

# 1 Arbitrary s.p.d. metric

## 1.1 Assumptions

- $Ind(M) = 0$
- $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = g_{11} (dx^1)^2 + 2g_{12} dx^1 dx^2 + g_{22} (dx^2)^2$  (s.p.d.)

## 1.2 General properties

$\alpha \in \Omega^p(M)$ ,  $\beta \in \Omega^q(M)$ ,  $\gamma \in \Omega^r(M)$ ,  $\vec{v} \in \mathcal{V}(M)$

### 1.2.1 Wedge product $\wedge$

- $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$  (**anti-/commutativ**)
- **associativ** ( $\alpha \wedge \beta \wedge \gamma$ )
- $(c_1 \alpha + c_2 \beta) \wedge \gamma = c_1 \alpha \wedge \gamma + c_2 \beta \wedge \gamma$  (**bilinear**)

### 1.2.2 Exterior derivative $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$

$\alpha \in \Omega^p(M)$

- $d \circ d = 0$  (**complex property**)
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$  (**product rule,  $\wedge$ -antiderivation**)

### 1.2.3 Hodge star $*$ : $\Omega^p(M) \rightarrow \Omega^{2-p}(M)$

- $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu$
- $*1 = \mu$  ( $*\mu = 1$ )
- $**\alpha = (-1)^p \alpha$
- $\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$

### 1.2.4 Contraction $i : (\mathcal{V} \times \Omega^p)(M) \rightarrow \Omega^{p-1}(M)$ (**inner product**)

- $i_{\vec{v}} \alpha (\vec{t}_1, \dots, \vec{t}_{p-1}) = \alpha(\vec{v}, \vec{t}_1, \dots, \vec{t}_{p-1})$
- $f i_{\vec{v}} \alpha = i_{f\vec{v}} \alpha = i_{\vec{v}} f \alpha$  (**bilinear**)
- $i_{\vec{v}}(\alpha \wedge \beta) = (i_{\vec{v}} \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_{\vec{v}} \beta)$  ( **$\wedge$ -antiderivation**)

### 1.2.5 Lie-derivative $\mathcal{L} : (\mathcal{V} \times \Omega^p)(M) \rightarrow \Omega^p(M)$

- $\mathcal{L}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\vec{v}}\alpha$  (**Cartans magic formular**)
- $\mathcal{L}_{f\vec{v}}\alpha = f\mathcal{L}_{\vec{v}}\alpha + \mathbf{d}f \wedge \mathbf{i}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}(\alpha \wedge \beta) = \mathcal{L}_{\vec{v}}\alpha \wedge \beta + \alpha \wedge \mathcal{L}_{\vec{v}}\beta$
- $\mathcal{L}_{\vec{v}}\mathbf{d}\alpha = \mathbf{d}\mathcal{L}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathcal{L}_{\vec{v}}\alpha$   
 $\Rightarrow \alpha \in \Omega^1(M) : \mathcal{L}_{\vec{v}}\langle \vec{v}^\flat, \alpha \rangle = \langle \vec{v}^\flat, \mathcal{L}_{\vec{v}}\alpha \rangle$
- $\mathcal{L}_{\vec{v}}\vec{w} = [\vec{v}, \vec{w}] = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}$  ((**Levi-Civita**)-**Con**ection  $\nabla$  is **Torsion-free**)

### 1.3 Wedge product $\wedge$

$f \in \Omega^0(M)$ ,  $\tilde{f} \in \Omega^0(M)$ ,  $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$ ,  $\beta := b_1 dx^1 + b_2 dx^2 \in \Omega^1(M)$ ,  
 $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $f\tilde{f} = f \wedge \tilde{f} = \tilde{f} \wedge f \in \Omega^0(M)$
- $f\alpha := f \wedge \alpha = \alpha \wedge f = fa_1 dx^1 + fa_2 dx^2 \in \Omega^1(M)$
- $\alpha \wedge \beta = -\beta \wedge \alpha = (a_1 b_2 - a_2 b_1) dx^1 \wedge dx^2 \in \Omega^2(M)$
- $f\omega := f \wedge \omega = \omega \wedge f = fw_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

