Chapter 2

THE EXTERIOR CALCULUS

"Let no one say that I have said nothing new;

the arrangement of the subject is new."

Blaise Pascal

1 Introduction

The exterior calculus of differential forms, also called Cartan's calculus, is one whose geometric underpinning is the exterior algebra. It is also known as Grassmann algebra, to honor Hermann Grassmann (1809-1877). He was a mathematical genius, who was also a superb linguist (he translated the Hindu *Rig Veda* into German verse), and an accomplished folklorist, musician and natural scientist (botany, crystallography, color mixing, electricity and acoustics, invented a heliostat, etc). All that was done in his spare time as a high school teacher. Exterior algebra is his most famous contribution to mathematics, but is only part of his great vision about algebra, the type of vision that is understood only long after the visionary is dead.

The late Professor Rota of the Massachussets Institute of Technology was a renown mathematician/philosopher whose opinion of Grassmann is relevant here, for it is also an opinion about the calculus of differential forms. In a paper he coauthored with Barnabei and Brini, mention is made of Grassmann's deep understanding of one of his mathematical discoveries. The authors go on to criticize most of what they said were the "mainstream algebraists of the time, such as Gordan, Capelli, Hermite, Cayley, Sylvester and even Hilbert", who, "while paying lip service to Grassmann in an occasional footnote, did not realize the sweeping extent of Grassmann discovery...". The authors (writing at the end of the twentieth century) went on to say "The epigons of invariant theory in this century, such mathematicians as "Turnbull, Aitken, Alfred Young, Littlewood and even Hermann Weyl" perpetuated the same sin of omission, and one finds in these authors scattered rediscoveries and partial glimpses of ideas that could have been made to bloom, had the authors used even only the notation of exterior algebra". Please observe that not even the famous Hilbert (1862-1943) is in the group of those who are credited at least for scattered rediscoveries and partial glimpses.

After giving some credit to Clifford, Schröder, A. N. Whitehead, Élie Cartan and specially Peano, the coauthors Barnabei, Brini and Rota credit Cartan specifically for realizing "the usefulness of the notion of exterior algebra in his theory of integral invariants, which was later to turn into the potent theory of differential forms". An interesting issue arises from this comment. Since differential forms supposedly have to do with differentiation and the concept of integral refers to integrations, it looks as if we are crossing the boundary from one sector of the calculus to the other. Viewing the exterior calculus in the particular way of this book, differentiation and integration come together in a much deeper way than in the prevalent way of looking af differential forms, which will be considered in scattered form several times in this book. In our way of looking at differential forms, these are meant to be integrated. An integral invariant is simply the integral of an on variant differential form.

Returning to the main theme, differential forms are underlined by exterior algebra, also known as Grassmann algebra. But that is a temporal accident, since differential forms was eventually endowed with a richer-than-exterior algebraic structure, of great relevance for quantum mechanics. On the other hand, exterior algebra has relevance beyond its use in the calculus of differential forms. Readers who find the remainder of this section too difficult should just move on.

Differential forms and the exterior calculus based on them were born through a paper of 1899 by Cartan. It would be appropriately called exterior calculus (here abbreviated *exteriorc*) or Cartan calculus (here abbreviated *Cartanc*), but only until 1922, as we shall explain below. In the 1899 paper, differential forms had been introduced without the fanfare that would normally accompany the birth of a new calculus. They were the subject of the introductory part of a paper devoted to his general theory of exterior differential systems, which he mainly developed during the first two decades of the twentieth century, and which Kähler generalized in 1934. For perspective, suffice to say that any partial differential equation or system thereof can be written as an exterior differential system. We must thus view *exteriorc* (equivalently *Cartanc*) not only as pertaining to the calculus but also to the main theory of systems of partial differential equations.

When the terms *exteriorc* and *Cartanc* are used, practitioners have in mind expressions called *scalar-valued* differential forms. They contain differentials ($dx, \ldots, dz$, $dr, ..., d\phi$, $\ldots$, df, *etc*) but not boldfaced factors. Thus differentiations of expressions such as $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ were not considered until Cartan did so starting in 1922, this extension remaining an exterior calculus but of more comprehensive differential forms. Though the terms *exteriorc* and *Car-*

tanc thus remain appropriate to describe the so extended calculus of differential forms, most practitioners still use these two terms to refer to the original, less comprehensive version. In this book, we shall not undertake a systematic study of the 1922 version of the calculus. We shall, however, made an incursion into it in the limited context of affine spaces (and Euclidean spaces in particular), the reasons being that in such spaces the extension is almost trivial and that it provides perspective on the issues that the 1922 extension deals with. the calculus of differential forms with bold faced coefficients (not only vectorial but also tensorial) resorts to the theory of exterior differential systems and constitutes modernly the main body of differential geometry. It deserves a separate book.

In 1960, the great mathematician Erich Kähler further enriched 1922 - *exteriorc* into what he called an *interior and exterior* calculus. He renamed it as interior calculus for no known reason in 1962. Kähler's post-exterior structure shall be considered in the next book in this series. Thus the term calculus of differential forms would have to be reserved for the Kähler calculus, except that Kähler did not consider for this extension all the generality of 1922-*exteriorc* (for the initiated into differential geometry, let us advance that he only considered the Levi-Civita connection). Even the extension of the 1899 calculus into a Kähler calculus already constitutes a formidable theory of Dirac equations which remains virtually unexplored to this day. It also has mathematical applications in the sense that parts of it constitute an extension to new realms of the theories of *de Rham* (of harmonicity) and *Hodges* (of strict harmonicity).

Our task in this chapter is the learning of the rudiments of 1899-*exteriorc*, which replaces, generalizes and supersedes most of the contents of the vector calculus. We shall do so without resorting to the bold-faced vectors, i.e. the linear combinations of **i**, **j**, **k**. The vector calculus, also known as multivariable calculus, is a bastard calculus in more than one way. For instance, it uses vector fields where it should not use them. We shall discuss it precisely to show its many failures. Vector products also will be absent from this book, except when we wish to show that they consist of a combination of the exterior product and a certain operation (duality) that complicates rather than simplifies matters. With this replacement, anything that we say about scalar-valued differental forms applies to spaces endowed with very little structure and of arbitrary positive integer number of dimensions. These generalizations will not be, however, our main focus. They will be treated in a separate book

To conclude, 1899-*exteriorc* is a first step towards much more mathematics, as well as a formidable platform for physics and for quantum theory in particular. Walking the present path, the reader will not have to learn new approaches for new mathematics (say for quantum mechanics), but simply extend the same approach to new fields.

2 Exterior Products

Exterior products is what the so called exterior algebra is about. One starts with a vector space and multiplies different numbers of them, and also by scalars, and adds such varied products, and multiplies them and so on. We shall approach these subjects in an easy, informal way. For compactness, the best approach to exterior algebra is to consider it as a quotient algebra of the general tensor algebra. We shall do that in a future chapter.

Exterior products are present, though hidden, in the integrands of multiple integrals. Those are the exterior products in which we are interested in this chapter. However, we shall start by considering exterior products of familiar boldfaced vectors, linear combinations of **i**, **j**, **k**, to draw from our experiences. Let us recall that, in E^4, we would need four unit vectors, $(\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l})$, to play the role that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ plays in three dimensions. What would the "vector product" of **i** and **j** be in four dimensions? It is not defined. If you claim that it must be **k** because it is so in particular in three dimensions, we ask then what is the vector product of **i** and **l**, of **j** and **l** and of **k** and **l**. The vector product is well defined in three dimensions because, given two vectors which are not collinear, there is only one direction which is perpendicular to it. But this is not the case in four or a higher number of dimensions: there is not a unique direction perpendicular to both **i** and **j**, since both **k** and **l** are perpendicular to **i** and **j**, and so are all the linear combinations of **k** and **l**. Vector products are a peculiarity of tree dimensions. One can concoct something of that sort in seven dimensions (see Lounesto), but it is highly artificial.

More importantly, we know that the volume of the parallelepiped whose sides are the vectors **a**, **b** and **c** is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. If $\mathbf{a} \times \mathbf{c}$ is not defined, how can we define the volume of the parallelepiped made by three vectors in four dimensions? We could restrict ourselves to the three-dimensional subspace spanned by the three vectors. But what about the "four-volume" of the four dimensional parallelepiped constructed upon four independent vectors? All these issues are addressed properly by the exterior product.

For simplicity, let us first consider three dimensions. Given two vectors **a** and **b** in $\mathbf{E}^3$, their exterior product, denoted $\mathbf{a} \wedge \mathbf{b}$, behaves in more or less the same way as its vector product for many but not all purposes. $\mathbf{a} \wedge \mathbf{b}$ is called a bivector, or linear combination of objects of the same nature represented with the symbols $(\mathbf{i} \wedge \mathbf{j})$, $(\mathbf{j} \wedge \mathbf{k})$ and $(\mathbf{k} \wedge \mathbf{i})$. There are only three linearly independent bivectors in three dimensions. But there are six of them (rather than four) in four dimensions and only one of them in two dimensions. So, again, three dimensions is a very special case: there are only as many independent bivectors as vectors.

The following geometric representation may be helpful. One may think of $\mathbf{i} \wedge \mathbf{j}$, $\mathbf{j} \wedge \mathbf{k}$ and $\mathbf{k} \wedge \mathbf{i}$ as the parallelograms whose sides are **i** and **j**, **j** and **k**, and **k** and **i** respectively. $\mathbf{a} \wedge \mathbf{b}$ is a paralelogram similarly constructed upon **a** and **b**. Because of the considerations at the very beginning and end of section 12 of Chapter I, it is more appropriate to think of $(1/2)\mathbf{a} \wedge \mathbf{b}$ as the triangle two of whose sides are the vectors **a** and **b**. A plane π that cuts three orthonormal axes determines a pyramid whose faces are a triangle on π and three other triangles on the three coordinate planes. Up to issues of sign, the first triangle may be represented as 1/2 the exterior product of any two of its sides. The same can be said of the other three triangles, but, for simplicity, we shall take the sides that lie on the coordinate axes. The bivector $\frac{1}{2}\mathbf{a} \wedge \mathbf{b}$ may then be viewed as a linear combination of the bivectors $\mathbf{i} \wedge \mathbf{j}$, $\mathbf{j} \wedge \mathbf{k}$ and $\mathbf{k} \wedge \mathbf{i}$. Needless to say that $\frac{1}{2}\mathbf{i} \wedge \mathbf{j}$, $\frac{1}{2}\mathbf{j} \wedge \mathbf{k}$ and $\frac{1}{2}\mathbf{k} \wedge \mathbf{i}$ represent triangles on the coordinate planes whose sides on the coordinate axes are of unit length.

Notice that, when defining $\mathbf{i} \wedge \mathbf{j}$, we have made no reference to $\mathbf{k}$ (in $\mathbf{E}^3$) or to $\mathbf{k}$ and $\mathbf{l}$ (in $\mathbf{E}^4$). To be more precise, let $\mathbf{a}$ and $\mathbf{b}$ be the vectors $\mathbf{a} \equiv a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} \equiv b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$. We must learn to "*exterior multiply*" $\mathbf{a}$ and $\mathbf{b}$, i.e. multiply them with the product "$\wedge$". This product is defined to have the properties:

$$\mathbf{a} \wedge \mathbf{b} = -(\mathbf{b} \wedge \mathbf{a}), \tag{2.1}$$

$$\begin{aligned} \mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}, \\ (\mathbf{b} + \mathbf{c}) \wedge \mathbf{a} &= \mathbf{b} \wedge \mathbf{a} + \mathbf{c} \wedge \mathbf{a}, \end{aligned} \tag{2.2}$$

$$(\alpha\mathbf{a}) \wedge \mathbf{b} = \mathbf{a} \wedge (\alpha\mathbf{b}) \equiv \alpha(\mathbf{a} \wedge \mathbf{b}). \tag{2.3}$$

An alternative way of writing the expressions in (2.3) is $\alpha\mathbf{a} \wedge \mathbf{b}$. Notice the limitations of property (2.3) on the a foregoing interpretation of bivectors. The triangles with sides $(\mathbf{i}, \mathbf{j})$, $\left(\frac{1}{2}\mathbf{i}, 2\mathbf{j}\right)$, $\left(2\mathbf{i}, \frac{1}{2}\mathbf{j}\right)$, etc yield the same bivector.

Rules (2.1) and (2.2) are called the anticommutative and distributive properties respectively. Rules (2.2) and (2.3) together state that the $\wedge$ product is linear in each of the factors. Rule (2.1), which of course applies to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in particular, readily implies that $\mathbf{a} \wedge \mathbf{a} = 0$.

Exercise 1 *Combine the anticommutative and distributive properties to show that:*

$$\mathbf{a} \wedge \mathbf{b} = (a_2b_3 - a_3b_2)\,\mathbf{j} \wedge \mathbf{k} + (a_3b_1 - a_1b_3)\,\mathbf{k} \wedge \mathbf{i} + (a_1b_2 - a_2b_1)\,\mathbf{i} \wedge \mathbf{j}. \tag{2.4}$$

The components of $\mathbf{a} \wedge \mathbf{b}$ with respect to *the basis* $(\mathbf{j} \wedge \mathbf{k}, \mathbf{k} \wedge \mathbf{i}, \mathbf{i} \wedge \mathbf{j})$ are precisely the components of what we know as the vector product. The bivector $(a_2b_3 - a_3b_2)\,\mathbf{j} \wedge \mathbf{k} + (a_3b_1 - a_1b_3)\,\mathbf{k} \wedge \mathbf{i} + (a_1b_2 - a_2b_1)\,\mathbf{i} \wedge \mathbf{j}$ may then be thought of as representing classes of oriented parallelograms or oriented triangles in 3-space. Again, to the extent that it helps, we may think of $(a_2b_3 - a_3b_2)\,\mathbf{j} \wedge \mathbf{k}$, $(a_3b_1 - a_1b_3)\,\mathbf{k} \wedge \mathbf{i}$ and $(a_1b_2 - a_2b_1)\,\mathbf{i} \wedge \mathbf{j}$.as representing parallelograms or triangles, but it does not have to be so. More than any specific figures, we should think of these bivectors as equivalence classes of figures that have the same oriented are in the planes of the respective bivectors. Correspondingly, $\mathbf{a} \wedge \mathbf{b}$ can also be thought as the class of oriented figures in the plane of $\mathbf{a}$ and $\mathbf{b}$ (the orientation being provided by a sense of rotation associated with the boundary of the figure) such that its projections on the coordinate planes have oriented areas given by $(a_2b_3 - a_3b_2)$, $(a_3b_1 - a_1b_3)$ and $(a_1b_2 - a_2b_1)$ respectively. Of course, when thinking about classes, it helps to think of any representative member of each class. These interpretations are just to help the reader get used to these products. It is still too early to discuss in depth the relation of differential 2-forms, which are function(al)s to be defined, to bivectors, which are classes of figures.

In order to avoid being distracted away from the main argument, we do not stop to define the magnitude of bivectors and other elements in the algebra (if and when vectors are endowed with the concept of magnitude in the first place). That goes beyond exterior algebra proper. Let us simply mention, in case the reader is curious, that the size of bivector (2.3) is the size of the paralelogram constructed on the vectors, namely

$$\left[\,(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2\,\right]^{1/2}$$

Notice that the exterior product of $\mathbf{i}$ and $\mathbf{j}$, for example, does not involve the perpendicular vector $\mathbf{k}$. Similarly, the exterior product of the specific vectors $\mathbf{a} \equiv a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} \equiv b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is what we have said it to be in Eq.(2.4) regardless of whether these vectors are said to belong to $\mathbf{E}^3$ or to a higher dimensional space.

We now obtain the volume of the parallelepiped without resort to the vector product, and to the formula $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ in particular. For this purpose, we first define the exterior product of several vectors through the properties of said product. Two of the properties are anticommutativity and multilinearly distributive:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \cdots \wedge \mathbf{m} \wedge \mathbf{n} \wedge \mathbf{p} = -\mathbf{a} \wedge \mathbf{n} \wedge \mathbf{c} \wedge \cdots \wedge \mathbf{m} \wedge \mathbf{b} \wedge \mathbf{p}, \tag{2.6}$$

$$\begin{aligned}\mathbf{a} \wedge (\lambda\,\mathbf{b}_1 + \mu\,\mathbf{b}_2) \wedge \mathbf{c}... &= \mathbf{a} \wedge \lambda\,\mathbf{b}_1 \wedge \mathbf{c}... + \mathbf{a} \wedge \mu\,\mathbf{b}_2 \wedge \mathbf{c}... = \\ &= \lambda\,\mathbf{a} \wedge \mathbf{b}_1 \wedge \mathbf{c}... + \mu\,\mathbf{a} \wedge \mathbf{b}_2 \wedge \mathbf{c}...,\end{aligned} \tag{2.7}$$

where λ and μ are numbers. We proceed similarly when $\mathbf{a}$, $\mathbf{c}, ...$are themselves linear combinations. Equation(2.6) is the generalization of the anticommutative property $ab = -ba$. It results from repeated application of (2.1) as follows:

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \cdots \wedge \mathbf{m} \wedge \mathbf{n} \wedge \mathbf{p} &= -\mathbf{a} \wedge \mathbf{c} \wedge \mathbf{b} \wedge \cdots \wedge \mathbf{m} \wedge \mathbf{n} \wedge \mathbf{p} = \\ &= (?)\ \mathbf{a} \wedge \mathbf{c} \wedge \cdots \wedge \mathbf{m} \wedge \mathbf{n} \wedge \mathbf{b} \wedge \mathbf{p},\end{aligned} \tag{2.8}$$

where the question mark stands for the fact that the sign depends on the number of times that we use (2.1) in (2.8). To be explicit, we move $\mathbf{b}$ to the right one place at a time until it is placed just behind $\mathbf{n}$. We then move $\mathbf{n}$ to where $\mathbf{b}$ was, in order to reach the expression on the right hand side of (2.6). This takes one less step than it took to move $\mathbf{b}$. Thus the total number of steps taken is odd. Finally, properties (2.6) and (2.7) are also required to work for all elements of the algebra, meaning the set of sums of any number of vector products of whatever grades.

