega, l, M)$. By Lemma 4.3, it suffices to show

$$|\partial^+(w, x, Q_n)| \le |\partial^+(w - q_s, x, Q_n)|,$$

for each $x \in Q_n$. Choose $p \in \partial^+(w, x)$, if this is not possible, we are done. Then, for each $y \in \bar{Q}_n$,

$$\begin{aligned} w(x) + (p - sx) \cdot (y - x) + \frac{1}{2}s(|y|^2 - |x|^2) &= w(x) + p \cdot (y - x) - sxy + s|x|^2 + \frac{1}{2}s|y|^2 - \frac{1}{2}s|x|^2 \\ &\geq w(y) + \frac{1}{2}s|x - y|^2 \\ &\geq w(y). \end{aligned}$$

And so rearranging, we get

$$w(x) - q_s(x) + (p - sx) \cdot (y - x) \geq w(y) - q_s(y),$$

meaning $p - sx \in \partial^+(w - q_s, x, Q_n)$. Since this holds for all $p \in \partial^+(w, x, Q_n)$, this implies

$$|\partial^+(w, x, Q_n)| \leq |\partial^+(w - q_s, x, Q_n)|.$$

The proof for μ^* is identical. □

In the next result, we show that if μ is strictly positive in the limit, then a subsolution must curve downwards in every direction.

Lemma 5.2. *Suppose that $\alpha := \mu(l_M, M) > 0$. For each ω in a set $\Omega_{l,M}$ of full probability and $0 < \beta < 1$, there exists a constant $C := C_{\eta_{\min}, l, M, d, \beta, \omega}$ so that the following holds. For every Lipschitz subset W, $W_n := nW \cap \mathbb{Z}^d$, there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, there exists $w_n \in S(W_n, \omega, l, M)$ so that for each $x_0 \in \{\Gamma_{w_n} = w_n\} \cap W_{(1-\beta)n}$ and $p_0 \in \partial^+(w_n, x_0, W_n)$*

$$w_n(y) \leq w_n(x_0) + p_0 \cdot (y - x) - Cn^2$$

for all $y \in \partial W_n$.

Proof. By rescaling and approximation, it suffices to prove the claim for $W_n = Q_{nm}$. As $\alpha > 0$, by the subadditive ergodic theorem, we can choose a set of full probability $\Omega_{l,M}$, so that for each $\omega \in \Omega_{l,m}$ there exists n_0, m so that for all $n \geq n_0$, there exists $w_n \in S(Q_{nm}, \omega, l, M)$ which satisfies

$$\frac{\alpha}{2} \leq \frac{|\partial^+(\Gamma_{w_n}, Q_m(x))|)}{|Q_m|} \leq \frac{\mu(Q_m(x))}{|Q_m|} \leq 2\alpha \text{ for all } x \text{ with } Q_m(x) \subset Q_{nm}, \tag{12}$$

(see for example Lemma 3.2 in [AS14c]).

In light of Lemma 4.1, we can assume $w \in S(Q_{nm}, \omega, l, M) \cap S^*(Q_{nm}, \omega, l, M)$. As $|\partial^+(w, Q_m)| > 0$, we can find $w_n(x_0) = \Gamma_{w_n}(x_0)$ with $|\partial^+(w_n, x_0)| > 0$. By a translation and affine transformation, we can suppose $\Gamma_{w_n}(x_0) = 0$, $0 \in \partial^+(\Gamma_{w_n}, x_0)$, and $x_0 = 0$. Then, it suffices to show

$$\Gamma_w(y) \leq -\alpha C n^2/m \tag{13}$$

for $y \in \partial Q_{nm}$.

Consider $\phi_n : B_{1-\beta} \to \mathbb{R}$ which we define as a rescaled version of the inner part of the concave envelope,

$$\phi_n := \frac{1}{n^2 m^2}\Gamma_{w_n}([nmx]), \text{ for } x \in Q_{nm\beta}.$$

By Lemma 4.6 applied everywhere, ϕ_n is equicontinuous and uniformly bounded. Hence, for some subsequence, ϕ_n converges uniformly to a continuous, differentiable concave function

ϕ with Lipschitz gradient. By the area formula for Lipschitz functions, this implies that for every Borel measurable $A \subset B_{1-\beta}$,

$$|\partial^+(\phi, A)| = \int_A \det -D^2\phi(x)dx.$$

Further, by weak convergence of the Monge-Ampere measures (Lemma 2.2 in [TW08]), (12), and the Lebesgue differentiation theorem,

$$C \geq \det -D^2\phi \geq C\alpha.$$

This also implies by Lemma 4.6 that $D^2\phi \leq m\text{Id}$ and so $D^2\phi \geq C\alpha/m\text{Id}$. Taking a further subsequence if necessary and undoing the scaling, we have (13). □

We next use Lemma 5.2 to show Lipschitz continuity of μ.

Lemma 5.3. *For each $\omega \in \Omega_{l,M}$, an event of probability 1, there exists a constant $C := C_{\eta_{\min},l,M,d,\beta,\omega}$ so that for all $n \geq n_0$ and $0 < s < 1$,*

$$\mu(Q_n, \omega, l, M) \leq \mu(Q_n, \omega, l - s, M) + sC|Q_n|.$$

Proof. Choose $\omega \in \Omega_{l,M}$ and C from Lemma 5.2. If $\mu(l, M) < Cs$, taking n larger if needed, we automatically have the bound by the ergodic theorem, so suppose not. We will show that after removing a portion of the square proportional to s, the set of slopes remaining must be in $\partial^+(w + q_s, Q_n)$ for all w close to achieving the supremum in $\mu(Q_n, \omega, l, M)$.Take

$$A := Q_{Csn},$$

so that we can apply Lemma 5.2 to all contact points $x \in A$. As a consequence, we can find $w \in S(Q_n, \omega, l, M)$ so that for every $x \in A$ with $\Gamma_w(x) = w(x)$ and $p \in \partial^+(w, x)$ for all $y \in \partial Q_n$,

$$w(x) + p \cdot (y - x) \geq w(y) + q_s(y). \tag{14}$$

Hence, the argument in the proof of Lemma 4.2 shows that $p \in \partial^+(w + q_s, Q_{nm})$ and since this applies for all such p,

$$\partial^+(w, A) \subseteq \partial^+(w + q_s, Q_n).$$

Further, using Lemma 4.4,

$$|\partial^+(w, Q_n \backslash A)| \leq sC|Q_n|,$$

which completes the proof. □

Due to Lemma 4.7, the above results show continuity of the limiting μ at each fixed l. Repeating this for every rational l in the interval specified by Lemma 4.4 and using the intermediate value theorem, we can choose the largest $l_M \in \mathbb{R}$ so that in the limit,

$$\mu(l_M, M) = \mu^*(l_M, M),$$

then define the *effective equation* uniquely as

$$\bar{F}_\eta(M) = l_M.$$

5.2. **Basic properties of the effective equation.** Here we show that the effective equation is bounded, degenerate elliptic, and Lipschitz continuous. This together with the fact any legal stabilizing toppling function has bounded Laplacian is enough to ensure that we have a comparison principle for solutions to the effective equation (see, for example, [RCC95] and the proof in Section 6.4).