### 1.4 Exterior derivative $\mathbf{d}$

$f \in \Omega^0(M)$ ,  $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$

- $\mathbf{d}f = \partial_1 f dx^1 + \partial_2 f dx^2$
- $(\mathbf{d}f)_\mu = \partial_\mu f$  (**Ricci**)
- $\mathbf{d}\alpha = (\partial_1 a_2 - \partial_2 a_1) dx^1 \wedge dx^2$
- $(\mathbf{d}\alpha)_{12} = (-1)^{\mu-1} \partial_\mu a_{\bar{\mu}}$  (**Ricci**)

### 1.5 Hodge star $*$

$f \in \Omega^0(M)$ ,  $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$ ,  $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $*f = f\mu = \sqrt{|g|} f dx^1 \wedge dx^2$
- $*\alpha = \sqrt{|g|} (- (a_1 g^{12} + a_2 g^{22}) dx^1 + (a_1 g^{11} + a_2 g^{12}) dx^2)$
- $(*a)_\mu = (-1)^\mu \sqrt{|g|} g^{\nu\bar{\mu}} a_\nu = (-1)^\mu \sqrt{|g|} a^{\bar{\mu}}$  (**Ricci**)
- $*\omega = \frac{w_{12}}{\sqrt{|g|}}$

## 1.6 Rising and lowering indices $\sharp / \flat$

$$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\alpha^\sharp = (g^{11}a_1 + g^{12}a_2) \partial_1 + (g^{12}a_1 + g^{22}a_2) \partial_2$
- $a^\mu = g^{\mu\nu} a_\nu$  (**Ricci**)
- $\vec{v}^\flat = (g_{11}v^1 + g_{12}v^2) dx^1 + (g_{12}v^1 + g_{22}v^2) dx^2$
- $v_\mu = g_{\mu\nu} v^\nu$  (**Ricci**)

## 1.7 Contraction $\mathbf{i}$

$$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\mathbf{i}_{\vec{v}}\alpha = \alpha(\vec{v}) = a_1 v^1 + a_2 v^2$
- $\mathbf{i}_{\vec{v}}\omega = w_{12} (-v^2 dx^1 + v^1 dx^2)$

## 1.8 Lie-derivative $\mathcal{L}$

$$f \in \Omega^0(M), \alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\mathcal{L}_{\vec{v}}f = v^1 \partial_1 f + v^2 \partial_2 f$
- $\mathcal{L}_{\vec{v}}\alpha = \sum_{i,k=1,2} (v^k \partial_k a_i dx^i + a_i \partial_k v^i dx^k)$
- $\mathcal{L}_{\vec{v}}\omega = (\partial_1 (w_{12} v^1) + \partial_2 (w_{12} v^2)) dx^1 \wedge dx^2$
- $\mathcal{L}_{\vec{v}}\omega = (w_{12} \partial_\mu v^\mu + v^\mu \partial_\mu w_{12}) dx^1 \wedge dx^2$  (**Ricci**)