Exercise 2 *Find out whether* $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d}$ *and* $\mathbf{b} \wedge \mathbf{d} \wedge \mathbf{a} \wedge \mathbf{c}$ *are equal or opposite for general vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

Exercise 3 *Show that* $\mathbf{a} \wedge (3\mathbf{c} + 2\mathbf{d}) \wedge \mathbf{c} \wedge \mathbf{d}$ *is zero.*

The next exercise constitutes an example of why exterior products of more than two vectors can be useful.

Exercise 4 *Show that the coefficient with respect to* $\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$ *of the exterior product* $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ *gives the volume of the parallelepiped determined by the three vectors* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$. *You should get*

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = \left[c_1\,(a_2b_3 - a_3b_2) + c_2\,(a_3b_1 - a_1b_3) + c_3\,(a_1b_2 - a_2b_1)\right]\mathbf{i}\wedge\mathbf{j}\wedge\mathbf{k}. \quad (2.9)$$

With the caveats mentioned above, it is worth thinking of $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ as the parallelepiped itself whose sides are given by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, or one can think of $(1/3!)\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ as the pyramid whose sides are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Consistently with previous remarks, $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is to be viewed as a class of figures having equal oriented volumes. The orientation of the volume is given by the sense of circulation imposed on its boundary.

Exercise 5 *Show that an exterior product of four or more vectors in a* 3*-dimensional vector space is always zero. Hint: express all vectors in terms of some basis and ask yourself what product of* 4 *factors, the factors being chosen among the three unit vectors* $\mathbf{i}, \mathbf{j}$ *and* $\mathbf{k}$, *could be non-zero, if any?*

It is a clear simple generalization of the result in the last exercise that the exterior product of a number m of vectors in a vector space of dimension n is zero if $m > n$.

We have not yet said everything needed to define, even informally, the exterior algebra. For instance, we have skipped the associative product. This is such a generalized property in structures of common interest, that the reader's mind is probably prompted to use it without question, rather than even thinking whether we have postulated it or not. We shall eventually provide a more elegant definition of exterior algebra as a quotient algebra. We do not yet have the concepts needed to do so. We now turn to a powerful application of exterior algebra to the calculus of integrals and, at the same time, become more familiar with exterior products.

3 Exterior Products of Differential 1-Forms

Consider the integral

$$\iiint f\big(x^2 + y^2 + z^2\big)dxdydz \quad (3.1)$$

over a ball of radius r. It is easiest to calculate it in spherical coordinates, both because it simplifies the integrand and because it simplifies the limits. For the purpose of performing this integral using spherical coordinates, we know that the volume element is $r^2 \sin\theta\, dr\, d\theta\, d\phi$ and that, therefore, the integral becomes

$$\iiint f(r^2)r^2 \sin\theta\, dr\, d\theta\, d\phi = \int 4\pi f(r^2)r^2\, dr, \quad (3.2)$$

with limits understood. In books on the calculus it is customary to use pictures to get to the expressions $r^2 \sin\theta \, dr \, d\theta \, d\phi$, and $4\pi r^2 \, dr$. But what is the volume element in terms of some coordinate system (a, b, c) where $a = f(x, y, z)$, $b = g(x, y, z)$ and $c = h(x, y, z)$? As we saw in Chapter I, one computes the Jacobian determinant or determinant of the matrix symbolically represented as $\partial(x, y, z)/\partial(a, b, c)$. One then inserts it into the integral in the coordinate system (a, b, c). We shall now obtain this determinant through coordinate substitution, which means that we do not need to use formulas, much less formulas which are not intuitive.

Let us start with a very simple example. Suppose that we tackle again, as we did in Chapter I, the problem of converting the integral $\iint dxdy$ to polar coordinates. Instead of $dxdy$, we shall write $dx \wedge dy$. After all, we just saw the relation of area to the exterior product, and we are used to think of $dxdy$ as area, regardless of whether it is advisable or not to do so. We then differentiate $x = \rho \cos\phi$ and $y = \rho \sin\phi$ and get $dx = \cos\phi \, d\rho - \rho \sin\phi \, d\phi$ and $dy = \sin\phi \, d\rho + \rho \cos\phi \, d\phi$. If we apply the rules that we have just learned above for the product "$\wedge$" as if $dx, dy, d\rho$ and $d\phi$ were bold faced vectors of previous examples, we have:

$$\begin{aligned} dx \wedge dy \quad = \quad & \cos\phi \sin\phi \; d\rho \wedge d\rho - \rho^2 \sin\phi \cos\phi \; d\theta \wedge d\theta \\ & -\rho \sin^2\phi \; d\phi \wedge d\rho + \rho \cos^2 \phi d\rho \wedge d\phi. \end{aligned} \tag{3.3}$$

The first two products are zero. This is so obvious that the reader will soon not write the terms with repeated factors. The last two terms can be put together using that $d\phi \wedge d\rho = -d\rho \wedge d\phi$. We thus get:

$$dx \wedge dy = \rho \, d\rho \wedge d\phi. \tag{3.4}$$

If we now ignore the symbol $\wedge$ in this expression, we recognize on the right hand side the element of area in polar coordinates. But, why should we ignore the symbol for such a powerful operation? We should rather get used to it. The concepts (to be given later) that $dx \wedge dy$, $d\rho \wedge d\phi$ and $\rho \, d\rho \wedge d\phi$ represent are called *exterior differential* 2*-forms*. They are integrands of double integrals. Similarly, we refer to $dx \wedge dy \wedge dz$ as a *differential* 3*-form* (an integrand of a triple integral). Differential 1-forms are dx, dy, dz, dr, $d\rho$, $d\theta$, $d\phi$...as well as their linear combinations, an example being

$$f(x, y, z) \, dx + g(x, y, z) \, dy + h(x, y, z) \, dz. \tag{3.5}$$

The triple $(dx,\, dy,\, dz)$ now plays the role that $(\mathbf{i}, \mathbf{j},\, \mathbf{k})$ played in the previous section. We shall later be more specific as to the vector-like space that $(dx,\, dy,\, dz)$ spans.

Exercise 6 *Knowing that the volume element is $dV = dx \wedge dy \wedge dz$ and knowing the relationship between rectilinear and spherical coordinates, find the volume element in spherical coordinates. You must differentiate $x = r \sin\theta \cos\phi$, $y = \ldots$, $z = \ldots$ and substitute the differentials in $dx \wedge dy \wedge dz$. After some practice, you will be able to write immediately the sole terms that do not vanish. They combine to yield*

$$dV = r^2 \sin\theta \, dr \wedge d\theta \wedge d\phi. \tag{3.6}$$

Incidentally, $r^2 \sin\theta$ *is the Jacobian* $|\partial(x, y, z)/\partial(r, \theta, \phi)|$ *but, who needs to write explicit Jacobians?*

Exercise 7 *Following similar steps, find the volume element*

$$dV = \rho \, d\rho \wedge d\phi \wedge dz \tag{3.7}$$

in cylindrical coordinates from $dV = dx \wedge dy \wedge dz$*, In this exercise, as in the previous one, observe the order of the differentials. Once again the coefficient,* ρ *in this case, is the corresponding Jacobian, namely* $|\partial(x, y, z)/\partial(\rho, \phi, z)|$.

Exercise 8 *Show that the volume element in terms of arbitrary coordinates* (a, b, c) *is*

$$\left| \frac{\partial(x, y, z)}{\partial(a, b, c)} \right| da \wedge db \wedge dc \tag{3.8}$$

We shall argue in Section 12 that is important to order the coordinates in systems thereof so that negative signs do not arise. We mean for example that, if $|\partial(x, y, z)/\partial(a, b, c)|$ were negative, we should use alter the order of the coordinates, say to $(a, c, b,)$

4 Further Considerations on Exterior Products

The contents of this paragraph will not be needed in this book. It is here for completeness purposes.

For a smooth introduction into the topic of exterior products, we considered in section 2 exterior products of vectors. We now view these products as elements of some structure, called a *exterior algebra.* This algebra contains products where, in principle, one term is a scalar, another is a vector, another is a bivector, yet another a trivector, and so on (In the same way as we use the term bivector for the exterior products of two vectors and their linear combinations, we use the term trivector to refer to exterior products of any three vectors and their linear combinations). Each of the terms is called an r-multivector, or multivector of grade r, where $r = 0$, 1, 2, ... A sum of terms of the same grade, say r, also is called an $r-$multivector. A sum of multivectors of different grades is called an inhomogeneous multivectors. The expansion in a power series of $f(x + dx,\ y + dy,\ z + dz)$ is an inhomogenous expansion, though it does not belong to an exterior algebra since we are not dealing there with antisymmetric products.

Let $\mathbf{B} \equiv \mathbf{a} \wedge \mathbf{b}$ and $\mathbf{T} \equiv \mathbf{m} \wedge \mathbf{n} \wedge \mathbf{p}$. Then, by definition:

$$\mathbf{B} \wedge \mathbf{T} \equiv (\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{m} \wedge \mathbf{n} \wedge \mathbf{p}) \equiv \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{m} \wedge \mathbf{n} \wedge \mathbf{p}. \tag{4.1}$$

As a result of exercise 3.3, the reader should readily understand by now that this product can be different from zero only in spaces of dimension 5 or higher.

This product will be called a multivector of grade 5. A ready application of this rule is that, in exercise 2.4, we could have computed $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ by computing $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$, since we already had $(\mathbf{a} \wedge \mathbf{b})$

Readers who feel lost may ignore it without major consequence. Let $\mathbf{B}_1 \equiv \mathbf{a}_1 \wedge \mathbf{b}_1$, $\mathbf{B}_2 \equiv \mathbf{a}_2 \wedge \mathbf{b}_2$, and $\mathbf{B}_3 \equiv \mathbf{a}_3 \wedge \mathbf{b}_3$, $\mathbf{C}_1 \equiv \mathbf{m}_1 \wedge \mathbf{n}_1 \wedge \mathbf{p}_1$ and $\mathbf{C}_2 \equiv \mathbf{m}_2 \wedge \mathbf{n}_2 \wedge \mathbf{p}_2$. The exterior product of multivectors are defined to have the distributive property:

$$\begin{aligned}&(\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3) \wedge (\mathbf{C}_1 + \mathbf{C}_2) = \\ &\quad = \mathbf{B}_1 \wedge \mathbf{C}_1 + \mathbf{B}_1 \wedge \mathbf{C}_2 + \mathbf{B}_2 \wedge \mathbf{C}_1 + \mathbf{B}_2 \wedge \mathbf{C}_2 + \mathbf{B}_3 \wedge \mathbf{C}_1 + \mathbf{B}_3 \wedge \mathbf{C}_2.\end{aligned} \tag{4.2}$$

Each term on the right hand side of this equation is a 5-vector and so is their sum. Operations in the algebra also include products such as, say,

$$(\mathbf{B}_1 + \mathbf{C}_1) \wedge (\mathbf{B}_2 + \mathbf{B}_3 + \mathbf{C}_2)\,, \tag{4.3}$$

$$(\lambda + \mathbf{C}_1) \wedge (\mu + \mathbf{B} + \mathbf{C}_2)\,, \tag{4.4}$$

where λ and μ are numbers, and so on. In other words, the algebra contains anything that the reader may imagine provided that we only have addittion and exterior multiplication operations. The product of two homogeneous multivectors of grades r and s is a homogeneous multivector of grade $r + s$, which is the reason why the exterior algebra is said to be a graded algebra.

As we shall see, the Maxwell system can be written in the form of just two equations involving homogeneous multivectors. One can also write the full system by means of just one equation involving inhomogeneous multivectors. For that, we need to introduce more structure on differential forms. That will be left for a future book. Inhomogeneous multivectors become essential in Kähler's general theory of Dirac equations.

We have dealt with exterior products of bold faced vectors, but, again, we can perform exterior products of differential forms. The $dx \wedge dy$, $dr \wedge d\theta$, $r\, dr \wedge d\theta$, etc. and the $d\rho \wedge d\phi \wedge dz$, $\rho\, d\rho \wedge d\phi \wedge dz$, $dx \wedge dy \wedge dz$ etc. are differential 2-forms and differential 3-forms respectively. This is not only to distinguish them from the bivectors and trivectors with boldface, for which we shall use those names, but also because the (dx, dy, dz) and their products belong to a structure called a module, rather than to a structure called a vector space. They are very similar structures, the module being more general (the interested reader may wish to refer to the appendix).

We have already explained how to multiply, say, a differential r-form, α_r, with a differential s-form, β_s. One just juxtaposes the differentials of the coordinates, with a "$\wedge$" symbol placed in between any two of them, and moves the coefficients to the front. The coefficients can be functions, but need not always be so. For instance, if $\alpha_r = 5\, dt$ and $\beta_s = \sin x\, dx \wedge dy \wedge dz$, we have

$$\alpha_r \wedge \beta_s = 5\, dt \wedge (\sin x\, dx \wedge dy \wedge dz) = 5\, \sin x\, dt \wedge dx \wedge dy \wedge dz. \tag{4.5}$$

The reader should understand by now that

$$5\, dt \wedge (sinx\, dx \wedge dy \wedge dz) = -(sinx\, dx \wedge dy \wedge dz) \wedge 5\, dt, \tag{4.6}$$

and that

$$\alpha_r \wedge \beta_s = (-1)^{rs} \beta_s \wedge \alpha_r, \tag{4.7}$$

since moving each of the s differential factors of β_s to the front of α_r yields a total of rs switches of sign. It is also clear that, if α_r and β_s have one or more factors in common, the product is zero (since for all i we have $dx^i \wedge dx^i = 0$). The product is a differential $(r+s)$-*form*, which can be zero in particular. If α_r is a sum of differential r-forms and β_s is a sum of differential s-forms, their product is defined by using the distributive property of the exterior product. One multiplies each of the terms in α_r by each of the terms in β_s without changing the order of the factors, as we did with the tangent multivectors.

It follows from Eq. (4.2) that α_r and β_s commute except if both r and s are odd numbers. In turn, it follows that the exterior product of a differential form of odd grade by itself is zero since we now have:

$$\alpha_r \wedge \alpha_r = (-1)^{rr} \alpha_r \wedge \alpha_r = -\alpha_r \wedge \alpha_r = 0 \qquad \text{for odd } r. \tag{4.8}$$

We can also have products of inhomogeneous differential forms, but they will rarely be written out explicitly in this book.

5 Integrands as Function(al)s

We shall use the term function(al) instead of function or functional. functional and functions are distinguish by their domains. A functional is a function whose domain is itself a set of functions. Hence, if we use the term function for a functional, it is not incorrect; it is simply less precise. However, the term itself may be a little bit more intimidating, and may prompt one to think of what the domain is. Of course, it is extremely important to know what the domain is, but we place a higher priority in that the reader should not be intimidated by any such issue until after becoming comfortable with the use of the term.

As may have already been noticed, we use the term integrand to refer to everything that goes under the integral sign. For instance, the integrand in

$$\int_a^b f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz \tag{5.1}$$

is

$$f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz, \tag{5.2}$$

and not just the coefficients $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ of the differential 1-forms dx, dy and dz.

The expression (5.2) is the symbol for a scalar-valued differential form. In three dimensions, the symbol for the most general vector-valued differential form would be, in abbreviated form,

$$\sum_{i=1}^{i=3} \sum_{j=1}^{j=3} a_i^{\cdot j}(x) dx^i \mathbf{a}_j. \tag{5.3}$$

It is not uncommon in the literature to write simply x for the multivariable argument. Thus $a_i^{\cdot j}(x)$ means in this case $a_i^{\cdot j}(x^1, x^2, x^3)$, and $a_i^{\cdot j}(x, y, z)$ in particular.

Let us introduce the notation known as the Kronecker delta, written δ_{ij}, or δ^{ij}, or $\delta_i^{\cdot j}$, or $\delta^i_{\cdot j}$. They all take the values of zero and one for $i \neq j$ and $i = j$, respectively. In general, the nature of the expressions where these symbols are used will speak of which one to use. Readers should be warned, however, that, in structures endowed with a dot product, the coefficients g^{ij}, $g^i_{\cdot j}$, $g_i^{\cdot j}$ and g_{ij} of the metric supersede the δ_{ij}, or δ^{ij}, or $\delta_i^{\cdot j}$, or $\delta^i_{\cdot j}$. It happens that $g^i_{\cdot j}$ always equals $\delta^i_{\cdot j}$and $g_i^{\cdot j}$, always equals $\delta_i^{\cdot j}$; corresponding statements do not hold for the other symbols, consistently with the fact that g^{ij} and g_{ij} need not take only the values of one and zero.

