Abelian Sandpiles, Random Walks, Monge-Ampère, and Supercritical Percolation

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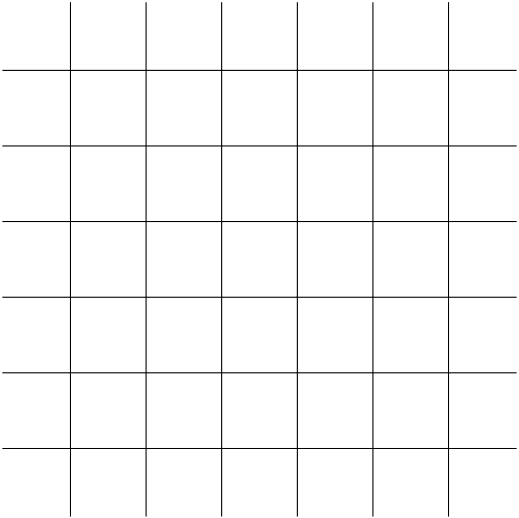
April 30, 2019

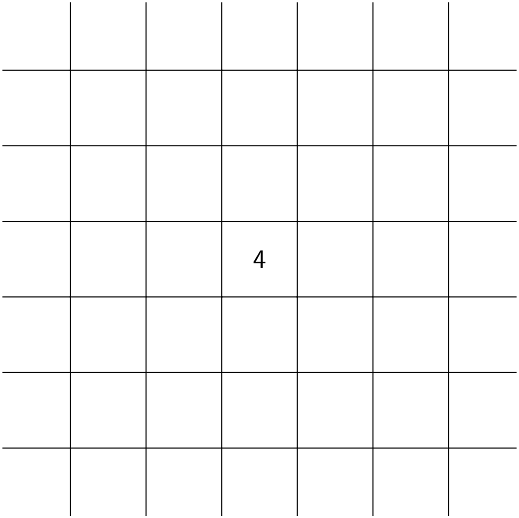
What is the Abelian Sandpile?

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

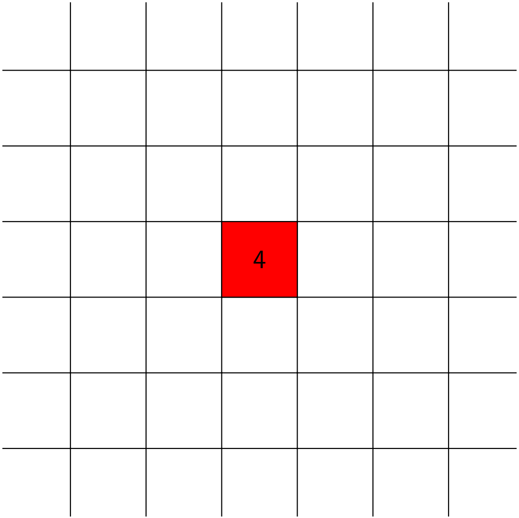
What is the Abelian Sandpile?



What is the Abelian Sandpile?



What is the Abelian Sandpile?



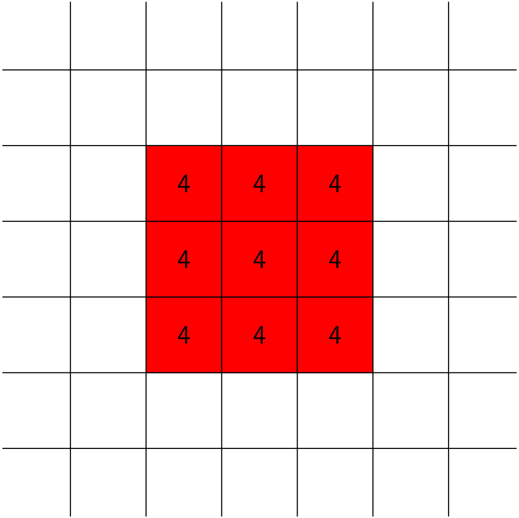
What is the Abelian Sandpile?

			1			
		1	0	1		
			1			

What is the Abelian Sandpile?

		4	4	4		
		4	4	4		
		4	4	4		

What is the Abelian Sandpile?



What is the Abelian Sandpile?

		1	1	1		
	1	2	3	2	1	
	1	3	4	3	1	
	1	2	3	2	1	
		1	1	1		

What is the Abelian Sandpile?

		1	1	1		
	1	2	3	2	1	
	1	3	4	3	1	
	1	2	3	2	1	
		1	1	1		

What is the Abelian Sandpile?

		1	1	1		
	1	2	4	2	1	
	1	4	0	4	1	
	1	2	4	2	1	
		1	1	1		

What is the Abelian Sandpile?

		1	1	1		
	1	2	4	2	1	
	1	4	0	4	1	
	1	2	4	2	1	
		1	1	1		

What is the Abelian Sandpile?

		1	2	1		
	1	4	0	4	1	
	2	0	4	0	2	
	1	4	0	4	1	
		1	2	1		

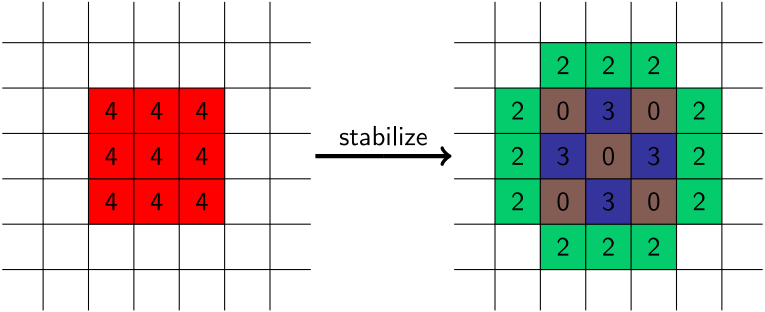
What is the Abelian Sandpile?

		1	2	1		
	1	4	0	4	1	
	2	0	4	0	2	
	1	4	0	4	1	
		1	2	1		

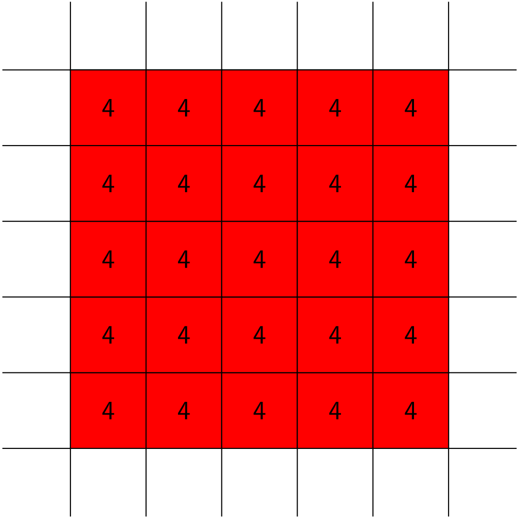
What is the Abelian Sandpile?

		2	2	2		
	2	0	3	0	2	
	2	3	0	3	2	
	2	0	3	0	2	
		2	2	2		

What is the Abelian Sandpile?



Convergence?



Convergence?

	2	3	3	3	2	
2	2	2	3	2	2	2
3	2	0	3	0	2	3
3	3	3	0	3	3	3
3	2	0	3	0	2	3
2	2	2	3	2	2	2
	2	3	3	3	2	

Convergence?

4	4	4	4	4	4	4
4	4	4	4	4	4	4
4	4	4	4	4	4	4
4	4	4	4	4	4	4
4	4	4	4	4	4	4
4	4	4	4	4	4	4
4	4	4	4	4	4	4

input square of side length $n = 6$

Convergence?

		1	1	1	1	1	1	1		
	2	1	3	3	3	3	3	1	2	
1	1	0	1	3	3	3	1	0	1	1
1	3	1	2	2	3	2	2	1	3	1
1	3	3	2	0	3	0	2	3	3	1
1	3	3	3	3	0	3	3	3	3	1
1	3	3	2	0	3	0	2	3	3	1
1	3	1	2	2	3	2	2	1	3	1
1	1	0	1	3	3	3	1	0	1	1
	2	1	3	3	3	3	3	1	2	
		1	1	1	1	1	1	1		

input square of side length $n = 6$

Convergence?

4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4

$n = 8$

Convergence?

		1	2	2	2	2	2	2	2	1		
	2	2	1	3	3	3	3	3	1	2	2	
1	2	2	2	2	3	3	3	2	2	2	2	1
2	1	2	2	0	3	3	3	0	2	2	1	2
2	3	2	0	2	2	3	2	2	0	2	3	2
2	3	3	3	2	0	3	0	2	3	3	3	2
2	3	3	3	3	3	0	3	3	3	3	3	2
2	3	3	3	2	0	3	0	2	3	3	3	2
2	3	2	0	2	2	3	2	2	0	2	3	2
2	1	2	2	0	3	3	3	0	2	2	1	2
1	2	2	2	2	3	3	3	2	2	2	2	1
	2	2	1	3	3	3	3	3	1	2	2	
		1	2	2	2	2	2	2	2	1		

$n = 8$

Convergence?

4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4

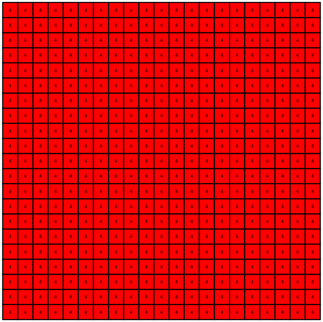
$n = 10$

Convergence?

