1 Arbitrary s.p.d. metric

1.1 Assumptions

- $Ind(M) = 0$
- $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}$ (**s.p.d.**)

1.2 General proberties

$\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $\gamma \in \Omega^r(M)$

1.2.1 Wedge product $\wedge$

- $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ (**anti-/commutativ**)
- **associativ** ($\alpha \wedge \beta \wedge \gamma$)
- $(c_1\alpha + c_2\beta) \wedge \gamma = c_1\alpha \wedge \gamma + c_2\beta \wedge \gamma$ (**bilinear**)

1.2.2 Exterior derivative $\mathbf{d}$

- $\mathbf{d} \circ \mathbf{d} = 0$ (**complex proberty**)
- $\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathbf{d}\beta$ (**product rule**)

1.2.3 Hodge star $*$

- $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu$
- $*1 = \mu \quad (*\mu = 1)$
- $**\alpha = (-1)^p \alpha$
- $\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$

1.3 Wedge product $\wedge$

$f \in \Omega^0(M)$, $\tilde{f} \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\beta := b_1 dx^1 + b_2 dx^2 \in \Omega^1(M)$, $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $f\tilde{f} = f \wedge \tilde{f} = \tilde{f} \wedge f \in \Omega^0(M)$
- $f\alpha := f \wedge \alpha = \alpha \wedge f = f a_1 dx^1 + f a_2 dx^2 \in \Omega^1(M)$
- $\alpha \wedge \beta = -\beta \wedge \alpha = (a_1 b_2 - a_2 b_1)\, dx^1 \wedge dx^2 \in \Omega^2(M)$
- $f\omega := f \wedge \omega = \omega \wedge f = f w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

1.4 Exterior derivative d

$f \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$

- $\mathbf{d}f = \partial_1 f dx^1 + \partial_2 f dx^2$
- $(\mathbf{d}f)_\mu = \partial_\mu f$ (**Ricci**)
- $\mathbf{d}\alpha = (\partial_1 a_2 - \partial_2 a_1)\, dx^1 \wedge dx^2$
- $(\mathbf{d}\alpha)_{12} = (-1)^{\mu-1} \partial_\mu a_{\bar{\mu}}$ (**Ricci**)

1.5 Hodge star $*$

$f \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $*f = f\mu = \sqrt{|g|} f dx^1 \wedge dx^2$
- $*\alpha = \sqrt{|g|} \left(-\left(a_1 g^{12} + a_2 g^{22}\right) dx^1 + \left(a_1 g^{11} + a_2 g^{12}\right) dx^2 \right)$
- $(*a)_\mu = (-1)^\mu \sqrt{|g|} g^{\nu\bar{\mu}} a_\nu = (-1)^\mu \sqrt{|g|} a^{\bar{\mu}}$ (**Ricci**)
- $*\omega = \frac{w_{12}}{\sqrt{|g|}}$

1.6 Rising and lowering indices $\sharp$ / $\flat$

$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\vec{v} := v^1 \partial_1 + v^2 \partial_2$

- $\alpha^\sharp = \left(g^{11} a_1 + g^{12} a_2\right) \partial_1 + \left(g^{12} a_1 + g^{22} a_2\right) \partial_2$
- $a^\mu = g^{\mu\nu} a_\nu$ (**Ricci**)
- $\vec{v}^\flat = \left(g_{11} v^1 + g_{12} v^2\right) dx^1 + \left(g_{12} v^1 + g_{22} v^2\right) dx^2$
- $v_\mu = g_{\mu\nu} v^\nu$ (**Ricci**)

1.7 Conclusions

$\vec{v} := v^1 \partial_1 + v^2 \partial_2$

- $$\begin{aligned} \mathrm{Div}\vec{v} = -\delta \vec{v}^\flat &= *\mathbf{d} * \vec{v}^\flat \\ &= \sum_{i=1,2} \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} v^i \\ &= \sum_{i=1,2} \frac{v^i}{\sqrt{|g|}} \partial_i \sqrt{|g|} + \partial_i v^i \end{aligned}$$