We may rewrite $dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ as $\delta_i^{\cdot j} dx^i \mathbf{a}_j$, where we sum over repeated indices. In other words, $\delta_i^{\cdot j} dx^i \mathbf{a}_j$ means

$$\sum_{i=1}^{i=3} \sum_{j=1}^{j=3} \delta_i^{\cdot j}(x) dx^i \mathbf{a}_j. \tag{5.4}$$

where dx^i now stands for (dx, dy, dz) and where $\mathbf{a}_j$ stands for $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. This suppressing of the summation signs is known as Einstein's convention. The presence of the summation sign is understood whenever an index is repeated, once as a subscript and a second time as a superscript in factors in the same term. In $\delta_i^{\cdot j} dx^i \mathbf{a}_j$ there are two repeated indices, corresponding to the two summation signs in (5.4). It is clear that $dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$ can also be written as $dx^i \mathbf{a}_i$ under the same convention. Although $dx^i \mathbf{a}_i$ is simpler than $\delta_i^{\cdot j} dx^i \mathbf{a}_j$, the latter expression helps us emphasize that $dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$ is a very special case of $a_i^{\cdot j}(x)\, dx^i \mathbf{a}_j$, namely the particular case where the matrix of the coefficients $a_i^{\cdot j}$ is the unit matrix. We already saw at the end of Section 8 of Chapter I the vector-valued 1-form $dx^i \mathbf{a}_i$ under the integral sign.

We now consider double integrals as evaluations (on surfaces) of differential 2-forms. We call evaluation of a differential 2-form what most but not all authors call integration of a differential 2-form; they use the term evaluation of a differential 2-form when one thinks of the symbol in question as representing an antisymmetric bilinear function(al) on two vectors. But the exterior calculus is the same regardless of whether one views the by now familiar symbols for $r-$forms as representing function(al)s of $r-$surfaces or of sets of vectors.

Expressions such as:

$$(a) \quad \iint_R f(x, y)\, dxdy, \qquad (b) \quad \int_a^b dy \int_{x_1(y)}^{x_2(y)} f(x, y)\, dx, \tag{5.5}$$

will be viewed as the evaluation in the domain R, which is more explicit in (b) than in (a). More precisely, we think of R as a domain of points in a space. We think of the limits in the second expression as defining the limits in a parametrization of that set of points. We shall use R to represent (in this case) a certain surface, meaning a continuous two-dimensional set of points on a given manifold, and whose existence is independent of any coordinates used to represent it. When preparing the expression (a) for integration, as in (b), we shall by then have specified the limits in terms of the coordinates used in the integrand

$$f(x,y)\,dx \wedge dy. \tag{5.6}$$

In this way, we achieve more flexibility of notation, though at the price of abusing it.

Expressions (5.5) correspond to

$$(a) \quad \int_R f(t)\,dt \qquad (b) \quad \int_a^b f(t)\,dt \tag{5.7}$$

in the calculus of one variable, where R now denotes an interval in the (real) line There is an essential difference, however. Under general conditions, $f(t)\,dt$ always has primitives under very general conditions, $F(t)+C$ (i.e. functions whose differential is the given $f dt$). In dimensions $n \geq 1$, however, the concept of primitive of an $r-$form does not exist in general (when it exists, it is an $(r-1)-$form). For this reason, there is not a theorem in the multivariable calculus that could be considered as a generalization of

$$\int_a^b f(t)\,dt = F(b) - F(a). \tag{5.8}$$

Thus, in general, the expression $\iint f(x,y)\,dxdy$, which would correspond to $f(t)dt$, does not represent something that we can compute starting with $f(x,y)$, in parallel to the computation of $F(t)+C$ starting with $f(t)$. In other words, there are not in general indefinite integrals in the multivariable calculus. However, the expression (a) in (5.5) will be computable in general, along the lines of (b) in (5.5) and thanks to the fact that $f(t)\,dt$ always has primitives.

Let us perform the conceptual transition from $f(x)\,dx$ to its generalization

$$f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz. \tag{5.9}$$

The evaluation point of this function is now a curve rather than an interval. This evaluation is equivalent to performing the integration given as (b) in (5.7) However, unlike $f(x)\,dx$, which has primitives whenever $f(x)$ is an integrable function (this is an extremely weak constraint for functions of one variable), the differential form (5.9) does not have a primitive in general. In order to understand this, we recall that evaluating (5.9) on a curve means integrating the pullback of the differential form to the curve, i.e.

$$\int_{t_1}^{t_2} \sigma[f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz], \tag{5.10}$$

where t_1 and t_2 are the values of the parameter of the curve at the end points of the same. More explicitly, the evaluation becomes

$$\int_{t_1}^{t_2} \{[f[x(t),y(t),z(t)]x' + g[x(t),y(t),z(t)]y' + h[x(t),y(t),z(t)]z'\}dt. \tag{5.11}$$

This yields a number when the functions f, g, h, x, y and z are given and the computations are performed. As we saw in section 8 of Chapter I, this number depends not only on the differential form but on the curve itself. But even if the result were independent of the curve (there are differential 1-forms which possess this property), the expression (5.11) does not yield the sought dependence on (x, y, z), but rather a function of t_1 and t_2. We retake this subject in the next section.

Finally, the expressions $f(t)\,dt$ and $f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$ live to be evaluated on curves. The differential as the increment of a function between two very close points of coordinates (x_0, y_0, z_0) and (x, y, z) is an approximation to the integration on a curve between two points. It is represented by $f\,(x_0, y_0, z_0)\,\Delta x + g\,(x_0, y_0, z_0)\,\Delta y + h\,(x_0, y_0, z_0)\,\Delta z$.

6 Differential 1-Forms and Potential Functions

We now deal with the issue of obtaining primitives for differential $1-$forms when they exist. Such an obtaining will be called in this book the integration of the differential form.

Since the differentials of the coordinates, say dx, dy, dz, dr, $d\rho$, $d\theta$, $d\phi$, *etc* are particular cases of differential 1-forms, the considerations of the previous section apply to them in particular. Now, it is clear that, in order to perform the evaluation of these differential forms, we do not need to perform their pullback to any curve. The result is independent of curve and the evaluation yields

$$\int_{x_1}^{x_2} dx = x_2 - x_1 \tag{6.1}$$

and, similarly, $y_2 - y_1$, $z_2 - z_1$, $r_2 - r_1$, *etc.* for the other differential forms. If we let the end point of the curve vary and if we take arbitrary curves from the same point, these results can then be written as $(x - x_1)$, $(y - y_1)$, $(z - z_1)$, $(r - r_1)$, $(\rho - \rho_1)$, $(\theta - \theta_1)$ and $(\phi - \phi_1)$. They do not depend on path between the points of coordinates (x_1, y_1, z_1) and (x, y, z) , etc.

Going one step further, the evaluation of the 1-form $\phi\,d\rho + \rho\,d\phi$ between a fixed point of coordinates (ρ_1, ϕ_1) and an arbitrary point of coordinates (ρ, ϕ) yields the path-independent result $\rho\phi - \rho_1\phi_1$. We may say that $\phi\,d\rho + \rho\,d\phi$ is the differential of $\rho\phi - \rho_1\phi_1$, also of $\rho\phi$. On the other hand, the evaluation of an expression as simple as $\rho d\phi$ does depend on the details of the path; if we evaluate $\rho\,d\phi$ along curves between the same end points represented by, say, the pairs of coordinates (ρ_1, ϕ_1) and (ρ_2, ϕ_2), we shall obtain in general different numbers for different curves. The integration is, therefore, not a function of the coordinates of the boundary points of the curve. This is not due to the fact that we are dealing with non-rectilinear coordinates, as we already saw with an example in section 8 of Chapter I.

In the vector calculus, the issue of whether the evaluation (on curves, that is) of a given differential 1-form gives a result which only depends on the end points of the curve is, in disguise, the problem of whether a vector field has a potential function. In the vector-calculus, one learns that a necessary and sufficient condition for the existence of a potential function φ of a vector field

$$\mathbf{F} = f(x,y,z)\,\mathbf{i} + g(x,y,z)\,\mathbf{j} + h(x,y,z)\,\mathbf{k} \tag{6.2}$$

−within regions whose nature we need not specify at this point− is that the curl of $\mathbf{F}$ be zero. In terms of components, this necessary and sufficient condition can be stated as:

$$f_{,y} = g_{,x}, \qquad f_{,z} = h_{,x}, \qquad g_{,z} = h_{,y}, \tag{6.3}$$

where these symbols, standard in the literature, mean

$$f_{,y} \equiv \frac{\partial f}{\partial y}, \qquad g_{,x} \equiv \frac{\partial g}{\partial x}, \qquad etc. \tag{6.4}$$

(This new notation for the partial derivative comes in very helpful when we encounter more complicated expressions and will be used increasingly often). If and only if the conditions (6.4) are satisfied, a function $\varphi(x,y,z)$, called a potential function, exists such that $\nabla\varphi = \mathbf{F}$ (in physics, one calls potential function the negative of φ). It then follows that

$$\int_Q^P f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz = \varphi(P) - \varphi(Q), \tag{6.5}$$

where the arguments P and Q of the function φ mean here the Cartesian coordinates of these points. The actual finding of φ is part of the contents of the vector calculus and will not be revisited until chapter VII. The important point that we have made is that in this approach to the problem of integration of line integrals, the related issues of existence of a potential function of a vector field and of independence of path of the integral

$$\int_Q^P f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz \tag{6.6}$$

for given end points have been tied to the vanishing of the curl of the original vector field, which is another vector field. And yet, the vector field and its curl, also a vector field, are an unnecessary distraction, not needed for solving the problem of when the integral (6.6) is or is not path dependent. They also are a potential source of confusion as we now show.

Suppose that we are solving some mathematical, physical or engineering problem where, because of its cylindrical symmetry, we use either polar coordinates (in two dimensions) or cylindrical coordinates (in three dimensions). Suppose further that we then encounter the integral

$$\int_Q^P \rho\, d\phi. \tag{6.7}$$

Does it depend on path for given end points? Since the vector calculus teaches us that consideration of the curl must be involved, one would be led to trying to solve this problem in one of the following two cumbersome ways.

First way:

- view this as a problem in three dimensions, where curls live,
- convert $\rho\, d\phi$ to Cartesian coordinates, namely

$$\int_{Q'}^{P'} \sqrt{x^2+y^2}\; d\left(\arctan\frac{y}{x}\right), \tag{6.8}$$

  where Q' and P' are used to emphasize the new limits, required by the change of coordinates, of actually the same points Q and P,
- after differentiating $\arctan\frac{y}{x}$, read from the expression obtained the vector field $\mathbf{F}_1$ such that $\mathbf{F}_1.d\mathbf{r}$ equals $\sqrt{x^2+y^2}\; d\left(\arctan\frac{y}{x}\right)$,
- check whether this vector field complies with the necessary and sufficient condition (6.4) and, if it does, proceed to obtain the function φ_1 whose curl is $\mathbf{F}_1$,
- perform in φ_1 a change of coordinates back to $(\rho,\, \phi)$ to obtain $\varphi_2(\rho,\, \phi)$, i.e. the pullback $\sigma\varphi_1$ by the coordinate transformation, so that one finally gets

$$\int_Q^P \rho\, d\phi = \varphi_2(P) - \varphi_2(Q), \tag{6.9}$$

  where Q and P denote now the pairs of coordinates $(\rho,\, \phi)$ at each of the two end points

Second way:

- by inspection, identify in terms of vector bases associated with cylindrical coordinates what is the vector field $\mathbf{F}_2$ that yields $\rho\, d\phi$ through the scalar multiplication $\mathbf{F}_2.(\sigma d\mathbf{r})$; where $\sigma d\mathbf{r}$ is $d\mathbf{r}$ in terms of cylindrical coordinates (i.e. the pullback to these coordinates of the $d\mathbf{r}$ in terms of Cartesian coordinates),
- develop the formula for the curl in cylindrical coordinates,
- check whether the curl of $\mathbf{F}_2$ is zero, etc.

The first of these two ways is simply laborious but straightforward. The second way requires learning about curls in cylindrical coordinates, which are very cumbersome in the extension of the vector calculus.

Let us now forget about vector fields and their curls and look at the issue of path independence from a new perspective. If a function $\varphi_2(\rho, \phi)$ exists such that $d\varphi_2(\rho,\phi) = \rho\, d\phi$, then $d\varphi_2(x,y) = x\, dy$. So, instead of finding whether the integral (6.7) depends on path, we need to find whether the integral

$$\int_Q^P x\, dy, \tag{6.10}$$

does, the limits being the same numbers as in (6.7), not as in (6.8). To be precise, (6.7) and (6.8) are the same integral with the same numbers in the positions for the limits; we are dealing now simply with a change of notation of the coordinates, not with a coordinate change. Although we could go back to the vector calculus to continue solving the problem of integrating (6.10), we do not advocate returning to such an inappropriate perspective. We continue to provide the right one, though relying when helpful on what we know from the vector calculus, after removing the clutter about vector fields and their curls. We return to notation for cylindrical coordinates instead of Cartesian coordinates because we want to emphasize the point that the geometric meaning of the variables is irrelevant. We still use cylindrical coordinates rather than arbitrary ones for continuity purposes, at a time when we have not yet defined arbitrary coordinate systems. The symbols (ρ,ϕ,z) in the example under discussion can be though of as representing an arbitrary system of coordinates, and not just the cylindrical coordinate system.

As we prove in a later chapter, the differential 1-form

$$m(\rho,\phi,z)d\rho + n(\rho,\phi,z)d\phi + p(\rho,\phi,z)dz, \tag{6.11}$$

is the differential of some function $\varphi(\rho,\phi,z)$,

$$d\varphi(\rho,\phi,z) = m(\rho,\phi,z)dr + n(\rho,\phi,z)d\phi + p(\rho,\phi,z)dz, \tag{6.12}$$

if and only if:

$$p_{,\phi} = n_{,z}, \qquad m_{,z} = p_{,\rho}\,, \qquad m_{,\phi} = n_{,\rho}. \tag{6.13}$$

The differential 1-form (6.11) is then called (locally) integrable, φ being one of its primitives. It is known as the potential function in the vector calculus. The path-independent result

$$\int_Q^P m(\rho,\phi,z)d\rho + n(\rho,\phi,z)d\phi + p(\rho,\phi,z)dz = F(P) - F(Q) \tag{6.14}$$

follows. The arguments P and Q in $F(P)$ and $F(Q)$ represent the cylindrical coordinates of the end points of the curve (or of whatever coordinate system is meant by (ρ,ϕ,z)). Returning to the example of expression (6.7), it is clear that the condition, $m_{,\phi} = n_{,r}$ is not satisfied since $m_{,\phi}$ is 0 and $n_{,r}$ is 1.

Exercise 9 *Show that no function exists whose differential is* $\sin\theta\, d\phi$.

The condition (6.13) for path independence, which we rewrite as

$$p_{,\phi} - n_{,z} = 0, \qquad m_{,z} - p_{,\rho} = 0, \qquad m_{,\phi} - n_{,\rho} = 0, \tag{6.15}$$

will later take a still easier form to remember. On the other hand, if we had used the curl of the vector calculus in cylindrical coordinates, we would get complicated expressions. After equating its components to zero and removing the clutter from the equations, we would end up with the simple equations (6.15) in a long winded way.

To summarize, potential functions should not be viewed as pertaining to vector fields but to differential 1-forms, and are then called primitives. Consistently with the economy of thought that we have just discussed, it is worth learning to think in terms of not only "vector fields of force" (given the thinking habits in which we have been trained) but simultaneously in terms of the differential 1-form of work. This would be given by the integrand in (6.6) if (f, g, h) were the components in Cartesian coordinates of the force field, and by (6.11) in terms of cylindrical or even arbitrary coordinates, as the case may be.

Finally, let us repeat that, when a differential 1-form is integrable (i.e. has a primitive), the obtaining of the primitive(s) will be referred to as *integrating the differential* 1*-form* rather than *evaluating it* (on a curve). Later in the chapter, we shall contrast this with the view of the same symbols as representing antisymmetric multilinear functions of vectors.

7 Differential r-Forms, r>1

In the same way as a differential 1-form is the integrand of a line integral, differential 2-forms are integrands of double and surface integrals, now considered as function(al)s of surfaces. We now explain this statement.

In dealing with surface integrals, one encounters sums of double integrals like

$$\iint_R A(x,y,z)dxdy + B(x,y,z)dydz + C(x,y,z)dzdx \tag{7.1}$$

on some surface $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ for some rectangular domain $D = [a \leq u \leq b, c \leq v \leq d]$ (One can always achieve that the domain be rectangular by a change of parameters). In the exterior calculus, this integral is viewed as the evaluation of the differential 2-form

$$A(x,y,z)dx \wedge dy + B(x,y,z)dy \wedge dz + C(x,y,z)dz \wedge dx \tag{7.2}$$

on the given surface. In general coordinates in 3-space, a surface integral would be represented by the differential 2-form

$$A(x)dx^1 \wedge dx^2 + B(x)dx^2 \wedge dx^3 + C(x)dx^3 \wedge dx^1, \tag{7.3}$$

where the argument x represents all three curvilinear coordinates.

