1 NSE on Riemannian manifold

Let M be a 2-dimensional compact, oriented and connected Riemannian Manifold embedded in $\mathbb{R}^3$ without a boundary. The Navier-Stokes equations on M read

$$\begin{aligned} \partial_t u + u \cdot \nabla u - \nu \left(\Delta^R u + 2Ku\right) + \nabla p &= F \\ \mathrm{div} u &= 0 \\ u\Big|_{t=0} &= u_0 \end{aligned} \tag{1}$$

where $u \in T^*M$ is a 1-form on M and K is the Gaussian curvature. When it is clear u also denote the asscociated vectorfield on M, because $T_p^*M \cong T_pM$ with the flat operator $\flat : T_pM \to T_p^*M$ resp. the sharp operator $\sharp : T_p^*M \to T_pM$ (u^i describe the components of a vector and u_i the components of the 1-form). With a Riemannian metric g and the utilization of the Levi-Civita connection is $u \cdot \nabla u = \nabla_u u$ the total covariant derivative of u along u (in vector presentation)

$$(\nabla_u u)^i = u^k \nabla_k u^i = u^k \left(\frac{\partial u^i}{\partial x_k} + \Gamma^i_{jk} u^j \right)$$

where $\nabla_k = \nabla_{\frac{\partial}{\partial x_k}}$. Γ^i_{jk} are the Christoffel symbols and we use the Einstein summation convention. The presure gradient (vector) is defined by $\nabla p = (\mathbf{d}p)^\sharp$. For a k-form $\mathbf{d} : \Omega^k \to \Omega^{k+1}$ is the exterior derivative. So we get for the components of the vector ∇p with metric g

$$(\nabla p)^i = g^{ij} \frac{\partial p}{\partial x_j}$$

$\Delta^R = -\mathbf{d}\delta - \delta\mathbf{d}$ is the Laplace-de Rham operator (see [1]). $\delta : \Omega^{k+1} \to \Omega^k$ is the formal L^2 adjoint of $\mathbf{d}$ and with the Hodge star operator it holds $\delta = - \star \mathbf{d} \star$.
In the following we assumed with no loss of generality that the given Riemann metric is diagonal

$$g_{ij} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \tag{2}$$

i.e. the local coordinate system is orthogonal.

The problem of the equations (1) is to handle the 1-forms, when we would like use global $\mathbb{R}^3$ standard coordinates to use FEM. It is easier to work with 0-form, resp. the asscociated scalar valued function. In the $\mathbb{R}^3$ (also in $\mathbb{R}^2$) flat case, it is possible to make use of the well known Helmholtz decomposition, where under certain conditions each vector field v can decomposed in a divergence free and a rotation free part.

$$v = \nabla\varphi - \mathrm{rot} A$$

If v is a vectorfield in $\mathbb{R}^2$, then A is to comprehend as vectorpotential normal to the $\mathbb{R}^2$-plane, i.e. A is describable with the normalcomponent and hence A is isomorphic to a scalarpotential in the normal space. It would be nice if we could also use this on a 2-manifold, but we need a generalization of the Helmholtz decomposition, that is the Hodge Decomposition Theorem (see [1]):

Theorem 1. Let $\omega \in \Omega^k(M)$, then there is an $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^{k+1}(M)$ and a harmonic (with respect to Δ^R) $\gamma \in \Omega^k(M)$ such that

$$\omega = \mathbf{d}\alpha + \delta\beta + \gamma$$

Furthermore $\mathbf{d}\alpha$, $\delta\beta$ and γ are mutually L^2-orthogonal and so are uniquely determined.

$\mathbf{d}$ is related to the gradient and δ to the rotation, because when we set $\star\beta = v^\flat$ and use the identity $\mathbf{d}v^\flat = \star(\mathrm{rot}v)^\flat$ we get

$$\delta\beta = -\star\mathbf{d}\star\beta = -\star\mathbf{d}v^\flat = -\mathrm{rot}v$$

Thus, it is also possible to decompose a vectorfield (or 1-form) on a 2-manifold.
In the following we develop a rotation operator, that we can use in $\mathbb{R}^3$-coordinates
Let $v \in \mathbb{R}^3$, than the rotation of v in fixed Cartesian coordinate system is

$$\mathrm{rot}v := \nabla \times v = \epsilon_{ijk} e_i \frac{\partial v^k}{\partial x_j}$$

ϵ_{ijk} denote the Levi-Civita symbols.
If $\psi \in TM^\perp$, $u \in TM$, $m \in M$ and $x \in \mathbb{R}^3$ it is useful to define

$$\begin{aligned} \mathrm{rot}\psi(m) &:= \mathrm{rot}\tilde{\psi}(x)\Big|_{x=m} \\ \mathrm{rot}_n u(m) &:= \mathrm{rot}_n \tilde{u}(x)\Big|_{x=m} := [n(x)\otimes n(x)]\mathrm{rot}\tilde{u}(x)\Big|_{x=m} \end{aligned} \tag{3}$$

where $\tilde{\psi}$, $\tilde{u}$ are some C^1-prolongation in a neighbourhood of M.

Remark 1. Definition (3) is correct and does not depend on the prolongation.

Proof. The general definition for the rotation is (see [1])

$$\mathrm{rot}v = \left[\star\mathbf{d}v^\flat\right]^\sharp \tag{4}$$

Now, we construct an extended coordinate system $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Let $B_\varepsilon(m_0) \subset \mathbb{R}^3$ be a ball around $m_0 \in M$. For $y \in B_\varepsilon(m_0)$ the orthogonal projection map $y \mapsto \mathrm{proj}_M^\perp y =: m(y)$ is well defined and smooth, if $\varepsilon > 0$ is small enough. We set $\tilde{x}^1 := x_1(m(y)), \tilde{x}_2 := x_2(m(y))$ and $\tilde{x}_3 := \|m(y) - y\|$. Under the condition (2) this coordinate system is orthogonal at m_0. In $B_\varepsilon(m_0) \cap M$ is $h_1(x)\Big|_{x=m_0} = \sqrt{g_{11}}$ and $h_2(x)\Big|_{x=m_0} = \sqrt{g_{22}}$ and we set $h_3(x)\Big|_{x=m_0} :\equiv 1$. Where h_i are the scalefactors for the curvlinear coordinates. Let $\tilde{v} := \tilde{v}^1 e_1 + \tilde{v}^2 e_2 + \tilde{v}^3 e_3$ a vectorfield, the e_i are the unitvectors of the curvlinear coordinate system. With the dual basis $\{d\tilde{x}_i\}$ of $\{e_i\}$, i.e. $d\tilde{x}_i(e_j) = \delta_{ij}$ (Kronecker delta), we get

