Tight triangulations of three manifolds

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Abstract.

Tight triangulations are exotic objects in combinatorial topology. Any piecewise linear embedding of a tight triangulation into euclidean space is as convex as allowed by the topology of the underlying space. Tight triangulations are also conjectured to be minimal, and proven to be so for dimensions two and three. Inspite of substantial theoretical results about such triangulations, there are precious few examples. Infact, apart from dimension two, we do not know if there are infinitely many of them in any dimension. In this paper, we present a computer friendly combinatorial scheme to obtain several three dimensional tight triangulations. While we still do not know if there are infinitely many of them, it does look like there are abundantly many, if we look for them the right way.

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1 Introduction

In this paper we present some tight triangulations of some three manifolds. We prove an adaptation of the construction method used in [5], which is more amenable to computer processing. The triangulations we obtain are neighborly members of Walkup's class $\mathcal{K}(d)$, with each vertex-link as a stacked sphere. All our triangulations triangulate connected sums of 1-handlebodies. Among the known constructions of such manifolds are: the unique 9-vertex triangulation of $(S^2 \times S^1)$ of Kühnel, apart from the two three dimensional members of the infinite family constructed in [5].

2 Preliminaries

All simplicial complexes considered here are finite and abstract. By a triangulated manifold/sphere/ball, we mean an abstract simplicial complex whose geometric carrier is a topological manifold/sphere/ball. We identify two complexes if they are isomorphic. A d-dimensional simplicial complex is called *pure* if all its maximal faces (called *facets*) are d-dimensional. A d-dimensional pure simplicial complex is said to be a *weak pseudomanifold* if each of its $(d-1)$-faces is in at most two facets. For a d-dimensional weak pseudomanifold X, the *boundary* ∂X of X is the pure subcomplex of X whose facets are those $(d-1)$-dimensional faces of X which are contained in unique facets of X. The *dual graph* $\Lambda(X)$ of a pure simplicial complex X is the graph whose vertices are the facets of X, where two facets are adjacent in $\Lambda(X)$ if they intersect in a face of codimension one. A *pseudomanifold* is a weak pseudomanifold with a connected dual graph. All connected triangulated manifolds are automatically pseudomanifolds.

If X is a d-dimensional simplicial complex then, for $0 \leq j \leq d$, the number of its j-faces is denoted by $f_j = f_j(X)$. The vector $f(X) := (f_0, \dots, f_d)$ is called the *face vector* of X and the number $\chi(X) := \sum_{i=0}^{d}(-1)^i f_i$ is called the *Euler characteristic* of X. As is well known, $\chi(X)$ is a topological invariant, i.e., it depends only on the homeomorphic type of $|X|$. A simplicial complex X is said to be *l-neighborly* if any l vertices of X form a face of X. A 2-neighborly simplicial complex is also called a *neighborly* simplicial complex.

A *standard d-ball* is a pure d-dimensional simplicial complex with one facet. The standard ball with facet σ is denoted by $\overline{\sigma}$. A d-dimensional pure simplicial complex X is called a *stacked d-ball* if there exists a sequence $B_1, \dots, B_m$ of pure simplicial complexes such that B_1 is a standard d-ball, $B_m = X$ and, for $2 \leq i \leq m$, $B_i = B_{i-1} \cup \overline{\sigma_i}$ and $B_{i-1} \cap \overline{\sigma_i} = \overline{\tau_i}$, where σ_i is a d-face and τ_i is a $(d-1)$-face of σ_i. Clearly, a stacked ball is a pseudomanifold. A simplicial complex is called a *stacked d-sphere* if it is the boundary of a stacked $(d+1)$-ball. A trivial induction on m shows that a stacked d-ball actually triangulates a topological d-ball, and hence a stacked d-sphere is a triangulated d-sphere. If X is a stacked ball then clearly $\Lambda(X)$ is a tree. So, a stacked ball is a pseudomanifold whose dual graph is a tree. But, the converse is not true (e.g., the 3-pseudomanifold X whose facets are $1234, 2345, 3456, 4567, 5671$ is a pseudomanifold for which $\Lambda(X)$ is a tree but $|X|$ is not a ball). The following results appear in [5].

