

Frank-Oseen energy density:

$$e[\vec{p}] = \frac{K_0}{2} \left(\|\text{Rot}\vec{p}\|^2 + \|\text{Div}\vec{p}\|^2 \right) \quad (1)$$

with $\|\vec{p}\| = 1$.

$*\vec{p} := (*\vec{p}^\flat)^\sharp$ is the Hodge dual of \vec{p} , i.e. $\vec{p} \perp (*\vec{p})$ and $\|*\vec{p}\| = \|\vec{p}\| = 1$.

$$\vec{q} := \cos \phi \vec{p} + \sin \phi (*\vec{p}) \quad (2)$$

is a length preserving linear combination of the orthonormal system $\{\vec{p}, *\vec{p}\}$, i.e. $\|\vec{q}\| = 1$, with a (space-)constant rotation angle ϕ , i.e. $\mathbf{d}\phi = 0$. Straight forward calculations implies

$$\|\text{Rot}(*\vec{p})\| = \left\| *\mathbf{d} * \vec{p}^\flat \right\| = \|\text{Div}\vec{p}\| \quad (3)$$

$$\|\text{Div}(*\vec{p})\| = \left\| *\mathbf{d} * *\vec{p}^\flat \right\| = \left\| *\mathbf{d}\vec{p}^\flat \right\| = \|\text{Rot}\vec{p}\| \quad (4)$$

$$\|\text{Rot}\vec{q}\|^2 = \left\| *\mathbf{d}\vec{q}^\flat \right\|^2 = \left\| \mathbf{d}\vec{q}^\flat \right\|^2 \quad (5)$$

$$= \cos^2 \phi \|\text{Rot}\vec{p}\|^2 + \sin^2 \phi \|\text{Div}\vec{p}\|^2 + 2 \cos \phi \sin \phi \left\langle \mathbf{d}\vec{p}^\flat, \mathbf{d} * \vec{p}^\flat \right\rangle \quad (6)$$

$$\|\text{Div}\vec{q}\|^2 = \left\| *\mathbf{d} * \vec{q}^\flat \right\|^2 = \left\| \mathbf{d} * \vec{q}^\flat \right\|^2 \quad (7)$$

$$= \cos^2 \phi \|\text{Div}\vec{p}\|^2 + \sin^2 \phi \|\text{Rot}\vec{p}\|^2 - 2 \cos \phi \sin \phi \left\langle \mathbf{d}\vec{p}^\flat, \mathbf{d} * \vec{p}^\flat \right\rangle \quad (8)$$

Finally we get

$$e[\vec{q}] = \frac{K_0}{2} \left(\|\text{Rot}\vec{q}\|^2 + \|\text{Div}\vec{q}\|^2 \right) = \frac{K_0}{2} \left(\|\text{Rot}\vec{p}\|^2 + \|\text{Div}\vec{p}\|^2 \right) = e[\vec{p}] \quad (9)$$