		1	2	3	3	3	3	3	3	3	2	1		
	2	3	3	1	3	3	3	3	3	1	3	3	2	
1	3	0	1	3	2	3	3	3	2	3	1	0	3	1
2	3	1	2	3	0	3	3	3	0	3	2	1	3	2
3	1	3	3	2	3	2	3	2	3	2	3	3	1	3
3	3	2	0	3	0	2	3	2	0	3	0	2	3	3
3	3	3	3	2	2	0	3	0	2	2	3	3	3	3
3	3	3	3	3	3	3	0	3	3	3	3	3	3	3
3	3	3	3	2	2	0	3	0	2	2	3	3	3	3
3	3	2	0	3	0	2	3	2	0	3	0	2	3	3
3	1	3	3	2	3	2	3	2	3	2	3	3	1	3
2	3	1	2	3	0	3	3	3	0	3	2	1	3	2
1	3	0	1	3	2	3	3	3	2	3	1	0	3	1
	2	3	3	1	3	3	3	3	3	1	3	3	2	
		1	2	3	3	3	3	3	3	3	2	1		

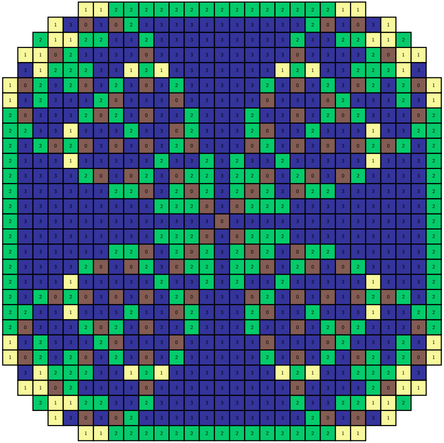
$n = 10$

Convergence?



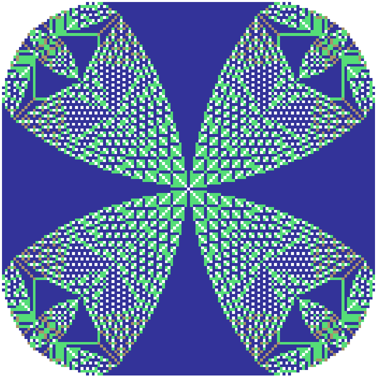
$n = 20$

Convergence?



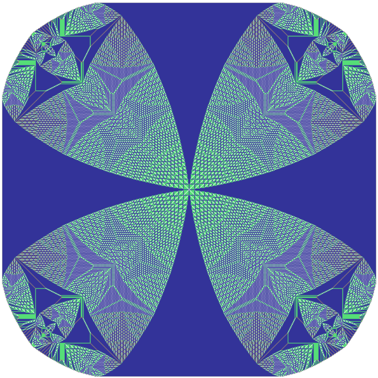
$n = 20$

Convergence?



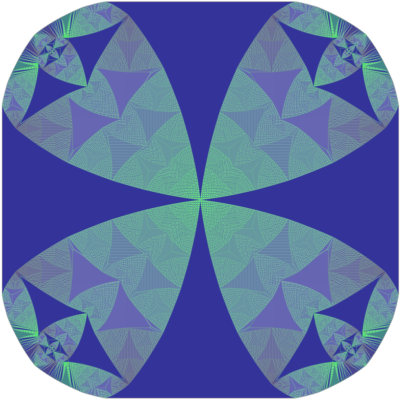
$n = 100$

Convergence?



$n = 200$

Convergence

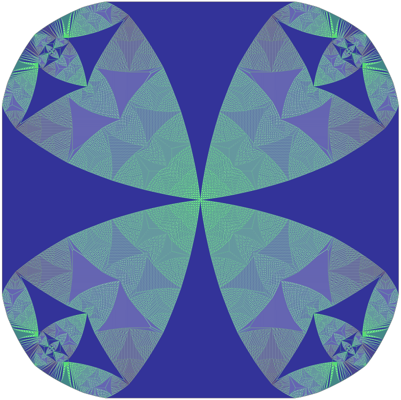


$n = 1,000$

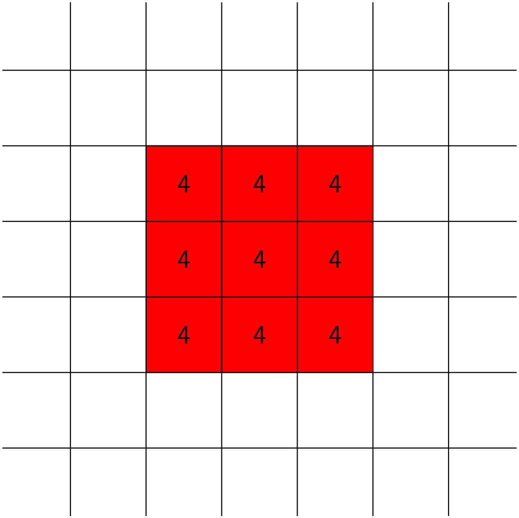
Convergence of the Abelian Sandpile

Theorem (Pegden-Smart 2011)

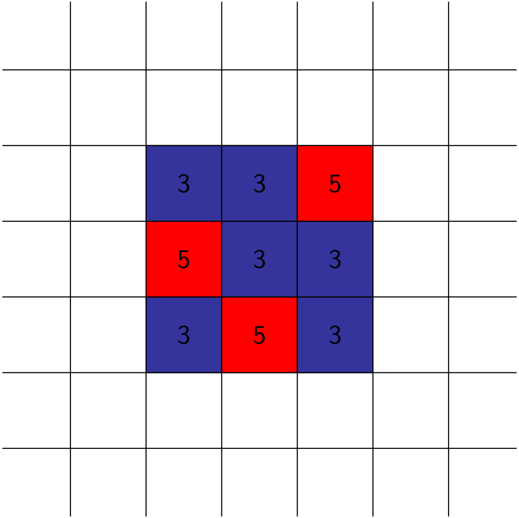
The scaling limit exists and is the Laplacian of the solution to an elliptic obstacle problem.



Introducing Randomness



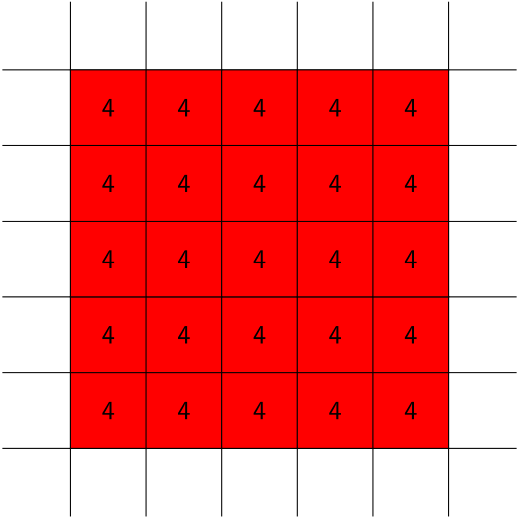
Random Initial State



Random Initial State

	1	1	1	
1	2	3	3	1
2	1	1	3	1
1	3	1	2	1
	1	2	1	

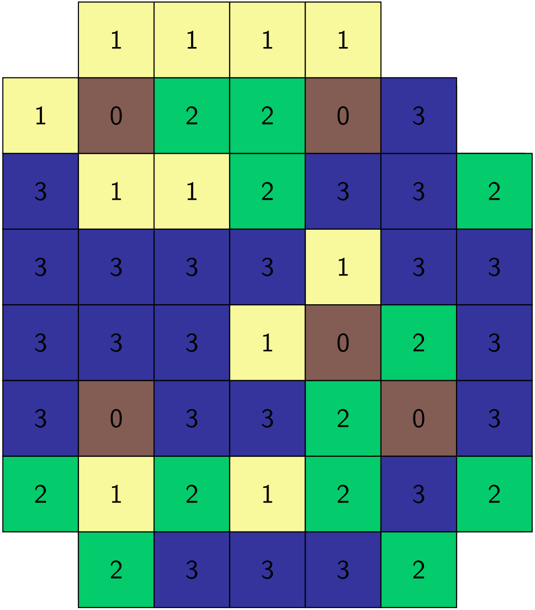
Random Initial State



Random Initial State

5	5	5	3	5
5	3	5	5	5
5	3	3	3	3
3	5	3	3	3
3	5	3	5	5

Random Initial State



Random Initial State

4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4

$n = 8$

Random Initial State

3	3	5	3	3	5	5	3	5
5	5	5	5	5	3	5	5	3
5	5	5	3	5	3	5	5	5
5	5	5	5	5	3	5	3	5
5	3	5	5	3	5	5	5	3
3	5	3	5	5	5	3	5	3
3	5	3	5	3	3	3	5	3
5	5	5	3	3	3	3	3	3
3	5	5	5	3	5	3	5	5

$n = 8$

Random Initial State

			1	2	3	3	3	3	3	2	1		
		3	3	3	2	3	3	3	1	2	2	2	
	3	0	1	2	1	2	2	2	3	3	3	2	1
1	0	3	2	3	3	2	2	2	3	1	3	2	2
1	2	3	2	1	3	1	3	2	2	3	3	1	3
1	3	1	2	3	1	2	2	2	0	3	2	3	3
1	3	1	2	1	1	3	2	0	2	1	1	3	3
1	2	3	0	2	3	2	2	3	3	1	3	2	3
1	0	3	3	3	3	1	3	3	1	3	3	0	3
	3	3	1	3	3	0	2	3	3	0	1	2	2
	1	2	3	2	0	2	3	2	2	3	3	2	1
		2	2	3	2	3	2	2	0	2	2	2	
			1	2	3	3	3	3	3	2	1		

$n = 8$

Convergence?