$$\begin{aligned}
\tilde{v}^\flat &= h_1\tilde{v}^1 d\tilde{x}_1 + h_2\tilde{v}^2 d\tilde{x}_2 + \tilde{v}^3 d\tilde{x}_3 \\
\mathbf{d}\tilde{v}^\flat &= \left(\frac{\partial h_2\tilde{v}^2}{\partial\tilde{x}_1} - \frac{\partial h_1\tilde{v}^1}{\partial\tilde{x}_2}\right) d\tilde{x}_1\wedge d\tilde{x}_2 \\
&+ \left(\frac{\partial\tilde{v}^3}{\partial\tilde{x}_2} - \frac{\partial h_2\tilde{v}^2}{\partial\tilde{x}_3}\right) d\tilde{x}_2\wedge d\tilde{x}_3 \\
&+ \left(\frac{\partial\tilde{v}^3}{\partial\tilde{x}_1} - \frac{\partial h_1\tilde{v}^1}{\partial\tilde{x}_3}\right) d\tilde{x}_1\wedge d\tilde{x}_3
\end{aligned}$$

The Hodge star of the 2-forms basis vectors are

$$\begin{aligned}
\star\left(d\tilde{x}_1\wedge d\tilde{x}_2\right) &= \frac{1}{h_1h_2} d\tilde{x}_3 \\
\star\left(d\tilde{x}_2\wedge d\tilde{x}_3\right) &= \frac{h_1}{h_2} d\tilde{x}_1 \\
\star\left(d\tilde{x}_1\wedge d\tilde{x}_3\right) &= -\frac{h_2}{h_1} d\tilde{x}_2
\end{aligned} \tag{5}$$

With (5) we get

$$\begin{aligned}
\star\mathbf{d}\tilde{v}^\flat &= \frac{h_1}{h_2}\left(\frac{\partial\tilde{v}^3}{\partial\tilde{x}_2} - \frac{\partial h_2\tilde{v}^2}{\partial\tilde{x}_3}\right) d\tilde{x}_1 \\
&- \frac{h_2}{h_1}\left(\frac{\partial\tilde{v}^3}{\partial\tilde{x}_1} - \frac{\partial h_1\tilde{v}^1}{\partial\tilde{x}_3}\right) d\tilde{x}_2 \\
&+ \frac{1}{h_1h_2}\left(\frac{\partial h_2\tilde{v}^2}{\partial\tilde{x}_1} - \frac{\partial h_1\tilde{v}^1}{\partial\tilde{x}_2}\right) d\tilde{x}_3
\end{aligned}$$

I.e. the rotation of $\tilde{v}$ on M is

$$\begin{aligned}
\mathrm{rot}\tilde{v}\Big|_{x=m_0} &= \left[\star\mathbf{d}\tilde{v}^\flat\right]^\sharp\Big|_{x=m_0} \\
&= \left[\frac{e_1}{h_2}\left(\frac{\partial\tilde{v}^3}{\partial\tilde{x}_2} - \frac{\partial h_2\tilde{v}^2}{\partial\tilde{x}_3}\right) + \frac{e_2}{h_1}\left(\frac{\partial h_1\tilde{v}^1}{\partial\tilde{x}_3} - \frac{\partial\tilde{v}^3}{\partial\tilde{x}_1}\right) + \frac{e_3}{h_1h_2}\left(\frac{\partial h_2\tilde{v}^2}{\partial\tilde{x}_1} - \frac{\partial h_1\tilde{v}^1}{\partial\tilde{x}_2}\right)\right]\Bigg|_{x=m_0}
\end{aligned} \tag{6}$$

If $\psi\in TM^\perp$, i.e. $\tilde{\psi}^1\Big|_M = \tilde{\psi}^2\Big|_M \equiv 0$ and $\tilde{\psi}^3\Big|_M = \psi$, we obtain from (6)

$$\mathrm{rot}\psi = \sqrt{g^{22}}e_1\frac{\partial\psi}{\partial x_2} - \sqrt{g^{11}}e_2\frac{\partial\psi}{\partial x_1} \tag{7}$$

In the same way, we have for $u\in TM$, i.e. $\tilde{u}^1\Big|_M = u^1, \tilde{u}^2\Big|_M = u^2$ and $\tilde{u}^3\Big|_M \equiv 0$

$$\mathrm{rot}\tilde{u}\Big|_{x=m_0} = \left[-\frac{e_1}{h_2}\frac{\partial h_2\tilde{u}^2}{\partial\tilde{x}_3} + \frac{e_2}{h_1}\frac{\partial h_1\tilde{u}^1}{\partial\tilde{x}_3} + \frac{e_3}{h_1h_2}\left(\frac{\partial h_2\tilde{u}^2}{\partial\tilde{x}_1} - \frac{\partial h_1\tilde{u}^1}{\partial\tilde{x}_2}\right)\right]\Bigg|_{x=m_0}$$

Apply the projection $[n \otimes n]$ to normal space, we obtain

$$\mathrm{rot}_n u = \sqrt{\det(g^{ij})} \left(\frac{\partial u^2 \sqrt{g_{22}}}{\partial x_1} - \frac{\partial u^1 \sqrt{g_{11}}}{\partial x_2} \right) n \tag{8}$$

So we can compute the rotation on M in local coordinates and it does not depend on the prolangations. □

We also see that in M with the local coordinate system and the Hodge star of the dual basis vectors

$$\begin{aligned} \star dx_1 &= \frac{h_2}{h_1} dx_2 \\ \star dx_2 &= -\frac{h_1}{h_2} dx_1 \end{aligned} \tag{9}$$

the rotation of scalar function can compute as

$$\mathrm{rot}\psi = -\left[\star \mathbf{d}\psi\right]^\sharp \tag{10}$$

and with the Hodge star of the basis vector of 2-forms

$$\star \left(dx_1 \wedge dx_2 \right) = \frac{1}{h_1 h_2} \tag{11}$$

the rotation of vectorfields can compute as

$$\mathrm{rot}_n u = \left[\star \mathbf{d} u^\flat\right]^\sharp \tag{12}$$

Remark 2. For $\psi \in TM^\perp$ and $u \in TM$ in M holds the equations

$$\mathrm{rot}\psi = -n \times \nabla \psi \tag{13}$$

and

$$\mathrm{rot}_n u = -\mathrm{div}(n \times u) n \tag{14}$$

where the righthand sides are the calculations on M in $\mathbb{R}^3$. When it is clear, then ψ means the vectorfunction or the scalarfunction (i.e the normalcomponent of the vectorfunction).