## 1.9 Levi-Civita-Connection (co-/contravariant derivatives)

- $\Gamma_{ij}^k = g^{kl} \Gamma_{ijl} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$  (**Christoffel symbols**)
- $\nabla_j v^i = v^i_{;j} = v^i|_j = \partial_j v^i + v^k \Gamma_{jk}^i$
- $\nabla \vec{v} := [\nabla_j v^i]_j^i$
- $[\nabla \vec{v}^b]_{ij} = [g(\nabla \vec{v})]_{ij} = \nabla_j v_i = v_{i;j} = v_{i|j} = \partial_j v_i - v_k \Gamma_{ij}^k = g_{il} \nabla_j v^l$
- $[\nabla^\sharp \vec{v}^\flat]_i^j = [g(\nabla \vec{v}) g^{-1}]_i^j = \nabla^j v_i = v_i^{;j} = v_i^{|j} = g^{jk} g_{il} \nabla_k v^l$
- $\nabla_i f = [\nabla f]_i = \partial_i f$
- $\nabla_{\vec{v}} f = \mathcal{L}_{\vec{v}} f = \langle \vec{v}, \nabla f \rangle = (\mathbf{d}f)(\vec{v}) = v^i \nabla_i f = v^i \partial_i f$

### 1.10 Shape-Operator $S$ , etc

- **Second fundamental form:**  
 $[II]_{ij} = [S^b]_{ij} = h_{ij} = -\partial_i \vec{N} \cdot \partial_j \vec{X} = -\left[\nabla \vec{N}\right]_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{X}$
- **Shape operator (Weingarten map):**  
 $[S]_j^i = g^{ik} h_{kj} = -\left[\nabla_\Gamma \vec{N}\right]^i \cdot \partial_j \vec{X} = -\left[\nabla_\Gamma \vec{N}\right]_j^i$
- **Inverse of second fundamental form:**  
 $b^{ij} = [II^{-1}]^{ij} = \frac{1}{|g|K} [II^{\text{Adj}}]^{ij}$
- $[S(\vec{v})]_i = -\left[\nabla_{\vec{v}} \vec{N}\right]_i = v^j h_{ij}$
- $S^T \alpha = \alpha S = S(\alpha^\sharp)$

### 1.11 Conclusions

$$\vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\text{Grad} f = \nabla_\Gamma f = \nabla^\sharp f = (\mathbf{d}f)^\sharp$   
 $[\text{Grad} f]^i = \nabla^i f = g^{ij} \nabla_j f = g^{ij} \partial_j f$
- $\text{Div} \vec{v} = -\delta \vec{v}^\flat = * \mathbf{d} * \vec{v}^\flat = \nabla_i v^i = \partial_i v^i + v^k \Gamma_{ik}^i = \partial_i v^i + v^k \partial_k \log \sqrt{|g|}$   

$$= \sum_{i=1,2} \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} v^i = \sum_{i=1,2} \frac{v^i}{\sqrt{|g|}} \partial_i \sqrt{|g|} + \partial_i v^i$$
- $\delta(f\alpha) = f\delta\alpha - \langle \mathbf{d}f, \alpha \rangle$
- $\text{Div}(f\vec{v}) = f\text{Div} \vec{v} + \nabla_{\vec{v}} f = f\nabla_i v^i + v^i \nabla_i f$
- **Laplace-Beltrami operator:**  
 $\Delta_B f = -\delta \mathbf{d}f = * \mathbf{d} * \mathbf{d}f = \text{Div Grad} f = \nabla_i \nabla^i f = \frac{1}{\sqrt{|g|}} \partial_j \left( g^{ij} \sqrt{|g|} \partial_i f \right)$
- **Laplace-de Rham operator:**  
 $\Delta_{dR} \alpha = (\delta \mathbf{d} + \mathbf{d} \delta) \alpha =: -(\Delta_B + \Delta_{CB}) \alpha$   
 $\Delta_{dR} \vec{v} = (\Delta_{dR} \vec{v}^\flat)^\sharp$
- **Giaquinta-Hildebrandt operator:**  
 $\Delta_{GH} f = \square f = \text{Div} (K II^{-1} \mathbf{d}f) = -\delta (KS^{-T} \mathbf{d}f) = \frac{1}{\sqrt{|g|}} \partial_j \left( \sqrt{|g|} K b^{ij} \partial_i f \right)$
- $-\delta(S^T \alpha) = -H \text{Div} \alpha^\sharp - \text{Div} (K II^{-1} \alpha) - \nabla_{\alpha^\sharp} H = H \delta \alpha + \delta (KS^{-T} \alpha) - \langle \alpha, \mathbf{d}H \rangle$   
 $-\delta(S^T \mathbf{d}f) = -H \Delta_B f - \Delta_{GH} f - \langle \mathbf{d}H, \mathbf{d}f \rangle$