A differential form or integrand of the type

$$B_1(x)dt \wedge dx^1 + B_2(x)dt \wedge dx^2 + B_3(x)dt \wedge dx^3 \tag{7.4}$$

will also correspond to a surface integral but in, at least, four dimensions. The most general integrand for a surface integral in four dimensions and arbitrary coordinates (one of them assumed to be time and represented as x^0) is of the form

$$\begin{aligned} F_1 dx^0 \wedge dx^1 + F_2 dx^0 \wedge dx^2 + F_3 dx^0 \wedge dx^3 + \\ + F_4 dx^2 \wedge dx^3 + F_5 dx^3 \wedge dx^1 + F_6 dx^1 \wedge dx^2. \end{aligned} \tag{7.5}$$

We shall practice with this 2-form later in the chapter, and shall deal at length in later chapters with 3-forms such as $dx \wedge dy \wedge dz$ and $r^2 \sin\theta \, dr \wedge d\theta \wedge d\phi$. They are functions of 3−dimensional surfaces, also to be denoted as 3−surfaces and 3−hypersurfaces. Again, they are evaluated by performing their integration on those regions. Let us proceed to motivate a most important 3-form in 4-dimensions.

Suppose that we have a distribution of moving charge represented by the density $\rho(t,\ x, y, z)$. Let the functions $u^i(t, x, y, z)$ denote the velocity field of the charge distribution. The integrand of the integral to be performed to obtain the total charge of the distribution is

$$\rho(t, x, y, z)\, dx \wedge dy \wedge dz, \tag{7.6}$$

where t is considered as constant in performing the integral. Suppose that from this differential form we obtained this other one:

$$\rho(t, x, y, z)\,(dx - u^x dt) \wedge (dy - u^y dt) \wedge (dz - u^z dt)\,. \tag{7.7}$$

We shall see in Chapter VI that these two forms are related in a very deep way. The exercise that follows is motivated by a result in that chapter.

Exercise 10 *Using the distributive property of the exterior product, show that the differential form* (7.7) *can also be written as:*

$$\begin{aligned} j = \rho\, dx \wedge dy \wedge dz - \rho\, u^x dt \wedge dy \wedge dz - \\ - \rho\, u^y dt \wedge dz \wedge dx - \rho\, u^z dt \wedge dx \wedge dy. \end{aligned} \tag{7.8}$$

In four dimensions, we also have differential 4-forms, all of them being f the form $f(t, x, y, z) dt \wedge dx \wedge dy \wedge dz$, which is often written as $f\left(x^0, x^1, x^2, x^3\right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ or $f(x^\mu) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ or still $f(x) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. Recall the Einstein convention, which for 1-forms in three dimensions reads

$$\sum_{i=1}^{3} A_i x^i = A_1 x^1 + A_2 x^2 + A_3 x^3 = A_i x^i. \tag{7.9}$$

In general, the A_i's are functions of the coordinates. The symbol for exterior product and the presence of other factors do not impede the use of Einstein's convention, as in

$$E_i dt \wedge dx^i \equiv E_1 dt \wedge dx^1 + E_2 dt \wedge dx^2 + E_3 dt \wedge dx^3. \tag{7.10}$$

Slightly more subtle is the notation $B_i dx^j \wedge dx^k$, to which we shall refer as *cyclic sum notation convention.* The convention is made clear by the definition

$$B_i dx^j \wedge dx^k \equiv B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2. \tag{7.11}$$

We shall denote the right hand side of (7.11) as $B_i dx^j \wedge dx^k$, meaning that we sum over the following three cyclic permutations: $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. When we do not want to have this summation, we choose three indices other than (i, j, k). The last convention is in particular a "*missing index*" notational convention, in the sense that the coefficient of $dx^j \wedge dx^k$ carries the missing index i. This convention is easily extended to $(n-1)-$forms in n dimensions,

Here is an example that combines the Einstein and the cyclic sum conventions:

$$F \equiv E_l (x^\mu) dx^0 \wedge dx^l - B_i (x^\mu) dx^j \wedge dx^k, \tag{7.12}$$

which, when expanded, reads

$$\begin{aligned} F \equiv\ & E_1 dx^0 \wedge dx^1 + E_2 dx^0 \wedge dx^2 + E_3 dx^0 \wedge dx^3 \\ & - B_1 dx^2 \wedge dx^3 - B_2 dx^3 \wedge dx^1 - B_3 dx^1 \wedge dx^2. \end{aligned} \tag{7.13}$$

An electromagnetic field is precisely written in this form when using the exterior calculus in electrodynamics. The form 7.12 should be memorized.

Exercise 11 *The differential form* $G \equiv B_i (x^\mu) dx^0 \wedge dx^i + E_i (x^\mu) dx^j \wedge dx^k$ *also is relevant in Maxwell's theory. Compare* F *and* G *and make yourself a rule to remember* G *(it helps to do so by comparison with the already memorized* F*).*

Exercise 12 *Consider in the plane the following Coulomb field:* $E = (q/r^2) dt \wedge dr$*. Express it in Cartesian coordinates Differentiate with respect to spacetime coordinates a special Lorentz transformation along the* x *axis, and substitute the differentials in the expression so obtained for the given Coulomb field. Read the magnetic field.*

Exercise 13 *Change differentials in Eq. (7.13) as pertains to a special Lorentz transformation. Read from the differentiated expression the components of the electromagnetic field in the primed frame in terms of the components of the same in the original frame. Your result for the previous exercise should be contained in this one. Verify it.*

It is clear that $dx \wedge dy \wedge dz$, $dt \wedge dy \wedge dz$, $dt \wedge dz \wedge dx$ and $dt \wedge dx \wedge dy$ constitute a basis of 3-forms in 4 dimensions (there is no other 3-form independent of these, and all four are independent of each other!). The notation for a general 3-form in four dimensions becomes clear from

$$\begin{aligned} j =\ & j_{123} dx \wedge dy \wedge dz \\ & + j_{023} dt \wedge dy \wedge dz + j_{031} dt \wedge dz \wedge dx + j_{012} dt \wedge dx \wedge dy. \end{aligned} \tag{7.14}$$

Comparing Eqs. (7.8) and (7.14), one gets the values of these components as $j_{123} = \rho$, $j_{023} = -\rho u_x$, $j_{031} = -\rho u_y$, $j_{012} = -\rho u_z$. Readers who know relativity will identify these four quantities as the (covariant) components of the relativistic 4-vector current. And yet, we have not used at all the Lorentz transformations in getting these components. Whereas the 4-vector current pertains only to relativity, the current 3-form j has universal validity; it is a valid concept in any theory whatsoever, including classical mechanics, because so are Eq. (7.6) and the rule used in the latter to obtain (7.8).

Notice that in (7.13) and (7.14) we have all the combinations of indices, but only they; hence, we do not have all the permutations. This is due to the antisymmetry of the exterior product. It is clear that $E_1 dx^0 \wedge dx^1$can be written as $Adx^0 \wedge dx^1 + Bx^1 \wedge dx^0$, since the second term can be immersed in the first by virtue of $dx^0 \wedge dx^1 = -dx^1 \wedge dx^0$. However, the equation $E_1 dx^0 \wedge dx^1 = Adx^0 \wedge dx^1 + Bx^1 \wedge dx^0$ does not determine A and B. This equation has an infinite number of solutions for A and B, if we do not impose some additional condition on these coefficients. Let us leave this for another section in order not to be overwhelmed at this point by matters of notation.

A final remark. We could have introduced differential 1-forms (and thus differential r-forms by exterior product) in a coordinate independent way, namely as $\sum_i f_i dg^i$ where the summation can be extended to an arbitrarily large number of terms, and where f, g and h are functions of several variables.

8 Exterior Derivatives: Motivation and Definition

At the end of section 16 of chapter 1, we concluded that the theorems of Gauss and Stokes may be viewed as being of the type

$$\int_R d\mu = \int_{\partial R} \mu. \tag{8.1}$$

We now provide the form of d. that makes this possible In section 10, we show that this equation is valid in any number of dimensions for any differential form. For this reason, (8.1) is known as the generalized Stokes theorem, or simply Stokes theorem if there is no possibility for confusion. As we shall see, the operator d has the extraordinary property among others of satisfying

$$d^2 \equiv dd = 0. \tag{8.2}$$

Our approach to d -there are others- will be one of minimum resistance. We shall produce the form of d, and verify that the theorems of Gauss and Stokes can then be written as Eq. (8.1). In the next section, we deal with the correspondence between theorems of the vector calculus and the calculus with differential forms in $\mathbf{E}^3$. Still another section will be used to show the general validity of Eq. (8.1), thus becoming the so called Stokes' theorem. We conclude this part of the chapter with a section on the derivation of some properties of

d. When we said above that we could approach d.in different ways, we meant that, for instance, one could start by postulating the properties or requirements to be satisfied by the sought operator and then showing that it exists, is unique and what form it takes.

The action of the operator d on a differential form μ is called exterior *differentiation*. $d\mu$ is called the exterior *derivative* of μ and, less frequently, the differential of μ. We require d to be linear, i.e.

$$d(\mu + \nu + ...\sigma) \equiv d\mu + d\nu + ... + d\sigma. \tag{8.3}$$

The differential forms $\mu, \nu ... \sigma$ may be of different grades. Property (8.3) reduces our problem to that of defining

$$d\left(f \ dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_r}\right), \tag{8.4}$$

We have used the series of indices $i_1, i_2 ... i_r$ instead of $1, 2...m$ for obvious reasons. The exterior derivative is then defined as:

$$\begin{aligned} & d\left(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r}\right) \equiv \\ & \left(\frac{\partial f}{\partial x^1}\, dx^1 \wedge \cdots \wedge \frac{\partial f}{\partial x^n}\, dx^n\right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}. \end{aligned} \tag{8.5}$$

We now give examples with Cartesian coordinates in E^3. We have

$$\begin{aligned} d\big[M(x,y,z)dx\big] &= \left(\frac{\partial M}{\partial x}\, dx + \frac{\partial M}{\partial y}\, dy + \frac{\partial M}{\partial z}\, dz\right) \wedge dx = \\ &= \frac{\partial M}{\partial y}\, dy \wedge dx + \frac{\partial M}{\partial z}\, dz \wedge dx \end{aligned} \tag{8.6}$$

for differential 1-forms, and

$$\begin{aligned} d\big[M(x,y,z)dy \wedge dz\big] &= \left(\frac{\partial M}{\partial x}\, dx + \frac{\partial M}{\partial y}\, dy + \frac{\partial M}{\partial z}\, dz\right) \wedge dy \wedge dz = \\ &= \frac{\partial M}{\partial x}\, dx \wedge dy \wedge dz \end{aligned} \tag{8.7}$$

for differential 2-forms, and

$$d\big[M(x,y,z)dx \wedge dy \wedge dz\big] = 0, \tag{8.8}$$

for differential 3-forms. The last result is obvious in view of the fact that the exterior derivative should be a 4-form. These are zero in three dimensions.

Exercise 14 *Show that*

$$\begin{aligned} d(Mdx + Ndy + Pdz) = & \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) dy \wedge dz + \\ & + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) dz \wedge dx + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge dy. \end{aligned} \tag{8.9}$$

Observe that the differential form on the right hand side of this equation and in the parenthesis on the left hand side are the integrands of the version of the Stokes theorem given as (*I*.**??**).

Exercise 15 *Show that:*

$$d\big(M\,dy\wedge dz+N\,dz\wedge dx+P\,dx\wedge dy\big)= \\ =\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right)dx\wedge dy\wedge dz. \tag{8.10}$$

Observe that the differential form on the right hand side of this equation and in the parenthesis on the left hand side are the integrands of the version of Gauss' theorem given as (*I*.**??**).

Computations with differential forms can be simplified by inspection of missing factors. Thus we have, for example,

$$d\,(f\,dt\wedge dx\wedge dy)=df\wedge dt\wedge dx\wedge dy=\left(\ldots+\frac{\partial f}{\partial z}\,dz\right)\wedge dt\wedge dx\wedge dy= \\ =-\frac{\partial f}{\partial z}\,dt\wedge dx\wedge dy\wedge dz, \tag{8.11}$$

Let A^x, A^y and A^z be the contravariant components(also covariant components since one raises and lowers indices in $3-$space without change of sign) of the $3-$vector potential. Let $A^0(=\phi)$ and A^i $(=A^x,\ A^y$ and $A^z)$ be the contravariant components of the $4-$vector potential, where we are using the most usual convention about signs in the metric. The covariant components are $A_0=A^0=\phi$, and $A_i=-A^i=-(A^x,\ A^y,\ A^z)=-(A_x,\ A_y,\ A_z)$. Let A be the differential $1-$form

$$A\equiv\phi\,dt-A_x dx-A_y dy-A_z dz. \tag{8.12}$$

Exercise 16 *Compute* dA*. Use the antisymmetry of exterior products to reduce the number of terms to six. Equate the coefficients of the differential* 2*-form so obtained to the coefficients of the differential* 2*-form* (7.13) *and get the equations* $E_i=-\phi_{,i}+A_{i,t}$ *and* $B_k=A_{i,j}-A_{j,i}$*, with cyclic* i,j,k*. Notice that, when the a foregoing relation between components in space and spacetime is taken into account, these equations correspond to the standard equations* $\mathbf{E}=-\mathbf{grad}\ \boldsymbol{\phi}-\mathbf{A}_{,t}$*, and* $\mathbf{B}=$*curl* $\mathbf{A}$.

Readers are advised advised to memorize the definition (8.12) of the potential differential $1-$form and the definition

$$F=dA \tag{8.13}$$

of the electromagnetic field $2-$form.

Exercise 17 *Show that*

$$dF = 0 \tag{8.14}$$

by explicit computation of ddA (obviously dF equals ddA).

The last exercise is contained in the theorem constituted by Eq. (8.2), which we now prove. We write (8.5) with Einstein's convention,

$$d\left(f\ dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_r}\right) = \frac{\partial f}{\partial x^\mu} dx^\mu \wedge dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_r}. \tag{8.15}$$

Further differentiation yields:

$$dd\left(f\ dx^{i_1} \wedge ... \wedge dx^{i_r}\right) = \frac{\partial f}{\partial x^\mu \partial x^\nu} dx^\nu \wedge dx^\mu \wedge dx^{i_1} \wedge ... \wedge dx^{i_r}. \tag{8.16}$$

Since $\partial f/\partial x^\mu \partial x^\nu$ is symmetric and $dx^\nu \wedge dx^\mu$ is antisymmetric with respect to (μ, ν), the right hand side of this equation vanishes, which completes the proof. The reader should be warned, however, of the following fact. One cannot use $dd = 0$ when applied to vector or tensor valued differential forms, i.e. when one or more of the factors in the expression to be differentiated is written in boldface (examples are $dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ and $d\theta \mathbf{e}_\theta + \sin\theta\, d\phi\, \mathbf{e}_\theta$. This is simply because in such cases the definition of the operator d is more general than in (8.7). The issue then arises of what is d in general if not $(\partial/\partial x^\mu)dx^\mu$ and why. This will be discussed in Chapter 5.

9 The Generalized Stokes Theorem in Euclidean 3-Space

We are now ready to show how the generalized Stokes' theorem, Eq. (8.1), encapsulates the usual theorems of Gauss and Stokes. After that, we shall prove the theorem in a general number of dimensions. If we use $Mdx + Ndy + Pdz$ for μ in Eq. (8.1) and take Eq. (8.9) into account, we immediately get the Stokes theorem in Euclidean 3-space, in the form given by Eq. (**??**), which we slightly modify to read:

$$\int_{\partial R} Mdx + Ndy + Pdz = \int \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) dy \wedge dz + \\ + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) dz \wedge dx + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge dy \tag{9.1}$$

Readers will have noticed by now that one does not use multiple integration signs to denote multiple integrals.

Similarly, if we use $M\, dy \wedge dz + N\, dz \wedge dx + P\, dx \wedge dy$ for μ in Eq. (8.1) and take Eq. (8.9) into account, we immediately get the theorem of Gauss in the form given by Eq. (**??**), which we slightly modify to read:

$$\int_{\partial R} M\, dy \wedge dz + N\, dz \wedge dx + P\, dx \wedge dy =$$

$$= \int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) dx \wedge dy \wedge dz. \tag{9.2}$$

The integrands on the left hand side of Eqs. (9.1) and (9.2) are respectively 1-forms and 2-forms. One would expect some similar result when μ is a $0-$form. Indeed, in order to make our point as clearly as possible, let us start by considering a solid. Its boundary is a closed surface. We cut the latter along a closed line to obtain two open surfaces. The boundary of each of these open figures of dimension 2 is a closed figure of dimension 1. We cut a closed curve at a pair of points to obtain two open figures of dimension 1, their boundaries being pairs of points. The difference between dimension zero and other dimensions is that a closed surface and a closed line can be cut in an infinite number of ways into two open surfaces and two open lines, respectively, but a pair of points can be "cut" in only one way into two single points.

We have thus been prompted to see a "pair of points" as a "closed figure of dimension zero". If R is an open figure of dimension one, written here for emphasis as γ, its boundary $\partial\gamma$ is a pair of points (Q, P). We define the evaluation of a function f on a boundary (Q, P) as $f(P) - f(Q)$. The analogous of the Stokes and Gauss theorems when μ is a $0-$form now becomes

$$\int_\gamma df = f(P) - f(Q), \tag{9.3}$$

where we have written $f(P) - f(Q)$ instead of $\int_{\partial\gamma} f$ since we are too used to see some differential in the integrand, which is not the case in $\int_{\partial\gamma} f$. Notice that the integrand on the left hand side of the last equation is not just any $1-$form, but one which happens to be the differential of a function, i.e. of a $0-$form. In the calculus one dimension, this becomes the fundamental theorem of the calculus, which also becomes a particular case of the generalized Stokes theorem since $g(t)dt$ can always be viewed as the differential of some f.

The reason for the minus sign in Eq. (8.16) can be seen by analogy with the graphical representations in figures 2.1 and 2.2.