Lemma 5.4. *For every $M, N \in \mathbf{S}^d$, the following hold.*

1. *Degenerate elliptic: If $M \leq N$, $\bar{F}_\eta(M) \geq \bar{F}_\eta(N)$.*
2. *Lipschitz continuous: $|\bar{F}_\eta(M) - \bar{F}_\eta(N)| \leq C|M-N|_2$.*
3. *Bounded: $|\bar{F}_\eta(M)| < \infty$.*

Proof. We show the first inequality. Suppose $N = M + A$ with $A \geq 0$. The proof of Lemma 5.1, using $q_A \geq 0$ in place of $q_s \geq 0$, shows that $\mu(l_M, M+A) \geq \mu(l_M, M)$ and $\mu^*(l_M, M+A) \leq \mu(l_M, M) = 0$. Hence, $l_{M+A} \leq l_M$ and so $\bar{F}_\eta(M+A) \leq \bar{F}_\eta(M)$.

For the second inequality, first rewrite,

$$\mu(l_M, M) = \mu(l_M, N + (M-N)) = \mu(l_M - |M-N|_2, N + (M-N) + |M-N|_2 I),$$

then observe that $(M-N) + |M-N|_2 I \geq 0$. Hence, by the argument in the first paragraph,

$$\mu(l_M - |M-N|_2, N + ((M-N) + |M-N|_2 I)), \geq \mu(l_M - |M-N|_2, N),$$

and so

$$0 = \mu(l_M, M) \geq \mu(l_M - |M-N|_2, N).$$

Similarly,

$$0 = \mu^*(l_M, M) \geq \mu^*(l_M + |M-N|_2, N),$$

meaning

$$|\bar{F}_\eta(M) - \bar{F}_\eta(N)| \leq 2|M-N|_2.$$

The last statement follows due to the construction of $\bar{F}_\eta$ using Lemma 4.4. □

6. Proof of the Theorem

For each n, recall that

$$v_n = \min\{v : \mathbb{Z}^d \to \mathbb{N} : \Delta_{\mathbb{Z}^d} v_n + \eta I(\cdot \in W_n) \leq 2d-1\},$$

is the odometer function for the sandpile on W_n with the *open* boundary condition and $\bar{v}_n = n^{-2} v_n([nx])$ is its rescaled linear interpolation. We start by showing that $\bar{v}_n$ is equicontinuous and bounded. Then, we show that the condition $\mathbf{E}(\eta(0)) > 2d-1$ implies $v_n \geq 1$ in $W_{n-o(n)}$, enabling an essential tool in the proof of Lemma 6.5, (Dhar's burning algorithm, Lemma 3.2). We then conclude by showing that every scaled subsequence converges to the same limit.

6.1. **An upper bound on the odometer function.**

Lemma 6.1. *For every subsequence $n_k \to \infty$ there is a subsequence n_{k_j} and a function $\bar{v} \in C(\mathbb{R}^d)$ so that $\bar{v}_{n_{k_j}} \to \bar{v}$ uniformly as $j \to \infty$.*

Proof. We show boundedness of $\bar{v}_n$ by constructing a toppling function which stabilizes $\eta_{\max}$ and hence η. Since $\eta_{\max}$ is constant, we can stabilize by toppling 'one dimension at a time', a trick from [FLP10], and restated below for the reader. (Note one could also compare to the divisible sandpile as in [LP09] to get a tighter bound).

Lemma 6.2 (Lemma 3.3 in [FLP10])**.** *Let $\ell \in \mathbb{N}$ be given. Pick $k \in \mathbb{N}$ so that $R_k := \eta_{\max} - (2d - k) = 2r$ for some $r \in \mathbb{N}$. Then, there exists $g : \mathbb{Z} \to \mathbb{N}$ so that*

$$\Delta_{\mathbb{Z}^d} g = f,$$

where $f : \mathbb{Z} \to \mathbb{Z}$ is given by

$$f(x) = \begin{cases} -R_k & \text{for } |x| \leq \ell \\ 2 & \text{for } \ell < |x| \leq \ell(r+1) \\ 1 & \text{for } \ell(r+1) < |x| \leq \ell(r+1) + r \end{cases}$$

Moreover, g is supported in $I = \{x \in \mathbb{Z} : |x| < \ell(r+1) + r\}$ and there exists $C := C_r$ for which

$$g(x) \leq Cx^2. \tag{15}$$

Cover W_n with a box of side length $C_{d,W}n$ for some $C_{d,W} \in \mathbb{N}$. Choose g from the above with $\ell = C_{d,W}n$ and for $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, define

$$u_i(x) = g(x_i),$$

and observe that by definition of g, $\Delta_{\mathbb{Z}^d} u_i + \eta_{\max} \leq 2d - 1$. Hence, by the least action principle, as $\min(u_1, \dots, u_d)$ is also stabilizing,

$$v_n(x) \leq \min(u_1(x), \dots, u_d(x)) \leq C_d |x|^2.$$

Hence, $\bar{v}_n \leq C_d$ and is supported in $Q_{C_{d,W}n}$. We have equicontinuity since $|\Delta_{\mathbb{Z}^d} v_n| \leq C_{d,\eta_{\min},\eta_{\max}}$ ([KT05]). The Arzela-Ascoli theorem now implies the claim. □

6.2. **A lower bound on the odometer function.**

In this subsection, we use a comparison principle argument to show that on an event of probability 1, $v_n \geq 1$ in $Q_{n-o(n)}$ As a corollary, this argument gives a quantitative proof of the (now classical) fact that if $\mathbf{E}(\eta(0)) > 2d - 1$ then η is almost surely *exploding*, (see [FMR09]). The technique takes inspiration from Lemma 4.2 in [LP09]. In essence, the proof is a comparison of v_n with the odometer function for the random divisible sandpile with threshold $2d-1$. See Section 7 for more on the random divisible sandpile, including a proof of convergence which uses Lemma 6.3.