Proof. We originate in the curvilinear coordinate system in the proof above. With (5) the cross product of two vectorfields v and w can compute as $v \times w = \left[\star \left(v^\flat \wedge w^\flat \right)\right]^\sharp$. Than

$$\begin{aligned} \left[-n \times \nabla \tilde{\psi}\right]^\flat &\overset{n^\flat = d\tilde{x}_3}{=} \star \left(d\tilde{\psi} \wedge d\tilde{x}_3 \right) \\ &= \star \left(\frac{\partial \tilde{\psi}}{\partial \tilde{x}_1} d\tilde{x}_1 + \frac{\partial \tilde{\psi}}{\partial \tilde{x}_2} d\tilde{x}_2 \right) \wedge d\tilde{x}_3 \\ &\overset{(5)}{=} -\frac{h_2}{h_1} \frac{\partial \tilde{\psi}}{\partial \tilde{x}_1} d\tilde{x}_2 + \frac{h_1}{h_2} \frac{\partial \tilde{\psi}}{\partial \tilde{x}_2} d\tilde{x}_1 \\ &\overset{|_M, (9)}{\rightarrow} -\star \left(\frac{\partial \psi}{\partial x_1} dx_1 + \frac{\partial \psi}{\partial x_2} dx_2 \right) \\ &= -\star \mathbf{d}\psi \overset{(10)}{=} \left[\mathrm{rot}\psi\right]^\flat \end{aligned}$$

Let v be a vectorfield in $\mathbb{R}^3$ on M, i.e. $v = v^i e_i$ with curvlinear basis vectors. With the Hodge star of the basis vector of 3-forms

$$\star\left(d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3\right) = \frac{1}{h_1 h_2} \tag{15}$$

we can calculate by means of the prolongation $\tilde{u}$ of $u \in TM$

$$\begin{aligned}
\left[-\mathrm{div}\left(n \times \tilde{u}\right)\right]^\flat &= \delta\left(n \times \tilde{u}\right)^\flat = \star\mathbf{d}\star\left(\tilde{u} \times n\right)^\flat \\
&\overset{\star\star\equiv\mathrm{id}}{=} \star\mathbf{d}\left(\tilde{u}^\flat \wedge d\tilde{x}_3\right) \\
&= \star\left(\frac{\partial h_2 \tilde{u}^2}{\partial \tilde{x}_1} - \frac{\partial h_1 \tilde{u}^1}{\partial \tilde{x}_2}\right) d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3 \\
&\overset{(15)}{=} \frac{1}{h_1 h_2}\left(\frac{\partial h_2 \tilde{u}^2}{\partial \tilde{x}_1} - \frac{\partial h_1 \tilde{u}^1}{\partial \tilde{x}_2}\right) \\
&\overset{|_M,(11)}{\to} \star\left(\frac{\partial h_2 u^2}{\partial x_1} - \frac{\partial h_1 u^1}{\partial x_2}\right) dx_1 \wedge dx_2 \\
&= \star\mathbf{d}u^\flat \overset{(12)}{=} \left[\mathrm{rot}_n u\right]^\flat
\end{aligned}$$

□

Remark 3. For $\psi \in TM^\perp$ it holds

$$\mathrm{rot}_n \mathrm{rot}\psi = -\Delta\psi n \tag{16}$$

and the righthand side can compute both in $\mathbb{R}^3$ or in local coordinate in M, i.e. $\Delta \equiv \Delta_{\mathrm{Beltrami}} \equiv \Delta_{\mathbb{R}^3}$ for scalarfunctions.

Proof.

(a) In $\mathbb{R}^3$:
With $n \times (n \times v) = -v$ we have

$$\mathrm{rot}_n \mathrm{rot}\psi \overset{(13),(14)}{=} \mathrm{div}(n \times (n \times \nabla\psi))n = -\mathrm{div}(\nabla\psi)n = -\Delta\psi n$$

(b) In M:

$$\mathrm{rot}_n \mathrm{rot}\psi \overset{(10),(12)}{=} \left[-\star\mathbf{d}\star\mathbf{d}\psi\right]^\sharp = \left[\delta\mathbf{d}\psi\right]^\sharp \overset{\text{see [1]}}{=} -\Delta_{\mathrm{Beltrami}}\psi$$

□

Remark 4. $\mathrm{rot}\nabla$ maps to the 0-function in $\mathbb{R}^3$.

Proof. Let v be a scalar function in $\mathbb{R}^3$. With (4) we can calculate

$$\mathrm{rot}\nabla v = \left[\star\mathbf{d}\left(\nabla v\right)^\flat\right]^\sharp = \left[\star\mathbf{d}\mathbf{d}v^\flat\right]^\sharp \overset{\mathbf{dd}\equiv 0}{\equiv} 0$$

□

Remark 5. If there exists for $u \in TM$ a $\psi \in TM^\perp$, so that $u = -\mathrm{rot}\psi$, than is $\mathrm{div}u = 0$.

Proof. We can calculate

$$\mathrm{div}u = -\mathrm{div}\mathrm{rot}\psi \overset{(13)}{=} \mathrm{div}(n \times \nabla\psi) \overset{(14)}{=} -(\mathrm{rot}_n \nabla\psi) \cdot n \overset{\text{Remark 4}}{=} 0$$

□

Remark 6. For all $f \in TM^\perp$ and $v \in TM$ it holds the product rule

$$\mathrm{rot}_n f v = f \mathrm{rot}_n v - v \cdot \mathrm{rot} f \tag{17}$$

Proof. With the prolongation $\tilde{v}$ of v and $\tilde{f}$ of f, we can calculate

$$\begin{aligned}
\mathrm{rot}_n f v &\overset{(14)}{=} -\mathrm{div}\left(f(n \times v)\right) \\
&= -f\mathrm{div}(n \times v) - (n \times v) \cdot \nabla f \\
&= -f\mathrm{div}(n \times v) + v \cdot (n \times \nabla f) \\
&\overset{(13),(14)}{=} f\mathrm{rot}_n v - v \cdot \mathrm{rot} f
\end{aligned}$$