Exercise 18 *Suppose that in the generalized Stokes theorem, μ were a $3-$form in three dimensions. Assuming the applicability of that theorem to this case also, what does it tell you about the issue of whether volumes in three dimensions have to be considered as open or closed figures?*

We now provide perspective on what differential forms we are actually talking about when we speak of curls and divergences. For arbitrary dimension, the differential or exterior derivative of a scalar-valued function will be called the *gradient 1-form*, since it has the same components as the vector field known

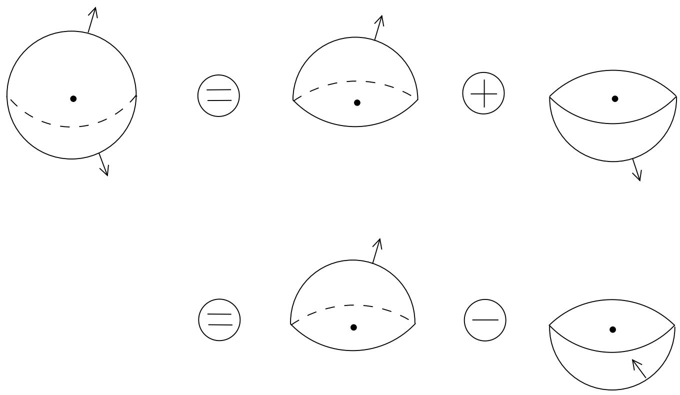

Figure 2.1: *Integral on a closed surface as a difference of integrals on open surfaces with positive projection of the normal on the vertical axis*

as the gradient. If (ρ, ϕ, z) represents some arbitrary system of coordinates in three dimensions, the gradient $1-$form is given by

$$df(\rho, \phi, z) = f_{,\rho} d\rho + f_{,\phi} d\phi + f_{,z} dz \tag{9.4}$$

The exterior derivative of a differential 1-form in any number of dimensions is called the *curl of the* 1*-form.* The exterior derivative of an $(n-1)$-form in an n-dimensional space is called the *divergence of the* $(n-1)$*-form.*

Exercise 19 *Show that the divergence of the particular* 3*-form* j *given by Eq.* (7.8) *is:*

$$dj = \left[\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_x)}{\partial x} + \frac{\partial (\rho u_y)}{\partial y} + \frac{\partial (\rho u_z)}{\partial z}\right] dt \wedge dx \wedge dy \wedge dz. \tag{9.5}$$

Notice that we are not defining the divergence as the coefficient (the bracket in the case of Eq.(9.2)), but as the differential form itself, as we also did with the gradient and the curl. By doing so, we can compute in arbitrary curvilinear coordinates, as well as change these in a straightforward way. This comment about straightforward change of coordinates applies not only to gradients, curls and divergences, but also to the exterior derivative of any differential form. We shall later be able to state the very important result that the pullback of the exterior derivative equals the exterior derivative of the pullback.

Physicists who work with the tensor calculus will find that different books treat these concepts differently. It is a jungle. In order to find one's way in this jungle, one must have a very clear concept of *when one can define what*, i.e. what type of structure is needed for each particular purpose. We have not said what our structure is at this point, since we have not adopted a formal approach in our exposition. We now compensate partially for that by stating

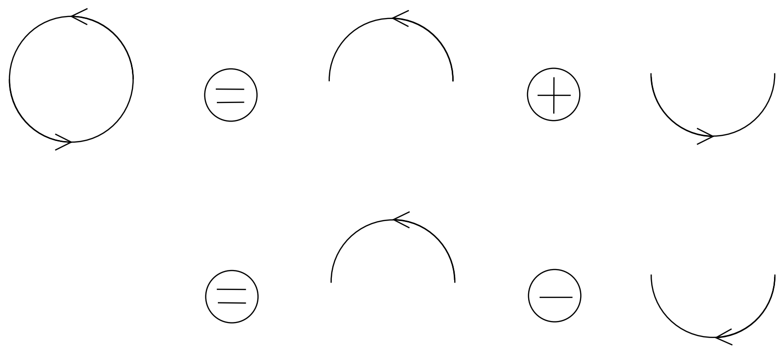

Figure 2.2: *Integral on a closed line as a difference of integral on open lines,both with negative projection of the tangent on the horizontal axis*

that all that which has been said so far about differential forms remains valid in any space which allows for differentiations. Such spaces are called differentiable manifolds. They will be defined in a later chapter. Let us now say what structural elements we have not yet used. We have not yet used tangent vectors and tensors (bold faced quantities) in this chapter, except for illustration of new algebraic concepts and for comparisons with the vector calculus. We have a exterior product of differential forms but we have not introduced a dot product of differential forms. The definitions introduced at the beginning of this section involve just the exterior derivative and are not made invalid by the introduction of additional structure. As for the future, we shall introduce not only a exterior derivative of a differential $r-$form, but also an interior derivative (of grade $r-1$) when we shall endow differential forms with a dot product in some future book, following Kähler. We shall abstain from calling them the curl and the divergence, as some authors do when defining similar concepts in the tensor calculus.

In order to make the point about the amount of structure needed to achieve a certain purpose, let us consider the example of the curl. We saw that its role in the exterior calculus is played by a differential $2-$form. The curl is, however, a quantity of grade one. One can get a differential $1-$form from a $2-$form (from $d\mu$ in this case, where μ is a differential $1-$form) by an operation which requires also an interior product. This operation is implicit in Euclidean space. We have not considered this additional structure in this chapter; it is not needed and we should not speak of curls of vector fields but of exterior derivatives of $1-$forms. The following comments respond to the same perspective of replacing jargon of the vector calculus with terminology from the exterior calculus

- The theorem which states that "a vector field is the gradient of a scalar-valued function if and only if its curl is zero" now reads: "a differential 1-form is the exterior derivative of a scalar-valued function if and only if its exterior derivative is zero". This result is local, in the sense explained in the previous chapter.

- The theorem which states that "a vector field is the curl of another vector field if and only if it is divergenceless" (i.e. it has zero divergence) now reads:"a 2-form is the exterior derivative of some 1-form if and only if its exterior derivative is zero". Again, this is a local result. For general number of dimensions, n, the theorem has greater generality as it reads: "an r-form is the exterior derivative of some $(r-1)$-form if and only if its exterior derivative is zero". The theorem contains as particular cases $r = 1$ (the previous theorem) and $r = n - 1$ (the next theorem).

- The theorem which states that any function is the divergence of a vector field now reads: "any n-form is the exterior derivative of some $(n-1)$-form". Notice that, unlike the previous two cases, we did not have the if and only if condition on the exterior derivative of the first form because it is superfluous, as the exterior derivative of an n-form is always zero. Once again, this is a local result.

All these theorems are particular cases of the one we are about to give. We first need a few definitions. A differential form β is said to be exact if another form α exists such that $\beta = d\alpha$. It is said to be closed if $d\beta = 0$. The announced theorem said: *locally, any closed form is exact.* In general, closed forms are not globally exact, i.e. exact. Hence, we are dealing with local as opposed to global equivalence. Global equivalence means equivalence which holds over the totality of a space without exceptions at any point (A simple example of something that is not global because it fails to hold -at two points of a space- is, say, the assignment of a North direction to the points of S^2; at the very North and South poles, directions N, S, E, W, NE, etc are not defined). Global equivalence of exactness and closeness is impeded whenever the components of a differential form do not have continuous derivatives. The regions of local validity to which the theorem refers are usually simply-connected regions that avoid those points. Sometimes, the theorem is said to apply to star-like regions. some versions of the theorem may be unnecessarily restrictive. We shall deal with that in chapter 7. Let us proceed now with just some general statements.

Relative to β, the form α is said to be a primitive. Like the potential of a vector field in the vector calculus, primitives are largely undefined. Let α_p be some primitive of β, i.e.

$$d\alpha_p = \beta, \tag{9.6}$$

where the subscript "p" stands for particular. Let α_g denote the most general primitive of the same β. It is clear that α_g can always be written as

$$\alpha_g \equiv \alpha_p + \mu, \tag{9.7}$$

where μ is any differential form that satisfies $d\mu = 0$. This is the case since the difference μ between α_p and any other particular primitive α_q satisfies

$$d\mu = d(\alpha_p - \alpha_q) = d\alpha_p - d\alpha_q = \beta - \beta = 0. \tag{9.8}$$

Let us put the essential equations together. Let $d^{-1}\beta$ denote primitive of β. We have, dropping the subscript in α_p,

$$d\beta = 0, \qquad \beta = d(\alpha + \mu), \qquad d^{-1}\beta = \alpha + \mu, \text{ with } d\mu = 0. \tag{9.9}$$

It should be noticed that if β is an r-form, α is an $(r-1)$-form, but μ need not be so, unless we require that the potential be of homogeneous rank.

Exercise 20 *The forms of highest rank in any given space, i.e. the n-forms, always have primitives. Why? Given a 3-form in three dimensions,*

$$\gamma = f(x, y, z) dx \wedge dy \wedge dz,$$

let (F, G, H) be scalar functions (i.e. 0-forms) such that $f = \partial F/\partial x = \partial G/\partial y = \partial H/\partial z$. The differential 2-forms $F\, dy \wedge dz$, $G\, dz \wedge dx$ and $H\, dx \wedge dy$ are all primitives of gamma. Check that the differential forms obtained by subtracting any two of them have zero exterior derivative.

10 Proof of the Generalized Stokes Theorem

We now prove the generalized Stokes theorem

$$\int_{\partial R} \mu = \int_R d\mu, \tag{10.1}$$

which is valid for any dimension, in any coordinate system and with differential forms of any grade.

Let μ be an r-form. It is to be evaluated on "r-surfaces": line ($r = 1$), surface ($r = 2$), volume ($r = 3$), etc. The differential 1-form $d\mu$ is then to be evaluated on, respectively, $(r + 1)$-surfaces: surfaces, volumes, 4-dimensional domains, etc., respectively. We assume, for obvious reasons that $r + 1 \leq n$.

A curve will now be given as

$$x^1 = x^1(u), \quad x^2 = x^2(u), \quad \ldots, \quad x^n = x^n(u). \tag{10.2}$$

Similarly, a surface will be given by

$$x^1 = x^1(u, v), \quad x^2 = x^2(u, v), \quad \ldots, \quad x^n = x^n(u, v). \tag{10.3}$$

An $(r + 1)$-domain will be specified as

$$x^1 = x^1\left(u^0, \ldots u^r\right), \quad x^2 = x^2\left(u^0, \ldots u^r\right), \quad \ldots \quad x^n = x^n\left(u^0, \ldots u^r\right). \tag{10.4}$$

The reason for starting with u^0 does not have to do at all with one of the coordinates being time, which need not be the case. It simply is a matter of "more transparent notation" for the computation that follows.

The differential form μ will be written as:

$$b_0\, du^1 \wedge \cdots \wedge du^r + b_1\, du^0 \wedge du^2 \wedge \cdots \wedge du^r + \ldots + b_r\, du^0 \wedge du^1 \wedge \cdots \wedge du^{r-1}, \tag{10.5}$$

with coefficients $b_0, \ldots, b_r$ which are functions of the $r+1$ parameters u. We proceed to prove that

$$\int_R d\big[b_0\left(u^0 \ldots u^r\right) \wedge du^1 \wedge \cdots \wedge du^r\big] = \int_{\partial R} b_0\, du^1 \wedge \cdots \wedge du^r, \tag{10.6}$$

where R is an $(r+1)$-surface in an $(r+1)$-dimensional "parameter space" (these are coordinates after all). An equivalent form of the last equation is

$$\int_R \frac{\partial b_0}{\partial u^0}\, du^0 \wedge du^1 \wedge \cdots \wedge du^r = \int_{\partial R} b_0\, du^1 \wedge \cdots \wedge du^r. \tag{10.7}$$

It is at this point that our choice of the subscript zero must have become obvious.

Let us proceed with the proof. The hypersurface ∂R is assumed to be contiguous, i.e. not made of two or more closed hyper-surfaces. If it is not contiguous, the argument applies to each contiguous part, and the theorem applies to the whole domain if it applies to the parts. Even if contiguous, the domain might not be u^0-simple, depending on parametrization. u^0-simple means that the u^0 coordinate lines (or parameter lines, if you will) cut the domain only in either two points, one (double) point or no point at all. If the contiguous domain is not u^0 simple, one divides it into u^0-simple domains and applies the argument to such subdomains. By a familiar argument of the vector calculus, the integration on the boundaries so created does not change the total integration over the boundaries, since they are created in pairs and their contribution to one subdomain cancels with the contribution to a contiguous subdomain.

Let $u^0 = f_2\left(u^1, \ldots, u^r\right)$ and $u^0 = f_1\left(u^1, \ldots, u^r\right)$ be the limits for the two parts of the boundary defined by the integration process in each u^0-simple domain. Let $f_1\left(u^1, \ldots, u^r\right) \leq f_2\left(u^1, \ldots, u^r\right)$. We then have

$$\begin{aligned}
&\int_R d\left(b_0\, du^1 \wedge \cdots \wedge du^r\right) = \int_{\partial R} du^1 \wedge \cdots \wedge du^r \int_{f_1}^{f_2} \frac{\partial b_0}{\partial u^0}\, du^0 = \\
&= \int_{\text{upper}\,\partial R} b_0\left[f_2\left(u^1, \ldots, u^r\right), u^1, \ldots, u^r\right]\, du^1 \wedge \cdots \wedge du^r \\
&- \int_{\text{lower}\,\partial R} b_0\left[f_1\left(u^1, \ldots, u^r\right), u^1, \ldots, u^r\right]\, du^1 \wedge \cdots \wedge du^r
\end{aligned} \tag{10.8}$$

The end points of the intersection of the lines u^0 with the domain constitute the boundary ∂R of R. As a result the last difference of integrals is nothing but

$$\int_{\partial R} b_0\, du^1 \wedge \cdots \wedge du^r. \tag{10.9}$$

The other terms of μ are done analogously. The completion of the proof requires some details about orientation of domains that will be considered in the next section.

It should be clear from the argument that it is immaterial whether the differential form that is being evaluated is given in Cartesian or curvilinear coordinates; integrations do not understand the nature of the coordinates. It is for this reason that we used notation appropriate for arbitrary systems of coordinates.

We return to Stokes' generalized theorem. One of its main uses is that the evaluation of an integral over a domain can be replaced by the evaluation of an integral over its boundary, and the other way around. There will be many examples of this in this book. Another main use has to do with conservation laws; these follow whenever a differential form satisfies $d\mu = 0$

In section 2, we showed that the second derivative of any differential form is zero. This result is related to the fact that the boundary of a boundary is zero. Let then μ be an $r-$form and let R be an $(r+2)-$surface. We have

$$\int_R dd\mu = \int_{\partial R} d\mu = \int_{\partial\partial R} \mu. \tag{10.10}$$

The first and last integrals are zero for different but intimately related reasons.

11 Properties of the Exterior Derivative

We now learn a few properties of exterior derivatives. They are useful in computations. We already saw the most important one, namely $d^2 = 0$. This equation is a first in mathematics, since it equates the square of an operator to zero. Next, let α_r and β_s be an r-form and an s-form respectively. The derivative of $\alpha_r \wedge \beta_s$ satisfies:

$$d\left(\alpha_r \wedge \beta_s\right) = d\alpha_r \wedge \beta_s + (-1)^r \, \alpha_r \wedge d\beta_s. \tag{11.1}$$

The proof of this statement uses the fact that the product $\alpha_r \wedge \beta_s$ is a sum of terms of the form

$$f\, dx^R + g\, dx^S, \tag{11.2}$$

where dx^R and dx^S stand for products of the type $dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ and $dx^{j_1} \wedge \cdots \wedge dx^{j_s}$. We then have

$$\begin{aligned} d\left(f\, dx^R \wedge g\, dx^S\right) &= d(fg)\, dx^R \wedge dx^S = \\ &= g\, df \wedge dx^R \wedge dx^S + f dg \wedge dx^R \wedge dx^S = \\ &= \left(df \wedge dx^R\right) \wedge g\, dx^S + f\, dx^R \wedge (-1)^r dg \wedge dx^S = \\ &= d\left(f\, dx^R\right) \wedge g\, dx^S + (-1)^r f\, dx^R \wedge d\left(g\, dx^S\right). \end{aligned} \tag{11.3}$$

The proof is virtually complete; we leave any remaining details for the reader.

Suppose we had three forms μ_m, ν_n and ρ_r. The following steps should by now be straightforward for the reader:

$$\begin{aligned} & d(\mu \wedge \nu \wedge \rho) = d\big[\mu \wedge (\nu \wedge \rho)\big] = d\mu \wedge \left[\nu \wedge \rho\right] + (-1)^m \mu \wedge d(\nu \wedge \rho) = \\ & = d\mu \wedge \nu \wedge \rho + (-1)^m \mu \wedge \left[d\nu \wedge \rho + (-1)^n \nu \wedge d\rho\right] = \\ & = d\mu \wedge \nu \wedge \rho + (-1)^m \mu \wedge d\nu \wedge \rho + (-1)^{m+n} \mu \wedge \nu \wedge d\rho. \end{aligned} \tag{11.4}$$

We now state for completeness certain equalities about pullbacks. The common theme is that the pullback of the different operations performed with differential forms in one coordinate system equals the operations performed on the pullbacks to another coordinate system. This is a manifestation of the point made that, as we said, differential forms transcend coordinate systems. Let σ denote again the pullback from one coordinate system to another. We have the following properties:

$$\sigma(\mu+\nu) = \sigma\mu + \sigma\nu, \tag{11.5a}$$

$$\sigma(\mu\wedge\nu) = \sigma\mu \wedge \sigma\nu, \tag{11.5b}$$

$$d(\sigma\mu) = \sigma(d\mu). \tag{11.5c}$$

Exercise 21 *Prove equations* (11.5a) *and* (11.5b).

