

1 Arbitrary s.p.d. metric

1.1 Assumptions

- $Ind(M) = 0$
- $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = g_{11} (dx^1)^2 + 2g_{12} dx^1 dx^2 + g_{22} (dx^2)^2$ (s.p.d.)

1.2 General properties

$\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $\gamma \in \Omega^r(M)$, $\vec{v} \in \mathcal{V}(M)$

1.2.1 Wedge product \wedge

- $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ (**anti-/commutativ**)
- **associativ** ($\alpha \wedge \beta \wedge \gamma$)
- $(c_1 \alpha + c_2 \beta) \wedge \gamma = c_1 \alpha \wedge \gamma + c_2 \beta \wedge \gamma$ (**bilinear**)

1.2.2 Exterior derivative $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$

$\alpha \in \Omega^p(M)$

- $d \circ d = 0$ (**complex property**)
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ (**product rule, \wedge -antiderivation**)

1.2.3 Hodge star $*$: $\Omega^p(M) \rightarrow \Omega^{2-p}(M)$

- $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu$
- $*1 = \mu$ ($*\mu = 1$)
- $**\alpha = (-1)^p \alpha$
- $\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$

1.2.4 Contraction $i : (\mathcal{V} \times \Omega^p)(M) \rightarrow \Omega^{p-1}(M)$ (**inner product**)

- $i_{\vec{v}} \alpha (\vec{t}_1, \dots, \vec{t}_{p-1}) = \alpha(\vec{v}, \vec{t}_1, \dots, \vec{t}_{p-1})$
- $f i_{\vec{v}} \alpha = i_{f\vec{v}} \alpha = i_{\vec{v}} f \alpha$ (**bilinear**)
- $i_{\vec{v}}(\alpha \wedge \beta) = (i_{\vec{v}} \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_{\vec{v}} \beta)$ (**\wedge -antiderivation**)

1.2.5 Lie-derivative $\mathcal{L} : (\mathcal{V} \times \Omega^p)(M) \rightarrow \Omega^p(M)$

- $\mathcal{L}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\vec{v}}\alpha$ (**Cartans magic formular**)
- $\mathcal{L}_{f\vec{v}}\alpha = f\mathcal{L}_{\vec{v}}\alpha + \mathbf{d}f \wedge \mathbf{i}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}(\alpha \wedge \beta) = \mathcal{L}_{\vec{v}}\alpha \wedge \beta + \alpha \wedge \mathcal{L}_{\vec{v}}\beta$
- $\mathcal{L}_{\vec{v}}\mathbf{d}\alpha = \mathbf{d}\mathcal{L}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathcal{L}_{\vec{v}}\alpha$
 $\Rightarrow \alpha \in \Omega^1(M) : \mathcal{L}_{\vec{v}}\langle \vec{v}^\flat, \alpha \rangle = \langle \vec{v}^\flat, \mathcal{L}_{\vec{v}}\alpha \rangle$
- $\mathcal{L}_{\vec{v}}\vec{w} = [\vec{v}, \vec{w}] = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}$ ((**Levi-Civita**)-**Con**ection ∇ is **Torsion-free**)

1.3 Wedge product \wedge

$f \in \Omega^0(M)$, $\tilde{f} \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$, $\beta := b_1dx^1 + b_2dx^2 \in \Omega^1(M)$,
 $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

- $f\tilde{f} = f \wedge \tilde{f} = \tilde{f} \wedge f \in \Omega^0(M)$
- $f\alpha := f \wedge \alpha = \alpha \wedge f = fa_1dx^1 + fa_2dx^2 \in \Omega^1(M)$
- $\alpha \wedge \beta = -\beta \wedge \alpha = (a_1b_2 - a_2b_1)dx^1 \wedge dx^2 \in \Omega^2(M)$
- $f\omega := f \wedge \omega = \omega \wedge f = fw_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

