# Sandpiles

Ahmed Bou-Rabee

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# Self-organized criticality

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# Self-organized criticality

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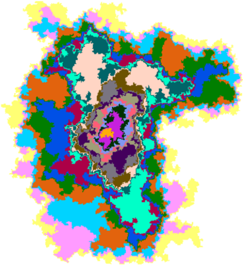
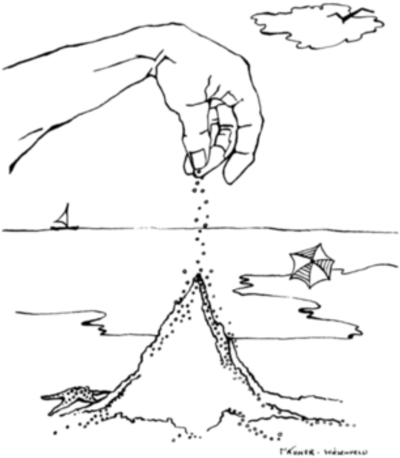
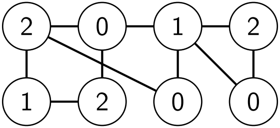
- how can complex behavior arise from simple rules?
  - **self-organized criticality:** nature perpetually self-organizes itself into a critical state, in which microscopic fluctuations can lead to macroscopic, complex changes
- in 1987, physicists Bak, Tang, Wiesenfeld, invented the **Abelian sandpile** as a simple prototype of self-organized criticality



# What is a sandpile?

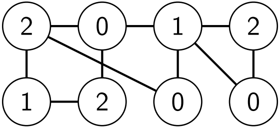


# What is an Abelian sandpile?



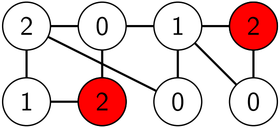

- collection of indistinguishable grains distributed among the vertices of a graph

# Sandpile dynamics




- one rule
  - a vertex is *unstable* if it has at least as many grains as its degree
  - an unstable vertex can *topple* sending one grain to each neighboring vertex

## Sandpile dynamics

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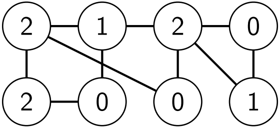


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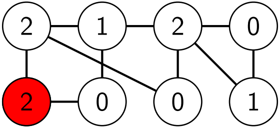


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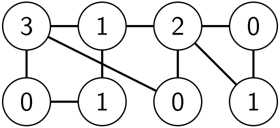


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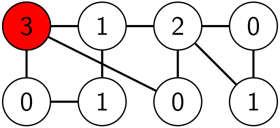


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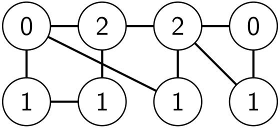
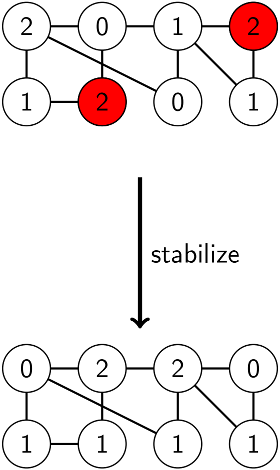


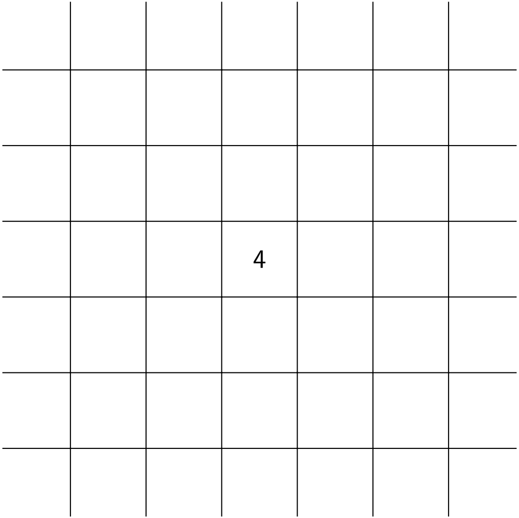

- one rule
  - a vertex is *unstable* if it has at least as many grains as its degree
  - an unstable vertex can *topple* sending one grain to each neighboring vertex
- if, eventually, all vertices are stable, the sandpile is *stabilizable*

# Stabilizing sandpiles



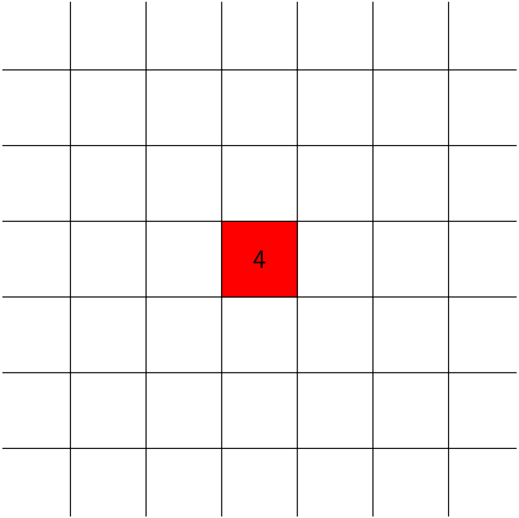

- if sandpile is stabilizable, order of topples doesn't change final sandpile - model is Abelian!
- sandpiles are not always stabilizable

## One way to ensure stabilization




- start with a finite number of grains on an infinite graph
- here: 4 grains at the origin in $\mathbf{Z}^2$ with nearest neighbor edges, $x \sim y$ if $|x - y| = 1$.