We start by briefly recalling the Green's function for simple random walk on $\mathbb{Z}^d$ and its estimates, these results can be found in [LL10, LP10]. Let S_n be simple random walk started at the origin in $\mathbb{Z}^d$ and for $d \geq 3$, let

$$G(x) = \frac{1}{2d} \mathbf{E} \sum_{n=0}^{\infty} I(S_n = x),$$

and for $d = 2$, let

$$a(x) = \frac{1}{4}\sum_{n=0}^{\infty}[P(S_n = 0) - P(S_n = x)].$$

Next, define for each $n \in \mathbb{N}$,

$$g_n(x) = \begin{cases} -a(x) + a(ne_1) & \text{for } d = 2 \\ G(x) - G(ne_1) & \text{for } d \geq 3, \end{cases}$$

so that

$$\Delta_{\mathbb{Z}^d} g_n = -\delta_0$$

We use the following asymptotic estimate on g_n

$$g_n(x) = \begin{cases} -C_d \log|x| - a(ne_1) + O(|x|^{-2}) & \text{for } d = 2 \\ C_d|x|^{2-d} + G(ne_1) + O(|x|^{-d}) & \text{for } d \geq 3, \end{cases} \tag{16}$$

and the following difference estimate,

$$|g_n(x) - g_n(x + e_i)| \leq C_d|x|^{1-d}. \tag{17}$$

Next, define for each n

$$r_n(x) := \sum_{y \in Q_n} g_n(x - y)\eta(y),$$

$$d_n(x) := \sum_{y \in Q_n} g_n(x - y)\,\mathbf{E}(\eta(0)).$$

The next lemma uses these estimates together with the ergodic theorem to show that r_n and d_n are identical in the scaling limit.

Lemma 6.3. *For each $\omega \in \tilde{\Omega}_0$, an event of probability 1, there is a constant $C := C_{d,\omega}$ so that the following holds. For each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$,*

$$\sup_{x \in \bar{Q}_n} |r_n(x) - d_n(x)| \leq \epsilon C n^2 \tag{18}$$

Proof. Let $\epsilon > 0$ be given. By the multiparameter ergodic theorem, on an event of full probability, $\tilde{\Omega}_0$, for $\omega \in \tilde{\Omega}_0$, we can find m and n_0 so that for all $n \geq n_0$,

$$|Q_m|\,(\mathbf{E}(\eta(0)) - \epsilon) \leq \sum_{y \in Q_m(x)} \eta(y) \leq |Q_m|\,(\mathbf{E}(\eta(0)) + \epsilon) \text{ for all } Q_m(x) \subset Q_{mn}. \tag{19}$$

By approximation we then consider Q_{nm} instead of Q_n so that

$$r_n(x) - d_n(x) = \sum_{Q_m \subset Q_{mn}} \sum_{y \in Q_m} g_{nm}(x - y)(\eta(y) - \mathbf{E}(\eta(0))), \tag{20}$$

where the outer sum is over a fixed partition of disjoint cubes of radius m which cover Q_{nm}. The rest of the argument is roughly the following. Imagine a non-random sandpile, η_{avg}, in which $\mathbf{E}(\eta(0))$ grains of sand are at each coordinate in $\mathbb{Z}^d$. In each small cube, Q_m, we try to rearrange the grains of sand in the random sandpile, η, to match the deterministic sandpile η_{avg}. It's possible that there aren't enough grains to do this, so at this point, we then add just enough grains to turn it into η_{avg}. By (19), we only need to add at most an additional $\epsilon|Q_m|$ grains of sand. Hence, by the asymptotic estimate, the total cost associated with adding grains is of order $\epsilon O(n^2)$, by the difference estimate, the total cost of rearranging grains is of order $o(n^2)$, leading to (18).

Here are the details, start by fixing $x \in \bar{Q}_{nm}$. As estimates for $g_n(\cdot)$ blow up near the origin, we start by removing a constant number of cubes which are close to x from consideration,

$$A_x = \{Q_m \subset Q_{mn} : \inf_{y \in Q_m} |x-y| \leq m\}.$$

We can provide a rough upper bound on the contribution from these cubes in (20), using $\sup_{x \in Q_m} g_{nm}(x) = C_d nm$ to get,

$$\sum_{Q_m \in A_x} \sum_{y \in Q_m} g_{nm}(x-y)(\eta(y) - \mathbf{E}(\eta(0))) \leq C_d nm|Q_m|(\eta_{max} + \mathbf{E}(\eta(0))) = o(n^2). \tag{21}$$

Next consider any cube, Q_m, in A_x^c and iterate (17) so that

$$\sup_{z,y \in Q_m} |g_{nm}(z) - g_{nm}(y)| \leq C_d m \sup_{z \in Q_m} |z|^{1-d}, \tag{22}$$

and note that an integral approximation of (16) yields

$$\sum_{y \in Q_{nm}} g_{nm}(x-y) = C_d n^2 \tag{23}$$

Putting this all together and making another integral approximation,

$$\begin{aligned}\sum_{Q_m \subset Q_{mn}} \sum_{y \in Q_m} g_{nm}(x-y)\eta(y) &\leq \epsilon C n^2 + Cnm + \sum_{Q_m \subset Q_{mn}} \sum_{y \in Q_m} g_{nm}(x-y)\, \mathbf{E}(\eta(0)) \\ &= \epsilon C n^2 + \sum_{Q_m \subset Q_{mn}} \sum_{y \in Q_m} g_{nm}(x-y)\, \mathbf{E}(\eta(0)).\end{aligned}$$

The other direction follows similarly.

□

We next use this to provide the desired lower bound on v_n.

Lemma 6.4. *For each $\omega \in \tilde{\Omega}_0$, an event of probability 1, and each $\epsilon > 0$, there exists n_0 so that for all $n \geq n_0$, the odometer function v_n for η in Q_n satisfies*

$$v_n \geq 1 \textit{ for all } x \in Q_{(1-\epsilon)n}$$

Proof. Let $\epsilon > 0$ be given and

$$\delta' := (\mathbf{E}(\eta(0)) - 2d - 1) > 0.$$

Choose $\tilde{\Omega}_0$, C, and n_0 from Lemma 6.3 with $\epsilon' > 0$ small to be chosen below. Let v_n be the odometer function for η restricted to B_n,

$$v_n \in \mathcal{L}(\eta, B_n) \cap \mathcal{S}(\eta, B_n) \text{ and } v_n = 0 \text{ on } \partial B_n.$$

Redefine,

$$r_n(x) := \sum_{y \in B_n} g_n(x-y)\eta(y),$$

$$d_n(x) := \sum_{y \in B_n} g_n(x-y)\, \mathbf{E}(\eta(0)),$$

so that the scaling limit of d_n is radially symmetric.