1.4 Exterior derivative \mathbf{d}

$f \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$

- $\mathbf{d}f = \partial_1f dx^1 + \partial_2f dx^2$
- $(\mathbf{d}f)_\mu = \partial_\mu f$ (**Ricci**)
- $\mathbf{d}\alpha = (\partial_1a_2 - \partial_2a_1)dx^1 \wedge dx^2$
- $(\mathbf{d}\alpha)_{12} = (-1)^{\mu-1}\partial_\mu a_{\bar{\mu}}$ (**Ricci**)

1.5 Hodge star $*$

$f \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$, $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

- $*f = f\mu = \sqrt{|g|}f dx^1 \wedge dx^2$
- $*\alpha = \sqrt{|g|}(- (a_1g^{12} + a_2g^{22})dx^1 + (a_1g^{11} + a_2g^{12})dx^2)$
- $(*a)_\mu = (-1)^\mu \sqrt{|g|}g^{\nu\bar{\mu}}a_\nu = (-1)^\mu \sqrt{|g|}a^{\bar{\mu}}$ (**Ricci**)
- $*\omega = \frac{w_{12}}{\sqrt{|g|}}$

- $1 - \frac{\langle \alpha, \beta \rangle^2}{\|\alpha\|^2 \|\beta\|^2} = \frac{\|\alpha \wedge \beta\|^2}{\|\alpha\|^2 \|\beta\|^2} = \frac{\langle \alpha, * \beta \rangle^2}{\|\alpha\|^2 \|\beta\|^2} \quad (\sim 1 - \cos^2 \phi = \sin^2 \phi)$
- $\langle \alpha, \beta \rangle^2 + \langle \alpha, * \beta \rangle^2 = \|\alpha\|^2 \|\beta\|^2$
- $\langle * \alpha, \beta \rangle = - \langle \alpha, * \beta \rangle$

1.6 Rising and lowering indices \sharp / \flat

$$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\alpha^\sharp = (g^{11} a_1 + g^{12} a_2) \partial_1 + (g^{12} a_1 + g^{22} a_2) \partial_2$
- $a^\mu = g^{\mu\nu} a_\nu$ (**Ricci**)
- $\vec{v}^\flat = (g_{11} v^1 + g_{12} v^2) dx^1 + (g_{12} v^1 + g_{22} v^2) dx^2$
- $v_\mu = g_{\mu\nu} v^\nu$ (**Ricci**)

1.7 Contraction \mathbf{i}

$$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\mathbf{i}_{\vec{v}} \alpha = \alpha(\vec{v}) = a_1 v^1 + a_2 v^2$
- $\mathbf{i}_{\vec{v}} \omega = w_{12} (-v^2 dx^1 + v^1 dx^2)$

1.8 Lie-derivative \mathcal{L}

$$f \in \Omega^0(M), \alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\mathcal{L}_{\vec{v}} f = v^1 \partial_1 f + v^2 \partial_2 f$
- $\mathcal{L}_{\vec{v}} \alpha = \sum_{i,k=1,2} (v^k \partial_k a_i dx^i + a_i \partial_k v^i dx^k)$
- $\mathcal{L}_{\vec{v}} \omega = (\partial_1 (w_{12} v^1) + \partial_2 (w_{12} v^2)) dx^1 \wedge dx^2$
- $\mathcal{L}_{\vec{v}} \omega = (w_{12} \partial_\mu v^\mu + v^\mu \partial_\mu w_{12}) dx^1 \wedge dx^2$ (**Ricci**)

1.9 Levi-Civita-Connection (co-/contravariant derivatives)

- $\Gamma_{ij}^k = g^{kl} \Gamma_{ijl} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ (**Christoffel symbols**)
- $\nabla_j v^i = v^i_{;j} = v^i_{|j} = \partial_j v^i + v^k \Gamma_{jk}^i$
- $\nabla \vec{v} := [\nabla_j v^i]_j^i$
- $[\nabla \vec{v}^\flat]_{ij} = [g(\nabla \vec{v})]_{ij} = \nabla_j v_i = v_{i;j} = v_{i|j} = \partial_j v_i - v_k \Gamma_{ij}^k = g_{il} \nabla_j v^l$