# One way to ensure stabilization

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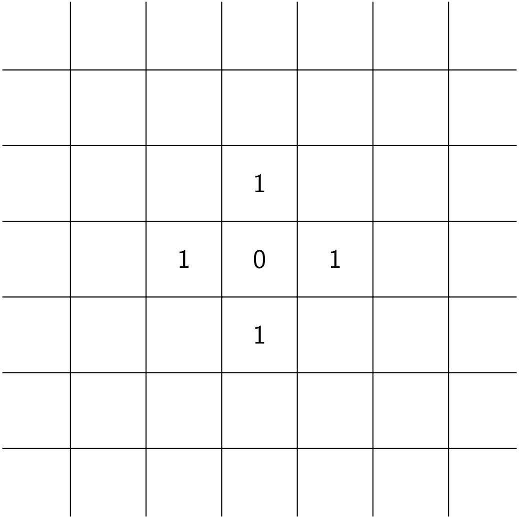


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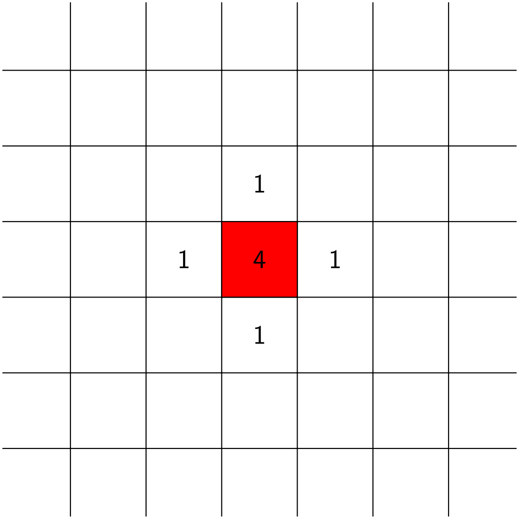
# One way to ensure stabilization

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | 1 | | | |
| | | 1 | 4 | 1 | | |
| | | | 1 | | | |
| | | | | | | |
| | | | | | | |

# One way to ensure stabilization



# One way to ensure stabilization

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | 2 | | | |
| | | 2 | 0 | 2 | | |
| | | | 2 | | | |
| | | | | | | |
| | | | | | | |

# One way to ensure stabilization

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | 2 | | | |
| | | 2 | 4 | 2 | | |
| | | | 2 | | | |
| | | | | | | |
| | | | | | | |

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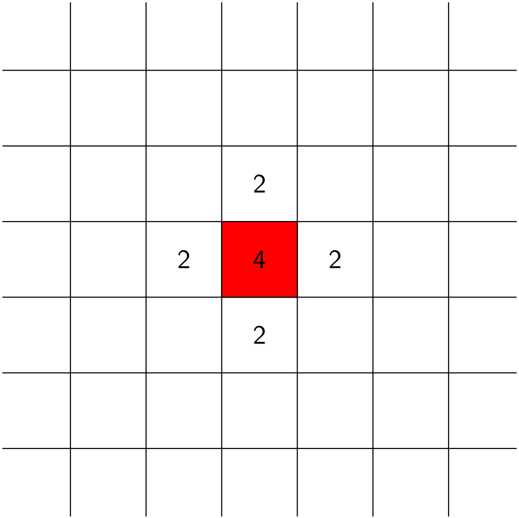

# One way to ensure stabilization

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | 3 | | | |
| | | 3 | 0 | 3 | | |
| | | | 3 | | | |
| | | | | | | |
| | | | | | | |

# One way to ensure stabilization

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | 3 | | | |
| | | 3 | 4 | 3 | | |
| | | | 3 | | | |
| | | | | | | |
| | | | | | | |

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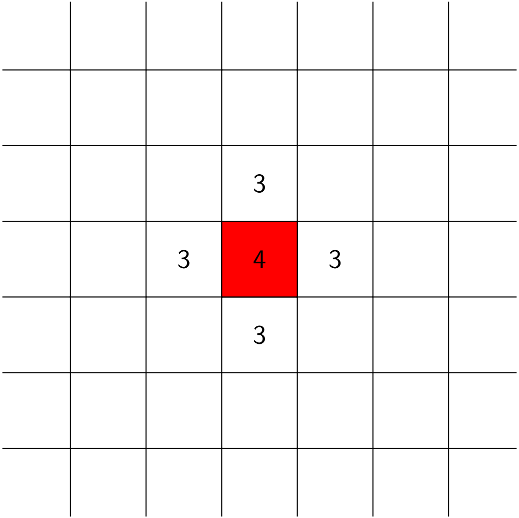

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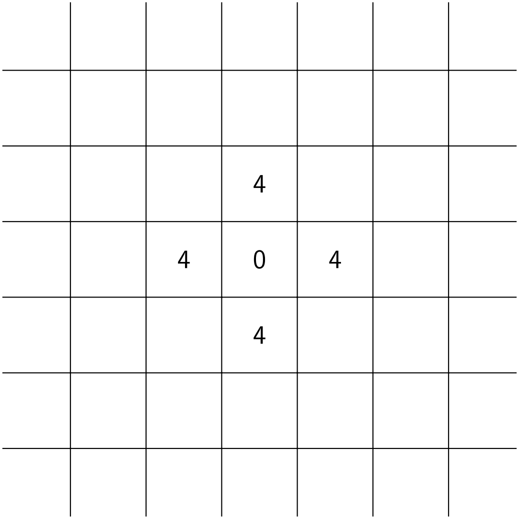