4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4

$n = 10$

Convergence?

3	3	5	5	3	5	5	5	3	3	3
5	5	3	3	5	3	5	5	5	5	5
3	5	3	3	3	5	5	3	3	3	3
5	5	3	3	5	3	5	3	3	3	3
3	3	3	3	3	3	3	3	5	5	3
5	5	5	3	3	5	5	5	5	3	5
3	5	3	3	5	5	5	3	3	5	5
5	3	5	3	5	5	3	5	5	5	3
5	5	5	5	5	5	3	5	3	5	3
5	3	3	5	5	5	3	5	3	5	5
5	5	5	3	5	5	5	5	5	3	5

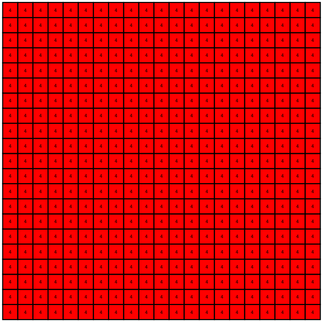
$n = 10$

Convergence?

				1	1	1	1	1	1	1	1	1				
		1	3	0	2	3	3	3	3	3	2	0	3	1		
	1	0	1	3	3	1	2	3	3	1	2	2	1	0	1	
	3	1	3	1	2	2	3	2	2	3	3	3	2	1	3	
1	0	3	3	1	0	2	3	0	2	1	1	3	1	3	0	1
1	2	3	0	3	3	1	3	3	2	3	2	2	2	2	2	1
1	3	1	3	0	2	2	1	3	3	3	2	1	3	0	3	1
1	3	2	3	2	2	3	2	2	1	3	3	0	3	2	3	1
1	3	3	2	2	3	3	3	1	1	2	3	3	2	3	3	1
1	3	3	1	2	3	1	3	3	1	1	3	0	3	3	3	1
1	3	3	1	3	3	1	1	3	3	3	0	2	3	2	3	1
1	3	1	2	3	1	0	3	3	2	2	3	3	3	0	3	1
1	1	1	2	3	3	3	1	2	1	1	3	1	2	1	1	1
	3	3	3	2	3	1	0	1	3	2	3	1	3	3	3	
	1	0	3	1	3	3	2	3	1	3	3	1	3	0	1	
		1	3	2	0	2	3	3	3	2	0	2	3	1		
				1	2	2	2	2	2	2	2	1				

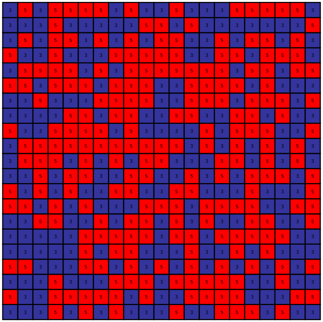
$n = 10$

Convergence?



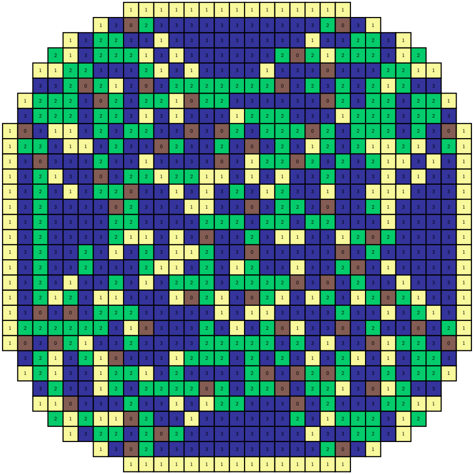
$n = 20$

Convergence?



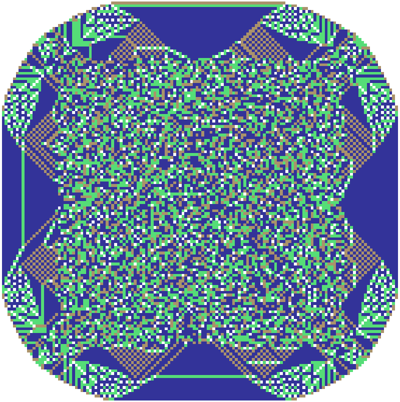
$n = 20$

Convergence?



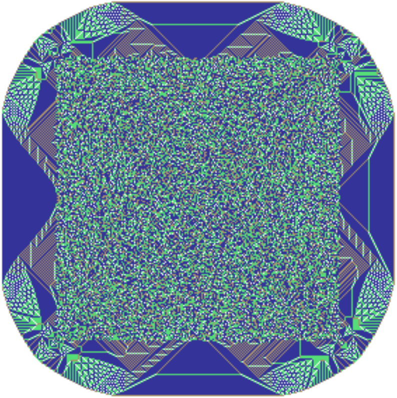
$n = 20$

Convergence?



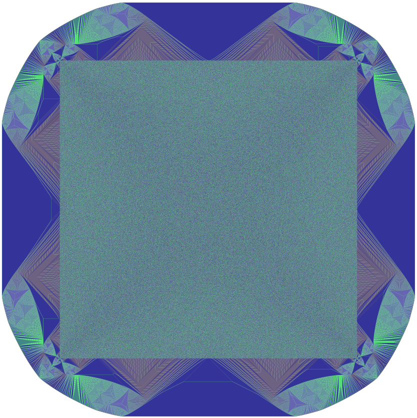
$n = 100$

Convergence?



$n = 200$

Convergence

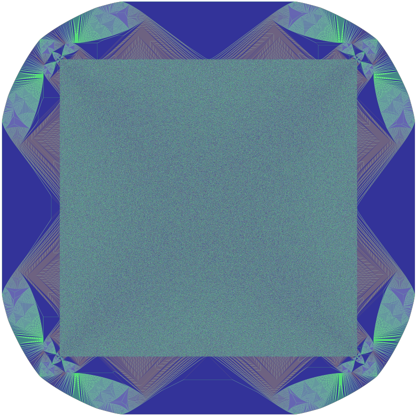


$n = 1,000$

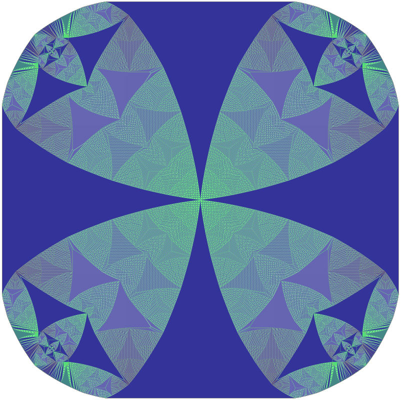
Convergence of the Random Abelian Sandpile

Theorem (B. 2019)

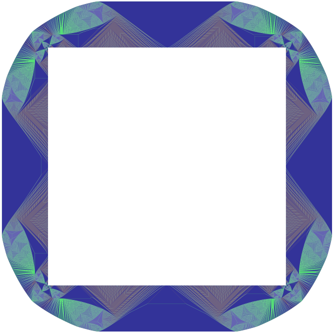
Almost surely, the scaling limit exists and is the Laplacian of the solution to an elliptic obstacle problem.

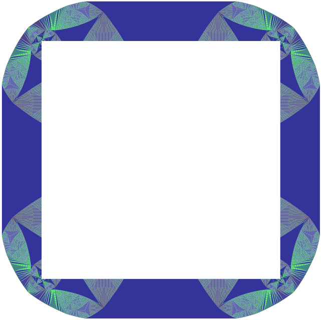


Is it the same limit?

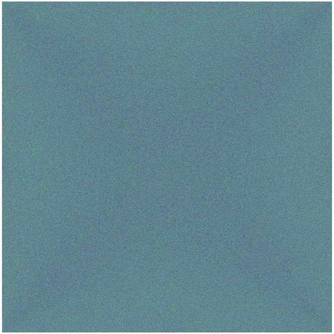


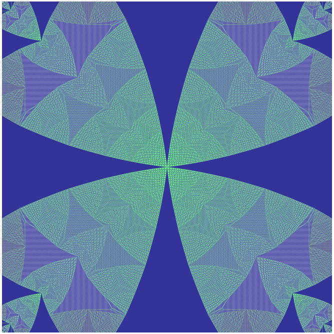
Outside?



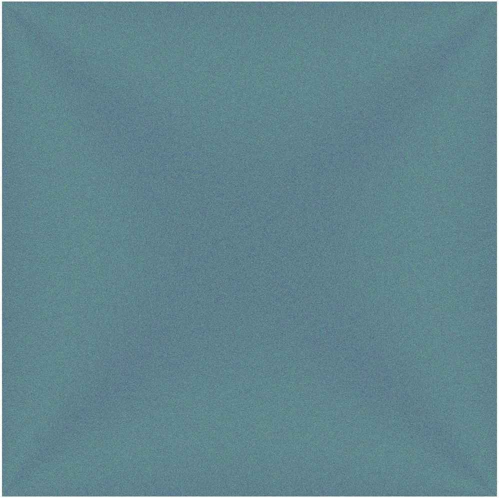


Inside?