□

Remark 7. For $f \in TM^\perp$ and $g \in TM^\perp$ we can calculate

$$\begin{aligned}
\mathrm{rot} f \cdot \mathrm{rot} g &= (n \times \nabla f) \cdot (n \times \nabla g) \\
&= (n \cdot n)(\nabla f \cdot \nabla g) - (\nabla f \cdot n)(\nabla g \cdot n) \\
&= \nabla f \cdot \nabla g
\end{aligned} \tag{18}$$

Remark 8. For all $u \in TM$ holds

$$\left[\Delta^R u^\flat\right]^\sharp = \left[-(\mathbf{d}\delta + \delta\mathbf{d})u^\flat\right]^\sharp = \nabla\mathrm{div}u - \mathrm{rot}\mathrm{rot}_n u \tag{19}$$

Proof. It is clear that $\left[-\mathbf{d}\delta u^\flat\right]^\sharp = \nabla\mathrm{div}u$ (see [1]). Let $u = u^1 e_1 + u^2 e_2$, where $e_i = \frac{\partial}{\partial x_i}$ are the unit vectors in M and $h_i = \sqrt{g_{ii}}$ $(i = 1, 2)$ like in Remark 1. By lowering the indices we get the asscociated 1-form of u

$$u^\flat = h_1 u^1 dx_1 + h_2 u^2 dx_2$$

and we can calculate the exterior derivative of $u^\flat$

$$\mathbf{d}u^\flat = \left(\frac{\partial h_2 u^2}{\partial x_1} - \frac{\partial h_1 u^1}{\partial x_2}\right) dx_1 \wedge dx_2$$

(11) allows us to compute

$$\star\mathbf{d}u^\flat = \frac{1}{h_1 h_2}\left(\frac{\partial h_2 u^2}{\partial x_1} - \frac{\partial h_1 u^1}{\partial x_2}\right) \overset{(8)}{=} \mathrm{rot}_n u =: r_u$$

The exterior derivative of the 0-form r_u is

$$\mathbf{d}r_u = \frac{\partial r_u}{\partial x_1}dx_1 + \frac{\partial r_u}{\partial x_2}dx_2$$

With (9) and by rising the indicies we get finally

$$\begin{aligned}\left[-\delta \mathbf{d} u^\flat\right]^\sharp &= \left[\star \mathbf{d} \star \mathbf{d} u^\flat\right]^\sharp = \left[\star \mathbf{d} r_u\right]^\sharp \\ &= \frac{1}{h_1}\frac{\partial r_u}{\partial x_1}e_2 - \frac{1}{h_2}\frac{\partial r_u}{\partial x_2}e_1 \\ &\stackrel{(7)}{=} -\mathrm{rot} r_u = -\mathrm{rot}\mathrm{rot}_n u\end{aligned}$$

□

In the following we will write $\Delta^R u$ for $u \in TM$ and mean $\left[\Delta^R u^\flat\right]^\sharp$, when there is no confusion.

Remark 9. For $u \in TM$ it holds

$$\nabla_u u = \frac{1}{2}\nabla(u \cdot u) - u \times \mathrm{rot}_n u \tag{20}$$

Proof. ∇ is the Levi-Civita connection, so it is possible to calculate the derivative for a prolongation $\tilde{u}$ in $\mathbb{R}^3$ on M and project to tangential space

$$\nabla_u u = \pi\left(\tilde{u} \cdot \nabla \tilde{u}\right)\Big|_M$$

Under utilization of the formula $\nabla \frac{\tilde{u}\cdot\tilde{u}}{2} = \tilde{u} \cdot \nabla\tilde{u} + \tilde{u} \times \mathrm{rot}\tilde{u}$ and the properties of the Levi-Civita connection we get

$$\nabla_u u = \nabla\frac{u \cdot u}{2} - \pi\left(\tilde{u} \times \mathrm{rot}\tilde{u}\right)\Big|_M$$

With (6) we can compute

$$\begin{aligned}\pi\left(\tilde{u} \times \mathrm{rot}\tilde{u}\right) &= \left[\frac{\tilde{u}^2}{h_1 h_2}\left(\frac{\partial h_2 \tilde{u}^2}{\partial \tilde{x}_1} - \frac{\partial h_1 \tilde{u}^1}{\partial \tilde{x}_2}\right) - \tilde{u}^3(\ldots)\right]e_1 \\ &\quad - \left[\frac{\tilde{u}^1}{h_1 h_2}\left(\frac{\partial h_2 \tilde{u}^2}{\partial \tilde{x}_1} - \frac{\partial h_1 \tilde{u}^1}{\partial \tilde{x}_2}\right) - \tilde{u}^3(\ldots)\right]e_2 \\ &\quad + [\ldots]\, e_3 \\ &\stackrel{|_M}{\to} \frac{u^2}{\sqrt{g}}\left(\frac{\partial\sqrt{g_{22}}u^2}{\partial x_1} - \frac{\partial\sqrt{g_{11}}u^1}{\partial x_2}\right)e_1 + \frac{u^1}{\sqrt{g}}\left(\frac{\partial\sqrt{g_{22}}u^2}{\partial x_1} - \frac{\partial\sqrt{g_{11}}u^1}{\partial x_2}\right)e_2 \\ &\stackrel{(7)}{=} u^2 \mathrm{rot}_n u e_1 - u^1 \mathrm{rot}_n u e_2 \\ &\stackrel{u^3=0}{=} u \times \mathrm{rot}_n u\end{aligned}$$

□

Thus, with the last Remarks the Navier-Stokes equations (1) becomes

$$\begin{aligned} \partial_t u + \frac{1}{2}\nabla(u\cdot u) - u\times \mathrm{rot}_n u + \nu\left(\mathrm{rotrot}_n u - 2Ku\right) + \nabla p &= F \\ \mathrm{div} u &= 0 \\ u\Big|_{t=0} &= u_0 \end{aligned} \tag{21}$$

Henceforward up to the pressure gradient we are able to formulate the Navier-Stokes equations on a 2D surface with $\mathbb{R}^3$ coordinates.