- $[\nabla^\sharp \vec{v}^\flat]_i^j = [g(\nabla \vec{v})g^{-1}]_i^j = \nabla^j v_i = v_i^j = v_i^j = g^{jk} g_{il} \nabla_k v^l$
- $\nabla_i f = [\nabla f]_i = \partial_i f$
- $\nabla_{\vec{v}} f = \mathcal{L}_{\vec{v}} f = \langle \vec{v}, \nabla_\Gamma f \rangle = (\mathbf{d}f)(\vec{v}) = v^i \nabla_i f = v^i \partial_i f$

1.10 Shape-Operator S , etc

- **Second fundamental form:**
 $[II]_{ij} = [S^\flat]_{ij} = h_{ij} = -\partial_i \vec{N} \cdot \partial_j \vec{X} = -[\nabla \vec{N}]_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{X}$
- **Shape operator (Weingarten map):**
 $[S]_j^i = g^{ik} h_{kj} = -[\nabla_\Gamma \vec{N}]^i \cdot \partial_j \vec{X} = -[\nabla_\Gamma \vec{N}]_j^i$
- **Inverse of second fundamental form:**
 $b^{ij} = [II^{-1}]^{ij} = \frac{1}{|g|K} [II^{\text{Adj}}]^{ij}$
- $[S(\vec{v})]_i = -[\nabla_{\vec{v}} \vec{N}]_i = v^j h_{ij}$
- $S^T \alpha = \alpha S = S(\alpha^\sharp)$

1.11 Conclusions

$$\vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\text{Grad} f = \nabla_\Gamma f = \nabla^\sharp f = (\mathbf{d}f)^\sharp$
 $[\text{Grad} f]^i = \nabla^i f = g^{ij} \nabla_j f = g^{ij} \partial_j f$
- $\text{Div} \vec{v} = -\delta \vec{v}^\flat = * \mathbf{d} * \vec{v}^\flat = \nabla_i v^i = \partial_i v^i + v^k \Gamma_{ik}^i = \partial_i v^i + v^k \partial_k \log \sqrt{|g|}$

$$= \sum_{i=1,2} \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} v^i = \sum_{i=1,2} \frac{v^i}{\sqrt{|g|}} \partial_i \sqrt{|g|} + \partial_i v^i$$
- $\delta(f\alpha) = f\delta\alpha - \langle \mathbf{d}f, \alpha \rangle$
- $\text{Div}(f\vec{v}) = f\text{Div} \vec{v} + \langle v, \nabla f \rangle = f\text{Div} \vec{v} + \nabla_{\vec{v}} f = f\nabla_i v^i + v^i \nabla_i f$
- $\text{Rot}(f\vec{v}) = f\text{Rot} \vec{v} + \langle v, \text{Rot} f \rangle$
- **Laplace-Beltrami operator:**
 $\Delta_B f = -\delta \mathbf{d}f = * \mathbf{d} * \mathbf{d}f = \text{Div Grad} f = \nabla_i \nabla^i f = \frac{1}{\sqrt{|g|}} \partial_j \left(g^{ij} \sqrt{|g|} \partial_i f \right)$
- **Laplace-de Rham operator:**
 $\Delta_{dR} \alpha = (\delta \mathbf{d} + \mathbf{d} \delta) \alpha =: -(\Delta_B + \Delta_{CB}) \alpha$
 $\Delta_{dR} \vec{v} = (\Delta_{dR} \vec{v}^\flat)^\sharp$

- **Giaquinta-Hildebrandt operator:**

$$\Delta_{GH} f = \square f = \text{Div} (KII^{-1} \mathbf{d}f) = -\delta (KS^{-T} \mathbf{d}f) = \frac{1}{\sqrt{|g|}} \partial_j \left(\sqrt{|g|} K b^{ij} \partial_i f \right)$$

- $-\delta (S^T \alpha) = -H \text{Div} \alpha^\sharp - \text{Div} (KII^{-1} \alpha) - \nabla_{\alpha^\sharp} H = H \delta \alpha + \delta (KS^{-T} \alpha) - \langle \alpha, \mathbf{d}H \rangle$
 $-\delta (S^T \mathbf{d}f) = -H \Delta_B f - \Delta_{GH} f - \langle \mathbf{d}H, \mathbf{d}f \rangle$