As $v_n - r_n - q_{2d-1}$ is superharmonic in B_n, for $x \in B_n$,

$$v_n(x) - r_n - q_{2d-1} \geq \min_{y \in \partial B_n} v_n(y) - r_n(y) - q_{2d-1},$$

then since $v_n = 0$ on ∂B_n, we have

$$v_n - r_n(x) - q_{2d-1} \geq -(2d-1)\frac{1}{2}n^2 + \min_{y \in \partial B_n} -r_n(y) + o(n^2).$$

Radial symmetry of the scaling limit of d_n and Lemma 6.3 implies that

$$\begin{aligned} v_n(x) &\geq d_n(x) + q_{2d-1} - (2d-1)\frac{1}{2}n^2 + \min_{y \in \partial B_n} -d_n(y) + \epsilon' C n^2 \\ &= d_n(x) + q_{2d-1} - (2d-1)\frac{1}{2}n^2 + \epsilon' C n^2 \end{aligned}$$

We also know that

$$d_n(x) + q_{2d-1+\delta'}$$

is superharmonic in B_n and so

$$d_n(x) + q_{2d-1+\delta'} \geq \min_{y \in \partial B_n} d_n(y) + (2d-1+\delta')n^2/2$$

Using again $d_n(y) = o(n^2)$ on ∂B_n,

$$v_n(x) \geq \delta'/2(n^2 - |x|^2) + \epsilon' C n^2.$$

In particular, we can choose ϵ' small and n large so that so that

$$v_n(x) \geq 1$$

for $x \in B_{(1-\epsilon)n}$. The extension to $Q_{(1-\epsilon)n}$ is done by a covering argument and the Abelian property. □

6.3. **A comparison principle in the limit.** In order to compare subsequential limits of odometer functions for different realizations of the random sandpile we must show that $\mu(l_M, M) = \mu^*(l_M, M) = 0$. The argument is roughly this: if both μ and μ^* are strictly positive in the limit, then there is a subsolution and supersolution whose difference bends upwards in every direction. However, when there are enough topples, this difference obeys a comparison principle on the microscopic scale, due to Proposition 3.2, and so this cannot happen.

Lemma 6.5. $\mu(l_M, M) = \mu^*(l_M, M) = 0$

Proof. We will show that it is impossible for both $\mu(l_M, M)$ and $\mu^*(l_M, M)$ to be strictly positive. Suppose for sake of contradiction that $\mu(l_M, M) = \mu^*(l_M, M) = \alpha > 0$. Then, by Lemma 4.1, there exist legal, stabilizing toppling functions u and v for which Lemma 5.2 holds. Moreoever, as μ is invariant under affine transformations, we can find affine functions L_u and L_v so that

$$\begin{aligned} &\inf_{x \in B_n} -(u - q_M + L_u)(x) = (u - q_M + L_u)(x_0) = 0 \\ &\inf_{x \in B_n} (v - q_M + L_v)(x) = (u - q_M + L_v)(x_0^*) = 0, \end{aligned} \tag{24}$$

for some $x_0, x_0^* \in Q_m \subset B_n$, where $m \in \mathbb{N}$ is large, and

$$-(u - q_M + L_u) \geq Cn^2 \text{ on } \partial B_n \tag{25}$$
$$(v - q_M + L_v) \geq Cn^2 \text{ on } \partial B_n.$$

Now, use the Abelian property, Proposition 3.3, to decompose u and v into the initial toppling of η and then topplings originating from the boundary, $u = u_1 + w$ and $v = v_1 + w$. Due to Lemma 6.4 and Proposition 3.4, (moving the boundary of the ball inwards if necessary and accumulating an $o(n^2)$ error), $\Delta_{\mathbb{Z}^d} w + \eta$ is recurrent. Now, approximate $L_v(x) = p \cdot x + r$ by

$$\tilde{L}_v(x) = [p] \cdot x + [r],$$

an integer valued function, (this approximation also incurs an $o(n^2)$ error). Repeat for L_u with $\tilde{L}_u$. Hence, by Proposition 3.2 and (25)

$$\begin{aligned}
\left((v + \tilde{L}_v - q_M) - (u + \tilde{L}_u - q_M)\right)(0) &= \left((v_1 + \tilde{L}_v) - (u_1 + \tilde{L}_u)\right)(0) \\
&\geq \inf_{y \in \partial B_n} \left((v_1 + \tilde{L}_v) - (u_1 + \tilde{L}_u)\right)(y) \\
&= \inf_{y \in \partial B_n} \left((v + L_v - q_M) - (u + L_u - q_M)\right)(y) - o(n^2) \\
&\geq Cn^2 - o(n^2)
\end{aligned}$$

However, this contradicts the Harnack inequality for n large. Indeed, due to (24) and

$$\max(|\Delta_{\mathbb{Z}^d}(v - q_M + L_v)|, |\Delta_{\mathbb{Z}^d}(u - q_M + L_u))| \leq C,$$

we can apply the Harnack inequality, Lemma 4.2, to see

$$\left((v + L_v - q_M) - (u + L_u - q_M)\right)(0) \leq Cm^2, \tag{26}$$

as $x_0, x_0^* \in Q_m$.

□

6.4. **Proof of Theorem 1.1.** Choose Ω_0 to be the intersection of $\Omega_{l,M}$ in Lemma 4.7 over all $l \in \mathbb{R}$ and $M \in \mathbf{S}^d$ with rational entries and $\tilde{\Omega}_0$ from Lemma 6.4. Pick $\omega, \omega' \in \Omega_0$ and choose respectively two subsequences $\bar{v}_n$ and $\bar{v}'_n$ which converge uniformly to v and v'. Suppose for sake of contradiction that $v \neq v'$. Since $v = v' = 0$ outside B_R for some $R > 0$, we may assume without loss of generality that

$$\sup_{B_R}(v - v') > 0 = \sup_{\partial B_R}(v - v')$$

We restate for the reader results contained in [PS13].