Proposition 2.1. *Let X be a pure d-dimensional simplicial complex.*

(i) *If $\Lambda(X)$ is a tree then $f_0(X) \leq f_d(X) + d$.*

(ii) *$\Lambda(X)$ is a tree and $f_0(X) = f_d(X) + d$ if and only if X is a stacked ball.*

Corollary 2.2. *Let X be a pure d-dimensional simplicial complex and let CX denote a* cone *over X. Then CX is a stacked $(d+1)$-ball if and only if X is a stacked d-ball.*

In [13], Walkup defined the class $\mathcal{K}(d)$ as the family of all d-dimensional simplicial complexes all whose vertex-links are stacked $(d-1)$-spheres. Clearly, all the members of $\mathcal{K}(d)$ are triangulated closed manifolds. Let $\mathcal{K}^*(d)$ be the class of 2-neighborly members of $\mathcal{K}(d)$. We know the following.

Proposition 2.3 (Bagchi and Datta [3])**.** *Let M be a connected closed triangulated manifold of dimension $d \geq 3$. Let $\beta_1 = \beta_1(M; \mathbb{Z}_2)$. Then the face vector of M satisfies:*

(a) $$f_j \geq \begin{cases} \binom{d+1}{j} f_0 + j\binom{d+2}{j+1}(\beta_1 - 1), & \textit{if } 1 \leq j < d, \\ d f_0 + (d-1)(d+2)(\beta_1 - 1), & \textit{if } j = d. \end{cases}$$

(b) $\binom{f_0 - d - 1}{2} \geq \binom{d+2}{2}\beta_1$.

When $d \geq 4$, the equality holds in (a) *(for some $j \geq 1$, equivalently, for all j) if and only if $M \in \mathcal{K}(d)$, and equality holds in* (b) *if and only if $M \in \mathcal{K}^*(d)$.*

The case $d = 4$ of the above proposition is due to Walkup [13] and Kühnel [10]. Part (b) of the above proposition is due to Novik and Swartz [9].

Proposition 2.4 (Kalai [8]). *For $d \geq 4$, a connected simplicial complex X is in $\mathcal{K}(d)$ if and only if X is obtained from a stacked d-sphere by $\beta_1(X)$ combinatorial handle additions. In consequence, any such X triangulates either $(S^{d-1}\times S^1)^{\#\beta_1}$ or $(S^{d-1}\times\!\!\!\!\_\, S^1)^{\#\beta_1}$ according as X is orientable or not.* (*Here* $\beta_1 = \beta_1(X)$.)

It follows from Proposition 2.4 that

$$\chi(X) = 2 - 2\beta_1(X) \text{ for } X \in \mathcal{K}(d). \tag{1}$$

For a field $\mathbb{F}$, a d-dimensional simplicial complex X is called *tight with respect to* $\mathbb{F}$ (or $\mathbb{F}$-*tight*) if (i) X is connected, and (ii) for all induced subcomplexes Y of X and for all $0 \leq j \leq d$, the morphism $H_j(Y;\mathbb{F}) \to H_j(X;\mathbb{F})$ induced by the inclusion map $Y \hookrightarrow X$ is injective. If X is $\mathbb{Q}$-tight then it is $\mathbb{F}$-tight for all fields $\mathbb{F}$ and called *tight* (cf. [4]).

A d-dimensional simplicial complex X is called *minimal* if $f_0(X) \leq f_0(Y)$ for every triangulation Y of the geometric carrier $|X|$ of X. We say that X is *strongly minimal* if $f_i(X) \leq f_i(Y)$, $0 \leq i \leq d$, for all such Y. We know the following.