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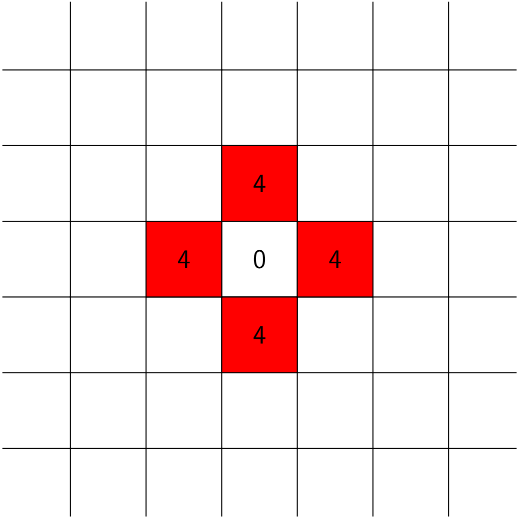


# One way to ensure stabilization

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | 1 | | | |
| | | 2 | 0 | 2 | | |
| | 1 | 0 | 4 | 0 | 1 | |
| | | 2 | 0 | 2 | | |
| | | | 1 | | | |
| | | | | | | |

# One way to ensure stabilization

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | 1 | | | |
| | | 2 | 0 | 2 | | |
| | 1 | 0 | 4 | 0 | 1 | |
| | | 2 | 0 | 2 | | |
| | | | 1 | | | |
| | | | | | | |

# One way to ensure stabilization

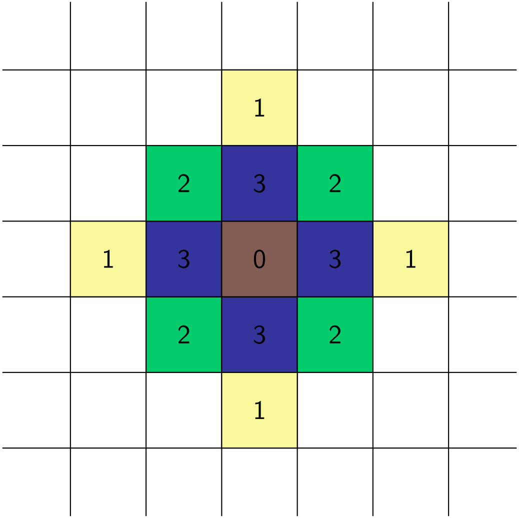
| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | 1 | | | |
| | | 2 | 1 | 2 | | |
| | 1 | 1 | 0 | 1 | 1 | |
| | | 2 | 1 | 2 | | |
| | | | 1 | | | |
| | | | | | | |

# Single-source sandpile on $\mathbf{Z}^2$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | 1 | | | |
| | | 2 | 3 | 2 | | |
| | 1 | 3 | 0 | 3 | 1 | |
| | | 2 | 3 | 2 | | |
| | | | 1 | | | |
| | | | | | | |

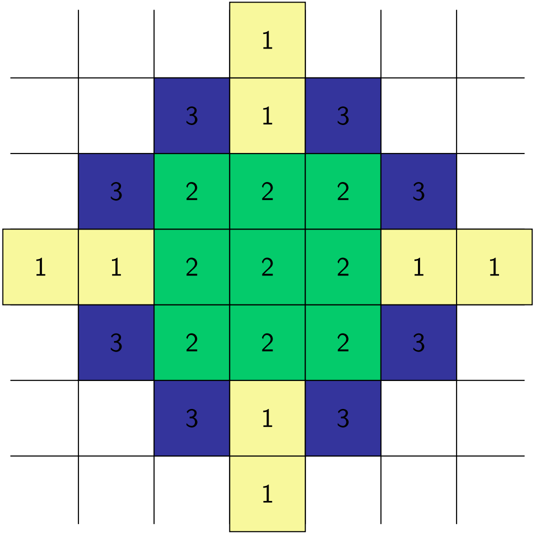
24 grains/grains at the origin

# Single-source sandpile on $\mathbb{Z}^2$ - coloring




24 grains/grains at the origin

# Larger and larger




50 grains of sand at the origin

# Larger and larger

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 1 | 2 | 1 | | | |
| | | 3 | 2 | 0 | 2 | 3 | | |
| | 3 | 0 | 3 | 2 | 3 | 0 | 3 | |
| 1 | 2 | 3 | 0 | 3 | 0 | 3 | 2 | 1 |
| 2 | 0 | 2 | 3 | 0 | 3 | 2 | 0 | 2 |
| 1 | 2 | 3 | 0 | 3 | 0 | 3 | 2 | 1 |
| | 3 | 0 | 3 | 2 | 3 | 0 | 3 | |
| | | 3 | 2 | 0 | 2 | 3 | | |
| | | | 1 | 2 | 1 | | | |

100 grains of sand at the origin

# Larger and larger

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 2 | 3 | 2 | 1 | | | |
| | | 3 | 2 | 3 | 1 | 3 | 2 | 3 | | |
| | 3 | 0 | 3 | 2 | 0 | 2 | 3 | 0 | 3 | |
| 1 | 2 | 3 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 1 |
| 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 |
| 3 | 1 | 0 | 3 | 3 | 0 | 3 | 3 | 0 | 1 | 3 |
| 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 |
| 1 | 2 | 3 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 1 |
| | 3 | 0 | 3 | 2 | 0 | 2 | 3 | 0 | 3 | |
| | | 3 | 2 | 3 | 1 | 3 | 2 | 3 | | |
| | | | 1 | 2 | 3 | 2 | 1 | | | |