Lemma 6.6. [PS13]

(1) *There exists $a \in \mathbb{R}^d$ either in W or outside the closure of W so that $v(a) > v'(a)$, both v and v' are twice differentiable at a and $D^2(v - v')(a) < -\delta I'$ for some $\delta > 0$.*
(2) *For each $\epsilon > 0$, if a is outside the closure of W, we may select $u : \mathbb{Z}^d \to \mathbb{Z}$ such that*

$$\Delta_{\mathbb{Z}^d} u(x) \leq 2d - 1 \text{ and } u(x) \geq \frac{1}{2} x^T (D^2 v(a) - \epsilon I) x \text{ for all } x \in \mathbb{Z}^d .$$

(3) *For each $\epsilon > 0$, if a is in W, we may select $u : \mathbb{Z}^d \to \mathbb{Z}$ such that*

$$\Delta_{\mathbb{Z}^d} u(x) \leq 2d - 1 \text{ and } u(x) \geq \frac{1}{2} x^T (D^2 v(a) - \epsilon I) x + o(|x|^2) \text{ for all } x \in \mathbb{Z}^d \ .$$

Proof. The first and second statements are Lemma 4.1 and Proposition 2.5 in [PS13]. We sketch the third. For each $\epsilon > 0$, the proof of Lemma 4.1 in [PS13] gives a function

$$u : \mathbb{Z}^d \to \mathbb{Z}$$

with

$$u(x) \geq \frac{1}{2} x^T (D^2 v(x_0) - \epsilon I) x.$$

and

$$\Delta_{\mathbb{Z}^d} u + \tilde{\eta} \leq 2d - 1$$

where $\tilde{\eta}$ is a periodic tiling of η in B_{rn} for some $r > 0$ and $n \in \mathbb{N}$ large. Due to the ergodic theorem, picking n larger if necessary, we have

$$\sum_{x \in B_{rn}} \eta(x) \geq 2d - 1$$

Hence, by Rossin's observation [LPS16], as a sandpile configuration on $\mathbb{Z}^d$, $\Delta_{\mathbb{Z}^d} u$ is stabilizable, and so by toppling it, we find a bounded $w : \mathbb{Z}^d \to \mathbb{N}$ so that

$$\Delta_{\mathbb{Z}^d} (u + w) \leq 2d - 1,$$

and $(u + w)(x) = q_{D^2 v(a) - \epsilon}(x) + o(|x|^2)$.

□

If $a \in \mathbb{R}^d$ is outside the closure of W, the argument in Theorem 4.2 in [PS13] which uses Lemma 6.6 leads to a contradiction. So, it suffices to suppose $a \in W$. In this case, we cannot use the same argument to compare v and v' as they stabilize (possibly) different random sandpiles. Instead, we use μ to compare the two.

Since v' is twice differentiable at a, by Taylor's theorem,

$$v'(x) = \phi(x) + o(|x - a|^2)$$

where

$$q_M + L_\phi := \phi(x) := v'(a) + Dv'(a) \cdot (x - a) + \frac{1}{2}(x - a)^T D^2 v'(a)(x - a)$$

Pick the unique $l := \bar{F}_\eta(D^2 v'(a)) \in \mathbb{R}$ so that

$$\mu(l, D^2 v'(a)) = \mu^*(l, D^2 v'(a)) = 0.$$

Then, by Lemma 4.2, (recalling that μ is invariant under affine transformations), for all small $r > 0$,

$$\max_{x \in B_{rn}(a)} (v_n - q_M - nL_\phi - q_l)(x) \leq \max_{y \in \partial B_{rn}(a)} (v_n - q_M - nL_\phi - q_l)(y) + C_d n \mu(B_{rn}, \omega, 0, M)^{1/d}.$$

And so, after rescaling,

$$\max_{x \in n^{-1} B_{rn}(a)} (\bar{v}_n - \phi - q_l)(x) \leq \max_{y \in \partial n^{-1} B_{rn}(a)} (\bar{v}_n - \phi - q_l)(y) + \left(\frac{C_d n \mu(B_{rn}, \omega, 0, M)^{1/d}}{n^2} \right)$$

which implies by uniform convergence of $\bar{v}_n \to v$ and the ergodic theorem,

$$\sup_{x \in B_r(a)} (v - \phi - q_l)(x) \leq \sup_{y \in \partial B_r(a)} (v - \phi - q_l)(y).$$

In particular,

$$(v - v' - q_l)(a) = (v - \phi - q_l)(a) \tag{27}$$
$$\leq \sup_{y \in \partial B_r(a)} (v - \phi - q_l)(y)$$
$$= \sup_{y \in \partial B_r(a)} (v - v' - q_l)(y) + o(r^2)$$

Suppose $l \leq 0$, then (27) implies that that

$$(v - v')(a) \leq \sup_{y \in \partial B_r(a)} (v - v')(y) + o(r^2) - l/2r^2$$

however, for small $r > 0$, this contradicts that $v - v'$ has a strict local maximum at a, and so $l > 0$. Then by the ergodic theorem applied to μ^* and the other part of Lemma 4.2,

$$(v' - \phi)(a) \geq \inf_{y \in \partial B_r(a)} (v' - \phi)(a) + l/2r^2 - o(r^2).$$

However, since v' is twice differentiable at a,

$$(v' - \phi)(a) \leq \inf_{y \in \partial B_r(a)} (v' - \phi)(a) + o(r^2),$$

a contradiction for small $r > 0$.

7. Convergence of the random divisible sandpile

One of the challenges involved in the Abelian sandpile model is the integrality constraint on the odometer function. In the *divisible sandpile* model, this constraint is relaxed and piles are allowed to topple a fractional number of times. This relaxation enables the use of Green's functions estimates which leads to a more direct proof of convergence.

7.1. Description of the divisible sandpile.

We briefly describe the divisible sandpile, referring the interested reader to [LP10, LMPU16] for more details. Begin with some, possibly fractional, distribution of sand and holes, $\eta_0 : \mathbb{Z}^d \to \mathbb{R}$. A site $x \in \mathbb{Z}^d$ is unstable whenever $\eta_0(x) > 1$, in which case the *excess mass*, $1 - \eta_0(x)$, is equally distributed among the neighbors of x until every site is stable. The odometer function, v_0, then counts the total mass emitted by each site. Here, the starting point is also a discrete obstacle problem: the *least action principle for the divisible sandpile.*

Proposition 7.1 ([LMPU16])**.** $v_0 = \min\{f : \mathbb{Z}^d \to \mathbb{R}^+ : \Delta_{\mathbb{Z}^d} f + \eta_0 \leq 1\}$.