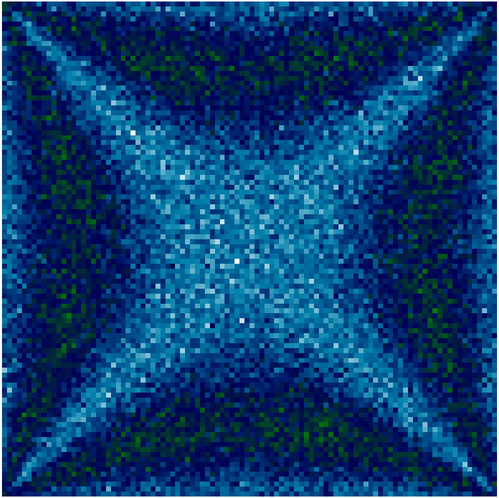


Inside?



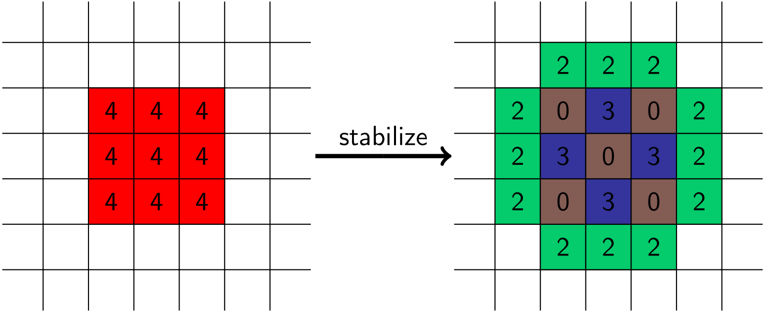
inner square for $n = 4,000$

Inside



averaged inner square for $n = 4,000$

How do you prove anything about the Abelian Sandpile?



Do not study the patterns

		4	4	4		
		4	4	4		
		4	4	4		

$\xrightarrow{\text{stabilize}}$

		2	2	2		
	2	0	3	0	2	
	2	3	0	3	2	
	2	0	3	0	2	
		2	2	2		

		2	2	2		
		2	3	2		
		2	2	2		

Study the odometer function

$$\begin{array}{|c|c|c|c|c|}\hline \ & & & & \\ \hline & 4 & 4 & 4 & \\ \hline & 4 & 4 & 4 & \\ \hline & 4 & 4 & 4 & \\ \hline \ & & & & \\ \hline \end{array} \; + \; \Delta^1 \begin{array}{|c|c|c|c|c|}\hline \ & & & & \\ \hline & 2 & 2 & 2 & \\ \hline & 2 & 3 & 2 & \\ \hline & 2 & 2 & 2 & \\ \hline \ & & & & \\ \hline \end{array} \; = \; \begin{array}{|c|c|c|c|c|}\hline & 2 & 2 & 2 & \\ \hline 2 & 0 & 3 & 0 & 2 \\ \hline 2 & 3 & 0 & 3 & 2 \\ \hline 2 & 0 & 3 & 0 & 2 \\ \hline & 2 & 2 & 2 & \\ \hline \end{array}$$

$$\eta_n + \Delta^1 v_n = s_n$$

Odometer Function

$$\eta_n + \Delta^1 v_n = s_n$$

Odometer Function

$$\eta_n + \Delta^1 v_n = s_n$$

- $\eta_n : \mathbf{Z}^2 \to \mathbf{Z}$ - initial configuration
- $s_n : \mathbf{Z}^2 \to \mathbf{Z}$, $s_n \leq 3$ - stable configuration
- $v_n : \mathbf{Z}^2 \to \mathbf{N}$ - number of topples per site when stabilizing
- $\Delta^1 v_n(x) = \sum_{y \sim x} (v_n(y) - v_n(x))$ - graph Laplacian

(More) General Framework

- sample a random background $\eta : \mathbf{Z}^2 \to \mathbf{Z}$ from a distribution which is
  - stationary, ergodic under spatial translations
  - uniformly bounded
  - high density: $\mathbf{E}(\eta(0)) > 3$
- motivating example: $\eta \sim$ Bernoulli$(3, 5)$.
- proof also works for

$$\eta(x) = \begin{cases} 2 & \text{with probability } p \\ 4 & \text{with probability } 1-p \end{cases}$$

  for any $p \in [0, 1]$
- $\eta_n(x) = \eta(x)$ for $x \in \square_n$ and 0 otherwise

Convergence of the Random Abelian Sandpile

Theorem (B. 2019)

- *There exists unique, compactly supported* $\bar{s} : \mathbf{R}^2 \to [0, 3]$ *and* $\bar{v} : \mathbf{R}^2 \to \mathbf{N}$ *so that almost surely*

$$n^{-2} v_n([nx]) \to \bar{v} \text{ locally uniformly}$$

$$s_n([nx]) \to \bar{s} \text{ weakly-*}$$

and

$$\bar{s}(x) = \Delta \bar{v}(x) + \mathbf{E}(\eta(0)) \cdot I(x \in \square_1)$$

- $\bar{v}$ *is the unique viscosity solution to the elliptic obstacle problem*

$$\min\{v \in C(\mathbf{R}^2) : v \geq 0, D^2 v \in \bar{\Gamma}_{random} \text{ in } \square_1 \text{ and } D^2 v \in \bar{\Gamma}_0 \text{ in } \mathbf{R}^2\}$$

$$\bar{\Gamma}_k = \{M \in S^2 \text{ so that there exists } u : \mathbf{Z}^2 \to \mathbf{Z}$$
$$\Delta^1 u + k \leq 3 \text{ and } u(y) \geq \tfrac{1}{2} x^T M x + o(|x|^2))\}$$

and $\bar{\Gamma}_{random} \subset \bar{\Gamma}_0$ *is nonrandom and downwards closed*

Convergence of the (Non-Random All-4s) Abelian Sandpile

Theorem (Pegden-Smart 2011)

- *There exists unique, compactly supported* $\bar{s} : \mathbf{R}^2 \to [0,3]$ *and* $\bar{v} : \mathbf{R}^2 \to \mathbf{N}$ *so that almost surely*

$$n^{-2} v_n([nx]) \to \bar{v} \text{ locally uniformly}$$

$$s_n([nx]) \to \bar{s} \text{ weakly-*}$$

and

$$\bar{s}(x) = \Delta \bar{v}(x) + 4 \cdot I(x \in \square_1)$$

- $\bar{v}$ *is the unique viscosity solution to the elliptic obstacle problem*

$$\min\{v \in C(\mathbf{R}^2) : v \geq 0, D^2 v \in \bar{\Gamma}_4 \text{ in } \square_1 \text{ and } D^2 v \in \bar{\Gamma}_0 \text{ in } \mathbf{R}^2\}$$

$$\bar{\Gamma}_k = \{M \in S^2 \text{ so that there exists } u : \mathbf{Z}^2 \to \mathbf{Z} \\ \Delta^1 u + k \leq 3 \text{ and } u(y) \geq \tfrac{1}{2} x^T M x + o(|x|^2))\}$$

Proof outline

discrete adaption of the program of Armstrong-Smart (2014) for stochastic homogenization

1. show convergence along subsequences by comparison with the divisible sandpile
2. find a subadditive quantity μ
   - show it controls the sandpile
   - show that it is nice
   - implicitly define $\bar{\Gamma}_{random}$ with μ and the ergodic theorem
3. conclude that every subsequential limit solves PDE defined by $\bar{\Gamma}_{random}$

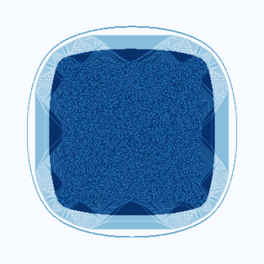
1. Subsequential Convergence

- the odometer function solves a discrete obstacle problem

$$v_n = \inf\{w : \mathbf{Z}^2 \to \mathbf{N} : \Delta^1 w + \eta_n \leq 3\},$$

  where $\eta_n : \mathbf{Z}^2 \to \mathbf{N}$ is the initial configuration at step n
- if you remove the integer constraint, you get the *divisible sandpile*

1. Subsequential Convergence



- bound v_n above and below by *divisible* sandpile odometers

$$w_n^{(l)} \leq v_n \lessapprox w_n^{(u)}$$

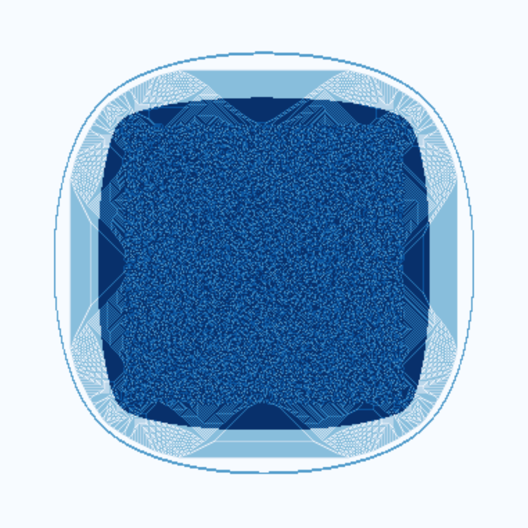
where

$$w_n^{(l)} = \inf\{w : \mathbf{Z}^2 \to \mathbf{R}^+ : \Delta^1 w + \eta_n \leq 3\}$$

and

$$w_n^{(u)} = \inf\{w : \mathbf{Z}^2 \to \mathbf{R}^+ : \Delta^1 w + \eta_n \leq 2\}$$

1. Subsequential Convergence



1. Subsequential Convergence

$$w_n = \inf\{w : \mathbf{Z}^2 \to \mathbf{R}^+ : \Delta^1 w + \eta_n \leq 3\}$$

- can control w_n using simple random walk, S_n on $\mathbf{Z}^2$.