We proceed to prove Eq.(11.5c). We first show that $d\sigma f = \sigma df$, where σ is again the symbol for pull-back under the coordinate transformation

$$\sigma x^i = x^i(y), \tag{11.6}$$

where f is a scalar function of x and where (y)=$(y^1, ..., y^n)$. We, therefore, have:

$$\begin{aligned} d[\sigma f(x)] &= df[x^i(y)] = \left(\sigma\frac{\partial f}{\partial x^i}\right)\frac{\partial x^i}{\partial y^j}dy^j = \\ &= \left(\sigma\frac{\partial f}{\partial x^i}\right)(\sigma dx^i) = \sigma\left(\frac{\partial f}{\partial x^i}dx^i\right) = \sigma df(x) \end{aligned} \tag{11.7}$$

since $\sigma\frac{\partial f}{\partial x^i}$ is $\frac{\partial f}{\partial x^i}$ itself with x^i replaced with $x^i(y)$. Notice customary abuses of notation like writing $d[\sigma f(x)]$ instead of $d[\sigma f]$ (one pulls the function, f, not its evaluation, $f(x)$) and other related abuses. In doing so, we minimize clutter in the intermediate steps in (11.7). The result just obtained,

$$d\sigma f = \sigma df, \tag{11.8a}$$

states in particular that

$$d\sigma x^i = \sigma dx^i, \tag{11.8b}$$

and, if we define dx^R as $dx^1 \wedge dx^2 \wedge ... \wedge dx^r$, it also implies

$$\sigma dx^R = \sigma dx^1 \wedge \sigma dx^2 \wedge ... \wedge \sigma dx^r = d\sigma x^1 \wedge d\sigma x^2 \wedge ... \wedge d\sigma x^r, \tag{11.9}$$

by virtue of (11.5b) and (11.8a). Clearly, from (11.9), we get

$$d\sigma(dx^R) = 0, \tag{11.10}$$

where the parenthesis is not necessary but contributes to immediate understanding.

Because of equation (11.5a), we only need to prove (11.5c) for fdx^R. Equations (11.5b) and (11.8a) together yield

$$\sigma d(fdx^R) = \sigma df \wedge \sigma dx^R = d\sigma f \wedge \sigma dx^R, \tag{11.11}$$

On the other hand, using Eq.(11.10)

$$d\sigma(f dx^I) = d\left[(\sigma f)\sigma dx^R\right] = d\sigma f \wedge \sigma dx^R + 0. \tag{11.12}$$

Comparison of equations (11.11) and (11.12) completes the proof of (11.5c).

A most important implication of (11.5c) is the explicit validity of Stokes' theorem under a change of coordinates,

$$\int_R \sigma d\mu = \int_R d(\sigma\mu) = \int_{\partial R} \sigma\mu, \tag{11.13}$$

where the numerical limits expressing the same physical sets R and ∂R of points of the space have changed in the process. Let us clarify these equalities with the simple example of the differential form $M\, dy \wedge dz + N\, dz \wedge dx + P\, dx \wedge dy$. Its pull-back to spherical coordinates is an expression of the type

$$R\, d\theta \wedge d\phi + S\, d\phi \wedge dr + T\, dr \wedge d\theta, \tag{11.14}$$

whose exterior derivative is

$$\left(\frac{\partial R}{\partial r} + \frac{\partial S}{\partial \theta} + \frac{\partial T}{\partial \phi}\right) dr \wedge d\theta \wedge d\phi. \tag{11.15}$$

Equations (11.13) become in this case:

$$\begin{aligned}
\int_R \sigma\left[d(M dy \wedge dz + N dz \wedge dx + P dx \wedge dy)\right] &= \\
\int_R \left(\frac{\partial R}{\partial r} + \frac{\partial S}{\partial \theta} + \frac{\partial T}{\partial \phi}\right) dr \wedge d\theta \wedge d\phi &= \\
\int_{\partial R} R\, d\theta \wedge d\phi + S\, d\phi \wedge dr + T\, dr \wedge d\theta.
\end{aligned} \tag{11.16}$$

Because one often says that the divergence is a scalar field (it is not), one might be tempted to think incorrectly that $\left(\frac{\partial R}{\partial r} + \frac{\partial S}{\partial \theta} + \frac{\partial T}{\partial \phi}\right)$ is the divergence in spherical coordinates. We rather have

$$\left[\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dx \wedge dy \wedge dz\right] = \left(\frac{\partial R}{\partial r} + \frac{\partial S}{\partial \theta} + \frac{\partial T}{\partial \phi}\right) dr \wedge d\theta \wedge d\phi. \tag{11.17}$$

The coefficients of these expressions are not equal since the pullback of $dx \wedge dy \wedge dz$ is not $dr \wedge d\theta \wedge d\phi$, but rather $r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi$.

The properties (11.5a)-(11.5c) express the fact that differential forms transcend coordinate systems. Consider the following equation:

$$dx = d\rho\, \cos\phi - \rho\, sin\phi\, d\phi. \tag{11.18}$$

It is as valid to say

$$\sigma_1(dx) = d\rho\, \cos\phi - \rho\, sin\phi\, d\phi, \tag{11.19a}$$

as to say

$$dx = \sigma_2(d\rho\,\cos\phi - \rho\, sin\phi\, d\phi), \tag{11.19b}$$

where σ_1 and σ_2 stand for pull backs in opposite directions. Equations (11.19a)-(11.19b) are equally meaningful. None of them is preferred unless accompanying circumstances in a particular problem make one of them so. A differential form is expressed in terms of coordinates, in the same way as coordinates are often used to define curves explicitly. But a curve is about functions whose domain is an interval and whose range is Euclidean space, or some other space; this map can be expressed in different coordinate system. The same can be said of the expressions of differential forms in terms of coordinates, since these are only an instrument. Fortunately, the statement (10.1) of the very important Stokes theorem (we no longer call it *generalized* Stokes theorem) is coordinate free.

12 Issues of Orientation

There are several issues of orientation that concern differential forms or the sypersurfaces on which they are evaluated. One of them has to do with the fact that, unlike a sheet of paper, there are surfaces with just one face, like a Möbius band or a Klein bottle. Dealing with global problems with such surfaces is a challenge, but only a very few mathematicians deal with such objects. They will not be considered in this book.

There is then the following issue. Whereas $f dy \wedge dx = -f dx \wedge dy$, one has

$$\iint_R f dx dy = \iint_R f dy dx. \tag{12.1}$$

Hence, we cannot simply think of $dxdy$ (respectively $dydx$) as $dx \wedge dy$ (respectively $dy \wedge dy$), since we encounter a contradiction in sign. One must, therefore, first consider how issues of orientation are or are not represented when writing expressions such as those on both sides of Eq. (12.1).

For simplicity of exposition, we shall chose the domain of integration to be rectangular; the argument not depending on the domain being so. Although it is seldom if ever explicitly said, the positioning of the limits in the two successive integrations

$$\iint_R f dx dy = \int_c^d dy \int_a^b f dx = \int_a^b dx \int_c^d f dy, \tag{12.2}$$

are chosen so that the integral is positive if f is a positive constant. Suppose now we use differential form notation. We define the convention

$$\iint_R f dx dy \equiv \iint_R f dx \wedge dy = \left(- \iint_R f dy \wedge dx \right). \tag{12.3}$$

To make our point absolutely clear, we write

$$\iint_R f dy \wedge dx = - \int_c^d dy \int_a^b f dx = - \int_a^b dx \int_c^d f dy. \tag{12.4}$$

Consider then the related issue of orientation of coordinate systems, exhibited with the simple example of the following change of coordinates: $x' = y$, $y' = x$. With the symbol for pull-back understood, we have

$$dx \wedge dy = dy' \wedge dx' = -dx' \wedge dy', \tag{12.5}$$

and

$$f(x, y)dx \wedge dy = -f(y', x')dx' \wedge dy'. \tag{12.6}$$

Needless to say that, when integrating, the limits of x' and y' are respectively those of y and x. In manipulations of this type there is the implicit convention that x' is the first coordinate of the new coordinate system. Hence, the integral of $f(x, y)dx \wedge dy$ would be expected to be the same as the integral of $-f(y', x')dx' \wedge dy'$ when the induced limits for (x', y') are taken into account. Thus the evaluation (integral) of (12.6) becomes

$$-\int_c^d dx' \int_a^b f(y', x')dy'. \tag{12.7}$$

Since it is immaterial what symbols we use for the integration variables, the expression (12.7) can be written as

$$-\int_c^d dy \int_a^b f(x, y)dx. \tag{12.8}$$

which is the opposite of the integral with which we started. We have reached a contradiction. This is simply because exterior products are sensitive to the ordering of the factors and integrals in the traditional sense (e.i. (12.1)) are not. The change of orientation under a coordinate transformation is reflected in the negative sign of the coefficient, i.e. of the Jacobian. In the usual coordinate systems, the ordering has been chosen so that the Jacobians in transformations among them are always positive. For this reason, we do not pursue this matter further. Interested readers should look for the key words *even* and *odd differential forms* in sophisticated books on integration.

Exercise 22 *Show that (r, θ, ϕ) and (ρ, ϕ, z) have positive orientation by relating it to the orientation of the (x, y, z) coordinate system. Do not compute Jacobians explicitly. Rather apply the pertinent coordinate transformations to $dx \wedge dy \wedge dz$*

Exercise 23 *Use your favorite visual technique to show the positive orientation of (r, θ, ϕ) and (ρ, ϕ, z).*

Since it is not possible in general to resort to visualizations, the first of the last two exercises is more relevant than the other.

We have discussed the issue of orientation of the coordinate system in connection with actual integrations. The Jacobian emerges. Consider, however, the issue of coordinate transformations of differential $r-$forms where $r < n$, say

$r = 2$ in 3 dimensions. A simple differential 2-form may go into a non simple differential 2-form, like the pull-back of $dx \wedge dy$ to spherical coordinates. There is not a Jacobian here. However, the best way to integrate 2-forms (or $r-$forms in general for $r < n$) is in the parametric form, in which case the issue of the Jacobian in n dimensions reduces to the issue of the Jacobian in $r-$dimensions associated with the change of parametrization. Readers who wish to pursue this issue further should refer to the 4th volume of the authoritative treatise on analysis by L. Schwartz, pp. 83-102 Notice also in Eqs. 13-7 to 13-10 of the previous chapter, that the issue of orientation of the parametrization (or of a change of parametrization from, say, (u, v) to, say, (u', v') defined by $u' = v$, $v' = u$) does not arise. The reason is as follows. Consider the magnitude of the parallelepiped determined by the vectors $(\mathbf{u}, \mathbf{v}, ...\mathbf{w})$. It is given by the absolute value of the determinant of the matrix of the components. We said *absolute value* since the order of the vectors makes a difference in the sign of the determinant. One can "remove" the sign of the determinant as follows. Multiply the matrix by its transpose. The product is the matrix

$$\begin{bmatrix} <\mathbf{u} \mid \mathbf{u}> & <\mathbf{u} \mid \mathbf{v}> & ... & <\mathbf{u} \mid \mathbf{w}> \\ <\mathbf{v} \mid \mathbf{u}> & <\mathbf{v} \mid \mathbf{v}> & ... & <\mathbf{v} \mid \mathbf{w}> \\ ... & ... & ... & ... \\ <\mathbf{w} \mid \mathbf{u}> & <\mathbf{w} \mid \mathbf{v}> & & <\mathbf{w} \mid \mathbf{w}> \end{bmatrix}, \tag{12.9}$$

whose determinant is the positive square root of the volume of the parallepiped. The choice of the right sign (i.e. positive) resides in the choice of the positive square root.

Of particular interest for the physicist are the differential 2-forms that involve dt, usually dx^0. We shall see in Section 3 of Chapter III that for consistency between the standard notation of the equations of electrodynamics in $3-$space and the spacetime version of Maxwell's equations in integral form, which uses exterior products, we have to adopt the convention

$$\iint_R dx^i \wedge dx^0. = \iint_R dx^0 dx^i = \iint_R dx^i dx^0 = -\iint_R dx^i \wedge dx^0. \tag{12.10}$$

This does not impede, however, that when we are not performing the evaluations of the differential forms we use $dx^0 \wedge dx^i$ instead of $dx^i \wedge dx^0$. We do so, for instance, in Eq. (7.13) in order to avoid a sea of negative signs.

For $n = 4$ and $r = 3$, a basis will be constituted by $dx \wedge dy \wedge dz$ and $dt \wedge \left(dx^i \wedge dx^j\right)$, where the $\left(dx^i \wedge dx^j\right)$ are as for $(n = 3, r = 2)$. See Eq. 1.7.14.

13 Subtleties and Notational Conventions

Differential r-forms involve exterior products of 1-forms. These are antisymmetric products. But the coefficients may not be born antisymmetric, say in defining the electromagnetic differential form F as dA:

$$F \equiv dA = A_{\mu,\nu} dx^\nu \wedge dx^\mu. \tag{13.1}$$

For later comparisons, notice the reversed order of the indices in $A_{\mu,\nu}$ relative to $dx^\nu \wedge dx^\mu$. There are 12 non-null terms in (13.1). They are not independent. We reduce that equation to independent terms:

$$F = A_{0,1} dx^1 \wedge dt - A_{1,0} dt \wedge dx^1 + ... = (A_{1,0} - A_{0,1}) dt \wedge dx^1 + ... \tag{13.2}$$

It is in the latest form that we can identify the coefficients of F with those in (7.13). For example

$$E_1 = A_{1,0} - A_{0,1} \ (= -A_{x,0} - \phi_{,x}) \tag{13.3}$$

(Recall that $A_0 = \phi$, $A_i = -A_x, -A_y, -A_z$).

Two ways among many other of writing F in terms of a basis with explicit components in terms of the derivatives of the components of dA are as follows

$$F = \sum_{\mu<\nu} (A_{\mu,\nu} - A_{\nu,\mu}) dx^\nu \wedge dx^\mu = \sum_{\mu>\nu} (A_{\mu,\nu} - A_{\nu,\mu}) dx^\nu \wedge dx^\mu. \tag{13.4}$$

This is equivalent to using just combinations rather than permutations of the indices. Corresponding to 13.4, we can introduce antisymmetric symbols $F_{\nu\mu}$,both, for $\mu > \nu$ and $\nu > \mu$. We then have

$$F = \sum_{\mu<\nu} F_{\nu\mu} dx^\nu \wedge dx^\mu = \sum_{\mu>\nu} F_{\nu\mu} dx^\nu \wedge dx^\mu. \tag{13.5}$$

Notice that $F_{\nu\mu}$ is introduced so that the ordering of the indices is the same as in the basis, $dx^\nu \wedge dx^\mu$, unlike in 13.1, where $A_{\mu,\nu}$ goes with $dx^\nu \wedge dx^\mu$. Of course, thee are more important differences, like the fact that the $A_{\mu,\nu}$'s are not components with respect to a basis of differential 2−forms. It is then clear that $F_{0i} = E_i$, and $F_{ij} = -B_k$.

Still one more way to write F in terms of the elements of a basis (i.e. combinations of indices) is to make the convention that $dx^\nu \wedge dx^\mu$ in parenthesis means precisely that. Thus

$$F = F_{\nu\mu} \, (dx^\nu \wedge dx^\mu) \tag{13.6}$$

is a summation over any six mutually independent differential 2−forms. If, say, F_{0i} is viewed as one of the components of the differential form F, then F_{i0} is not so at the same time. We still maintain some kind of order, however, so that if, for example, we choose one of the terms to be $F_{01} dt \wedge dx$, we do not choose also $F_{20} dy \wedge dt$ but rather $F_{02} dt \wedge dy$). Because both (13.4) and (13.5) express F in terms of bases of differential forms (i.e. only six rather than twelve terms), they imply

$$F_{\nu\mu} = A_{\mu,\nu} - A_{\nu,\mu}. \tag{13.7}$$

We also readily get, combining equations (13.4) and using (13.7),

$$F = \frac{1}{2} (A_{\mu,\nu} - A_{\nu,\mu}) dx^\nu \wedge dx^\mu = \frac{1}{2} F_{\nu\mu} dx^\nu \wedge dx^\mu, \tag{13.8}$$

where we have regained Einstein's convention of summation over repeated indices, as in (13.1), but now with antisymmetric coefficients.