7.2. Convergence of the odometer function.

As in Section 2, we consider a stationary, ergodic, probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with Ω the set of all bounded backgrounds,

$$\eta_0 : \mathbb{Z}^d \to \mathbb{R}$$

for which

$$\sup_{x \in \mathbb{Z}^d} \eta_0(x) < \infty.$$

In this case, we do not require η to be high density, but we do assume for simplicity uniform boundedness: there exists $\eta_{\min_0}, \eta_{\max_0} \in \mathbb{R}$ so that for every $x \in \mathbb{Z}^d$,

$$\mathbf{P}\left[\eta_{\min_0} \leq \eta_0(x) \leq \eta_{\max_0}\right] = 1. \tag{28}$$

Let $W \subset \mathbb{R}^d$ be a bounded Lipschitz subset. For each $n \in \mathbb{N}$, let $W_n = \mathbb{Z}^d \cap nW$ denote the discrete approximation of V. Initialize the sandpile according to $\eta_0(x)$ in W_n and let v_{n_0} be its odometer function. Next, consider the averaged initial sandpile,

$$\eta_{avg} := \mathbf{E}\, \eta(0),$$

and the corresponding odometer function, v_{avg_n} for η_{avg} in W_n. For the reader's convenience, we restate Lemma 6.3. Let

$$r_n(x) := \sum_{y \in W_n} g_n(x-y)\eta(y),$$

$$d_n(x) := \sum_{y \in W_n} g_n(x-y)\, \mathbf{E}(\eta(0)).$$

Lemma 7.1. *There exists a constant $C := C_d$ so that on an event of full probability, for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$,*

$$\sup_{x \in \bar{W}_n} |r_n(x) - d_n(x)| \leq \epsilon C_d n^2 \tag{29}$$

Levine and Peres showed in [LP10] that $\bar{v}_{avg_n}$ converges uniformly to the solution of a certain continuous obstacle problem. So, in order to show that $\bar{v}_n$ has a scaling limit, it suffices to show that it stays close to $\bar{v}_{avg_n}$ for all large n. Most of the work for this proof is done in Lemma 7.1, all that's left is a use of the least action principle for the divisible sandpile.

Theorem 7.2. *On an event of full probability, as $n \to \infty$, the rescaled functions $\bar{v}_n := n^{-2}v_n([nx])$ and $\bar{v}_{avg_n} := n^{-2}v_{avg_n}([nx])$ converge uniformly together,*

$$\sup_{x \in n^{-1}\bar{W}_n} |\bar{v}_n(x) - \bar{v}_{avg_n}(x)| \to 0.$$

Proof. By definition,

$$\Delta_{\mathbb{Z}^d} v_n + \eta \leq 1,$$

which can be rewritten as

$$\Delta_{\mathbb{Z}^d}(v_n - (r_n - d_n)) + \eta_{avg} \leq 1.$$

Let $\epsilon > 0$ be given. For n large, Lemma 7.1 implies $-(r_n - d_n) + \epsilon C n^2$ is positive in W_n. Hence, by the least action principle in W_n,

$$v_n - (r_n - d_n) + \epsilon C n^2 \geq v_{avg_n}$$

and so,

$$v_n - v_{avg_n} \geq (r_n - d_n) - \epsilon C n^2.$$

Scale and invoke Lemma 7.1 again to see that

$$\bar{v}_n - \bar{v}_{avg_n} \geq \epsilon C.$$

The other direction is identical. $\square$

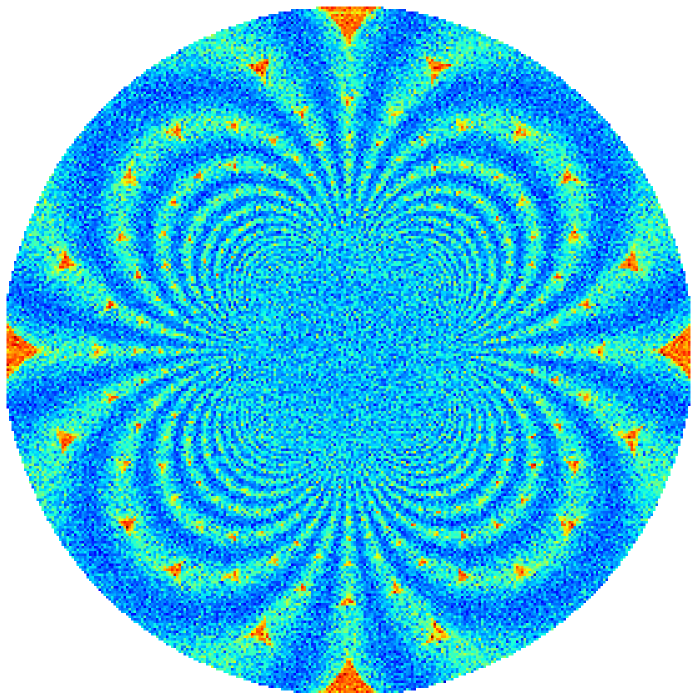

FIGURE 3. Start with 10^7 chips at the origin in $\mathbb{Z}^2$ with an iid Bernoulli(0,-1,1/2) background and stabilize. What's displayed is an approximation of the weak-* limit.

8. Concluding remarks

We conclude with some straightforward extensions of our results and open questions.

8.1. **Single-source sandpile on a random background.** Arguments as in [PS13] and this paper show that single-source sandpiles on random backgrounds also have scaling limits. See Figure 3 for an example.

Theorem 8.1. *Let v_n be the odometer function for the sandpile with n chips on the origin and an almost surely bounded, not exploding, stationary ergodic random background, η:*

$$v_n \in \mathcal{L}(\eta + n\delta_0, \mathbb{Z}^d) \cap \mathcal{S}(\eta + n\delta_0, \mathbb{Z}^d).$$

Almost surely, as $n \to \infty$, the rescaled functions $\bar{v}_n := n^{-2/d} v_n([n^{1/d} x])$ converge locally uniformly away from the origin to the unique solution $\bar{v} \in C(\mathbb{R}^d \backslash \{0\})$ of the obstacle problem

$$\bar{v} := \min\{w \in C(\mathbb{R}^d \backslash \{0\}) | w \geq 0, \Delta w + \delta_0 \leq 2d - 1, \text{and } \bar{F}_\eta(D^2 w) \leq 0\}$$

where $\bar{F}_\eta$ is a unique degenerate elliptic operator, the minimum is taken point-wise, and the differential inclusion is interpreted in the viscosity sense.

8.2. **Sandpiles with closed boundaries.** The same argument given in this paper also works for sandpiles with the closed boundary condition.