Proposition 2.5 (Effenberger [6], Bagchi and Datta [3]). *Every $\mathbb{F}$-orientable member of $\mathcal{K}^*(d)$ is $\mathbb{F}$-tight for $d \neq 3$. An $\mathbb{F}$-orientable member of $\mathcal{K}^*(3)$ is $\mathbb{F}$-tight if and only if $\beta_1(X) = (f_0(X)-4)(f_0(X)-5)/20$.*

Proposition 2.6 (Bagchi and Datta [3]). *Every $\mathbb{F}$-tight member of $\mathcal{K}(d)$ is strongly minimal.*

Let $\overline{\mathcal{K}}(d)$ be the class of all d-dimensional simplicial complexes all whose vertex-links are stacked $(d-1)$-balls. Clearly, if $N \in \overline{\mathcal{K}}(d)$ then N is a triangulated manifold with boundary and satisfies

$$\mathrm{skel}_{d-2}(N) = \mathrm{skel}_{d-2}(\partial N). \tag{2}$$

Here $\mathrm{skel}_j(N) = \{\alpha \in N : \dim(\alpha) \leq j\}$ is the j-skeleton of N. We know the following.

Proposition 2.7 (Bagchi and Datta [4]). *For $d \geq 4$, $M \mapsto \partial M$ is a bijection from $\overline{\mathcal{K}}(d+1)$ to $\mathcal{K}(d)$.*

From the above proposition, we have the following:

Corollary 2.8. *For $d \geq 4$, if $M \in \overline{\mathcal{K}}(d+1)$ then* $\mathrm{Aut}(M) = \mathrm{Aut}(\partial M)$.

Let G be a graph and $\mathcal{T} = \{T_i\}_{i\in\mathcal{I}}$ be a family of induced subtrees of G, such that every vertex of G is contained in exactly $d+1$ trees and any two adjacent vertices appear together in exactly d trees. We define the pure d-dimensional simplicial complex $\mathcal{K}(G,\mathcal{T})$ by the following facet complex:

$$\mathcal{K}(G,\mathcal{T}) := \{\{i : u \in T_i\} : u \in V(G)\}. \tag{3}$$

We will denote the facet $\{i : u \in T_i\}$ by $\hat{u}$ for $u \in V(G)$. Our constructions are based on following result from [5].

Proposition 2.9. *Let G be a graph and $\mathcal{T} = \{T_i\}_{i=1}^n$ be a family of $(n-d)$-vertex induced subtrees of G, any two of which intersect. Suppose that* (i) *each vertex of G is in exactly $d+1$ members of $\mathcal{T}$ and* (ii) *for any two vertices $u \neq v$ of G, u and v are together in exactly d members of $\mathcal{T}$ if and only if uv is an edge of G. Then $\mathcal{K}(G,\mathcal{T})$ is a neighborly member of $\overline{\mathcal{K}}(d)$, with $\Lambda(\mathcal{K}(G,\mathcal{T})) \cong G$.*

3 Examples of tight three manifolds

In this section we present examples of tight three dimensional manifolds obtained as boundaries of tight four manifolds with boundary. More explicitly for each $n = 20k+9, 2 \leq k \leq 5$, we construct n-vertex triangulations using Proposition 2.9 which have $\mathbb{Z}_n$ as the automorphism group. We need some notation to describe our construction.

Let $k \geq 1$, $n = 20k + 9$ and $d_0, d_1, \ldots, d_k$ be invertible elements in $(\mathbb{Z}_n, \cdot)$. Let $G := G(k; d_0, \ldots, d_k)$ denote the graph on $n(4k + 1)$ vertices with the vertex set $V(G)$ given by $V(G) := \{v_{i,j} : 0 \leq i \leq 4k, j \in \mathbb{Z}_n\}$. The edge set $E(G)$ of G consists of edges $(v_{i,j}, v_{i+1,j})$ for $0 \leq i < 4k, j \in \mathbb{Z}_n$, and edges $(v_{4i,j}, v_{4i,j+d_i})$ for $0 \leq i \leq k$ and $j \in \mathbb{Z}_n$ (Note that the second subscript is read modulo n). In the graph $G(k; d_0, \ldots, d_k)$ let P_j denote the path $v_{0,j}v_{1,j} \cdots v_{4k,j}$ for $j \in \mathbb{Z}_n$. Also let C_i denote the subgraph spanned by the edges $\{(v_{4i,j}, v_{4i,j+d_i}) : j \in \mathbb{Z}_n\}$ for $0 \leq i \leq k$. Note that when $d_0, \ldots, d_k$ are relatively prime to n, each of the subgraphs C_i is an n-cycle. We also note the following automorphism of the graph $G(k; d_0, \ldots, d_k)$:

$$\varphi := \prod_{i=0}^{4k} (v_{i,0}, v_{i,1}, \ldots, v_{i,n-1}). \tag{4}$$

The automorphism φ generates the automorphism group of G isomorphic to $\mathbb{Z}_n$. To construct neighborly members of $\overline{\mathcal{K}}(4)$, we exhibit a family of subtrees of G which satisfy the conditions in Proposition 2.9.

3.1 Construction of induced subtrees of G

To describe the subtrees of G, we introduce some terminology. A collection $\mathcal{D} = \{(\sigma_i, \tau_i) : 1 \leq i \leq k\}$ where σ_i, τ_i are permutations of the set $\{0, 1, 2, 3\}$ is called a *k-deck* of permutations. For a vertex $v_{i,j} \in V(G)$, we call the subpath $v_{i-l,j}P_jv_{i,j}$ of P_j to be the *upward* path of length l at $v_{i,j}$. Similarly we call the subpath $v_{i,j}P_jv_{i+l,j}$ of P_j to be the *downward* path of length l at $v_{i,j}$.

Definition 3.1. For a k-deck $\mathcal{D} = \{(\sigma_i, \tau_i)\}_{i=1}^{k}$, let $\mathcal{D}T_j$ be the induced subgraph of G spanned by the following:

(i) The paths $v_{4i,j}v_{4i,j+d_i} \cdots v_{4i,j+4\cdot d_i}$ for $0 \leq i \leq k$.

(ii) The path P_j.

(iii) Upward paths of length $\tau_i(t)$ at vertex $v_{4i,j+(t+1)d_i}$ for $1 \leq i \leq k$ and $0 \leq t \leq 3$.

(iv) Downward paths of length $\sigma_{i+1}(t)$ at vertex $v_{4i,j+(t+1)d_i}$ for $0 \leq i \leq k-1$ and $0 \leq t \leq 3$.

We notice that $\mathcal{D}T_j = \varphi^j(\mathcal{D}T_0)$. For a permutation pair $(\sigma_i, \tau_i) \in \mathcal{D}$, we define its *span* $sp(\sigma_i, \tau_i)$ to be the subset of $\mathbb{Z}_n$ given by $\{\pm((p + 1)d_{i-1} - (q + 1)d_i) : 0 \leq p, q \leq 3, \sigma_i(p) + \tau_i(q) \geq 4\}$. We define the *span* of $\mathcal{D}$, denoted by $sp(\mathcal{D})$ as:

$$sp(\mathcal{D}) = \bigcup_{i=1}^{k} sp(\sigma_i, \tau_i) \cup \{\pm t d_i : 0 \leq t \leq 4, 0 \leq i \leq k\}. \tag{5}$$

Lemma 3.2. *If $sp(\mathcal{D}) = \mathbb{Z}_n$, then the subgraph $\mathcal{D}T_j$ is a tree for all $j = 0, \ldots, n-1$.*