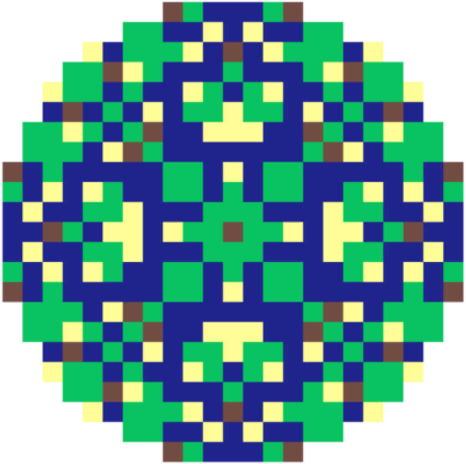
200 grains of sand at the origin

# Larger and larger

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 3 | 2 | 2 | 1 | 3 | 1 | 2 | 2 | 3 | 1 | | |
| | 2 | 1 | 3 | 3 | 3 | 2 | 0 | 2 | 3 | 3 | 3 | 1 | 2 | |
| 1 | 1 | 2 | 2 | 0 | 3 | 2 | 3 | 2 | 3 | 0 | 2 | 2 | 1 | 1 |
| 3 | 3 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 3 | 3 |
| 2 | 3 | 0 | 3 | 0 | 1 | 1 | 3 | 1 | 1 | 0 | 3 | 0 | 3 | 2 |
| 2 | 3 | 3 | 3 | 1 | 2 | 3 | 3 | 3 | 2 | 1 | 3 | 3 | 3 | 2 |
| 1 | 2 | 2 | 3 | 1 | 3 | 2 | 1 | 2 | 3 | 1 | 3 | 2 | 2 | 1 |
| 3 | 0 | 3 | 3 | 3 | 3 | 1 | 0 | 1 | 3 | 3 | 3 | 3 | 0 | 3 |
| 1 | 2 | 2 | 3 | 1 | 3 | 2 | 1 | 2 | 3 | 1 | 3 | 2 | 2 | 1 |
| 2 | 3 | 3 | 3 | 1 | 2 | 3 | 3 | 3 | 2 | 1 | 3 | 3 | 3 | 2 |
| 2 | 3 | 0 | 3 | 0 | 1 | 1 | 3 | 1 | 1 | 0 | 3 | 0 | 3 | 2 |
| 3 | 3 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 3 | 3 |
| 1 | 1 | 2 | 2 | 0 | 3 | 2 | 3 | 2 | 3 | 0 | 2 | 2 | 1 | 1 |
| | 2 | 1 | 3 | 3 | 3 | 2 | 0 | 2 | 3 | 3 | 3 | 1 | 2 | |
| | | 1 | 3 | 2 | 2 | 1 | 3 | 1 | 2 | 2 | 3 | 1 | | |