1. Subsequential Convergence

$$w_n = \inf\{w : \mathbf{Z}^2 \to \mathbf{R}^+ : \Delta^1 w + \eta_n \le 3\}$$

- for right choice of $A_n \subset \mathbf{Z}^2$, least action principle, random walk estimates, and the ergodic theorem imply

$$\begin{aligned} w_n(x) + o(n^2) &= \tilde{w}_n(x) \\ &= \sum_{y \in A_n} G_{A_n}(x-y)\,(\eta_n(y) - 3) \\ &= \sum_{y \in A_n} G_{A_n}(x-y)\,(\mathbf{E}(\eta(0)) - 3) + o(n^2) \end{aligned}$$

where $G_{A_n}(x) = \mathbf{E}\left(\sum_{n=0}^{(\tau_{A_n}-1)} 1(S_n = y)\right)$ and τ_{A_n} is first exit time from A_n.

2. Monge-Ampère

- for $A \subset \mathbf{Z}^2$, $v : \bar{A} \to \mathbf{R}$, let

$$\partial^-(v, A) = \{p \in \mathbf{R}^2 \text{ so that for some } x \in A$$
$$v(x) + p \cdot (y - x) \leq v(y) : \text{ for all } y \in \bar{A}\}$$

  denote the subgradient set of v in A.
- let $q_M(x) = \frac{1}{2} x^T M x$ and $q_\ell(x) = \frac{1}{2} \ell |x|^2$, for $M \in \mathbf{S}^2$ and $\ell \in \mathbf{R}$
- the monotone quantity is

$$\mu(A, \ell, M) = \sup\{|\partial^-(v - q_M - q_\ell, A)|\},$$

  supremum is taken over all $v : \bar{A} \to \mathbf{Z}$ with $\Delta^1 v + \eta \leq 3$ in A.
  - subadditive, stationary by construction
  - convergence of $\frac{\mu(A_n)}{|A_n|}$ then follows by multiparameter subadditive ergodic theorem of Akcoglu and Krengel

A wrong subadditive quantity

- let
$$\nu(A) = \sup\{\sum_{x \in A} \big(\Delta^1 v(x) + \eta(x)\big)\}$$
supremum is taken over all $v : \bar{A} \to \mathbf{Z}$ with $\Delta^1 v + \eta \le 3$ in A.
- useless since no matter what η is, $\nu(A) = 3|A|$

2. Alexandroff-Bakelman-Pucci

Theorem (Lawler 1982)

For all v so that $\Delta^1 v_n + \eta \leq 3$ in $\square_n$,

$$\inf_{x \in \partial \square_n} (v_n - q_M - q_\ell)(x) \leq \inf_{x \in \square_n} (v_n - q_M - q_\ell)(x) + Cn\mu(\square_n, l, M)^{1/2}$$

- quantitative comparison principle
- in the limit,

$$\inf_{x \in \partial \square_1} (\bar{v} - q_M - q_\ell)(x) \leq \inf_{x \in \square_1} (\bar{v} - q_M - q_\ell)(x) + C\mu(l, M),$$

  where $\mu(l, M)$ is the (non-random) limit from the ergodic theorem
- dual quantity μ^* provides control in the other direction,

$$\sup_{x \in \partial \square_1} (\bar{v} - q_M - q_\ell)(x) \geq \sup_{x \in \square_1} (\bar{v} - q_M - q_\ell)(x) + C\mu^*(l, M),$$

2. μ is nice

$$\inf_{x\in\partial\Box_1}(\bar{v}-q_M-q_\ell)(x) \le \inf_{x\in\Box_1}(\bar{v}-q_M-q_\ell)(x)+C\mu(\ell,M),$$

$$\sup_{x\in\partial\Box_1}(\bar{v}-q_M-q_\ell)(x) \ge \sup_{x\in\Box_1}(\bar{v}-q_M-q_\ell)(x)+C\mu^*(\ell,M),$$

- suffices to find ℓ_M so that $\mu(\ell_M,M)=\mu^*(\ell_M,M)=0$ to establish a comparison principle in the limit.
  - $\mu(\ell,M)-\mu^*(\ell,M)$ is continuous, bounded in ℓ, and in $[-c,c]$.
  - when $\mu(\ell_M,M)=\mu^*(\ell_M,M)$, both must be zero.
- technical details involve regularity theory of the discrete Monge-Ampère equation
- key tool is a comparison principle on the microscopic scale, hidden in the sandpile

2. Definition of $\bar{\Gamma}_{random}$

- choose ℓ_M so that $\mu(\ell_M, M) = \mu^*(\ell_M, M) = 0$, then

$$\inf_{x \in \partial \square_1} (\bar{v} - q_M - q_{\ell_M})(x) \leq \inf_{x \in \square_1} (\bar{v} - q_M - q_{\ell_M})(x)$$

$$\sup_{x \in \partial \square_1} (\bar{v} - q_M - q_{\ell_M})(x) \geq \sup_{x \in \square_1} (\bar{v} - q_M - q_{\ell_M})(x)$$

- define the set

$$\bar{\Gamma}_{random} = \{M \in \mathbf{S}^2 : \mu(\ell_M) = \mu^*(\ell_M) = 0, \ell_M \geq 0\}$$

3. Convergence

- remains to check every subsequence solves

  $$\min\{v \in C(\mathbf{R}^2) : v \geq 0, D^2 v \in \bar{\Gamma}_{random} \text{ in } \Box_1 \text{ and } D^2 v \in \bar{\Gamma}_0 \text{ in } \mathbf{R}^2\}$$

  in the viscosity sense.
- same method as in Pegden-Smart 2011 shows that $D^2 \bar{v} \in \bar{\Gamma}_0$
- $\bar{\Gamma}_{random}$ is constructed so that $D^2 \bar{v} \in \bar{\Gamma}_{random}$

What is $\bar{\Gamma}_{random}$?

- can characterize $\bar{\Gamma}_{random}$ as

$$\begin{aligned}\bar{\Gamma}_{random} = \{M \in S^2 \text{ so that for almost all } \eta \text{ there exists } u : \mathbf{Z}^2 \to \mathbf{Z} \\ \Delta^1 u + \eta \leq 3 \text{ and } u(x) \geq q_M(x) + o(|x|^2))\}\end{aligned}$$

- random (but not-random!) analogue of

$$\begin{aligned}\bar{\Gamma}_k = \{M \in S^2 \text{ so that there exists } u : \mathbf{Z}^2 \to \mathbf{Z} \\ \Delta^1 u + k \leq 3 \text{ and } u(x) \geq q_M(x) + o(|x|^2))\}\end{aligned}$$

What is $\bar{\Gamma}_k$?

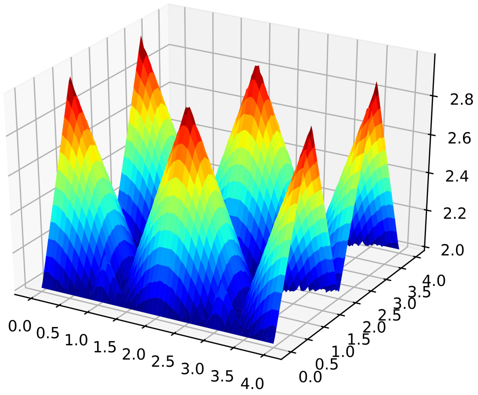
$$\bar{\Gamma}_k = \{M \in S^2 \text{ so that there exists } u : \mathbf{Z}^2 \to \mathbf{Z}$$
$$\Delta^1 u + k \leq 3 \text{ and } u(x) \geq q_M(x) + o(|x|^2))\}$$

- look at the boundary $\partial\bar{\Gamma}_0$ with a computer algorithm
- parameterize $M \in \mathbf{S}^2$ by

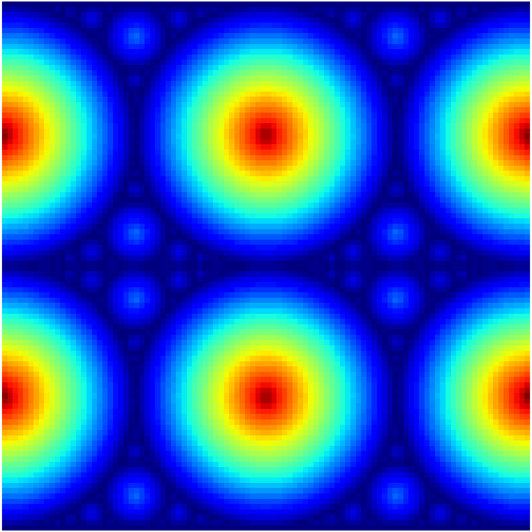
$$M(a,b,c) = \frac{1}{2}\begin{bmatrix} c-a & b \\ b & c+a \end{bmatrix}$$

  and view $\partial\bar{\Gamma}_0$ as a surface in $\mathbf{R}^3$
- translate between different $\bar{\Gamma}_k$ using the integer-valued function $q(x_1, x_2) = \frac{1}{2}x_1(x_1+1)$

What is $\bar{\Gamma}_0$?