1.12 Moving Surfaces $M(t)$

$\vec{V} := \vec{v} + v_n \vec{N} = \partial_t \vec{X}$ (surface velocity), $\vec{X} : M \rightarrow E^3$ (parametrization)

- $\partial_i \vec{V} \cdot \partial_j \vec{X} = g_{jk} \nabla_i v^k - v_n h_{ij} = \left[(\nabla \vec{v} - v_n S)^\flat \right]_{ij}$
- (rate-of-deformation tensor \mathbf{d})
 $\frac{d}{dt} g = \left(\nabla \vec{v}^\flat \right) + \left(\nabla \vec{v}^\flat \right)^T - 2v_n II = \mathcal{L}_{\vec{V}} g = 2\mathbf{d}$
 $\frac{d}{dt} g_{ij} = g_{ik} \nabla_j v^k + g_{jk} \nabla_i v^k - 2v_n h_{ij} = 2d_{ij}$
- $\left\langle \frac{d}{dt} \vec{p}^\flat, \alpha \right\rangle = \dot{\vec{p}} \alpha + 2\vec{p} \mathbf{d} \alpha^\sharp = \dot{\vec{p}} \alpha + \vec{p} \left((\nabla \vec{v})^T + \left(\nabla^\sharp \vec{v}^\flat \right) - 2 * v_n S^T \right) \alpha$
 $= \dot{p}^i \alpha_i + p^j (\nabla_j v^i + \nabla^i v_j - 2v_n h_j^i) \alpha_i$
- $\frac{d}{dt} \alpha^\sharp = \left[\dot{\alpha} + \left(2v_n S^T - (\nabla \vec{v})^T - \left(\nabla^\sharp \vec{v}^\flat \right) \right) \alpha \right]^\sharp$
- $\frac{d}{dt} * \omega = * [\dot{\omega} - (\text{Div} \vec{v} + v_n H) \omega]$
- $\frac{d}{dt} * \vec{p}^\flat = * \left[\dot{\vec{p}} + (\text{Div} \vec{v} + v_n H) \vec{p} \right]^\flat$
- $\frac{d}{dt} * \alpha = * \left[\dot{\alpha} + \left(2v_n S^T - (\nabla \vec{v})^T - \left(\nabla^\sharp \vec{v}^\flat \right) \right) \alpha + (\text{Div} \vec{v} + v_n H) \alpha \right]$
- $\frac{1}{2} \frac{d}{dt} \|\alpha\|^2 = \langle \dot{\alpha} + v_n S^T \alpha - (\nabla \vec{v})^T \alpha, \alpha \rangle = \dot{\alpha} \alpha^\sharp + \alpha (v_n S - \nabla \vec{v}) \alpha^\sharp$
 $= \dot{\alpha}_i \alpha^i + v_n \alpha_i h_j^i \alpha^j - \alpha_i (\nabla_j v^i) \alpha^j$
- $\frac{1}{2} \frac{d}{dt} \|\omega\|^2 = \frac{1}{2} \frac{d}{dt} (*\omega)^2 = \langle \dot{\omega}, \omega \rangle - (\text{Div} \vec{v} + v_n H) \|\omega\|^2$
- $\frac{1}{2} \frac{d}{dt} \|\delta \alpha\|^2 = \left\langle \delta \left[\dot{\alpha} + \left(2v_n S^T - (\nabla \vec{v})^T - \left(\nabla^\sharp \vec{v}^\flat \right) \right) \alpha + (\text{Div} \vec{v} + v_n H) \alpha \right], \delta \alpha \right\rangle - (\text{Div} \vec{v} + v_n H) \|\delta \alpha\|^2$
- $\frac{1}{2} \frac{d}{dt} \|\delta \vec{p}^\flat\|^2 = \left\langle \delta \left[\dot{\vec{p}} + (\text{Div} \vec{v} + v_n H) \vec{p} \right]^\flat, \delta \vec{p}^\flat \right\rangle - (\text{Div} \vec{v} + v_n H) \|\delta \vec{p}^\flat\|^2$
- $\frac{d}{dt} \int_{M(t)} f \mu = \int_{M(t)} \dot{f} + f (\text{Div} \vec{v} + v_n H) \mu$