Theorem 8.2. *Let W be a bounded Lipschitz subset and let v_n be the odometer function for the sandpile $W_n := \mathbb{Z}^d \cap nW$ with the closed boundary condition:*

$$v_n \in \mathcal{L}(\eta, W_n) \cap \mathcal{S}(\eta, W_n) \text{ and } v_n = 0 \text{ on } \partial W_n.$$

Almost surely, as $n \to \infty$, the rescaled functions $\bar{v}_n := n^{-2} v_n([nx])$ converge uniformly to the unique viscosity solution $\bar{v} \in C(\mathbb{R}^d)$ of the deterministic equation

$$\begin{cases} \bar{F}_\eta(D^2 \bar{v}) = 0 & \text{in } W \\ \bar{v} = 0 & \text{on } \partial W \end{cases}$$

where $\bar{F}_\eta$ is a unique degenerate elliptic operator.

Note that $\bar{F}_\eta$ is the *same* operator appearing in the limit for the open boundary sandpile. For example, if the background is is the product Bernoulli measure, simulations reveal interesting pictures. These may help in characterizing $\bar{F}_\eta$ - see Figure 4.

8.3. **Sandpiles with $\mathbf{E}(\eta(0)) \leq 2d - 1$.** The high density assumption, $\mathbf{E}(\eta(0)) > 2d - 1$, was used in two places in the argument. The first was to ensure that after stabilizing η, in a sufficiently large initial domain, what we get is very close to a recurrent configuration. The second was to show that solutions to $\bar{F}_\eta(D^2 \bar{v}) = 0$ also satisfy $D^2 \bar{v} \in \bar{\Gamma}$. For the first usage, we can replace the assumption on $\mathbf{E}(\eta(0))$ by one that states that after toppling in nested volumes η is close to a recurrent configuaration. For example, for each $p \in [0, 1]$, the following random sandpile on $\mathbb{Z}^2$ has a scaling limit by our argument as it is always recurrent,

$$\eta(x) = \begin{cases} 2 \text{ with probability } p \\ 4 \text{ with probability } 1 - p \end{cases}.$$

For the second usage, it suffices to use the weaker bound $\mathbf{E}(\eta(0)) \geq d$. And in fact, if $\mathbf{E}(\eta(0)) < d$, the sandpile is almost surely stabilizable. This implies, by conservation of density, (Lemma 2.10 in [FMR09], Lemma 3.2 in [LMPU16]), that the stable sandpiles converges weakly* to $\mathbf{E}(\eta(0))$ and so $\bar{v}_n \to 0$. However, this still leaves unaddressed sandpiles with $\mathbf{E}(\eta(0)) \in [d, 2d - 1]$ which are not stabilizable, but also not close to a recurrent configuration. We believe, but cannot prove, that all such sandpiles have odometer functions with subquadratic growth. See Figure 5 for an example of what could be such a sandpile.

8.4. **Characterizing the effective equation.** Recently L. Levine, W. Pegden, and C. Smart characterized $\bar{F}_0$ on $\mathbb{Z}^2$ as the downwards closure of an Apollonian circle packing [LPS16, LPS17]. Then, W. Pegden and C. Smart explained the micro-scale structure of the sandpile on $\mathbb{Z}^2$ by establishing a rate of the convergence to the continuum obstacle problem and showing pattern stability [PS17].

Analogous results for $\bar{F}_\eta$ are currently out of reach. Numerical evidence indicates that $\bar{F}_\eta$ is not the Laplacian; one reason for this may be the extra log factor in the mixing time of the sandpile Markov chain, see [HJL19]. Lemma 6.6 also shows that solutions to $\bar{F}_\eta(D^2 v) \geq 0$

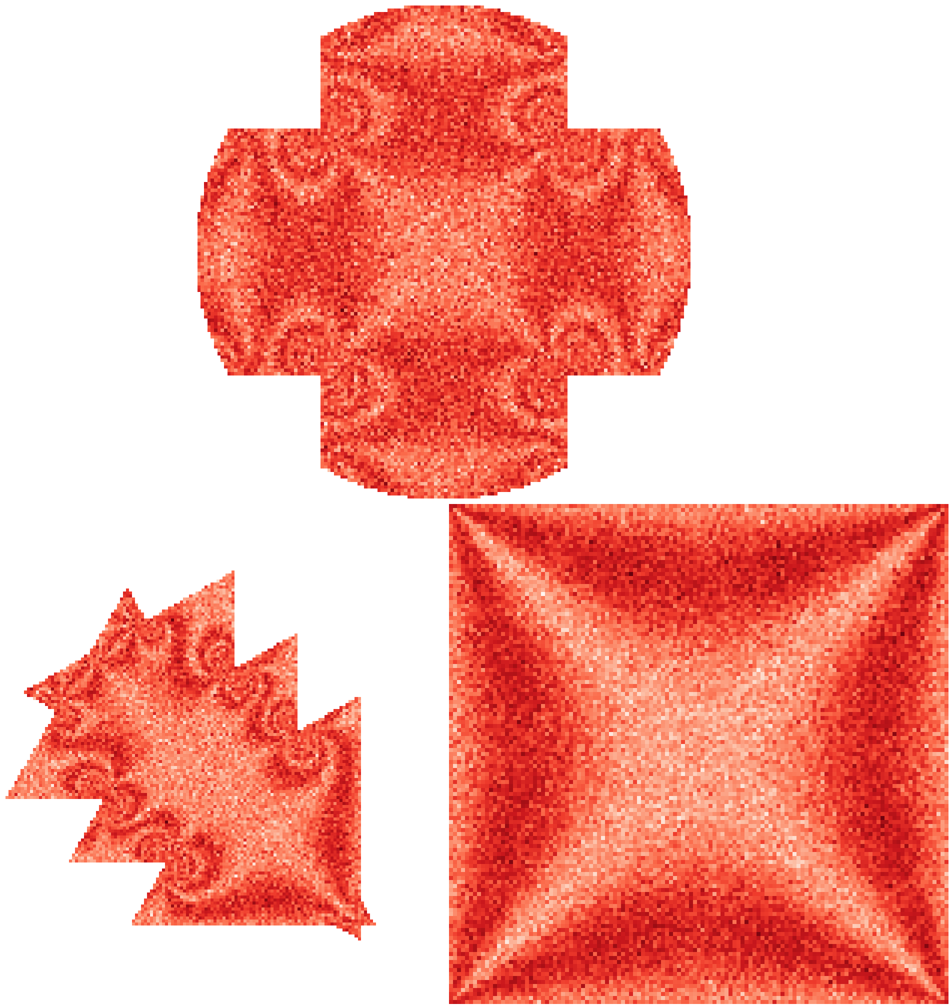

FIGURE 4. Start with an iid Bernoulli(3,5,1/2) sandpile configuration and stabilize with the closed boundary condition. Darker reds are closer to 2 while lighter reds are closer to 3. The displays are approximations of the weak-* limits.

also satisfy $\bar{F}_0(D^2 v) \geq 0$. However, a finer characterization of the effective equation is yet to be seen.