2 Vorticity equation (VE)

First we set the form

$$\begin{aligned} J : TM^\perp \times TM^\perp &\to TM^\perp \\ (a,b) &\mapsto J(a,b) := (n\times\nabla a)\cdot\nabla b = -\mathrm{rot}a\cdot\nabla b \end{aligned}$$

J is the triple product (or box product) of $(n,\nabla a,\nabla b)\in\{n\}\times TM\times TM$, i.e.

$$J(a,b) = -J(b,a) \tag{22}$$

Remark 10. It holds

$$J(a,b) = \mathrm{rot}_n\left(an\times\mathrm{rot}b\right) \tag{23}$$

Proof.

$$\begin{aligned} -\mathrm{rot}_n\left(an\times(n\times\nabla b)\right) = \mathrm{rot}_n\left(a\nabla b\right) &\overset{(17)}{=} a\mathrm{rot}_n\nabla b - \nabla b\cdot\mathrm{rot}a \\ &\overset{\text{Remark 4}}{=} J(a,b) \end{aligned}$$

□

Now, we will substitude $u = -\mathrm{rot}\psi$ and apply rot_n term by term in (21)

$$\begin{aligned} \mathrm{rot}_n\partial_t u &= -\partial_t\mathrm{rot}_n\mathrm{rot}\psi \overset{(16)}{=} \partial_t\Delta\psi \\ \mathrm{rot}_n\nabla_u u &\overset{(20),\text{Remark 4}}{=} -\mathrm{rot}_n(u\times\mathrm{rot}_n u) \\ &= -\mathrm{rot}_n(-\mathrm{rot}\psi\times(-\mathrm{rot}_n\mathrm{rot}\psi)) \\ &\overset{(16)}{=} -\mathrm{rot}_n(\Delta\psi n\times\mathrm{rot}\psi) \overset{(23)}{=} -J(\Delta\psi,\psi) \\ &\overset{(22)}{=} J(\psi,\Delta\psi) \\ \mathrm{rot}_n\Delta u &\overset{(19),\text{Remark 4}}{=} \mathrm{rot}_n(-\mathrm{rotrot}_n u) \\ &\overset{(16)}{=} \Delta\mathrm{rot}_n u = -\Delta\mathrm{rot}_n\mathrm{rot}\psi \\ &\overset{(16)}{=} \Delta^2\psi \\ \mathrm{rot}_n Ku &\overset{(17)}{=} K\mathrm{rot}_n u - u\cdot\mathrm{rot}K \\ &\overset{(16),(18)}{=} K\Delta\psi + \nabla K\cdot\nabla\psi \end{aligned}$$

With $f := \mathrm{rot}_n F$ and Remark 5 and 4 we obtain the vorticity equation

$$\begin{aligned} \partial_t \Delta\psi + J(\psi, \Delta\psi) - \nu\left(\Delta^2\psi + 2K\Delta\psi + 2\nabla\psi \cdot \nabla K\right) &= f \\ \psi\Big|_{t=0} &= -\mathrm{rot}^{-1} u_0 =: \psi_0 \end{aligned} \tag{24}$$

2.1 Numerical Simulations

Henceforward J_a is the linear first order operator, which describe the Form J with fixed first argument, i.e.

$$J(a, \cdot) =: J_a : TM^\perp \to TM^\perp.$$

With a simple time discretization and rewriting (24) in a system of PDEs of order 2, (24) becomes

$$\begin{aligned} \frac{1}{\tau_i}\varphi_i + J_{\psi_{i-1}}\varphi_i - \nu\left(\Delta\varphi_i + 2K\varphi_i + 2\nabla K \cdot \nabla\psi_i\right) &= \frac{1}{\tau_i}\varphi_{i-1} + f_i \\ \Delta\varphi_i - \psi_i &= 0 \end{aligned} \tag{25}$$

in the i-th timestep. To solve (25) we use the FE-Toolbox AMDiS (see [6]). The surface is discretized in a polyhedron, particulary we use a surface triangulation to represent the geometry. The testfunctions are only linear, because it is not a good idea to use interpoints on the elements. Only the vertices of the element are on the surface, so we commit a additional error by evaluating some prolongation outside the manifold. The numerical solution in the i-th timestep is $\psi_{h,i}$ (resp. $\varphi_{h,i}$), this is a linear combination of the testfunctions.

$$\begin{aligned} \psi_i(x) &\approx \psi_{h,i}(x) = v^j_{\psi_i} v^j(x) \\ \varphi_i(x) &\approx \varphi_{h,i}(x) = v^j_{\varphi_i} v^j(x) \end{aligned}$$

where $v^j(x_k) = \delta_{jk}$ for all $j, k \in \{0, 1, \ldots, N_i - 1\} =: I_i$ (with N_i degrees of freedom (DOFs) in the i-th timestep). With the DOF vector $v_i := \left[\left\{v^j_{\psi_i}\right\}_{j\in I_i}, \left\{v^j_{\varphi_i}\right\}_{j\in I_i}\right]^T$ we get the linear system

$$\begin{bmatrix} \tilde{A}^{1,1}_{h,i} & \tilde{A}^{1,2}_{h,i} \\ \tilde{A}^{2,1}_{h,i} & \tilde{A}^{2,2}_{h,i} \end{bmatrix} v_i = \tilde{r}_{h,i} \tag{26}$$

But we see, that (25) as well as (26) can not uniquely solve. So, we require a additional condition on (26). Therewith the solution $\psi_{h,i}$ stay in a reasonable range we set the l-th DOF zero

$$\underbrace{\begin{bmatrix} \tilde{A}^{1,1}_{h,i} & \tilde{A}^{1,2}_{h,i} \\ \tilde{A}^{2,1}_{h,i} & \tilde{A}^{2,2}_{h,i} \\ e_l & \mathcal{O} \end{bmatrix}}_{=:A_{h,i}} v_i = \begin{bmatrix} \tilde{r}_{h,i} \\ 0 \end{bmatrix}$$

where $e_l \in \mathbb{R}^{N_i}$ is the l-th unit (row) vector. To solve this $(2N_i+1)\times(2N_i)$ linear system we use the method of least squares

$$B_{h,i}v_i = r_{h,i} \tag{27}$$

where

$$\begin{aligned} B_{h,i} &:= A_{h,i}^T A_{h,i} \\ r_{h,i} &:= A_{h,i}^T \begin{bmatrix} \tilde{r}_{h,i} \\ 0 \end{bmatrix} \end{aligned}$$

The drawback of (27) is that the systemmatrix $B_{h,i}$ have about 12 non-zero entries more than the systemmatrix of (26) (they have about 40 non-zero entries). It is also possible to require

$$\int_M \psi_{h,i} dA = 0 \tag{28}$$

instead $\psi_{h,i}(x_l) = 0$, but then (27) becomes to a dense linear system. When we still want the condition (28), then we can easely scale the solution, where the condition $\psi_{h,i}(x_l) = 0$ was required.

$$\hat{\psi}_{h,i} := \psi_{h,i} - \frac{1}{\mathcal{A}} \int_M \psi_{h,i} dA$$

where $\mathcal{A} = \int_M dA$. So $\hat{\psi}_{h,i}$ comply with condition (28).