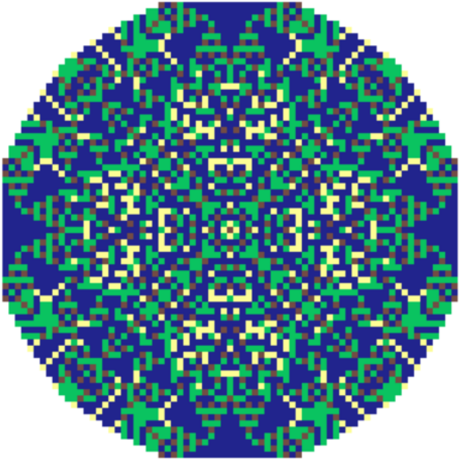
500 grains of sand at the origin

# Larger and larger



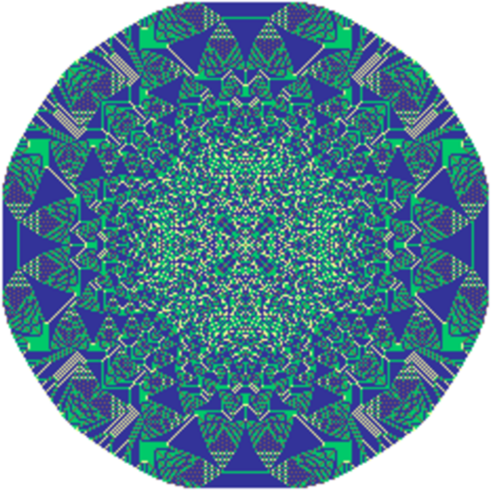
$10^3$ grains of sand at the origin

# Convergence?



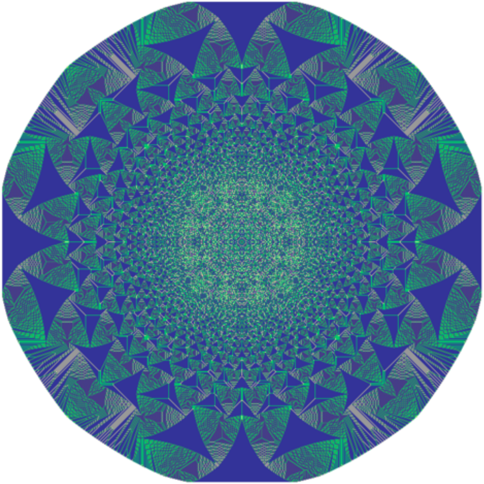
$10^4$ grains

# Convergence?



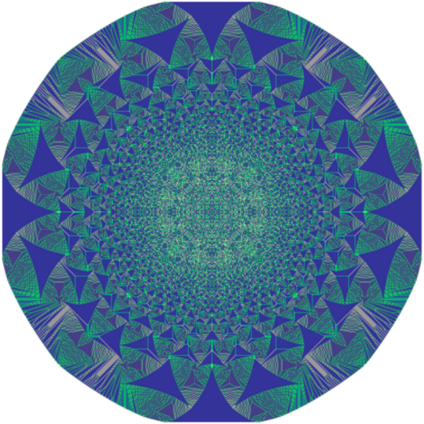
$10^5$ grains

# Convergence?



$10^6$ grains

## Compactness

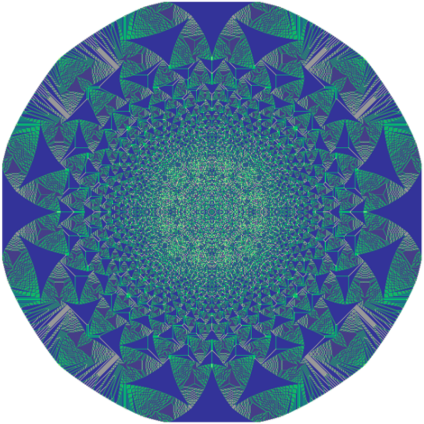


### Theorem (Fey-Levine-Peres 2009)

*The $n$-grain single-source sandpile on $\mathbf{Z}^2$ has growth rate $n^{1/2}$,*

- *it contains a ball of radius $cn^{1/2}$ and is contained within a ball of radius $Cn^{1/2}$*

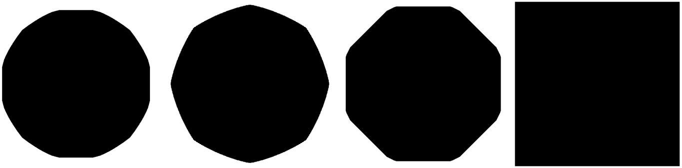
# Convergence



## Theorem (Pegden-Smart 2013)

*The scaling limit of the single source sandpile exists and is the Laplacian of the solution to a nonlinear PDE*

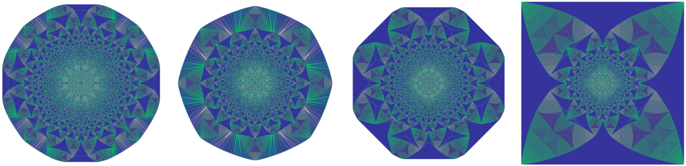
# Growth rates of sandpiles - general version



Theorem (Fey-Levine-Peres 2010)

*Starting with $n$ grains at the origin on a background $\eta_{\min} \leq \eta \leq (2d-2)$ in $\mathbf{Z}^d$, the diameter of the set of sites which topple has order $n^{1/d}$*

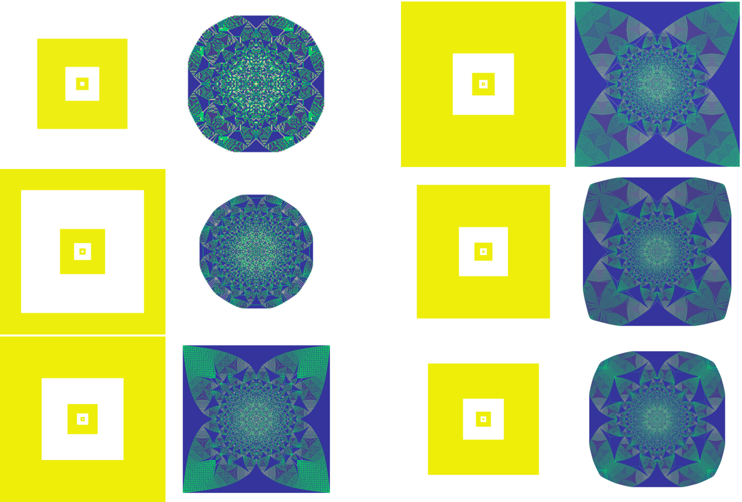
## Convergence of the Abelian sandpile - periodic backgrounds



Theorem (Pegden-Smart 2013)

*If* $\eta_{\min} \leq \eta \leq (2d-2)$ *and* $\eta$ *is periodic, the scaling limit of* $(n \cdot \delta_0 + \eta)$ *exists and is the Laplacian of the solution to a nonlinear PDE.*

# Why periodic?



If the background doesn't converge - the stable sandpile won't either

# Introducing randomness

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 |

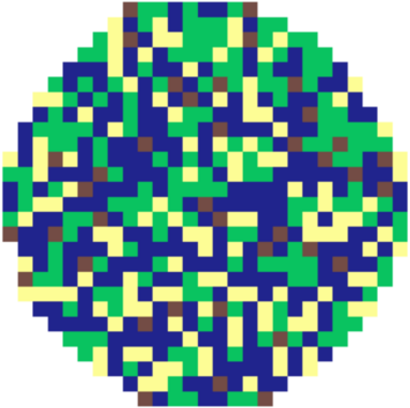
$P(\eta(x) = 0) = P(\eta(x) = 1) = 1/2 \qquad \eta$ i.i.d

# Random initial state

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 1 | 0 | 1 |
| 0 | 2 | 3 | 2 | 2 | 3 | 1 |
| 2 | 2 | 2 | 2 | 3 | 3 | 1 |
| 2 | 3 | 2 | 1 | 1 | 3 | 2 |
| 1 | 3 | 3 | 2 | 3 | 2 | 1 |
| 0 | 3 | 3 | 2 | 2 | 2 | 0 |
| 1 | 0 | 1 | 3 | 2 | 0 | 1 |