References

[AS14a] Scott N Armstrong and Charles K Smart. Quantitative stochastic homogenization of elliptic equations in nondivergence form. *Archive for Rational Mechanics and Analysis*, 214(3):867–911, 2014.

[AS14b] Scott N Armstrong and Charles K Smart. Regularity and stochastic homogenization of fully nonlinear equations without uniform ellipticity. *The Annals of Probability*, 42(6):2558–2594, 2014.

[AS14c] Scott N Armstrong and Charles K Smart. Stochastic homogenization of fully nonlinear uniformly elliptic equations revisited. *Calculus of Variations and Partial Differential Equations*, 50(3-4):967–980, 2014.

[BTW87] Per Bak, Chao Tang, and Kurt Wiesenfeld. Self-organized criticality: An explanation of the 1/f noise. *Physical review letters*, 59(4):381, 1987.

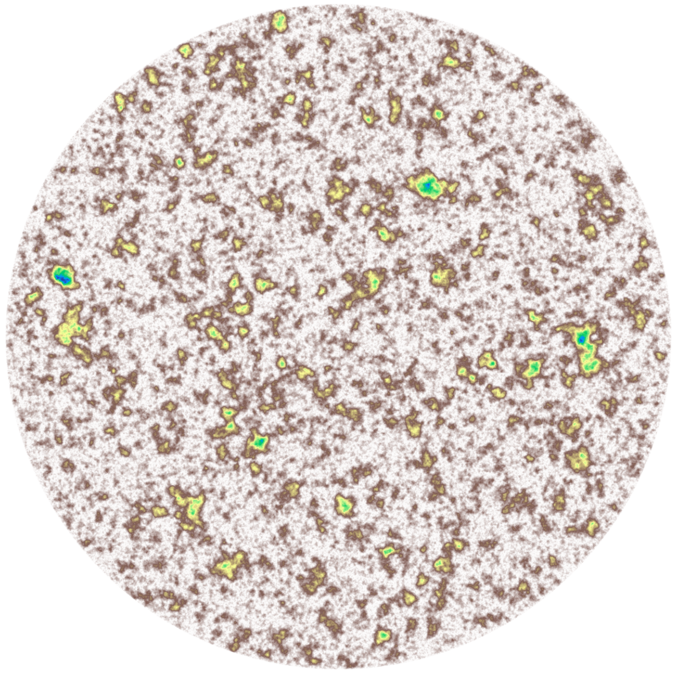

FIGURE 5. A heat map of the odometer function for a Bernoulli(0,4,1/2) initial sandpile with $p = 0.528$ started in a circle of radius $6 \cdot 10^3$ with the open boundary condition. Note that $\mathbf{E}(\eta(0)) = 2.122 < 17/8$.

[Caf99] Luis A Caffarelli. A note on nonlinear homogenization. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 52(7):829–838, 1999.

[CP18] Scott Corry and David Perkinson. *Divisors and Sandpiles: An Introduction to Chip-Firing*, volume 114. American Mathematical Soc., 2018.

[CSW05] Luis A Caffarelli, Panagiotis E Souganidis, and Lihe Wang. Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 58(3):319–361, 2005.

[FLP10] Anne Fey, Lionel Levine, and Yuval Peres. Growth rates and explosions in sandpiles. *Journal of Statistical Physics*, 138(1-3):143–159, 2010.

[FMR09] Anne Fey, Ronald Meester, and Frank Redig. Stabilizability and percolation in the infinite volume sandpile model. *The Annals of Probability*, 37(2):654–675, 2009.

[GB01] Cristian E Gutiérrez and Haim Brezis. *The Monge-Ampere equation*, volume 44. Springer, 2001.

[HJL19] Bob Hough, Dan Jerison, and Lionel Levine. Sandpiles on the square lattice. *Communications in Mathematical Physics*, 367(1):33–87, 2019.

[HLM+08] Alexander E Holroyd, Lionel Levine, Karola Mészáros, Yuyal Peres, James Propp, and David B Wilson. Chip-firing and rotor-routing on directed graphs. In *In and Out of Equilibrium 2*, pages 331–364. Springer, 2008.

[Jár18] Antal A Járai. Sandpile models. *Probability Surveys*, 15(243–306), 2018.

[KT05] Hung-Ju Kuo and Neil S Trudinger. Estimates for solutions of fully nonlinear discrete schemes. In *Trends in partial differential equations of mathematical physics*, pages 275–282. Springer, 2005.

[LL10] Gregory F Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123. Cambridge University Press, 2010.

[LMPU16] Lionel Levine, Mathav Murugan, Yuval Peres, and Baris Evren Ugurcan. The divisible sandpile at critical density. In *Annales Henri Poincaré*, volume 17, pages 1677–1711. Springer, 2016.

[LP09] Lionel Levine and Yuval Peres. Strong spherical asymptotics for rotor-router aggregation and the divisible sandpile. *Potential Analysis*, 30(1):1, 2009.

[LP10] Lionel Levine and Yuval Peres. Scaling limits for internal aggregation models with multiple sources. *Journal d'Analyse Mathématique*, 111(1):151–219, 2010.

[LPS16] Lionel Levine, Wesley Pegden, and Charles K Smart. Apollonian structure in the abelian sandpile. *Geometric and functional analysis*, 26(1):306–336, 2016.

[LPS17] Lionel Levine, Wesley Pegden, and Charles K Smart. The apollonian structure of integer superharmonic matrices. *Annals of Mathematics*, pages 1–67, 2017.

[PS13] Wesley Pegden and Charles K Smart. Convergence of the abelian sandpile. *Duke Mathematical Journal*, 162(4):627–642, 2013.

[PS17] Wesley Pegden and Charles K Smart. Stability of patterns in the abelian sandpile. *arXiv preprint arXiv:1708.09432*, 2017.

[RCC95] Luis A Roberts, Luis A Caffarelli, and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43. American Mathematical Soc., 1995.

[Red05] Frank Redig. Mathematical aspects of the abelian sandpile model. *Les Houches lecture notes*, page 18, 2005.

[TW08] Neil S Trudinger and Xu-Jia Wang. The monge-ampere equation and its geometric applications. *Handbook of geometric analysis*, 1:467–524, 2008.