Proof. First we show $sp(\mathcal{D}) = \mathbb{Z}_n$ implies that the sets in the union in (5) are mutually disjoint. Note that $|sp(\sigma_i, \tau_i)| \leq 6$. Then the number of non-zero elements in $sp(\mathcal{D})$ is at most $2 \cdot 6k + 2 \cdot 4(k+1) = 20k + 9 = n - 1$. Thus all the sets in the union must be mutually disjoint to acheive $sp(\mathcal{D}) = \mathbb{Z}_n$. Now we proceed to show that $\mathcal{D}T_j$ is a tree for all $j = 0, \ldots, n-1$. Notice that the steps (i) and (ii) of the construction in Definition 3.1, yeild a tree consisting of the path P_j and arcs of length 4 of cycles C_i for $0 \leq i \leq k$. In the steps (iii) and (iv) we attach upward and downward paths to the arcs attached in step (i). The resulting graph is clearly connected. Since the new vertices introduced in steps (iii) and (iv) have an edge only if belong to the same path P_j for some j, for a cycle to occur, two paths added in steps (iii) and (iv) must be subpaths of the same path P_j for some j. But then we must have $j + (t+1)d_i = j + (t'+1)d_i$, or $j + (t+1)d_{i-1} = j + (t'+1)d_i$ for some $0 \leq t, t' \leq 3$. The former implies $t = t'$, in which case the two paths are the same. In the latter case, we have $(t+1)d_{i-1} = (t'+1)d_i$. But then the sets $\{\pm td_i : 0 \leq t \leq 4\}$ and $\{\pm td_{i-1} : 0 \leq t \leq 4\}$ are not disjoint, a contradiction. Thus the resulting graph is connected, and without a cycle. Hence it is a tree. $\square$

We have the following:

Lemma 3.3. *For $j \in sp(\mathcal{D})$, the subgraphs $\mathcal{D}T_0$ and $\mathcal{D}T_j$ intersect.*

Proof. First assume that $j = \pm td_i$ for some $0 \leq t \leq 4$. For $j = td_i$, $0 \leq t \leq 4$, the vertex v_{4i,td_i} is common to both $\mathcal{D}T_0$ and $\mathcal{D}T_j$. For $j = -td_i$, $0 \leq t \leq 4$, the vertex $v_{4i,0}$ is common to both $\mathcal{D}T_0$ and $\mathcal{D}T_j$. Hence the subgraphs intersect. Next assume that $j = \pm((p+1)d_{i-1} - (q+1)d_i)$ where $0 \leq p, q \leq 3$ and $\sigma_i(p) + \tau_i(q) \geq 4$. If $j = (p+1)d_{i-1} - (q+1)d_i$, let $r = j + (q+1)d_i = (p+1)d_{i-1}$. Then the subgraph $\mathcal{D}T_0$ contains a downward path of length $\sigma_i(p)$ at $v_{4(i-1),r}$, and the subgraph $\mathcal{D}T_j$ contains an upward path of length $\tau_i(q)$ at $v_{4i,r}$. Since $\sigma_i(p) + \tau_i(q) \geq 4$, the two paths intersect. Similarly, it can be shown that $\mathcal{D}T_0$ and $\mathcal{D}T_j$ intersect when $j = -(p+1)d_{i-1} + (q+1)d_i$. $\square$

Lemma 3.4. *Let $\mathcal{D} = \{(\sigma_i, \tau_i)\}_{i=1}^k$ be a deck of permutations such that:*

(a) $sp(\mathcal{D}) = \mathbb{Z}_n$.

(b) $\sigma_i(t) + \tau_{i-1}(t) \geq 1$ *for* $2 \leq i \leq k, 0 \leq t \leq 2$.

(c) $\sigma_i(3), \tau_i(3) \geq 1$ *for* $1 \leq i \leq k$.

Then $\mathcal{K}(G, \mathcal{D}T)$ is a neighborly member of $\overline{\mathcal{K}}(4)$ where $\mathcal{D}T = \{\mathcal{D}T_j\}_{j=0}^{n-1}$.

To prove Lemma 3.4, we will first show an equivalent of Proposition 2.9.