3 Examples

3.1 Rotating sphere

Let $M = S_2$. In [4] it is shown, that we can add the tangent component of the Coriolis acceleration $ln\times u$ on the lefthand side of (21). Thus, the vorticity equation (24) becomes

$$\begin{aligned} \partial_t \Delta\psi + J(\psi, \Delta\psi + l) - \nu\left(\Delta\varphi_i + 2K\varphi_i\right) &= f \\ \psi\Big|_{t=0} &= -\mathrm{rot}^{-1} u_0 =: \psi_0 \end{aligned} \tag{29}$$

where $l = 2\omega \sin\phi$ is the Coriolis parameter, ω is the angular frequency and ϕ is the latitude. The Gaussian curvature ist constant ($K = R^{-2}$, with radius R).
If we use semi-implicite time-discretization and rewriting as a system of second order PDEs, (29) becomes

$$\begin{aligned} \frac{1}{\tau}\varphi_i + J_{\psi_{i-1}}\varphi_i - J_l\psi_i - \nu\left(\Delta\varphi_i + 2K\varphi_i\right) &= \frac{1}{\tau}\varphi_{i-1} + f_i \\ \varphi_i - \Delta\psi_i &= 0 \\ \psi_0 := \psi\Big|_{t=0}, \quad \varphi_0 &:= \Delta\psi_0 \end{aligned}$$

and we can solve this with the finite-elements-toolbox AMDiS (for example see Fig.1).

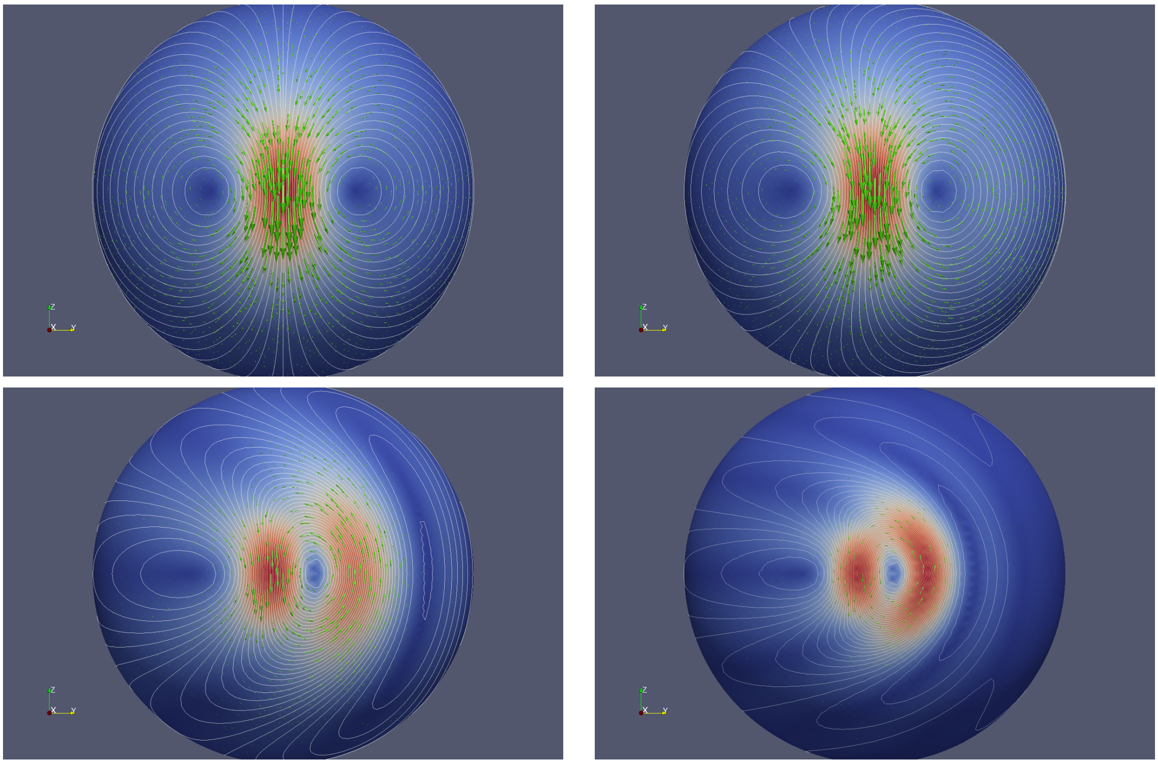

Figure 1: $\psi_0 \equiv 0$, $R = 1$, $\nu = 0.05$, $l = 2\omega z$, $\omega \in \{0, 1, 10, 50\}$, if $\sqrt{y^2 + z^2} < 0.2$ and $x > 0$ then $f(x, y, z) = 20y$ else $f \equiv 0$, $\tau = 0.1$, ω is increasing from left to right and top to bottom, figures show velocity, velocityfield and streamline after 10 timesteps.

3.1.1 Harmonic Waves

A stationary solution for no viscosity and no forces, i.e.

$$\partial_t \Delta\psi + J(\psi, \Delta\psi + l) = 0$$

is given by [4]

$$\psi(\phi, \lambda) = A\sin(m\lambda)P_n^m(\sin\phi) - \frac{2\omega}{n(n+1)-2}R^2\sin\phi$$

where $A \in \mathbb{R}$, $m, n \in \mathbb{N}$ arbitrary, with $m \leq n$, ϕ is the latitude, λ is the longitude and P_n^m is a associated Legendre polynomial. m can assign to the number of waves on the sphere.