Given that we have a summation over all values of μ and ν in (13.8), one might think that the electromagnetic tensor has components $\frac{1}{2}F_{\mu\nu}$. It is certainly true that the electromagnetic form can be viewed as a 2-tensor (see a later chapter on tensors) with components $\frac{1}{2}F_{\mu\nu}$, where $F_{0i} = E_i$ and $F_{ij} = -B_k$. However, in the tensor formulation of electrodynamics (clearly distinguished from the differential form formulation, which we are reproducing in this book), one gets the right result by using a tensor with components $F_{\mu\nu}$, not $\frac{1}{2}F_{\mu\nu}$. The reason for this difference is as follows. The left hand side of the tensorial equation for the inhomogeneous pair of Maxwell's equations is $F^{\mu\nu}{}_{,\nu}$ in terms of t, x, y, z (the components $F^{\mu\nu}$ are, up to signs, the same ones as the components $F_{\mu\nu}$). Hence, there are on the left hand side three partial derivatives per value of μ for a total of 12 terms which are partial derivatives of the electric and magnetic fields' components, up to minus one factors. On the other hand, when using the exterior calculus of differential forms, the left hand side of the same equations is constituted by dG, where $G = \frac{1}{2}G_{\mu\nu}dx^\nu \wedge dx^\mu$ and ($G_{0i} = B_i$, $G_{ij} = E_k$). The twelve non-null terms in $\frac{1}{2}G_{\mu\nu}dx^\nu \wedge dx^\mu$ carry the coefficient $\frac{1}{2}$. They can be grouped as usual into six terms with coefficient one. Upon exterior differentiation, each of these six terms gives rise to two, for a total of twelve terms. Exterior differentiation doubles the number of terms in this case; partial differentiation does not. One must, therefore, make clear what one means when one says that so and so are the components of the electromagnetic tensor; unless it becomes necessary, it is better not to refer to differential forms as (antisymmetric) tensors, even if they are so depending on how one defines tensors.

In a certain sense, we can already deal in terms of components with issues which have some of the flavor of the decomposition of $2-$tensors into symmetric and antisymmetric parts. Consider the differential form $H = H_{\nu\mu}dx^\nu \wedge dx^\mu$, where $H_{\nu\mu}$ is such that is not born necessarily antisymmetric (see for example Eq. (13.1)). $H_{\mu\nu}$ can always be written as

$$H_{\mu\nu} = A_{\mu\nu} + S_{\mu\nu}, \tag{13.9}$$

where $A_{\mu\nu}$ and $S_{\mu\nu}$ are defined as its respectively antisymmetric and symmetric parts:

$$\begin{aligned} A_{\mu\nu} &= \frac{1}{2}\left(H_{\mu\nu} - H_{\nu\mu}\right), \\ S_{\mu\nu} &= \frac{1}{2}\left(H_{\mu\nu} + H_{\nu\mu}\right). \end{aligned} \tag{13.10}$$

It follows that

$$H = A_{\mu\nu}\, dx^\mu \wedge dx^\nu + S_{\mu\nu}\, dx^\mu \wedge dx^n u = A_{\mu\nu}\, dx^\mu \wedge dx^\nu. \tag{13.11}$$

The symmetric part of $H_{\mu\nu}$ has vanished. We could have obtained H as $H = A'_{\mu\nu} + S'_{\mu\nu}$, i.e. in terms of some other set of coefficients. It is clear then clear that $A_{\mu\nu} = A'_{\mu\nu}$, but that we cannot conclude whether $S_{\mu\nu}$ equals $S'_{\mu\nu}$, as both these two sets of coefficients add nothing to the differential form. In other words, if two forms H and J are said to be equal but their coefficients are not antisymmetric, we can only conclude that

$$H_{\nu\mu} - H_{\mu\nu} = J_{\nu\mu} - J_{\mu\nu}, \tag{13.12}$$

since only the antisymmetric part of the coefficients defines a differential form. The interplay of $dx^\nu \wedge dx^\mu$ with $dx^\mu \wedge dx^\nu$ picks the antisymmetric part of the coefficients and ignores the rest.

For purposes of differentiation in implicit form (i.e. without writing out all the terms explicitly), 2-forms (and r-forms in general) are best written with summation over permutations rather than combinations of the indices. Thus

$$dH = H_{\nu\mu,\lambda} dx^\lambda \wedge dx^\nu \wedge dx^\mu. \tag{13.13}$$

If the 2−form H is a exterior derivative, like F is, we would have:

$$dF = A_{\mu,\nu,\lambda} dx^\lambda \wedge dx^\nu \wedge dx^\mu = \frac{1}{2} F_{\nu\mu,\lambda} dx^\lambda \wedge dx^\nu \wedge dx^\mu. \tag{13.14}$$

We would first reduce terms on the right hand side of (13.13) and (13.14) in order to read *the* coefficients (i.e. relative to a basis) of dH and dF. In general, dH is not zero but dF is. Indeed, the symmetry of $A_{\mu,\nu,\lambda}$ in (ν, λ) combines with the antisymmetry of $dx^\lambda \wedge dx^\nu$ to cause dF to vanish identically.

Differential r-forms of higher grade cannot be split into a completely symmetric and a completely antisymmetric part. It is still possible, however, to define complete antisymmetry, i.e. antisymmetry with respect to any pair of indices. In the same way as antisymmetry of the products of the differentials picks the antisymmetric part $A_{\mu\nu}$ of $H_{\mu\nu}$, it also picks the completely antisymmetric part of the coefficients of differential forms of higher grade. The next three exercises together exemplify a variety of circumstances with respect to the issue of antisymmetry.

Exercise 24 *Let G be a 2-form $G_{lm}(x^1, x^2, x^3) dx^l \wedge dx^m$ in three dimensions. Show that*

$$\begin{aligned} dG = \big[\, &(G_{12,3} - G_{21,3}) + (G_{23,1} - G_{32,1}) + \\ &+ (G_{31,2} - G_{13,2}) \,\big] dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \tag{13.15}$$

and that the sole coefficient of this differential form is completely antisymmetric, i.e. antisymmetric with respect to the exchange of any pair of indices. Show further that if G_{lm} is antisymmetric (13.10) *reduces to*

$$dG = 2\,(G_{12,3} + G_{23.1} + G_{31,2})\, dx^1 \wedge dx^2 \wedge dx^3. \tag{13.16}$$

Exercise 25 *Show that*

$$A_{123} dx^1 \wedge dx^2 \wedge dx^3 = \frac{1}{3!} A_{lmp} dx^l \wedge dx^m \wedge dx^p, \tag{13.17}$$

if A_{lmp} is defined to be completely antisymmetric.

Exercise 26 *Let G be defined as*

$$G = \frac{1}{2} G_{\lambda\mu}\left(t, x^1, x^2, x^3\right) dx^\lambda \wedge dx^\mu, \quad (\lambda, \mu = 0, 1, 2, 3). \tag{13.18}$$

Show that the exterior derivative $\frac{1}{2}G_{\lambda\mu,\nu}dx^\nu \wedge dx^\lambda \wedge dx^\mu$ (equivalently $\frac{1}{2}G_{\lambda\mu,\nu}dx^\lambda \wedge dx^\mu \wedge dx^\nu$) can be written as

$$dG = \frac{1}{2}\left(G_{\lambda\mu,\nu} + G_{\mu\nu,\lambda} + G_{\nu\lambda,\mu} - G_{\mu\lambda,\nu} - G_{\lambda\nu,\mu} - G_{\nu\mu,\lambda}\right)(dx^\lambda \wedge dx^\mu \wedge dx^\nu), \tag{13.19}$$

where the summation takes place only over the four different elements of a basis of 3-forms in four dimensions. Verify explicitly the complete antisymmetry of the coefficients. Show further that, if $G_{\lambda\mu} = -G_{\mu\lambda}$, dG can further be written as

$$dG = \left(G_{\lambda\mu,\nu} + G_{\mu\nu,\lambda} + G_{\nu\lambda,\mu}\right)(dx^\lambda \wedge dx^\mu \wedge dx^\nu). \tag{13.20}$$

Notice the cyclic sum in (13.13) and the double cyclic sum in (13.14). There are times in four dimensions when it is convenient to split differential forms into two parts, one which does not depend on dt and one which does:

$$G = G_1 + G_2, \tag{13.21}$$

where

$$G_1 = \frac{1}{2} G_{lm}\, dx^l \wedge dx^m, \qquad G_2 = \frac{1}{2} G_{0m}\, dt \wedge dx^m + \frac{1}{2} G_{m0}\, dx^m \wedge dt, \tag{13.22}$$

with $G_{\mu\nu} = -G_{\nu\mu}$. Comparison of (13.21)-(13.22) with the alternative way of giving a standard way of giving a differential 2-form in terms of a basis of 2-forms

$$G = H_m dt \wedge dx^m + D_i dx^j \wedge dx^k. \tag{13.23}$$

yields

$$H_m = G_{0m}\,, \qquad D_i = G_{jk}. \tag{13.24}$$

It is instructive to show by explicit calculation that dG computed by using (13.21)-(13.22) coincides with dG given by (13.20), valid only for $G_{\lambda\mu} = -G_{\mu\lambda}$. Indeed we have:

$$dG_1 = \frac{1}{2} G_{lm,0} dt \wedge dx^l \wedge dx^m + \frac{1}{2} G_{lm,p} dx^p \wedge dx^l \wedge dx^m, \tag{13.25}$$

$$dG_2 = \frac{1}{2} G_{0m,l} dx^l \wedge dt \wedge dx^m + \frac{1}{2} G_{m0,l} dx^l \wedge dx^m \wedge dt. \tag{13.26}$$

In (13.24), λ is l (=1,2,3) and zero, π is p and zero, and μ is m and zero. In the sum of dG_1 and dG_2 using equations (13.25) and (13.26), we combine the three terms where dt is not a factor. The completion of the proof is now straightforward.

Exercise 27 *Let j be given by Eq. (7.8). and G defined by $G \equiv B_m dt \wedge dx^m + E_i dx^j \wedge dx^k$. Show that $dG = j$ yields Maxwell's equations in terms of just the fields E and B. Notice that differentiation of $dG = j$ yields $dj = 0$.*

Exercise 28 *Develop $dj = 0$ to obtain the* continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u^i)}{\partial x^i} = 0, \tag{13.27}$$

which expresses the more familiar form of conservation of charge. The equation $dj = 0$ is more fundamental. We shall understand why in the next chapter and in the chapter on integral invariants.

14 Interpretation of the Exterior Derivative

We proceed to interpret the exterior derivative. We first recall the interpretation of the divergence and the curl in the vector calculus.

Exercise 29 *Justify the standard interpretation of the divergence at a point as the flux per unit volume of a vector field out of a small three dimensional region surrounding the point. Hint: resort to Gauss' theorem.*

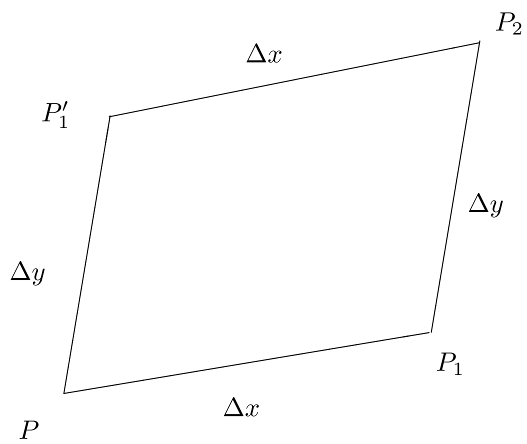

Figure 2.3: *Parallelogram determined by translations along two rectilinear coordinate lines, not necessarily Cartesian*

In a similar vein, the curl of a vector field at a point is the circulation per unit area on a small closed path close to the point. This quantity depends on the inclination of the surface of the path. Hence, the details involve components of the curl.

Correspondingly, consider the differential form

$$\alpha = f(x, y, z, \ldots)\, dx + g(x, y, z, \ldots)\, dy + h(x, y, z, \ldots)\, dz + \ldots. \tag{14.1}$$

and the two paths rom P to P_2in figure 2.3. The dimensionality of the space is irrelevant. The coordinate lines need not be mutually orthogonal and they do not even need to be rectilinear. We may forget all terms on the right hand side of (14.1) except for the first two. The reason is that only these two terms contribute to the evaluation of the differential form α on these paths.

The interpretation of $d\alpha$ is "the function which, when evaluated, it gives us the difference of the evaluations of α on two different path between the same two points". Notice that integrating on a path $P \to P_1 \to P_2 \to P_1' \to P$ is the same as substracting the integrations on $P \to P_1 \to P_2$ and $P \to P_1' \to P_2$. This is a simple application of the fact that

$$\int_a^b F(x)\,dx = -\int_b^a F(x)\,dx, \tag{14.2}$$

and of other trivial facts.

We assume that Δx and Δy are small enough. We then have

$$\begin{aligned}
\int_P^{P_1} f\,dx + g\,dy &= \int_P^{P_1} f\,dx \simeq f\,\Delta x,\\
\int_{P_1}^{P_2} f\,dx + g\,dy &= \int_{P_1}^{P_2} g(x+dx,y,\ldots)\,dy \simeq g(x+\Delta x,y,\ldots)\,\Delta y \simeq\\
&\simeq g(x,y,\ldots)\,\Delta y + \frac{\partial g}{\partial x}(x,y,...)\,\Delta x\,\Delta y,\\
\int_P^{P_1'} f\,dx + g\,dy &= \int_P^{P_1'} g\,dy \simeq g\,\Delta y,\\
\int_{P_1'}^{P_2} f\,dx + g\,dy &= \int_{P_1'}^{P_2} f(x,y+\Delta y,\ldots)\,dx \simeq\\
&\simeq f(x,y+\Delta y,\ldots)\,\Delta x + \frac{\partial f}{\partial y}(x,y,...)\,\Delta x\,\Delta y.
\end{aligned} \tag{14.3}$$

Using these equations, we further get

$$\begin{aligned}
\int_P^{P_1} + \int_{P_1}^{P_2} + \int_{P_2}^{P_1'} + \int_{P_1'}^{P} &= \int_P^{P_1} + \int_{P_1}^{P_2} - \left[\int_P^{P_1'} + \int_{P_1'}^{P_2}\right] \simeq\\
&\simeq \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\Delta x\,\Delta y \simeq\\
&\simeq \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy.
\end{aligned} \tag{14.4}$$

where the line integrands refer to the evaluation of α. With the last step, we have reproduced the generalized Stokes' theorem as it applies to this situation. It is then clear that

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \simeq \frac{1}{\Delta x \Delta y}\left[\int_P^{P_1} + \int_{P_1}^{P_2} + \int_{P_2}^{P_1'} + \int_{P_1'}^{P}\right]. \tag{14.5}$$

In words, the coefficient of the $dx \wedge dy$ component in the exterior derivative of the differential 1-form (14.1) is, per unit area, the circulation of α (i.e. the valuation of α) on closed paths in the $x - y$ plane, as the area of the surface enclosed by the path goes to zero.

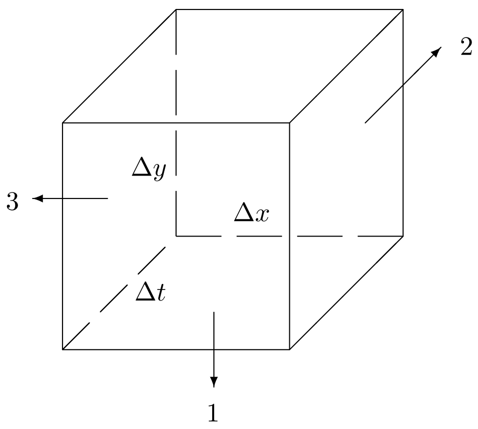

Figure 2.4: *Cube pertaining to the integration of $dx \wedge dy \wedge dt$*

In general, one interprets components of $d\mu$, i.e. its coefficients relative to simple differential forms (i.e. products of differential 1-forms, but not sums of such products). Let $\phi = f dx^{i_1} \wedge ... \wedge dx^{i_{r+1}}$ be one of the terms in the expansion of the $(r+1)$-form $d\mu_r$. It is evaluated on domains of the $(r+1)$-surface spanned by the coordinate lines $x^{i_1}, ...x^{i_{r+1}}$. Take a small region on this hypersurface. The coefficient f of the differential form ϕ then is, per unit of the $(r+1)$-surface, the differential form μ_r's "flux" through that surface's boundary. In order to make sure that the concept is firmly established let us assume that μ is a differential 2-form β, not simple in general. In this case, the exterior derivative is a 3-form. The role previously played by the four lines that enclosed a parallelogram is now played by the six faces that enclose a parallelipipedic 3-dimensional domain. Let us consider the term $dt \wedge dx \wedge dy$ of the exterior derivative and the corresponding volume (Fig. 2.4). We arrange the six surfaces into just two, one of them constituted by the faces $1, 2$ and 3. We would single out all the terms in β whose differentiation yields $dt \wedge dx \wedge dy$ and not, say, $dx \wedge dy \wedge dz$. We thus write β as

$$\begin{aligned}\beta = f(t,x,y,z,\ldots)\, dt \wedge dx + g(t,x,y,z,\ldots)\, dx \wedge dy +& \\ + h(t,x,y,z,\ldots)\, dy \wedge dt + \ldots .&\end{aligned} \tag{14.6}$$

Let "4", "5" and "6" denote the faces opposite to "1", "2" and "3". The evaluation of β on the first of these two "composite" surfaces of the differential form β is

$$\int_1 f\, dt \wedge dx + \int_2 g\, dx \wedge dy + \int_3 h\, dy \wedge dt. \tag{14.7}$$

The evaluation on the other surface is

$$\begin{aligned}\int_4 f(t,x,y+\Delta y,\ldots)\,dt\wedge dx + \int_5 g(t+\Delta t,x,y,\ldots)\,dx\wedge dy + \\ + \int_6 h(t,x+\Delta x,y,\ldots)\,dy\wedge dt.\end{aligned} \tag{14.8}$$

For small Δx, Δy, Δz, the last expression is approximately equal to

$$\begin{aligned}\int_1 f\,dt\wedge dx + \Delta y\int_1 \frac{\partial f}{\partial y}\,dt\wedge dx + \int_2 g\,dx\wedge dy + \\ + \Delta t\int_2 \frac{\partial g}{\partial t}\,dx\wedge dy + \int_3 h\,dy\wedge dt + \Delta x\int_3 \frac{\partial g}{\partial t} dy\wedge dt.\end{aligned} \tag{14.9}$$

Exercise 30 *Complete the argument for the interpretation of the $dt\wedge dx\wedge dy$ component of $d\beta$.*

15 The Arena of the Exterior Calculus

We mentioned in passing that a space requires very little structure in order to support the calculus that we have considered so far in this chapter. For instance, one does not need a metric or a connection, etc. One can easily justify why it works on manifolds, meaning at this point to continuous sets with relatively little structure, which will be made explicit in some future opportunity. The reason can be easily understood if we simply assume that what has been said is assumed to be valid at least in E^n for arbitrary n, since one easily understands that the dimensionality of the Euclidean space is irrelevant in the arguments.