Lemma 3.5. *Let G be a graph and $\mathcal{T} = \{T_i\}_{i=1}^n$ be a family of $(n-d)$-vertex induced subtrees of G, any two of which intersect. Suppose that* (i) *each vertex of G is in exactly $d+1$ members of $\mathcal{T}$,* (ii) *any two adjacent vertices u and v occur together in exactly d members of $\mathcal{T}$ and* (iii) *for a vertex $u \in V(T), T \in \mathcal{T}$, we have $d_G(u) - d_T(u) \leq 1$. Then $\mathcal{K}(G, \mathcal{T})$ is a neighborly member of $\overline{\mathcal{K}}(d)$ with $\Lambda(\mathcal{K}(G, \mathcal{T})) \cong G$.*

Proof. Let $T_i \in \mathcal{T}$ be a tree. For a vertex $r \in T_i$, define the oriented tree $T_i(r)$ with directed edges $\overrightarrow{uv}$ where $uv \in T_i$ and v is closer to r than u. Define *label* $l(\overrightarrow{uv})$ to be the unique element of $\hat{u} \backslash \hat{v}$ (follows from conditions (i) and (ii)). We prove that all edges in $T_i(r)$ have distinct labels. Now, there are d other trees that intersect T_i in r. Let $T_j \in \mathcal{T}$ be a tree that does not intersect T_i in r. Since any two trees in $\mathcal{T}$ intersect, there is a vertex $w \neq r$

which is common to T_i and T_j. Then we see that one of the edges in the w-r path in $T_i(r)$ must have the label j. Since there are $n-d-1$ such trees and also $n-d-1$ edges in $T_i(r)$, we conclude that all labels must be distinct. Further the labels are different from the ones seen at r.

We now prove that $(G, \mathcal{T})$ satisfy the conditions in Proposition 2.9. Essentially, we need to show that $|\hat{u} \cap \hat{v}| = d$ implies that uv is an edge in G. Suppose u, v are vertices in G such that $|\hat{u} \cap \hat{v}| = d$. Assume uv is not an edge of G. Let T_i be one of trees containing both u and v. Let w be an internal vertex of the u-v path in T_i.

Claim: $\hat{u} \cap \hat{v} \subseteq \hat{u} \cap \hat{w}$.

Proof: If possible, let $j \in (\hat{u} \cap \hat{v}) \backslash (\hat{u} \cap \hat{w})$. Then $j \in \hat{u}, \hat{v}$ but $x \notin \hat{w}$. Hence, in the oriented tree $T(w)$, there exist edges on paths uv and vw in $T(w)$ with label j. But this contradicts the uniqueness of labels on the edges of $T(w)$. Hence the claim.

Let u, z, w be the first three vertices on the u-v path in the tree T_i. Let j and k be the labels of edges $\overrightarrow{uz}$ and $\overrightarrow{zw}$ respectively. We show that $\{j, k\} \in \hat{u} \backslash \hat{w}$. Since $j \notin \hat{z}$, by the previous claim $j \notin \hat{w}$. Also, we have $k \notin \hat{w}$. It remains to show that $k \in \hat{u}$. Suppose, on the contrary that $k \notin \hat{u}$. But then $k \in \hat{z}$, but $k \notin \hat{u}, \hat{w}$. But then $d_G(z) \geq d_{T_k}(z) + 2$, a contradiction to the assumption (iii). Thus $\{j, k\} \subseteq \hat{u} \backslash \hat{w}$, or $\hat{u} \cap \hat{v} \subseteq \hat{u} \cap \hat{w} = \hat{u} \backslash (\hat{u} \cap \hat{w}) \subseteq \hat{u} \backslash \{j, k\}$. Thus $|\hat{u} \cap \hat{v}| \leq d-1 < d$, a contradiction. This proves the lemma. □

We are now in a position to prove Lemma 3.4.

Proof of Lemma 3.4. We show that $(G, \mathcal{T})$ satisfy the conditions in Lemma 3.5 for $d = 4$.