- $\frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\alpha\|^2 \mu = \int_{M(t)} \langle \dot{\alpha}, \alpha \rangle + v_n \left\langle S^T \alpha + \frac{1}{2} H \alpha, \alpha \right\rangle + \left\langle \frac{1}{2} (\text{Div} \vec{v}) \alpha - (\nabla \vec{v})^T \alpha, \alpha \right\rangle \mu$
- $\frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\omega\|^2 \mu = \int_{M(t)} \langle \dot{\omega}, \omega \rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \|\omega\|^2 \mu$
- $\begin{aligned} \frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\mathbf{d}\vec{p}^b\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^b, \delta \mathbf{d}\vec{p}^b \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \|\mathbf{d}\vec{p}^b\|^2 \mu \\ &= \int_{M(t)} \dot{\vec{p}} \delta \mathbf{d}\vec{p}^b + \vec{p} \left((\nabla \vec{v})^T + (\nabla^\# \vec{v}^b) - 2v_n S^T \right) \delta \mathbf{d}\vec{p}^b \\ &\quad - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \|\mathbf{d}\vec{p}^b\|^2 \mu \end{aligned}$
- $\begin{aligned} \frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\delta \vec{p}^b\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^b, \mathbf{d}\delta \vec{p}^b \right\rangle + \left\langle \left(2v_n S^T - (\nabla \vec{v})^T - (\nabla^\# \vec{v}^b) \right) \vec{p}^b, \mathbf{d}\delta \vec{p}^b \right\rangle \\ &\quad + \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^b, \mathbf{d}\delta \vec{p}^b \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \|\delta \vec{p}^b\|^2 \mu \\ &= \int_{M(t)} \dot{\vec{p}} \delta \mathbf{d}\vec{p}^b + (\text{Div} \vec{v} + v_n H) \vec{p} \mathbf{d}\delta \vec{p}^b - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \|\delta \vec{p}^b\|^2 \mu \end{aligned}$
- $\begin{aligned} \frac{d}{dt} \int_{M(t)} \frac{1}{2} \left(\|\mathbf{d}\vec{p}^b\|^2 + \|\delta \vec{p}^b\|^2 \right) \mu &= \frac{d}{dt} \int_{M(t)} \frac{1}{2} \left(\|\text{Rot} \vec{p}\|^2 + \|\text{Div} \vec{p}\|^2 \right) \mu \\ &= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^b, \Delta_{dR} \vec{p}^b \right\rangle - \frac{\text{Div} \vec{v} + v_n H}{2} \left(\|\mathbf{d}\vec{p}^b\|^2 + \|\delta \vec{p}^b\|^2 \right) \\ &\quad + \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^b, \mathbf{d}\delta \vec{p}^b \right\rangle \\ &\quad - \left\langle \left((\nabla \vec{v})^T + (\nabla^\# \vec{v}^b) - 2v_n S^T \right) \vec{p}^b, \mathbf{d}\delta \vec{p}^b \right\rangle \mu \\ &= \int_{M(t)} \left\langle \dot{\vec{p}}, \Delta_{dR} \vec{p} \right\rangle - \frac{\text{Div} \vec{v} + v_n H}{2} \left(\|\text{Rot} \vec{p}\|^2 + \|\text{Div} \vec{p}\|^2 \right) \\ &\quad + (\text{Div} \vec{v} + v_n H) \vec{p} \mathbf{d}\delta \vec{p}^b \\ &\quad + \vec{p} \left((\nabla \vec{v})^T + (\nabla^\# \vec{v}^b) - 2v_n S^T \right) \delta \mathbf{d}\vec{p}^b \mu \end{aligned}$
- $\begin{aligned} \frac{d}{dt} \int_{M(t)} \frac{1}{4} \left(\|\vec{p}\|^2 - 1 \right)^2 \mu &= \int_{M(t)} \left(\|\vec{p}^b\|^2 - 1 \right) \left[\left\langle \frac{d}{dt} \vec{p}^b, \vec{p}^b \right\rangle + \left\langle \left(v_n S^T - (\nabla \vec{v})^T \right) \vec{p}^b, \vec{p}^b \right\rangle \right. \\ &\quad \left. + \frac{1}{4} \left(\|\vec{p}^b\|^2 - 1 \right) (\text{Div} \vec{v} + v_n H) \right] \mu \\ &= \int_{M(t)} \left(\|\vec{p}^b\|^2 - 1 \right) \left[\left\langle \dot{\vec{p}}, \vec{p} \right\rangle + \left\langle \left((\nabla^\# \vec{v}^b)^T - v_n S \right) \vec{p}, \vec{p} \right\rangle \right. \\ &\quad \left. + \frac{1}{4} \left(\|\vec{p}\|^2 - 1 \right) (\text{Div} \vec{v} + v_n H) \right] \mu \end{aligned}$