Let us start by considering the main idea behind the argument about to be made. Consider S^2. It is not Euclidean, but lives in E^3. In spherical coordinates, we choose r to be constant and get S^2. We further set dr to zero to specialize all the equations valid in E^3 to S^2. Suppose now that you had the same sphere in an n-dimensional Euclidean space, E^n with $n > 3$ (S^2 is contained in E^3 which in turn is contained in some E^5). One can choose two of the five coordinates to be (θ, ϕ) and all other coordinates to be constant. So, in reality, it does not matter what is the dimensionality of the Euclidean space in which our sphere S^2 is contained. The computations on S^2 reduce to the computations on the Euclidean space with $x^1 = \theta$ and $x^2 = \phi$, and where we have

$$dx^3 = dx^4 = dx^5 = 0.$$

Does any curved space of dimension n (i.e. describable with as few as n coordinates but not less than n) live in some Euclidean space? A theorem proposed by Schlaefli in 1871-73 and proven by Janet (1926) and Cartan (1927) states that one can immerse a Riemannian space (broadly speaking, a curved manifold) in a Euclidean space of dimension n(n+1)/2. The result is local (i.e. not necessarily of global validity), but this is all that is needed for our purposes. The theorem guarantees that there always be finite-dimensional Euclidean spaces that contain the given Riemannian manifold. Hence, the theory that we have presented applies to any "curved spaces", except perhaps for peculiar spaces that, in general, take an out-of-the-ordinary mathematician to dream of. In fact, the Riemannian structure (meaning a quadratic form, and a concomitant length of curves) is already more structure than one actually needs for an exterior calculus. Kähler showed that if one endows differential forms with an interior product also, the added structure allows one to do relativistic quantum mechanics with differential forms, without gamma matrices.

Although we have invoked Schlaefli's theorem for expediency purposes, this is not the only way in which the calculus of exterior differential forms on differentiable manifolds of an arbitrary dimension n can be justified and is usually developed. Such manifolds can be studied in themselves without resort to the Euclidean space of n(n+1)/2.dimensions.

16 The Bourbaki Concept of Differential Form

The exterior calculus may be viewed as a calculus of integrands or as a calculus of antisymmetric multilinear functions of vectors. There are several very strong reasons for a physicist to prefer the first point of view. They will become obvious in the course of this book.

An exterior form is an antisymmetric multilinear function of vectors. Most authors use the terms differential form and exterior differential form to refer to a field of exterior forms, i.e. an exterior form at each point of a (differentiable) manifold. This is not the way in which the quantities in the exterior calculus were born. The modern concept of differential form just mentioned appears to have its origin in the Bourbaki school. But that is not the point of view in which we are interested, namely the point of view of É. Cartan, Kähler and Rudin (see next section). The concept of differential form has evolved with time in ways that this author, who has the Kähler calculus in mind, does not welcome. But it is not the only relevant concept for physicists that has lost its original "physical character" in Bourbaki's time. The same is the case with the concep of vector field, which at some point became a differential operator acting on functions. This makes mathematics unnecessarily complicated for physicists, and brakes with centuries of tradition for no useful purpose. Both themes, vector fields and differential forms are related..

For illustration, consider the tangent planes to S^2. Each of them has a special point, the point of contact with S^2. On each plane, we identify that point with the zero of a vector space V^2. Thus the tangent planes become tangent vectors spaces (over the reals, in this book). For reasons that will be dealt with in later in this series, one cannot identify in general the different tangent vector spaces (which is the reason why one speaks of parallel transport, specially in the general relativity literature).

Some of the authors that define differential forms as antisymmetric multilinear functions of vectors state in the same breath that differential forms are integrands of multivariable integrals. Most readers who have taken a course in the multivariable calculus have not even heard of multilinear functions of vectors, even if they have learned that the theorems of Gauss and Stokes have been written in the form of the generalized Stokes theorem. They do not see in the statement of that theorem multilinear functions of vectors. To all those readers, defining a differential $r-$form as an $r-$linear function(al) of vector fields and, at the same time, stating that an $r-$form is an integrand in an $r-$dimensional integral must be confusing, at the very least. Those books will eventually speak of the evaluation of the differential form (on an ordered set of vector fields) and of the integration of the same (on an $r-$surface). In other words, the same symbol is denoting two different functionals. Hence we are dealing here with two functionals being represented by the same symbol.

Suppose for simplicity that μ is a differential $1-$form in the Bourbaki sense, i.e. a linear function of vectors. Then $\int_\gamma \mu$ represents the integration of μ on some specific curve γ. We said *specific* to bring to the fore the fact that, if μ does not admit a potential differential form (which, if existing, it would be a $0-$form in this case), there is nothing to be computed until a curve is given on which to integrate. Those authors may view $\int$ (no curve is specified) as the map that takes us from a linear functional of vectors, μ, to a linear functional of curves, $\int \mu$, which they call a $1-$cochain (If μ, were an $r-$differential form, $\int \mu$ would be a functional of $r-$surfaces called an $r-$cochain). The *evaluation* of the $r-$cochain.$\int \mu$ on a $r-$surface is the *integration* of the differential form μ. Hence, there is a two step process in the so called integration of the Bourbaki differential form μ. But, we repeat, there is nothing to compute in the step $\mu \rightarrow \int \mu$ (unless μ is a total differential, in which case case $\int \mu$ may be said to be the primitives of μ). The computation lies in the step $\int \mu \rightarrow \int_R \mu$. For this reason, the exterior calculus of antisymmetric multilinear functions of vectors and of cochains is the same. An example of this situation is provided in this regard by the symbol $ydx - xdy$, which we have chosen because of its simplicity and, yet, its having a primitive.

The Bourbaki point of view just exposed is not, however, the only one among those who, rightly, have put Grassmann in a well deserved pedestal. For instance, there is the alternative point of view that I shall call the Rotta point of view, in honor of the late algebraist and mathematical philosopher Gian Carlo Rotta. After pointing out that the exterior product is a progressive (the grade goes up) product, he blames É. Cartan and the Bourbaki group of attaching to the exterior structure the duality operation, like the evaluation of a $1-$form on a vector field (which brings the grade of the form down from 1 to zero). That is, of course, the simple example of evaluations of differential forms on sets of vectors. Rotta rather advocates using also a regressive (the grade goes down) product instead of duality operations. Hence Rotta's critique implies a critique of manifold theory (and calculus if you will) based on antisymmetric multilinear functions of vector fields. This author, who is enamored of É. Cartan's methods, does not share, however, the association of É. Cartan with the Bourbaki group. This is an issue that we shall have the opportunity to review repeatedly in our series of books on calculus and geometry.

Consider finally the theory of manifolds not endowed with an affine connection. Representative works are De Rham's book on differentiable manifolds, and (Henri) Cartan and Eilenberg's monograph on homological algebra (which actually is about homology and cohomology in a unified way). When the exterior products of differentials are introduced by De Rham's, they are just symbols. It soon becomes clear, however, that he means antisymmetric multilinear functions of vectors. However, there is no aparent reason why one could not ignore their nature and treat them simply as symbols, since what matters is not their nature but how one operates with them and specifically the property $dd = 0$. One operates with those symbols as one does with the scalar-valued differential $r-$forms of Cartan and Kaehler (which are functions of $r-$surfaces), not with $r-$linear functions of vectors. The importance of this remark is clear once one observes in the book on homological algebra by Cartan and Eilenberg that they do not even introduce multilinear functions of vectors. The key concept here is that of a complex, and, after defining it, they state: "Note that we are using the word "complex" to denote what is usually called a "cochain complex" (p. 58).

17 The Cartan-Kähler-Rudin-Clifton Concept of Differential Form

Let us now discuss the Cartan-Kähler-Rudin-Clifton (and this author's) view of differential forms. Élie Cartan was not very clear as to what he meant by dx and, thus, by the exterior products of such differentials.

Let us start at the birth of the concept in a 1899 paper by É. Cartan, and where the precursors of the concept are briefly mentioned. There we read: "Given n variables x_1, $x_2, ..., x_n$, considerons **purely symbolic** expressions ω which are obtained with the help of a finite number of *signs* of addition or multiplication from the n differentials dx_1, $dx_2, ...,$ dx_n..." (Italics in original; bold face emphasis has been added). Cartan does not say what he means by the dx_i, if he had meant more than just symbols. In a 1922 paper, he considers quadratic differential forms (the metric), which are not the exterior differential $2-$forms. Hence, differential form is a concept which transcends in this case the antisymmetry. Since the metric is a bilinear map on a vector field, one can see here a connection with the concept of (exterior) differential form as antisymmetric multilinear function of vectors. This is not quite so, however, in view of what he says after obtaining the so called "square root of the metric", ω^i (in reality, one should rather say that the metric is the product of $d\mathbf{r}$ by itself; there is here a dot product of the vectors and a tensor product of the differential forms). He refers to the ω^i as the components of an infinitesimally small translation. In a book of the same year, he deals with the linear combinations of differentials of coordinates in a purely symbolic form, as when he obtains the "square root" of a symbolic symmetric quadratic form. In a book on integral invariants of the same year, he refers to the symbols under the integral sign as differential forms, "where there are variables, dependent and independent and their differentials".

They are not necessarily exterior (i.e. antisymmetric), or the tensor products just mentioned, since one of the examples given is precisely an integrand which is the square root of a quadratic expression on the differentials. In a later chapter, he considers the exterior ones, with the following preamble: "The formes which we are about to consider are those which occur under the integral sign when one considers the differentials as variables. These are forms which have special rules of calculus, which will not hurt to insist upon." In another part of the same book, he considers the evaluation of an $r-$form on what he calls an infinitesimal operator to obtain an $(r-1)-$form. These infinitesimal operators correspond to the tangent vectors of the Bourbaki school, i.e. differential operators acting on functions. This is the point where É. Cartan comes closest to the Bourbaki concept of differential form. It is to be noted, however, the concept of vector field in Cartan (or Kähler for that matter). In 1945, in his book on exterior differential systems, we find a first chapter devoted to exterior forms, like $\frac{1}{2}a_{ij}[u^i u^j]$ (Cartan uses square brackets for the exterior product), and a second chapter devoted to exterior differential forms, like $\frac{1}{2}a_{ij}(x)[dx^i dx^j]$. He views the second as being like the first but where the variables are now the differentials, and the coefficients being functions of the coordinates. Hence, there are several changes here. It is not just a matter of going from $[u^i u^j]$ to $[dx^i dx^j]$, for, in that case, we would simply get $\frac{1}{2}a_{ij}(x)[dx^i dx^j]$. There is also the fact that the coefficients are functions, and further more that the dx^i are not just differentials of any variables, but of the independent variables for those functions. However, virtually everywhere else in Cartan's work, those symbols represent the cochains in the Bourbaki perspective. This is clear when he proceeds to differentiate with an operator which becomes the connection when applied to tensor-valued (excepting scalars) differential 0-forms, and the exterior derivatives when applied to (his) differential forms of scalar-valuedness. This operator is presented in a very formal way by Flanders in his paper on the extended exterior differential calculus; the reduction to the two cases is the subject of his theorem 7.1 of that paper. To conclude, Cartan views the symbolic forms that are the subject of this chapter and book as symbols to which he may or may not attach a meaning. When he does, they can mean anything. But, in his geometric work, they are Bourbaki's cochains. Incidentally, the concept of cochain appears to have been created by De Rham, but he was not a member of any of original Bourbakees, much less of successive generations.

Consider now the introduction of differential forms by Kähler in his work on the extension of the exterior calculus, extension appropriate to deal with quantum mechanics. He also introduced differential forms as the aforementioned symbols, but with coefficients which are tangent and cotangent tensors, though he does not use these terms himself. Given his rule for differentiation of the cotangent tensors, it is clear that the multilinear functions of vectors fit here; they have covariant rather than exterior derivatives in the Kähler calculus. The components of his tensorial differentials have (in addition to a series of superscripts), two series of subscripts, one each for multilinear functions of vectors and cochains. It is clear that his differential forms are the cochains. His is a calculus where functions of vectors have covariants derivatives and where cochains have exterior (and interior) derivatives.

As for Rudin (see his "Principles of Mathematical Analysis"), he defines differential $r-$form as a functional that assigns through integration a number to each $r-$surface, i.e. what Bourbaki names $\int \mu$, and which Rudin simply names as μ. In accordance with this point of view, the evaluation should be written as per the standard notation for functions, $\mu(R)$, but, following custom, one still uses $\int_R \mu$.

As for Clifton, this author knows his view from collaborating with him (We shall see in other books in this series his tremendous overlooked contribution to the method of the method of the moving frame, which makes his view relevant in this discussion). He advised this author to compute everything in the Cartan way, which is much easier, and then write in the modern way for publication. It is ironic that, being the one who has best formalized and made rigorous the method of the moving frame, he made that recommendation. He must have given up convincing others.

Following the published work of Kähler and Rudin, and less explicitly Cartan, we shall use the term differential forms for what the Bourbaki school and most modern practitioners of manifold theory call co-chains. When this definition is mentioned to Bourbaki mathematicians, some of them react as follows: Oh, you mean a current! The answer is no. Let us first deal with the concept of current. It was created by De Rham (See his classic book *Variétées Differentiables*, where his source papers are referenced). Except for details that concern the even-odd issue, we follow the more elaborate definition of Laurent Schwartz in the last chapter of his book on the theory of distributions, chapter which deals precisely with the subject of currents on manifolds. A $p-$current is a continuous linear form on an $(n-p)-$differential form. Hence, if we fix R and let μ vary, we have the current $\int_R$. De Rham's first example of the concept of current is as follows. He says that a chain R defines a current, $R(\mu) = \int_R \mu$. He actually uses the notation $c[\varphi] = \int_c \varphi$ and goes on to say "We shall say that this *current is equal to the chain c*" (emphasis in the original). Returning to our notation, notice that the functional is R, thus the current, is R in $R(\mu)$. The domain is a set of differential forms in the Bourbaki sense. In the Rudin presentation of the exterior calculus, on the other hand, (a) the functional is μ rather than R, (b) μ is understood as a cochain, and not as an antisymmetric multilinear function of vectors and (c) the evaluation "point" of the functional is a hypersurface R (which we treat as a term interchangeable with chain c for economy of concepts which does not affect the main point of the argument). Hence, the differential forms of Rudin are cochains but not currents, since the domain is not a vector space (or a module) of differential forms in the sense of Bourbaki (nor, for that matter, of the differential forms of Rudin). De Rham himself considers the evaluation of a cochain f on a chain c, which he writes as $c.f$. With more standard notation for evaluations, we would write this as $c(f)$. However, in view of what has been said above about chains (hypersurface) and cochains (functions thereof), this means that we evaluate $\int \mu$ on R. This would be written as $(\int \mu)(R)$, where the functional $\int \mu$ goes in parenthesis for greater clarity, even if it is not necessary. Notice that a first action by μ, followed by an action by $\int$would not make sense since μ acts on sets of vector fields.

The calculus with cochains is the same as with differential forms precisely because $\int \mu$ does not involve any computation. The point would then appear to be mute, but it is not. There are very strong reasons to think of the exterior calculus of differential forms as pertaining to functionals of $r-$surfaces rather than antisymmetric multilinear functions of vectors. One has to do with electrodynamics, which is about cochains. Another one has to do with Cartan's theory of affine and Euclidean connections, viewed as a calculus of tensor-valued cochains, which we shall thus refer to as tensor-valued differential forms. Finally, Kähler's calculus for geometry, relativity and quantum mechanics also is about tensor-valued cochains, which he calls tensor-valued differential forms.

As a last remark, if we had to define the dx^i's in terms of which the differential $r-$forms are defined by exterior product, we would define then as functionals such that $\int_Q^P dx = x_P - x_Q$.

18 Maxwell's Equations: Point and Integral Forms

Consider finally the Maxwell system of equations:

$$dF = 0, \qquad dG = j, \tag{18.1}$$

where F and G are the two differential 2-forms previously introduced. There is no essential difference between these equations and

$$\iiint dF = 0, \qquad \iiint dG = \iiint j. \tag{18.2}$$

They are two views of the same information. Equations (18.1) express the equality of the functionals on the left and on the right of each equation. Equations (18.2) express the equality of the values taken by those functions. Using Stokes theorem, we further have

$$\oint \oint F = 0, \qquad \oint \oint G = \iiint j. \tag{18.3}$$

The integration domains, which are open, for the triple integrals are arbitrary. If the differential forms F and G are differentiable, one considers arbitrarily small integration domains, which gives rise to the usual point form of Maxwell's equations (as opposed to their form involving integrals).