What is $\bar{\Gamma}_0$?



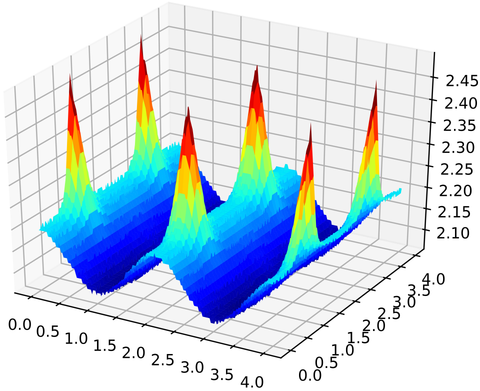
$\partial\bar{\Gamma}_0$ is an Appolonian circle packing (Levine, Pegden, Smart 2017)

What is $\bar{\Gamma}_{random}$?

$$\bar{\Gamma}_{random} = \{M \in S^2 \text{ so that for almost all } \eta \text{ there exists } u : \mathbf{Z}^2 \to \mathbf{Z}$$
$$\Delta^1 u + \eta \leq 3 \text{ and } u(x) \geq q_M(x) + o(|x|^2))\}$$

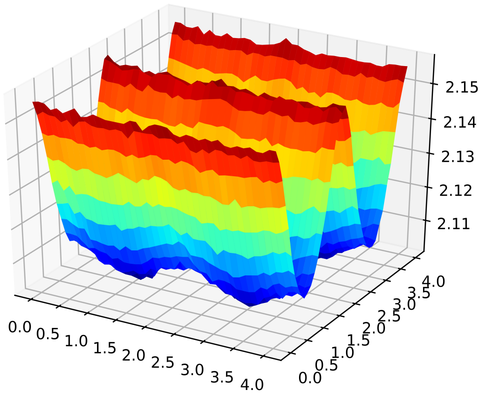
- can also look at the boundary $\partial\bar{\Gamma}_{random}$ with a computer
- will depend on the distribution of η

What is $\bar{\Gamma}_{random}$?



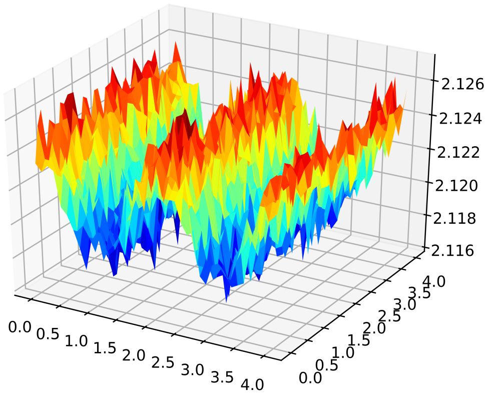
$\eta \sim$ Bernoulli$(3, 4)$

What is $\bar{\Gamma}_{random}$?



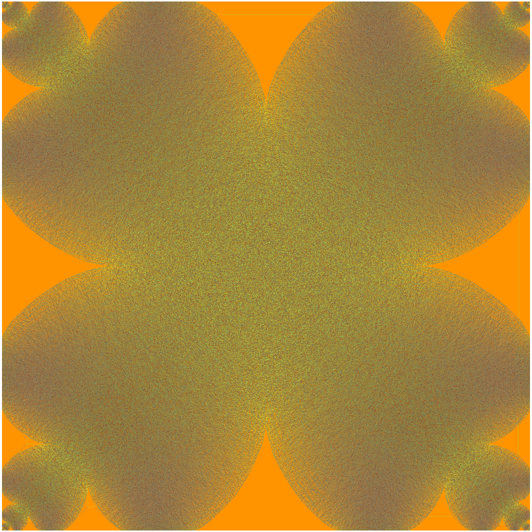
$\eta \sim \text{Bernoulli}(3, 5)$

What is $\bar{\Gamma}_{random}$?



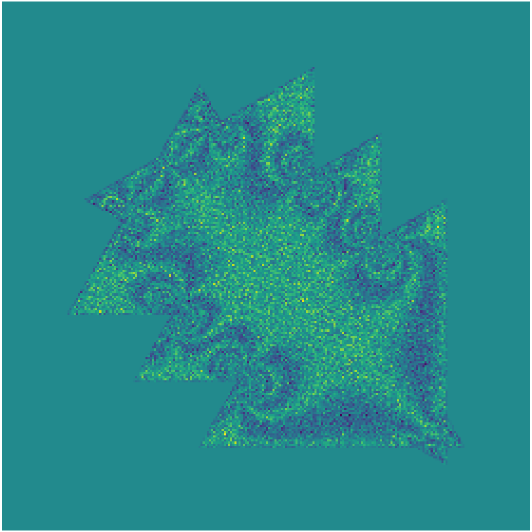
$\eta \sim \text{Bernoulli}(2, 6)$

Convergence of the Random Abelian Sandpile



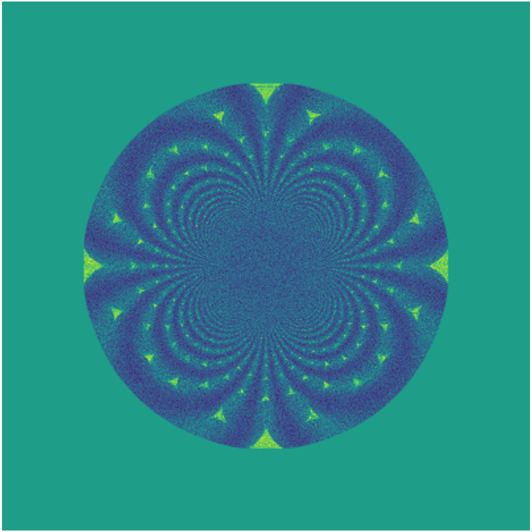
Dirichlet problem on square domain $\eta \sim \text{Bernoulli}(3, 4)$

Convergence of the Random Abelian Sandpile



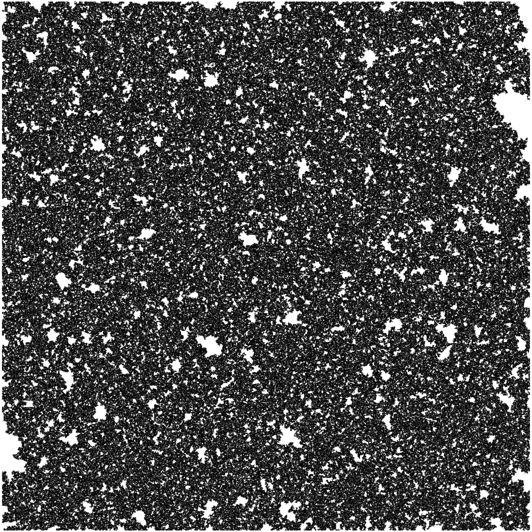
Dirichlet problem on stingray domain $\eta \sim \text{Bernoulli}(3, 5)$

Convergence of the Random Abelian Sandpile



free boundary problem with random background $\eta \sim \text{Bernoulli}(0, -1)$

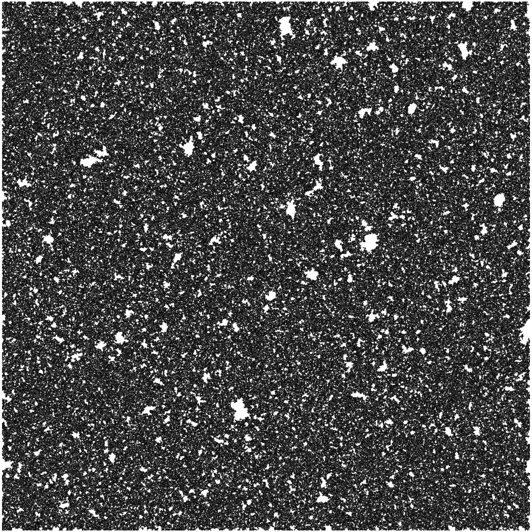
Other types of randomness



Other types of randomness?

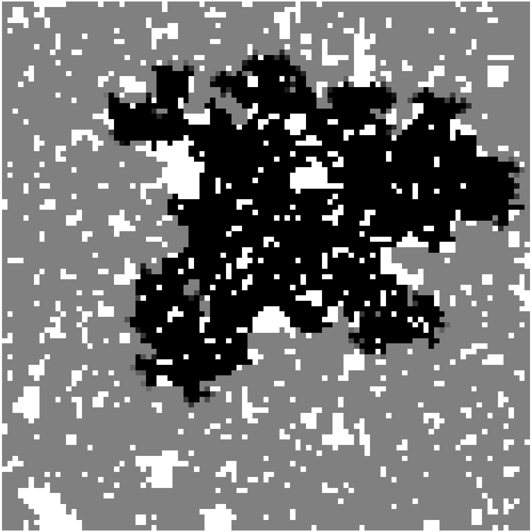
- generate a supercritical percolation cluster $C_\infty \subset \mathbf{Z}^2$, condition on $\{0 \in C_\infty\}$.
- study the sandpile on C_∞ with a *non-random* initial condition
  - whenever a site has more grains of sand than it has neighbors it topples, sending one grain to each of its neighbors
- proof for random initial states does not show convergence

Divisible sandpile on C_∞.