### 1.12 Moving Surfaces $M(t)$

$\vec{V} := \vec{v} + v_n \vec{N} = \partial_t \vec{X}$  (surface velocity),  $\vec{X} : M \rightarrow E^3$  (parametrization)

- $\partial_i \vec{V} \cdot \partial_j \vec{X} = g_{jk} \nabla_i v^k - v_n h_{ij} = \left[ (\nabla \vec{v} - v_n S)^b \right]_{ij}$
- (rate-of-deformation tensor  $\mathbf{d}$ )  

$$\frac{d}{dt} g = \left( \nabla \vec{v}^b \right) + \left( \nabla \vec{v}^b \right)^T - 2v_n \mathbf{II} = \mathcal{L}_{\vec{V}} g = 2\mathbf{d}$$

$$\frac{d}{dt} g_{ij} = g_{ik} \nabla_j v^k + g_{jk} \nabla_i v^k - 2v_n h_{ij} = 2d_{ij}$$
- $\left\langle \frac{d}{dt} \vec{p}^b, \alpha \right\rangle = \dot{\vec{p}} \alpha + 2\vec{p} \mathbf{d} \alpha^\sharp = \dot{\vec{p}} \alpha + \vec{p} \left( (\nabla \vec{v})^T + \left( \nabla^\sharp \vec{v}^b \right) - 2 * v_n S^T \right) \alpha$   

$$= \dot{p}^i \alpha_i + p^j \left( \nabla_j v^i + \nabla^i v_j - 2v_n h_j^i \right) \alpha_i$$
- $\frac{d}{dt} \alpha^\sharp = \left[ \dot{\alpha} + \left( 2v_n S^T - (\nabla \vec{v})^T - \left( \nabla^\sharp \vec{v}^b \right) \right) \alpha \right]^\sharp$
- $\frac{d}{dt} * \omega = * [\dot{\omega} - (\text{Div} \vec{v} + v_n H) \omega]$
- $\frac{d}{dt} * \vec{p}^b = * \left[ \dot{\vec{p}} + (\text{Div} \vec{v} + v_n H) \vec{p} \right]^b$
- $\frac{d}{dt} * \alpha = * \left[ \dot{\alpha} + \left( 2v_n S^T - (\nabla \vec{v})^T - \left( \nabla^\sharp \vec{v}^b \right) \right) \alpha + (\text{Div} \vec{v} + v_n H) \alpha \right]$
- $\frac{1}{2} \frac{d}{dt} \|\alpha\|^2 = \langle \dot{\alpha} + v_n S^T \alpha - (\nabla \vec{v})^T \alpha, \alpha \rangle = \dot{\alpha} \alpha^\sharp + \alpha (v_n S - \nabla \vec{v}) \alpha^\sharp$   