$$\begin{aligned}
& \bullet \int_{M(t)} \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu \\
& \quad = \int_{M(t)} \mathcal{L}_{\vec{p}} \left\langle \vec{p}^\flat, \mathbf{d} \delta \vec{v}^\flat + \mathbf{d} (v_n H) \right\rangle + \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu
\end{aligned}$$

2 Tensors

2.1 Flat / Sharp

- $t := t^i_j \partial_i \otimes dx^j$
- ${}^b t = gt = g_{ik} t^k_j dx^i \otimes dx^j = t_{ij} dx^i \otimes dx^j$
- $t^\sharp = tg^{-1} = t^i_k g^{kj} \partial_i \otimes \partial_j = t^{ij} \partial_i \otimes \partial_j$
- ${}^b t^\sharp = gtg^{-1} = g_{ik} t^k_l g^{lj} dx^i \otimes \partial_j = t_i^j dx^i \otimes \partial_j$

2.2 Product / Contraction

- $(s \cdot t)_i^j := s_{ik} t^{kj}$
- $s : t := s_{ij} t^{ij} = \text{Tr}(s \cdot t^T) = \text{Tr}(s^T \cdot t) = \dots$

2.3 Conclusions

$$\alpha = \vec{v}^\flat = {}^b \vec{v}, \vec{w} = \beta^\sharp = {}^\sharp \beta, s = \vec{v} \otimes \beta:$$

- t symmetric ($t_{12} = t_{21}$ resp. $t^{12} = t^{21}$): ${}^b t^\sharp = t^T$
- $\alpha t \vec{w} = \vec{v} {}^b t \vec{w} = \alpha t^\sharp \beta = \vec{v} {}^b t^\sharp \beta$ (Associativity referring to arguments)

$$\Rightarrow t(\alpha, \vec{w}) = {}^b t(\vec{v}, \vec{w}) = t^\sharp(\alpha, \beta) = {}^b t^\sharp(\vec{v}, \beta)$$

- $s = \vec{v} \otimes \beta$: ${}^b s = \alpha \otimes \beta$, $s^\sharp = \vec{v} \otimes \vec{w}$, ${}^b s^\sharp = \alpha \otimes \vec{w}$ (Associativity referring to factors, tensor product is metric compatible)
- $t^T := {}^\sharp ({}^b t)^T = ((t^\sharp)^T)^\flat = g^{-1}(gt)^T = (tg^{-1})^T g = \{t_j^i\}^i_j \partial_i \otimes dx^j$
- $(t^T)^T = t^T$
- $tt^T = (tt^T)^T = t^T t$
- $|t| = |t^T| = |{}^b t^\sharp| = |g| |t^\sharp| = \frac{|{}^b t|}{|g|}$
- $(\alpha \otimes \beta)^T = \beta \otimes \alpha$
- $\text{Tr}[\alpha \otimes \beta] = \langle \alpha, \beta \rangle$
- $(\alpha \otimes \beta) \gamma^\sharp = \langle \beta, \gamma \rangle \alpha$