Each tree has $n-4$ vertices: From Definition 3.1, the number of vertices in a tree is:

$$\begin{aligned} &4(k+1) + (4k+1) && [(i) + (ii)] \\ &+ 2k(0+1+2+3) && [(iii) + (iv)] \\ &= 20k + 5 = n - 4 = n - d \end{aligned}$$

Any two trees intersect: From Lemma 3.3, it follows that any two trees in $\mathcal{DT}$ intersect.

Each vertex appears in 5 trees, each edge in 4-trees: Next, we calculate the number of trees that cover a particular vertex $v \in V(G)$. Since $T_j = \varphi^j(T_0)$, we see that $|\{j : v \in V(T_j)\}| = |\{j : \varphi^j(v) \in V(T_0)\}|$, which is same as the number of vertices in T_0 from φ-orbit of v. Without loss of generality, assume $v = v_{i,0}$ for some $0 \leq i \leq 4k$. Clearly, when $v = v_{4i,0}$ for some $0 \leq i \leq k$, T_0 contains 5 vertices from φ-orbit of v. Now let $v = v_{4i+t,0}$ where $1 \leq t \leq 3$. Let φ_v denote the φ-orbit of v. Note that T_0 intersects φ_v at v. Other intersections between T_0 and φ_v occur along the downward and upward paths in T_0 from vertices in C_i and C_{i+1} respectively. Now downward path of length $\sigma_{i+1}(p)$ at a vertex in C_i intersects φ_v if $\sigma_{i+1}(p) \geq t$. Similarly an upward path of length $\tau_{i+1}(q)$ at a vertex in C_{i+1} intersects φ_v if $\tau_{i+1}(q) \geq 4-t$. Recalling that σ_{i+1}, τ_{i+1} are permutations of $\{0, 1, 2, 3\}$, and that the respective paths are distinct, we get that T_0 contains $1 + 4 - (4-t) + 4 - t = 5$ vertices from φ_v. Similarly, it can be shown that T_0 contains 4 edges from each edge orbit under φ. As in the vertex case, this implies that each edge is covered by exactly 4 trees in $\mathcal{T}$.

For a vertex $v \in V(T_i)$, $d_G(v) - d_{T_i}(v) \leq 1$: Via the automorphism φ, it is sufficient to prove that for $v \in V(T_0)$, $d_G(v) - d_{T_0}(v) \leq 1$. As $d_G(v_{i,j}) = 2$ for $i \not\equiv 0 \pmod 4$, there is nothing to prove for those vertices. For remaining vertices we have,

$$d_G(v_{4i,j}) = \begin{cases} 3 & \text{if } i \in \{0, k\}, \\ 4 & \text{otherwise.} \end{cases} \tag{6}$$

First consider the case when $i \in \{0, k\}$. Note that $v_{4i,j} \in V(T_0)$ for $j = 0, d_i, \ldots, 4d_i$. The vertices $v_{4i,0}$ have at least two neighbors in T_0 (one on the path P_0, and other on the cycle C_i). Thus the condition holds for these vertices. Similarly the vertices $v_{4i,d_i}, v_{4i,2d_i}$ and $v_{4i,3d_i}$ have degree at least two in T_0 (being internal vertices of a path). The vertex $v_{4i,4d_i}$ has a neighbor $v_{4i,3d_i}$ on T_0, and an additional one on the downward (resp., upward) path at $v_{4i,3d_i}$ when $i = 0$ (resp., k). The preceeding claim holds because $\sigma_1(3) \geq 1$ and $\tau_k(3) \geq 1$. Now consider the case when $i \notin \{0, k\}$. Then for $t = 0, 1, 2$ the vertices $v_{4i,(t+1)d_i}$ are internal vertices of a path in T_0. It has a further neighbor on an upward or a downward path as $\sigma_i(t) + \tau_{i-1}(t) \geq 1$ for $t = 0, 1, 2$. Thus the vertices $v_{4i,(t+1)d_i}$ for $t = 0, 1, 2$ have degree at least 3 in T_0, and hence they satisfy the condition. The vertices $v_{4i,4d_i}$ have a neighbor $v_{4i,3d_i}$ and one neighbor each on upward and downward paths at $v_{4i,4d_i}$ as the condition (c) implies both the paths are non-trivial.