$$= \dot{\alpha}_i \alpha^i + v_n \alpha_i h_j^i \alpha^j - \alpha_i (\nabla_j v^i) \alpha^j$$
- $\frac{1}{2} \frac{d}{dt} \|\omega\|^2 = \frac{1}{2} \frac{d}{dt} (*\omega)^2 = \langle \dot{\omega}, \omega \rangle - (\text{Div} \vec{v} + v_n H) \|\omega\|^2$
- $\frac{1}{2} \frac{d}{dt} \|\delta \alpha\|^2 = \left\langle \delta \left[ \dot{\alpha} + \left( 2v_n S^T - (\nabla \vec{v})^T - \left( \nabla^\sharp \vec{v}^b \right) \right) \alpha + (\text{Div} \vec{v} + v_n H) \alpha \right], \delta \alpha \right\rangle - (\text{Div} \vec{v} + v_n H) \|\delta \alpha\|^2$
- $\frac{1}{2} \frac{d}{dt} \|\delta \vec{p}^b\|^2 = \left\langle \delta \left[ \dot{\vec{p}} + (\text{Div} \vec{v} + v_n H) \vec{p} \right]^b, \delta \vec{p}^b \right\rangle - (\text{Div} \vec{v} + v_n H) \|\delta \vec{p}^b\|^2$
- $\frac{d}{dt} \int_{M(t)} f \mu = \int_{M(t)} \dot{f} + f (\text{Div} \vec{v} + v_n H) \mu$
- $\frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\alpha\|^2 \mu = \int_{M(t)} \langle \dot{\alpha}, \alpha \rangle + v_n \left\langle S^T \alpha + \frac{1}{2} H \alpha, \alpha \right\rangle + \left\langle \frac{1}{2} (\text{Div} \vec{v}) \alpha - (\nabla \vec{v})^T \alpha, \alpha \right\rangle \mu$
- $\frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\omega\|^2 \mu = \int_{M(t)} \langle \dot{\omega}, \omega \rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \|\omega\|^2 \mu$

- $$\begin{aligned}
\frac{d}{dt} \int_{M(t)} \frac{1}{2} \left\| \mathbf{d}\vec{p}^\flat \right\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^\flat, \delta \mathbf{d}\vec{p}^\flat \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \mathbf{d}\vec{p}^\flat \right\|^2 \mu \\
&= \int_{M(t)} \dot{\vec{p}} \delta \mathbf{d}\vec{p}^\flat + \vec{p} \left( (\nabla \vec{v})^T + \left( \nabla^\sharp \vec{v}^\flat \right) - 2v_n S^T \right) \delta \mathbf{d}\vec{p}^\flat \\
&\quad - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \mathbf{d}\vec{p}^\flat \right\|^2 \mu
\end{aligned}$$
- $$\begin{aligned}
\frac{d}{dt} \int_{M(t)} \frac{1}{2} \left\| \delta \vec{p}^\flat \right\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle + \left\langle \left( 2v_n S^T - (\nabla \vec{v})^T - \left( \nabla^\sharp \vec{v}^\flat \right) \right) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle \\
&\quad + \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu \\
&= \int_{M(t)} \dot{\vec{p}} \mathbf{d} \delta \vec{p}^\flat + (\text{Div} \vec{v} + v_n H) \vec{p} \mathbf{d} \delta \vec{p}^\flat - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu
\end{aligned}$$
- $$\begin{aligned}
\frac{d}{dt} \int_{M(t)} \frac{1}{2} \left( \left\| \mathbf{d}\vec{p}^\flat \right\|^2 + \left\| \delta \vec{p}^\flat \right\|^2 \right) \mu &= \frac{d}{dt} \int_{M(t)} \frac{1}{2} \left( \left\| \text{Rot} \vec{p} \right\|^2 + \left\| \text{Div} \vec{p} \right\|^2 \right) \mu \\
&= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^\flat, \Delta_{dR} \vec{p}^\flat \right\rangle - \frac{\text{Div} \vec{v} + v_n H}{2} \left( \left\| \mathbf{d}\vec{p}^\flat \right\|^2 + \left\| \delta \vec{p}^\flat \right\|^2 \right) \\
&\quad + \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle \\
&\quad - \left\langle \left( (\nabla \vec{v})^T + \left( \nabla^\sharp \vec{v}^\flat \right) - 2v_n S^T \right) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle \mu \\
&= \int_{M(t)} \left\langle \dot{\vec{p}}, \Delta_{dR} \vec{p} \right\rangle - \frac{\text{Div} \vec{v} + v_n H}{2} \left( \left\| \text{Rot} \vec{p} \right\|^2 + \left\| \text{Div} \vec{p} \right\|^2 \right) \\
&\quad + (\text{Div} \vec{v} + v_n H) \vec{p} \mathbf{d} \delta \vec{p}^\flat \\
&\quad + \vec{p} \left( (\nabla \vec{v})^T + \left( \nabla^\sharp \vec{v}^\flat \right) - 2v_n S^T \right) \delta \mathbf{d}\vec{p}^\flat \mu
\end{aligned}$$
- $$\begin{aligned}
\frac{d}{dt} \int_{M(t)} \frac{1}{4} \left( \left\| \vec{p} \right\|^2 - 1 \right)^2 \mu &= \int_{M(t)} \left( \left\| \vec{p}^\flat \right\|^2 - 1 \right) \left[ \left\langle \frac{d}{dt} \vec{p}^\flat, \vec{p}^\flat \right\rangle + \left\langle \left( v_n S^T - (\nabla \vec{v})^T \right) \vec{p}^\flat, \vec{p}^\flat \right\rangle \right. \\
&\quad \left. + \frac{1}{4} \left( \left\| \vec{p}^\flat \right\|^2 - 1 \right) (\text{Div} \vec{v} + v_n H) \right] \mu \\
&= \int_{M(t)} \left( \left\| \vec{p}^\flat \right\|^2 - 1 \right) \left[ \left\langle \dot{\vec{p}}, \vec{p} \right\rangle + \left\langle \left( \left( \nabla^\sharp \vec{v}^\flat \right)^T - v_n S \right) \vec{p}, \vec{p} \right\rangle \right. \\
&\quad \left. + \frac{1}{4} \left( \left\| \vec{p} \right\|^2 - 1 \right) (\text{Div} \vec{v} + v_n H) \right] \mu
\end{aligned}$$
- $$\begin{aligned}
\int_{M(t)} \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu \\
= \int_{M(t)} \mathcal{L}_{\vec{p}} \left\langle \vec{p}^\flat, \mathbf{d} \delta \vec{v}^\flat + \mathbf{d} (v_n H) \right\rangle + \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu
\end{aligned}$$