- $(*\alpha) \otimes (*\beta) + \beta \otimes \alpha = \langle \alpha, \beta \rangle g$
- $(*\alpha) \otimes (*\alpha) + \alpha \otimes \alpha = \|\alpha\|^2 g$
- $\alpha \otimes (*\beta) + (*\beta) \otimes \alpha = \langle \alpha, \beta \rangle E$
- $\alpha \otimes (*\alpha) - (*\alpha) \otimes \alpha = \|\alpha\|^2 E$

2.4 Levi-Civita-Tensor

- Levi-Civita-Symbols: $\epsilon_{ij} = \epsilon^{ij} = \begin{cases} 1 & \text{if } (i, j) = (1, 2) \\ -1 & \text{if } (i, j) = (2, 1) \\ 0 & \text{else} \end{cases}$
- Levi-Civita-Tensor: $E_{ij} = \sqrt{|g|} \epsilon_{ij}$
- $E_{ij} = -E_{ji}$
- $\nabla E = \nabla g = \mathcal{O}$
- $E \otimes E = |g| \epsilon \otimes \epsilon$
- $E_{ij} E_{kl} = g_{ik} g_{jl} - g_{il} g_{jk}$
- $E_{ij} E_k{}^j = g_{ik}$
- $(*\alpha)_i = -E_{ij} \alpha^j = \alpha^j E_{ji}$
- $|t| E_{ij} = E^{kl} t_{ik} t_{jl}$
- $|t| = \frac{1}{2} E_{ij} E_{kl} t^{ik} t^{jl} = \frac{1}{2} ((\text{Tr} t)^2 - \text{Tr} t^2)$
- $0 = t^2 - (\text{Tr} t)t + |t|g$

2.5 Covariant Derivative $\nabla_\bullet = g_{\bullet i} \nabla^\bullet$

- $\nabla_k f = \partial_k f$
- $\nabla_k \sqrt{|g|} = \sqrt{|g|} \Gamma_{kl}{}^l$
- $\nabla_k v^i = \partial_k v^i + \Gamma_{kl}{}^i v^l$
- $\nabla_k v_i = \partial_k v_i - \Gamma_{ki}{}^l v_l$
- $\nabla_k t^i{}_j = \partial_k t^i{}_j + \Gamma_{kl}{}^i t^l{}_j - \Gamma_{kj}{}^l t^i{}_l$
- $\nabla_k t_i{}^j = \partial_k t_i{}^j - \Gamma_{ki}{}^l t_l{}^j + \Gamma_{kl}{}^j t_i{}^l$
- $\nabla_k t^{ij} = \partial_k t^{ij} + \Gamma_{kl}{}^i t^{lj} + \Gamma_{kl}{}^j t^{il}$
- $\nabla_k t_{ij} = \partial_k t_{ij} - \Gamma_{ki}{}^l t_{lj} - \Gamma_{kj}{}^l t_{il}$

- $\nabla * \alpha = -E \nabla \alpha^\sharp \rightsquigarrow (*\alpha)_{i|k} = -E_{ij} \alpha^j{}_{|k}$
- $*\nabla_\beta \alpha = \nabla_\beta * \alpha$
- $\nabla(\alpha \otimes \beta) = \alpha \otimes \nabla \beta + (\beta \otimes \nabla \alpha)^T$, $(t_{ijk}^T = t_{jik})$, $\rightsquigarrow (\alpha_i \beta_j)_{|k} = \alpha_i \beta_{j|k} + \beta_j \alpha_{i|k}$
- $\text{Rot}(f) = -\frac{1}{\sqrt{|g|}} g_{ik} \epsilon^{kl} \partial_l f dx^i = -\frac{1}{\sqrt{|g|}} g \epsilon \mathbf{d}f = -\sqrt{|g|} \epsilon \nabla f = -E \cdot \nabla f$
- $\text{Div}(v) = \nabla_i v^i = \nabla^i v_i = \text{Tr}(\nabla \vec{v}) = \text{Tr}({}^b \nabla^\sharp \alpha) = \sqrt{\nabla \vec{v} : g \otimes g : \nabla \vec{v}}$
- $\text{Rot}(v) = (\sqrt{|g|})^{-1} \epsilon^{ki} \nabla_k v_i = (\sqrt{|g|})^{-1} \text{Tr}((\nabla \alpha) \epsilon) = \sqrt{\nabla \vec{v} : E \otimes E : \nabla \vec{v}}$
- $\text{Rot}(v) = -\text{Tr}(\nabla * \alpha) = \text{Tr}(E \nabla v) = E_{ki} \nabla^k v^i$
- $\text{Rot}(v)^2 = \nabla_j v^i (\nabla^j v_i - \nabla_j v^i) = |\nabla v| + \text{Tr}(\nabla v (\nabla v)^T) - \text{Div}(v)^2$
- $\text{Div}(t) = \nabla^j t^i{}_j \partial_i = \nabla_j t_i{}^j dx^i = \nabla_j t^{ij} \partial_i = \nabla^j t_{ij} dx^i$