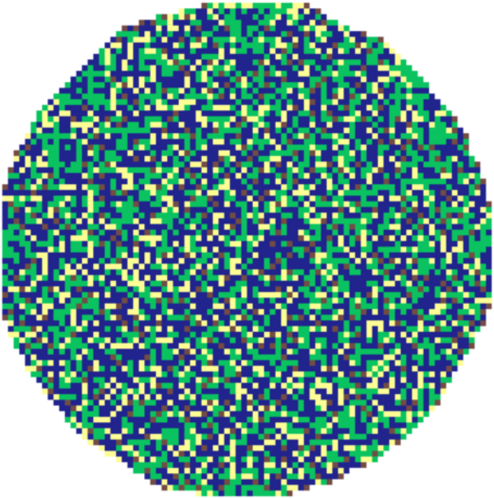
50 grains at the origin

# Random initial state



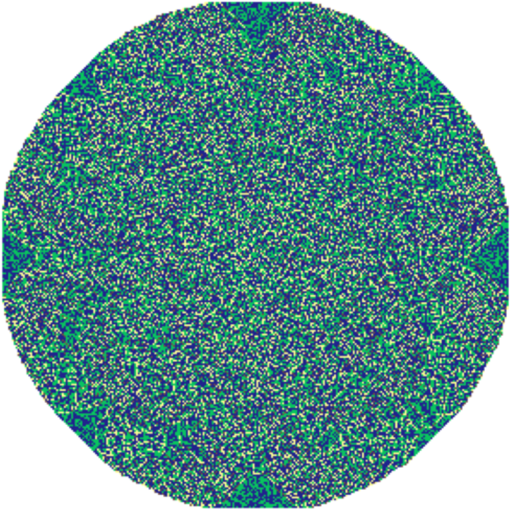
1000 grains at the origin

# Random initial state



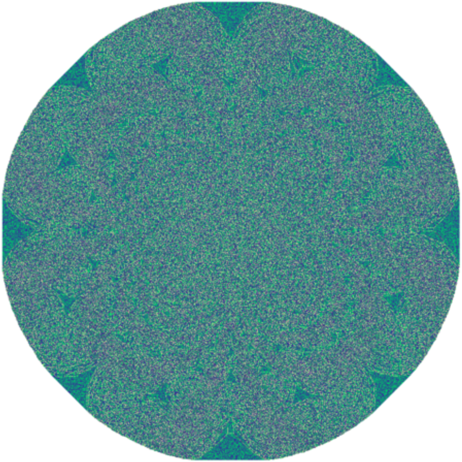
$10^4$ grains at the origin

# Random initial state – scaling limit?



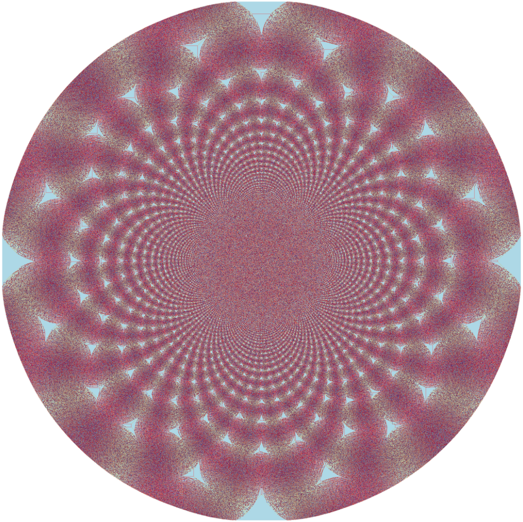
$10^5$ grains at the origin

# Random initial state - scaling limit



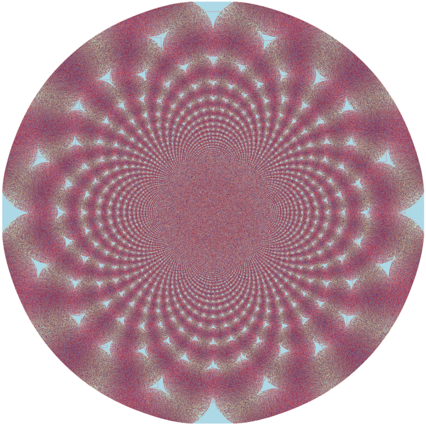
$10^6$ grains at the origin

# Random initial state - scaling limit



$10^7$ grains at the origin, averaged

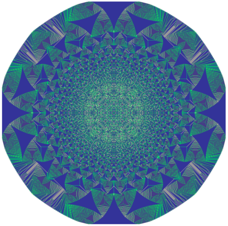
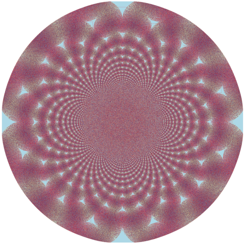
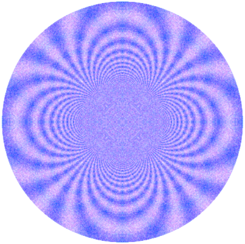
# Convergence of the random Abelian sandpile



## Theorem (B. 2021)

*The scaling limit of the single-source sandpile on a random background exists and is the Laplacian of the solution to a different nonlinear PDE.*