For $R = 1, n = 7, m = 6$ we get

$$\psi(\phi, \lambda) = \frac{1}{27}\sin\phi\left(T\cos^6\phi\sin(6\lambda) - \omega\right) \tag{30}$$

where $T := 3648645A$. With the definition of the Laplace-Beltrami Operator and the Riemannian metric

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\phi \end{pmatrix}$$

we can calculate

$$\varphi(\phi, \lambda) = \Delta\psi(\phi, \lambda) = -\frac{2}{27}\sin\phi\left(28T\cos^6\phi\sin(6\lambda) - \omega\right)$$

With the inclusion map $\mathbf{x} : S_2 \to \mathbb{R}^3$

$$\mathbf{x}(\phi, \lambda) = \begin{pmatrix} x(\phi, \lambda) \\ y(\phi, \lambda) \\ z(\phi, \lambda) \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix} \tag{31}$$

it is possible to formulate ψ and ϕ in $\mathbb{R}^3$ standard coordinates

$$\begin{aligned} \psi(x, y, z) = \tilde{\psi}(x, y, z)\Big|_{S_2} &= \frac{1}{27}z\left(2Txy(x^2 - 3y^2)(3x^2 - y^2) - \omega\right)\Big|_{S_2} \\ \varphi(x, y, z) = \tilde{\varphi}(x, y, z)\Big|_{S_2} &= -\frac{2}{27}z\left(56Txy(x^2 - 3y^2)(3x^2 - y^2) - \omega\right)\Big|_{S_2} \end{aligned}$$

(Note: $\tilde{\psi}$ and $\tilde{\phi}$ are only one of infinity possible prolongations.) This solution is plotted in Fig. 2. In the test case we set $\omega = 10$ and $T = 10$. After 1 timeunit we look at the

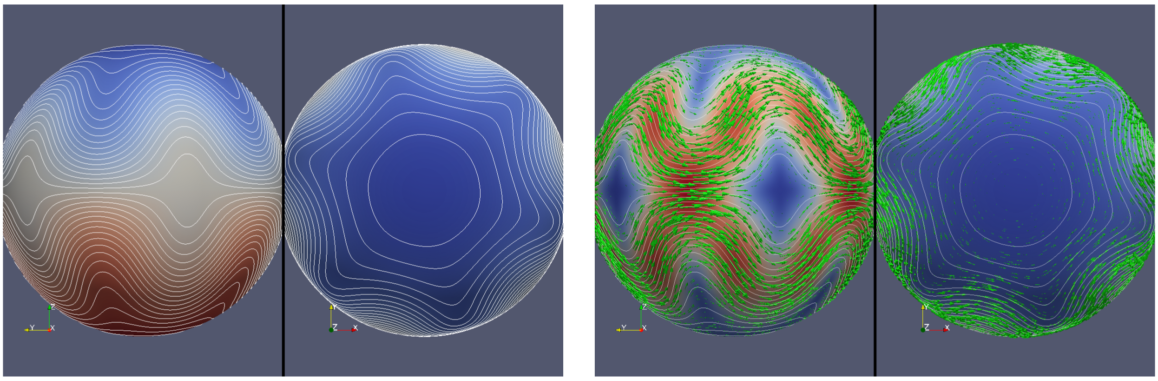

Figure 2: left: vorticity with streamlines; right: streamline, velocity and velocityfield of the exact solution ψ; $\omega = 10$ and $T = 10$.

maximum error $e_{N,\tau}$, where N is the number of DOFs and τ is the timestepwidth.

N	$e_{N,0.1}$	$e_{N,0.01}$	$e_{N,0.001}$
770	0.0274569	0.0245939	0.0248309
1538	0.0156816	0.0119981	0.0121428
3074	0.00792934	0.00662822	0.00668264
6146	0.00447738	0.00339522	0.00331579
12290	0.00215659	0.00180018	0.00181839
24578	0.0011472	0.000880489	0.000860363
49154	0.000550031	0.000461713	0.000465302
98306	0.000288885	0.000222212	0.000216912
196610	0.000139095	0.000117188	
N	$EOC_{N,0.1}$	$EOC_{N,0.01}$	$EOC_{N,0.001}$
1538	0.80961388525875	1.0374396099005	1.03397396678589
3074	0.984724076944796	0.856916645160185	0.862420309929852
6146	0.824933243548767	0.965569665228393	1.01154033301633
12290	1.05415031028891	0.915578828861833	0.866894492651742
24578	0.910741909782413	1.03188541883864	1.07977102443038
49154	1.06059434886204	0.931363297854444	0.8868301514022
98306	0.929044904431654	1.05509059332495	1.1010898381
196610	1.05444020949545	0.923125429411429	

3.1.2 24 gyre on the sphere

Inspired from the section above, we use the solution and its derivations

$$\begin{aligned} \psi(\phi,\lambda,t) &= \sin\phi\cos^6\phi\sin 6\lambda g(t) \\ \Delta\psi(\phi,\lambda,t) = \varphi(\phi,\lambda,t) &= -56\sin\phi\cos^6\phi\sin 6\lambda g(t) \\ \Delta\varphi(\phi,\lambda,t) &= 3136\sin\phi\cos^6\phi\sin 6\lambda g(t) \end{aligned}$$

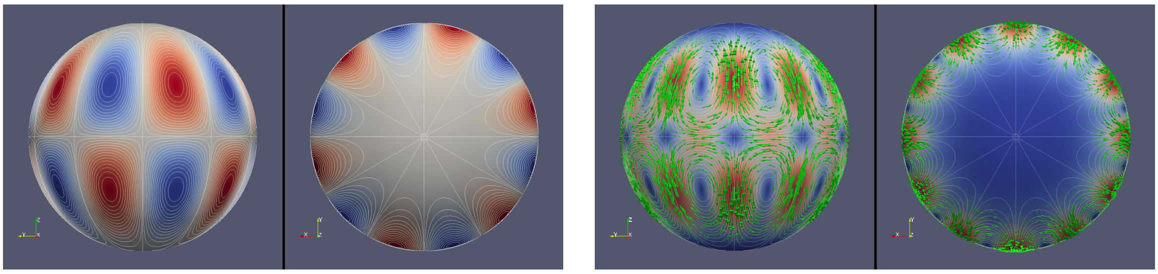

Figure 3: left: vorticity with streamlines; right: streamline, velocity and velocityfield of the exact solution ψ; $t = 0$, $\nu = 0.1$, $\omega = 0$ and $T = 27$; at $t = 1$ ψ becomes $-\psi$.

for the problem (29), where $g(t) := -2t + 1$. This is (30) with $\omega = 0$ and $T = 27$ (see Fig.3). It hold $J(\psi, \varphi) = 0$, because $\hat{\psi}$ is a stationary solution for $\partial_t \hat{\varphi} + J(\hat{\psi}, \hat{\varphi}) = 0$, where $\psi = \hat{\psi} \cdot g$. So we get for the right-hand side

$$f(\phi, \lambda, t) = -56 \sin\phi \cos^6\phi \sin 6\lambda \left(\dot{g}(t) + 54\nu g(t)\right)$$

and with the inclusion map (31)

$$\tilde{f}(x, y, z, t) = -112xyz\left(x^2 - 3y^2\right)\left(3x^2 - y^2\right)\left(\dot{g}(t) + 54\nu g(t)\right)$$

Like in the section above, we compare the exact solution with the numerical solution at $t = 1$. The timestepwidth is given by $\tau = 0.01$. $e := |\psi - \psi_h|$ is the error function after one time unit, i.e. after 100 timesteps.