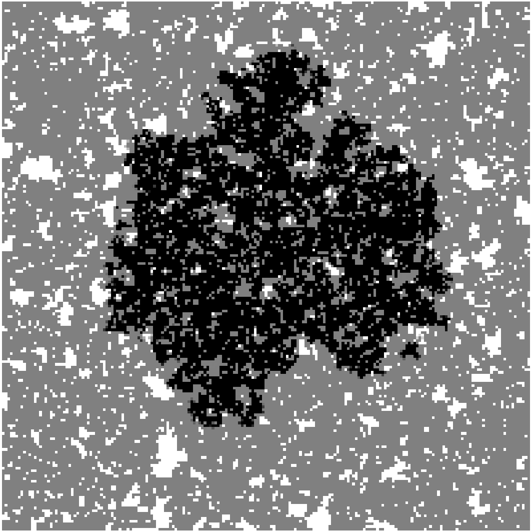
C_∞ with edge-removal probability $p = 0.47$

Divisible sandpile on C_∞.



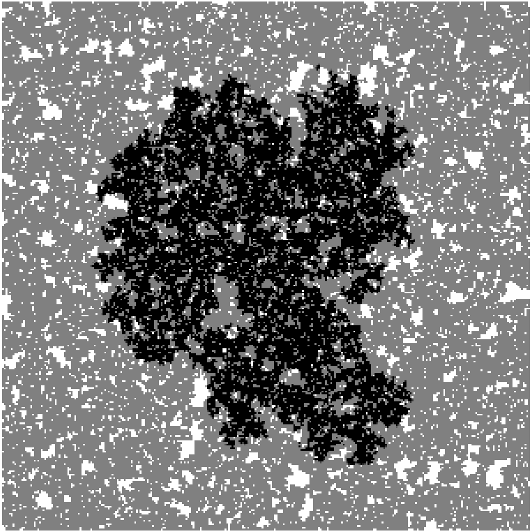
$n = 100^2$ grains at the origin in C_∞,

Divisible sandpile on C_∞.



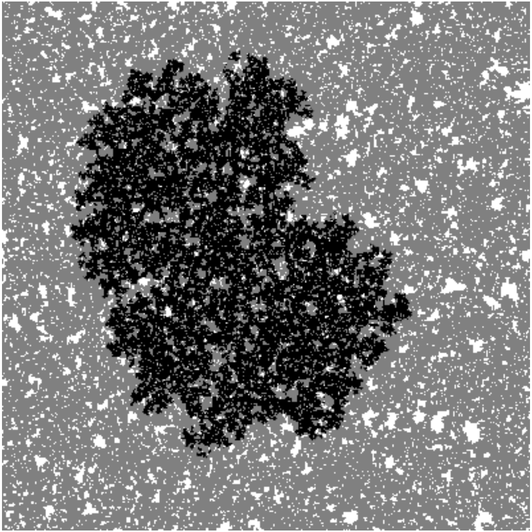
$n = 200^2$

Divisible sandpile on C_∞.



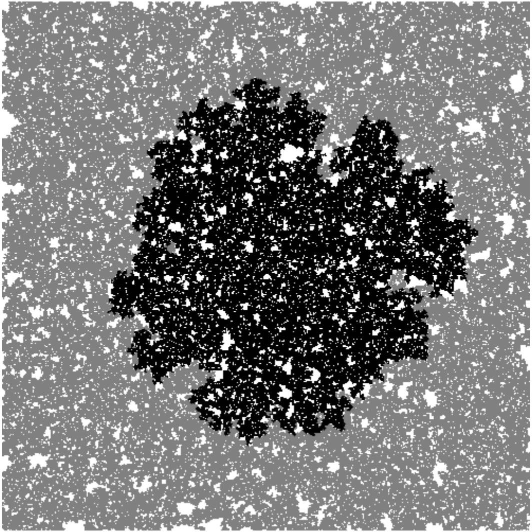
$n = 300^2$

Divisible sandpile on C_∞.



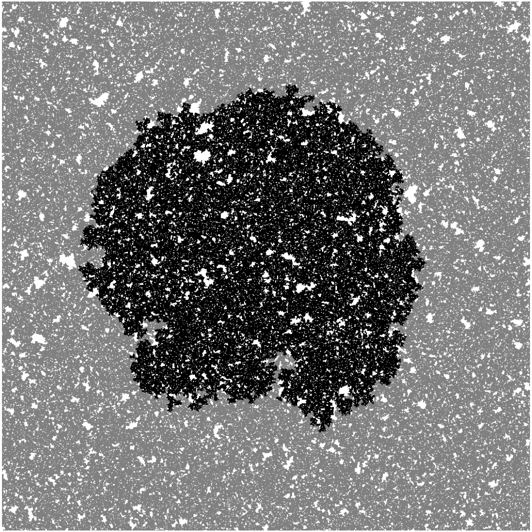
$n = 400^2$

Divisible sandpile on C_∞.



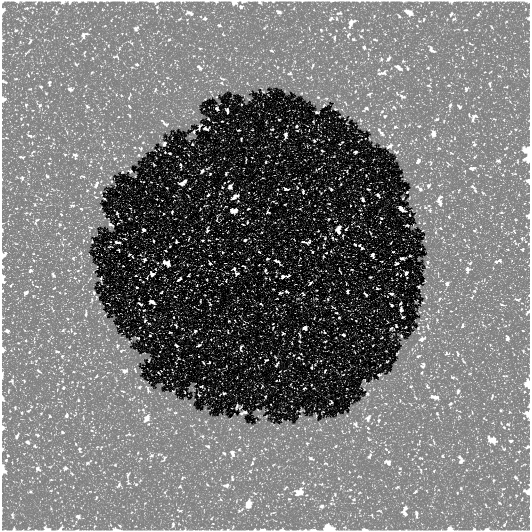
$n = 500^2$

Divisible sandpile on C_∞.



$n = 1000^2$

Divisible sandpile on C_∞.



$n = 2000^2$

Convergence of the divisible sandpile on C_∞.

Theorem (B. 2019)

Almost surely, for each $\epsilon > 0$, there exists M sufficiently large so that for all $n \geq M$,

$$B_{(1-\epsilon)R} \subset \{v_n > 0\} \subset B_{(1+\epsilon)R}$$

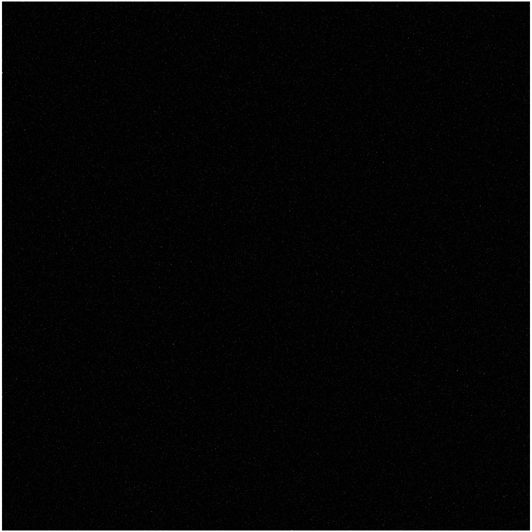
where $R = \sigma_p n^{1/2}$ and $B_r = \{x \in C_\infty : |x| < r\}$.

proof is recognizing

$$v_n(x) \approx \frac{1}{4}\left(nG_R(x) - \mathbf{E}(\tau_R(x))\right),$$

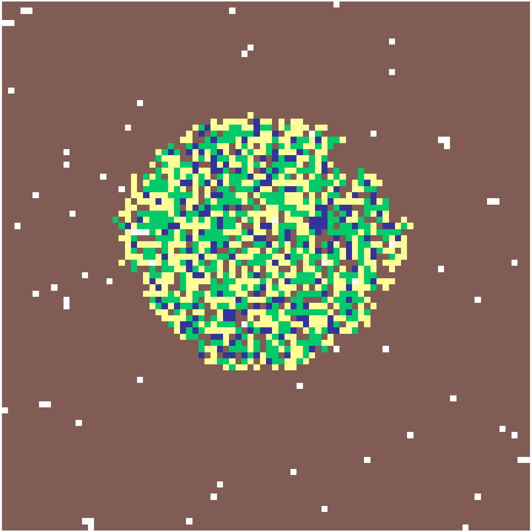
$G_R(x) = \mathbf{E}\left(\sum_{n=0}^{(\tau_R - 1)} 1(S_n = y)\right)$ and τ_R is first exit time from B_R, and S_n is the *blind* random walk on C_∞

Abelian sandpile on C_∞.