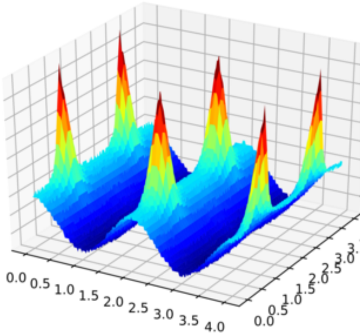
# Convergence of the random Abelian sandpile - general framework

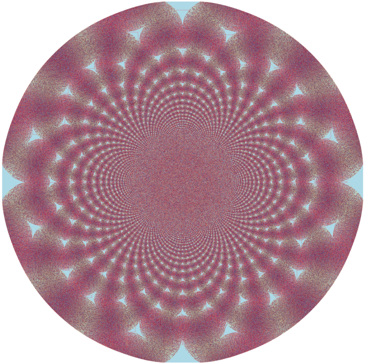




## Theorem (B. 2021)

- *sample a random background* $\eta : \mathbf{Z}^d \to \mathbf{Z}$ *from a probability space* $(\Omega, \mathcal{F}, \mathrm{P})$ *which is*
    1. *Stationary, ergodic under spatial translations*
    2. *Bounded,* $\mathrm{P}(\eta_{\min} \leq \eta \leq (2d-2)) = 1$
- *add* $n$ *grains at the origin to* $\eta$
- *after a rescaling, the single source sandpile converges almost surely to a deterministic scaling limit*

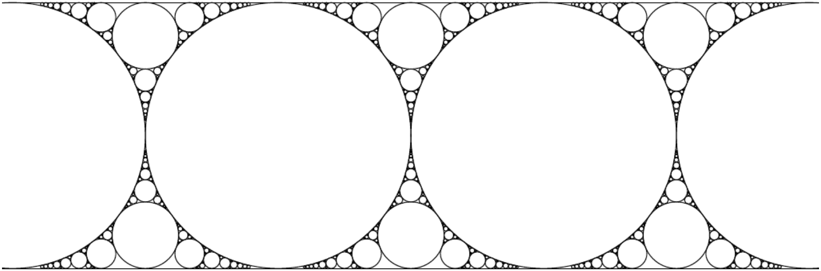
# What are the limit PDEs?



- open: characterize the scaling limits for any random $\eta$ when $d \geq 2$
- difficult because it is also completely open when $\eta \equiv 0$ for other lattices!

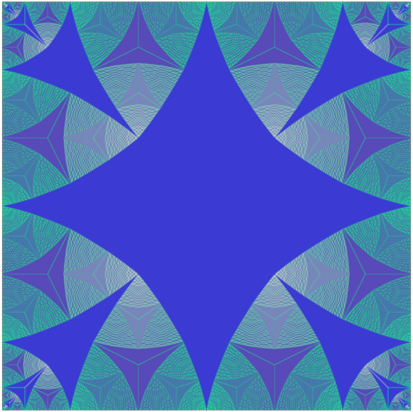
# Apollonian circle packing



Theorem (Levine-Pegden-Smart 2017)

*The limit PDE of the single-source Abelian sandpile for* $\eta \equiv 0$ *on* $\mathbf{Z}^2$ *is characterized by an Apollonian circle packing.*

# Explicit solutions



Theorem (Levine-Pegden-Smart 2016)

*The 'sandpile PDE' for $\eta \equiv 0$ on $\mathbf{Z}^2$ admits fractal solutions.*

# $F$-lattice

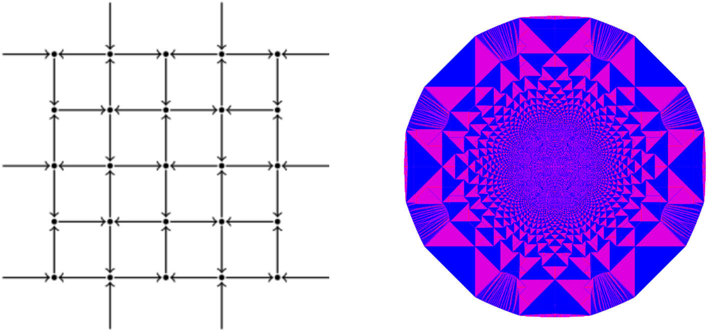


Figure: A $5 \times 5$ section of the $F$-lattice and the single-source sandpile on it.

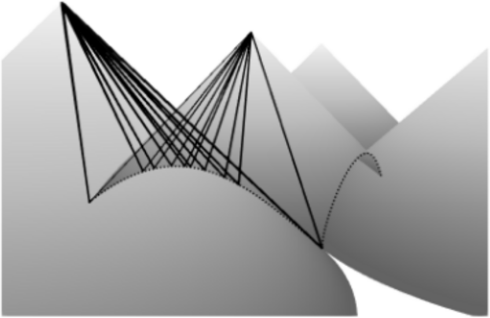
the $F$-lattice is a directed periodic planar graph $(\mathbf{Z}^2, E)$, where

$$\begin{cases} (x \pm e_1, x) \in E & \text{if } x_1 + x_2 \equiv 0 \pmod 2 \\ (x \pm e_2, x) \in E & \text{otherwise,} \end{cases}$$