N	$\|e\|_{L^\infty}$	$\|e\|_{L^2}$	$\|e\|_{H^1}$	EOC_{L^∞}	EOC_{L^2}	$EOC_{H^1(\text{seminorm})}$
386	0.0562224	0.0606727	0.55999			
770	0.027538	0.0304006	0.376774	1.033589362	1.0006927107	0.5738492263
1538	0.0184179	0.017913	0.255598	0.5814052556	0.7645261932	0.560874419
3074	0.00843349	0.00807341	0.188349	1.1279653034	1.1508367843	0.4408818703
6146	0.00472532	0.0046504	0.121191	0.8361102161	0.7961970791	0.6364245461
12290	0.00234895	0.00205221	0.0936494	1.008632811	1.180453565	0.3720282436
24578	0.00120345	0.00117363	0.0595035	0.9649530795	0.8062953663	0.6543720394
49154	0.000638018	0.000515249	0.0467371	0.9155609148	1.1877056676	0.3484267641
98306	0.000301583	0.000294029	0.0296189	1.0810737495	0.8093350997	0.6580694715
196610	0.000172838			0.803146452		

4 Appendix

4.1 Gaussian curvature

A elegant way to compute the Gaussian curvature K for surfaces in R^3 was shown in [5]. Let $x : M \to \mathbb{R}^3$ the inclusion map, then we define 1-forms ω_1 and ω_2 on M by

$$\mathbf{d}x = \omega_1 e_1 + \omega_2 e_2 \tag{32}$$

where e_1 and e_2 are the orthonormal unit vectors tangential to M. Now, we define a ω_{12} by (Structure Equation)

$$\begin{aligned} \mathbf{d}\omega_1 &= \omega_{12} \wedge \omega_2 \\ \mathbf{d}\omega_2 &= \omega_1 \wedge \omega_{12} \end{aligned} \tag{33}$$

($\omega_{ij}(U)$ tells us how fast e_i is twisting towards e_j at P as we move with velocity U, $\omega_{ij} = \omega_{ji}$.) When we use the approach

$$\omega_{12} := P\omega_1 + Q\omega_2 \tag{34}$$

ω_{12} is determine with (33) by

$$\begin{aligned} \mathbf{d}\omega_1 &= P\omega_1 \wedge \omega_2 \\ \mathbf{d}\omega_2 &= Q\omega_1 \wedge \omega_2 \end{aligned} \tag{35}$$

With the Gauss equation

$$\mathbf{d}\omega_{12} = -\omega_{13} \wedge \omega_{23}$$

we get

$$\mathbf{d}\omega_{12} = -K dA \tag{36}$$

where $dA = \omega_1 \wedge \omega_2$.

4.1.1 Torus

We define the inclusion map $x : M \to \mathbb{R}^3$ by

$$x(u, v) := ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)^T$$

which allow us to discribe the torus as a surface with parameter u and v, where $u, v \in [0, 2\pi)$. R is the distance from the center of the tube to the center of the torus and r is the radius of the tube. Therefore,we can compute and define the unit vectors

$$\begin{aligned} \frac{\partial x}{\partial u} &= r \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix} =: r e_1 \\ \frac{\partial x}{\partial v} &= (R + r\cos u) \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix} =: (R + r\cos u) e_2 \end{aligned}$$

and we see that e_1 and e_2 are orthonormal unit vectors tangential to M. With (32) we get

$$\mathbf{d}x = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv = e_1 r du + e_2 (R + r\cos u) dv$$

i.e the 1-forms ω_1 and ω_2 are

$$\omega_1 = rdu$$
$$\omega_2 = (R + r\cos u)dv$$

The exterior derivative of these 1-forms are

$$\mathbf{d}\omega_1 = r\mathbf{d}du = 0 = 0\omega_1 \wedge \omega_2$$
$$\mathbf{d}\omega_2 = -r\sin u du \wedge dv = -\frac{\sin u}{R + r\cos u}\omega_1 \wedge \omega_2$$

so, we get with the approach (34) and (35)

$$\omega_{12} = -\sin u dv$$

With the exterior derivative of ω_{12} ($\mathbf{d}\omega_{12} = -\cos u du \wedge dv$) and (36) we get the Gaussian curvature in local coordinates and global coordinates $(x_1, x_2, x_3)^T := x(u, v)$

$$K = \frac{cosu}{r(R + r\cos u)} = \frac{\sqrt{r^2 - x_3^2}}{r^2\left(R + \sqrt{r^2 - x_3^2}\right)} \qquad\qquad a, b > 0$$

References

[1] Abraham, Ralph, Jerrold E. Marsden, and Tudor Ratiu: *Manifolds, Tensor Analysis, and Applications*. Applied Mathematical Sciences 75. Springer, 2nd edition, 1988.

[2] Boyer, Franck: *A theoretical and numerical model for the study of incompressible mixture flows*. Computers & Fluids, 31:41–68, 2002.

[3] Kay, David, Vanessa Styles, and Richard Welford: *Finite element approximation of a cahn-hilliard-navier-stokes system*. Interfaces and free Boundaries, 10(1):15–43, 2008.

[4] Neamtan, S. M.: *The motion of harmonic waves in the atmosphere*. Journal of Meteorology, 3(2):43–56, 1946.

[5] Shifrin, Theodore: *Differential geometry: A first course in curves and surfaces*. Preliminary Version. Lecture notes for a course in differential geometry, University of Georgia., 2010.

[6] Vey, Simon: *Adaptive finite elements for systems of PDEs: Software concepts, multilevel techniques and parallelization.* Dresden: Univ. Dresden, Fakultät für Mathematik und Naturwissenschaften(Diss.). 215 p. , 2008.