Thus $(G, \mathcal{T})$ fufill the requirements of Lemma 3.5, and hence yeild tight neighborly four manifolds with boundary. □

4 Implementation notes

We describe an optimized algorithm to search for tight triangulations using Lemma 3.5. Given $k \geq 2$ and $n = 20k + 9$, our task is the following:

(i) Find distinct invertible elements $d_0, \ldots, d_k$ in $\mathbb{Z}_n$. Without loss of generality, we may choose $d_0 = 1$.

(ii) For a choice of $d_i : 0 \leq i \leq k$, search for k pairs of permutations $\{(\sigma_i, \tau_i)\}_{i=1}^k$ of the set $\{0, 1, 2, 3\}$ satisfying the conditions in Lemma 3.4.

We explore the sequence $(d_0, \ldots, d_k)$ in a depth-first manner. To prune the search tree, we use the following fact, which follows from the proof of Lemma 3.2.

$$\{\pm td_i : t = 1, 2, 3, 4\} \cap \{\pm td_j : t = 1, 2, 3, 4\} = \emptyset \text{ for } i \neq j. \tag{7}$$

Having determined a sequence $d_0, \ldots, d_k$, we look for a deck of k permutation pairs $\{(\sigma_i, \tau_i)\}_{i=1}^k$. Again, we explore the permutation pairs in a depth-first manner. The observations below help to ecnomize the search. From Lemma 3.4, it follows that a valid permutation σ should satisfy $\sigma(t) = 0$ for $t \in \{0, 1, 2\}$. Accordingly, we call σ to be of type 0, 1 or 2 depending on whether $\sigma(0) = 0$, $\sigma(1) = 0$ or $\sigma(2) = 0$. Similarly, we call a permutation pair (σ, τ) to be of type (l, m) if σ is of type l and τ is of type m. Since there are 6 permutations of each type, there are $6 \times 6 = 36$ permutation pairs of each type. As there are 9 types of permuation pairs, we have $36 \times 9 = 324$ permutations to consider at each level. However, we can substantially reduce this number. We call permutation pairs of type (l, m) and (l', m') to be *compatible* if $m \neq l'$. Observe that the adjacent permutation pairs (σ_i, τ_i) and $(\sigma_{i-1}, \tau_{i-1})$ in a k-deck satisfying Lemma 3.4 must be compatible. Thus, apart from the first level, we only have to consider 216 compatible permutation pairs. To enable faster access to compatible permutations at each level, we do a pre-processing step of storing them by their type. We store 36 permutations of each type in a contiguous block. Then we stack 9 such blocks to form a linear array of 324 permutation pairs. The blocks are stacked following the lexicographic ordering of the type of the permutation pairs they contain. It can be seen that in this scheme, all the permutation pairs compatible with a given permutation pair, occur as contiguous blocks, possibly wrapping around at the end of the array. Finally, we store a permutation pair (σ, τ) as the following set, which we call its *treetype*.

$$\mathit{treetype}(\sigma, \tau) = \{(p+1, q+1) : \sigma(p) + \tau(q) \geq 4\}. \tag{8}$$

Given a set of six tuples S, there is at most one permutation pair (σ, τ) such that $\mathit{treetype}(\sigma, \tau) = S$. The nomenclature "treetype" denotes the fact that (σ, τ) determine the shape of the tree at a particular level.

5 Results

Following table summarizes the number of solutions for different values of k. The d-vectors and treetypes of the solutions appear in the appendix.

k	n	#(solutions)	Remarks
0	9	1	Kühnel's twisted torus
1	29	6	Includes examples in [5]
2	49	1	New
3	69	15	"
4	89	41	"
5	109	12	"

Table 1: Examples of tight three manifolds

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