## 2 Tensors

### 2.1 Flat / Sharp

- $t := t^i_j \partial_i \otimes dx^j$
- ${}^b t = gt = g_{ik} t^k_j dx^i \otimes dx^j = t_{ij} dx^i \otimes dx^j$
- $t^\sharp = tg^{-1} = t^i_k g^{kj} \partial_i \otimes \partial_j = t^{ij} \partial_i \otimes \partial_j$
- ${}^b t^\sharp = gtg^{-1} = g_{ik} t^k_l g^{lj} dx^i \otimes \partial_j = t_i^j dx^i \otimes \partial_j$

#### 2.1.1 Conclusions

$\alpha = \vec{v}^b = {}^b \vec{v}$ ,  $\vec{w} = \beta^\sharp = {}^\sharp \beta$ ,  $s = \vec{v} \otimes \beta$ :

- $t$  symmetric ( $t_{12} = t_{21}$  resp.  $t^{12} = t^{21}$ ):  ${}^b t^\sharp = t^T$
- $\alpha t \vec{w} = \vec{v} {}^b t \vec{w} = \alpha t^\sharp \beta = \vec{v} {}^b t^\sharp \beta$  (Associativity referring to arguments)

$$\Rightarrow t(\alpha, \vec{w}) = {}^b t(\vec{v}, \vec{w}) = t^\sharp(\alpha, \beta) = {}^b t^\sharp(\vec{v}, \beta)$$

- $s = \vec{v} \otimes \beta$ :  ${}^b s = \alpha \otimes \beta$ ,  $s^\sharp = \vec{v} \otimes \vec{w}$ ,  ${}^b s^\sharp = \alpha \otimes \vec{w}$  (Associativity referring to factors, tensor product is metric compatible)
- $t^T := {}^\sharp ({}^b t)^T = ((t^\sharp)^T)^b = g^{-1}(gt)^T = (tg^{-1})^T g = \{t_j^i\}^i_j \partial_i \otimes dx^j$
- $(t^T)^T = t^T$
- $tt^T = (tt^T)^T = t^T t$
- $|t| = |t^T| = |{}^b t^\sharp| = |g||t^\sharp| = \frac{|{}^b t|}{|g|}$