# Sandpiles on the $F$-lattice

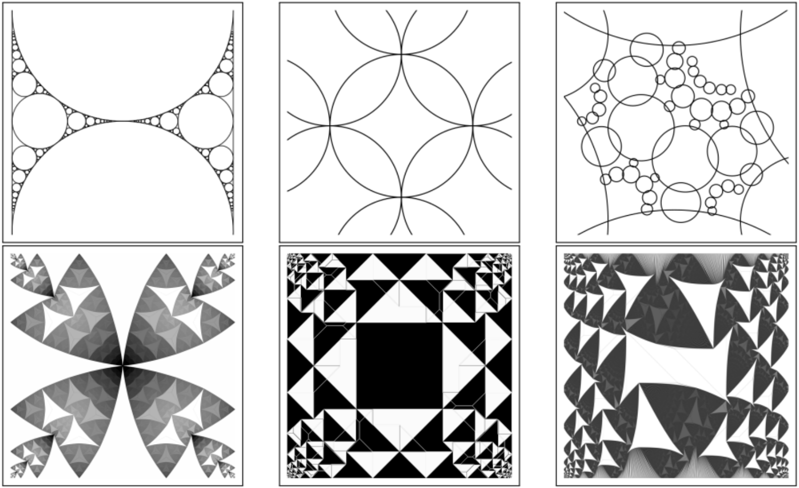
## Theorem (B. 2021)

*On the $F$-lattice, the sandpile PDE is characterized by an overlapping circle packing.*



- the proof involves recursively associating certain 'efficient' sandpiles to rational points on a hyperbola

# Other lattices?



sandpiles PDEs in general appear to be described by 'Kleinian bugs' a generalization of Apollonian circle packings recently introduced by Kapovich-Kontorovich (2021)

# Disorder in the graph

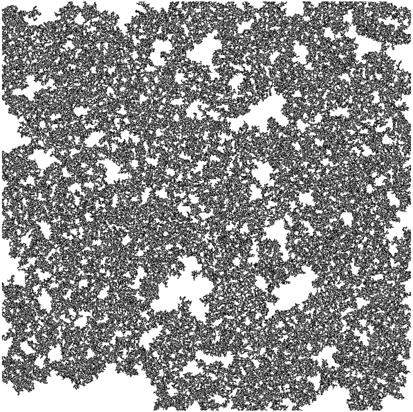


Figure: a supercritical percolation cluster on $\mathbf{Z}^2$

# Single-source sandpile on the supercritical cluster

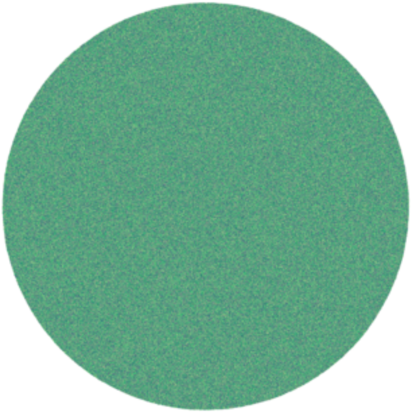
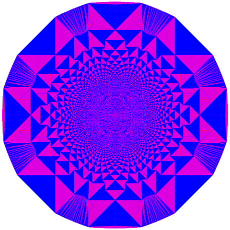
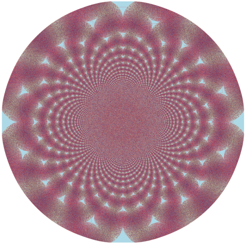
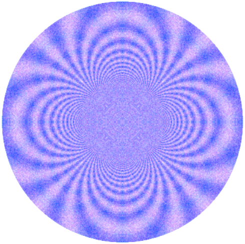


Figure: $10^6$ grains at the origin on the supercritical percolation cluster with $\mathfrak{p} = 0.9$

existence of a scaling limit for random graphs is still completely open.

# Thank you for listening!

[arXiv:1909.07849] and [arxiv:2110.07556]

# Appendix: 3D single source sandpiles

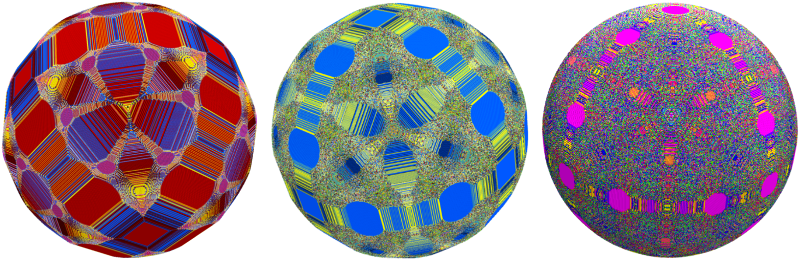


Figure: $2^{30}$ grains in $\mathbf{Z}^3$ on an empty background, a Bernoulli(0,1) background, and a Bernoulli(-1,1) background

# Appendix: 4D single source sandpile

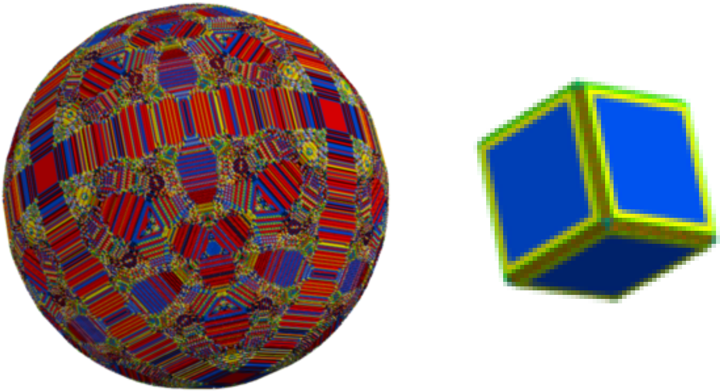


Figure: $2^{32}$ grains in $\mathbf{Z}^4$ on an empty background, two three-dimensional slices