2.6 Conclusion Stuff

- $\nabla \|\alpha\|^2 = \mathbf{d} \|\alpha\|^2 = 2\alpha \cdot \nabla \alpha \rightsquigarrow \partial_k(\alpha_i \alpha^i) = 2\alpha^i \alpha_{i|k}$
- $\nabla \langle \alpha, \beta \rangle = \mathbf{d} \langle \alpha, \beta \rangle = \alpha \cdot \nabla \beta + \beta \cdot \nabla \alpha \rightsquigarrow \partial_k(\alpha_i \beta^i) = \alpha^i \beta_{i|k} + \beta^i \alpha_{i|k}$
- $*(\beta \cdot \nabla \alpha) = (\text{Div} \alpha)(*\beta) - \nabla_{*\beta} \alpha = \beta \cdot \nabla * \alpha + (\text{Rot} \alpha) \beta + (\text{Div} \alpha)(*\beta)$
 $\rightsquigarrow -E_i{}^k \beta^j \alpha_{j|k} = (*\beta)_i \alpha_k{}^{|k} - (*\beta)^k \alpha_{i|k} = \beta^l (*\alpha)_{l|i} + E^{kj} \alpha_{j|k} \beta_i + \alpha_k{}^{|k} (*\beta)_i$
- $\beta \cdot \nabla \alpha = \nabla_{*\beta} * \alpha + (\text{Div} \alpha) \beta = \nabla_\beta \alpha - (\text{Rot} \alpha)(*\beta)$
 $\rightsquigarrow \beta^i \alpha_{i|k} = (*\beta)^k (*\alpha)_{i|k} + \beta_i \alpha_k{}^{|k} = \beta^i \alpha_{k|i} - E_{ik} E_{lj} \beta^i \alpha^j{}_{|l}$

2.7 Rotation $R : T_p M \rightarrow (T_p M)'$

- $Rv := v' := \cos(\varphi)v + \sin(\varphi)(*v) = \cos(\varphi)v + \sin(\varphi)(*v)^\sharp$
- $R = \cos(\varphi)I + \frac{\sin(\varphi)}{\sqrt{|g|}} \begin{bmatrix} -g^{12} & -g^{22} \\ g^{11} & g^{12} \end{bmatrix} = R^i{}_j \partial_i \otimes dx^j$ **indices unten? nutze levi-cevita tensor!**
- $R = \cos(\varphi)g - \sin(\varphi)E \in \mathcal{T}^0_2$
- $R^{-1} = R^T$
- $\langle v', w' \rangle = \langle v, w \rangle \Rightarrow R \in O(T_p M)$
- $\nabla v' = R \nabla v$

2.7.1 Push-forward $R_* = R^{*T}$

- $R_*v = Rv = v'$
- $R_*\alpha = \alpha R^T = \alpha'$
- $R_* \bullet t \bullet = R \bullet t \bullet R^T = \bullet t' \bullet$
- $R_*(\bullet \otimes \bullet) = (R_*\bullet) \otimes (R_*\bullet)$