### 2.2 Levi-Civita-Tensor

- $\epsilon^\sharp = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{dx^i \otimes dx^j}$
- ${}^b \epsilon = |g| \epsilon^\sharp$

### 2.3 Covariant Derivative $\nabla_\bullet = g_{\bullet i} \nabla^\bullet$

- $\nabla_k f = \partial_k f$
- $\nabla_k v^i = \partial_k v^i + \Gamma_{kl}^i v^l$
- $\nabla_k v_i = \partial_k v_i - \Gamma_{ki}^l v_l$
- $\nabla_k t^i_j = \partial_k t^i_j + \Gamma_{kl}^i t^l_j - \Gamma_{kj}^l t^i_l$

- $\nabla_k t_i^j = \partial_k t_i^j - \Gamma_{ki}^l t_l^j + \Gamma_{kl}^j t_i^l$
- $\nabla_k t^{ij} = \partial_k t^{ij} + \Gamma_{kl}^i t^{lj} + \Gamma_{kl}^j t^{il}$
- $\nabla_k t_{ij} = \partial_k t_{ij} - \Gamma_{ki}^l t_{lj} - \Gamma_{kj}^l t_{il}$
- $\text{Rot}(f) = -\frac{1}{\sqrt{|g|}} g_{ik} \epsilon^{kl} \partial_l f dx^i = -\frac{1}{\sqrt{|g|}} g \epsilon^\sharp \mathbf{d}f = -\sqrt{|g|} \epsilon^\sharp \nabla f$
- $\text{Div}(v) = \nabla_i v^i = \nabla^i v_i = \text{Tr}(\nabla \vec{v}) = \text{Tr}({}^b \nabla^\sharp \alpha)$
- $\text{Rot}(v) = (\sqrt{|g|})^{-1} \epsilon^{ki} \nabla_k v_i = (\sqrt{|g|})^{-1} \text{Tr}((\nabla \alpha) \epsilon^\sharp)$
- $\text{Rot}(v)^2 = \nabla_j v^i (\nabla^j v_i - \nabla_j v^i) = |\nabla v| + \text{Tr}(\nabla v (\nabla v)^T) - \text{Div}(v)^2$
- $\text{Div}(t) = \nabla^j t^i_j \partial_i = \nabla_j t_i^j dx^i = \nabla_j t^{ij} \partial_i = \nabla^j t_{ij} dx^i$

## 2.4 Rotation $R: T_p M \rightarrow (T_p M)'$

- $Rv := v' := \cos(\varphi)v + \sin(\varphi)(*v) = \cos(\varphi)v + \sin(\varphi)(*v^\flat)^\sharp$
- $R = \cos(\varphi)I + \frac{\sin(\varphi)}{\sqrt{|g|}} \begin{bmatrix} -g^{12} & -g^{22} \\ g^{11} & g^{12} \end{bmatrix} = R^i_j \partial_i \otimes dx^j$  **indices unten? nutze levi-cevita tensor!**
- $R^{-1} = R^T$
- $\langle v', w' \rangle = \langle v, w \rangle \Rightarrow R \in O(T_p M)$
- $\nabla v' = R \nabla v$

### 2.4.1 Push-forward $R_* = R^{*T}$

- $R_* v = Rv = v'$
- $R_* \alpha = \alpha R^T = \alpha'$
- $R_* \bullet^\flat = R \bullet^\flat R^T = \bullet'^\flat$
- $R_*(\bullet \otimes \bullet) = (R_* \bullet) \otimes (R_* \bullet)$