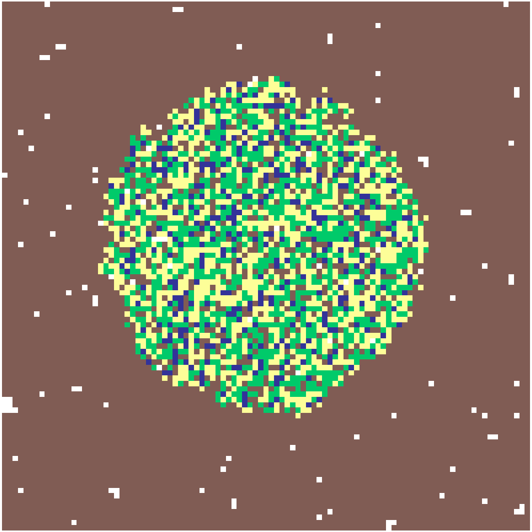
C_∞ with edge-removal probability $p = 0.1125$

Abelian sandpile on C_∞.



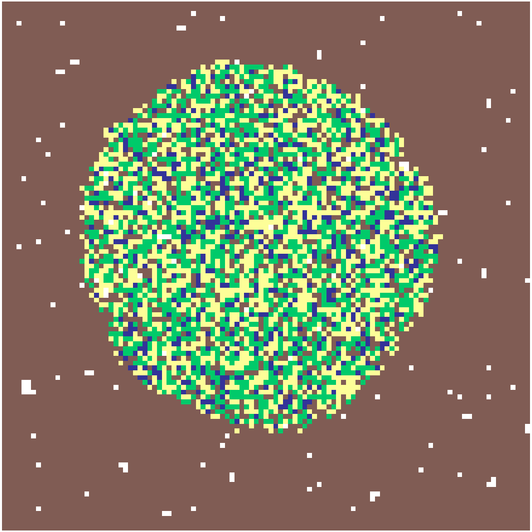
$n = 100$

Abelian sandpile on C_∞.



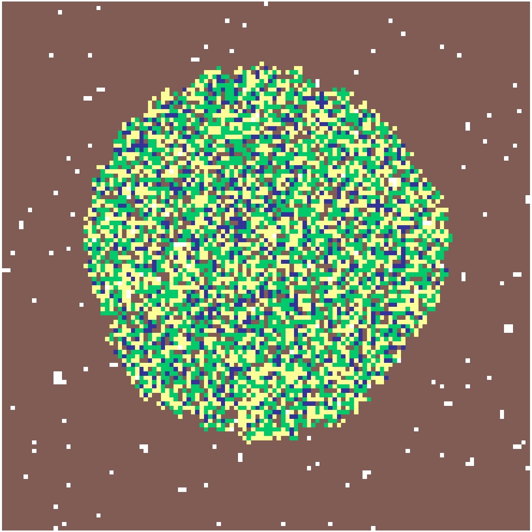
$n = 200$

Abelian sandpile on C_∞.



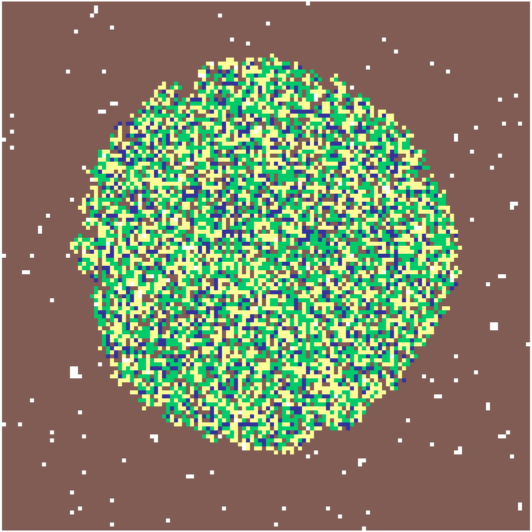
$n = 300$

Abelian sandpile on C_∞.



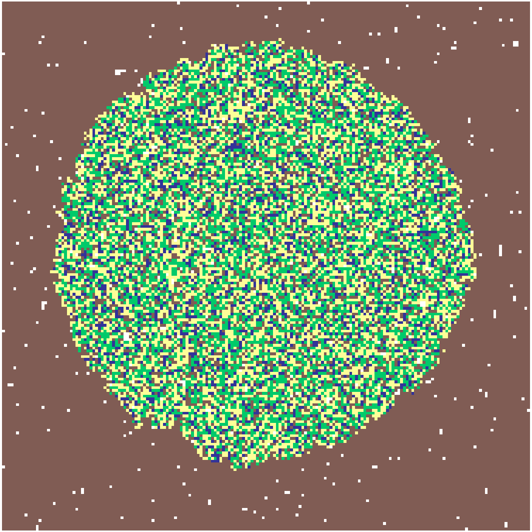
$n = 400$

Abelian sandpile on C_∞.



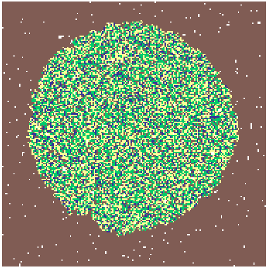
$n = 500$

Abelian sandpile on C_∞.



$n = 1000$

Convergence of the Abelian sandpile on the supercritical percolation cluster.



Conjecture

Almost surely, for each $\epsilon > 0$, there exists M sufficiently large so that for all $n \geq M$,

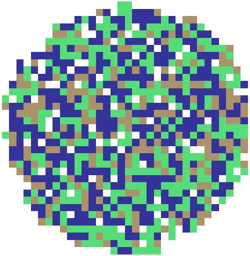
$$B_{(1-\epsilon)R} \subset \{v_n > 0\} \subset B_{(1+\epsilon)R}$$

where $R = \sigma_p n^{1/2}$ and $B_r = \{x \in C_\infty : |x| < r\}$.

The hardest kind of randomness

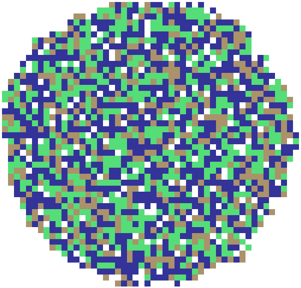
- start with a deterministic, finite initial configuration on $\mathbf{Z}^2$
- when a site has more than 4 grains of sand it *stochastically topples*
  - uniformly at random pick a neighbor 4 times and give it a grain
- cannot show subsequential convergence!
- named the Manna sandpile model

Manna Sandpile



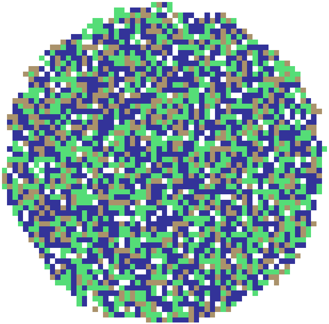
$n = 100$ grains at the origin in $\mathbf{Z}^2$,

Manna Sandpile



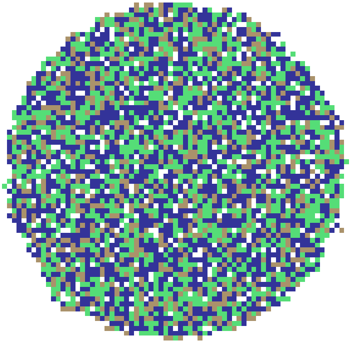
$n = 200$ grains at the origin in $\mathbf{Z}^2$,

Manna Sandpile



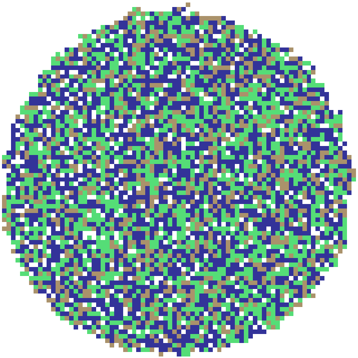
$n = 300$ grains at the origin in $\mathbf{Z}^2$,

Manna Sandpile



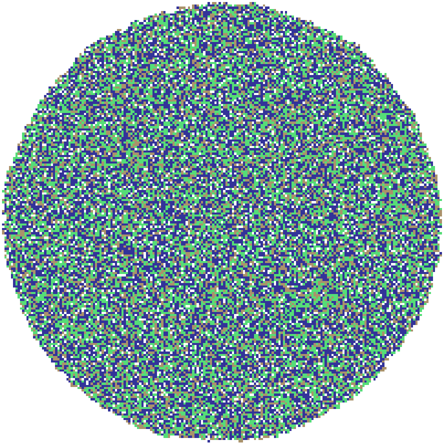
$n = 400$ grains at the origin in $\mathbf{Z}^2$,

Manna Sandpile



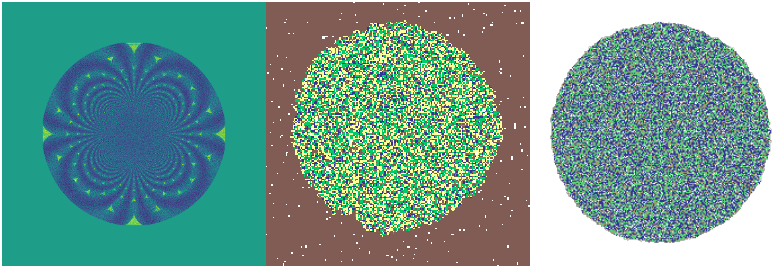
$n = 500$ grains at the origin in $\mathbf{Z}^2$,

Manna Sandpile



conjecture that spherical asymptotics also hold for the Manna model

Random Sandpiles



Thank you!