Notes On Nonic Surfaces Experiment

February 12, 2016

1 Surface Descriptions

We are starting with the standard parametrization of the unit sphere $\mathbb{S}^2$ with local coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, i.e.,

$$\mathbf{x}_{\mathbb{S}^2}(\theta, \phi) = \sin\theta \cos\phi \boldsymbol{e}^x + \sin\theta \sin\phi \boldsymbol{e}^y + \cos\theta \boldsymbol{e}^z \,. \tag{1}$$

For stretching the unit sphere by a displacement function $f : [-1, 1] \to \mathbb{R}$ in the x-direction depending on the z-positions and pressing to the x-z-plane by a press factor $B \in [0, 1)$, we obtain the surface

$$\mathbf{x}_{f,B}(\theta, \phi) := \mathbf{x}_{\mathbb{S}^2}(\theta, \phi) + f(\cos\theta)\boldsymbol{e}^x - B \sin\theta \sin\phi \boldsymbol{e}^y \tag{2}$$

and with $B \nearrow 1$ the surface becomes flat. We choose for the displacement function f a double well function, which should break the symmetry referring to the x-y-plane, so that the north pole ($z = 1$) of the initial sphere is shifting right in x-direction by $C > 0$ and the south pole ($z = -1$) by $r \cdot C$ with the proportion factor $0 \leq r < 1$. This implies

$$f(z) := f_{C,r}(z) = \frac{1}{4} C z^2 \left[(z + 1)^2 (4 - 3z) + r(z - 1)^2 (4 + 3z) \right] \tag{3}$$

where the double well conditions $f(1) = C$, $f(-1) = r \cdot C$ and $f'(1) = f'(0) = f'(-1) = 0$ are fulfilled, see for example Figure 1. For the immersion $\mathbf{x}_{B,C,r} := \mathbf{x}_{f,B} : [0, \pi] \times [0, 2\pi) \to \mathbb{R}^3$ the surface family $\mathcal{S}_{B,C,r} := \mathrm{Im}(\mathbf{x}_{B,C,r})$ can also expressed implicitly by the 0-Levelset of the function

$$\varphi_{B,C,r}(x, y, z) := (x - f_{C,r}(z))^2 + \frac{1}{(1 - B)^2} y^2 + z^2 - 1 \tag{4}$$

defined in a smooth neighbourhood of the surface. We call $\mathcal{S}_{B,C,r}$ a **Nonic Surface**, because $\varphi_{B,C,r}$ is a polynomial of degree 10. The gradient

$$\nabla \varphi_{B,C,r}(x, y, z) = 2 \begin{bmatrix} x - f_{C,r}(z) \\ \frac{y}{(1-B)^2} \\ z - (x - f_{C,r}(z)) f'_{C,r}(z) \end{bmatrix} , \tag{5}$$

restricted to the surface, points in the direction of the outer surface normals.

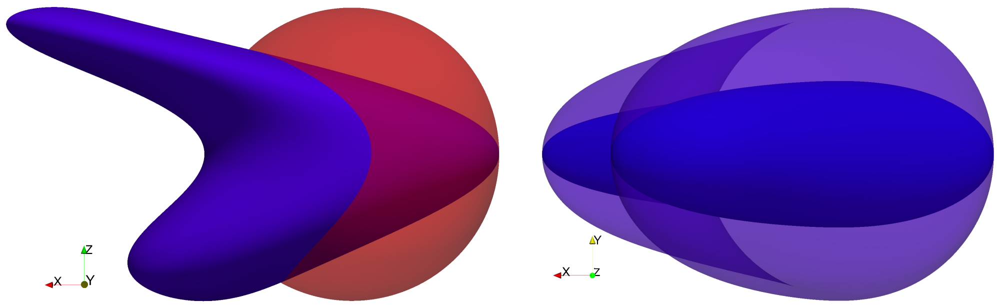

Figure 1: Nonic Surface with parameters $r = 0.5$, $C = 2$ and $B = 0.5$. The left figure shows the stretching of the unit sphere in the x-direction. Hence, by the choice of the parameter, the north pole ($z = 1$) is shifting by $C = 2$ and the south pole ($z = -1$) by $r \cdot C = 1$ units of length to the left. The right figure shows the pressing of the resulting surface to the x-z-plane by the press factor $B = 0.5$.

2 Initial Solutions Construction for the Frank-Oseen-Equations

To solve the director field evololutions in paper **NUMERICAL METHODS FOR ORIENTATIONAL ORDER ON SURFACES**, we have to assign initial fields $\mathbf{p}$, $\boldsymbol{\alpha} = \mathbf{p}^\flat$ respectively, with $\|\mathbf{p}\| = \|\boldsymbol{\alpha}\| = 1$ a.e..

2.1 4 Defect Init

The 4 defect configuration, 3 with positive charge at the bulges and 1 with negative charge at the saddle point, is potentially stable depending on the choice of the surface parameter. The proportion factor $r \in [0, 1)$ prevent a metastable solution, because the resulting symmetry break induce different dynamics for 2 defect locations on the bulges. This implies, that the defect on the smaller bulge and the saddle point defect will mutually annihilate, if the 4 defect configuration is not pure stable, see e. g., Figure 2. For the initial solution $\boldsymbol{\alpha}^0$ we can use the x-coordinate potential, i. e.,

$$\boldsymbol{\alpha}^0 = \frac{\mathbf{d}x}{\|\mathbf{d}x\|_\varepsilon}, \tag{6}$$

where

$$\|\mathbf{q}\|_\varepsilon = \begin{cases} \infty & \text{if } \|\mathbf{q}\| < \varepsilon \\ \|\mathbf{q}\| & \text{else} \end{cases} \tag{7}$$

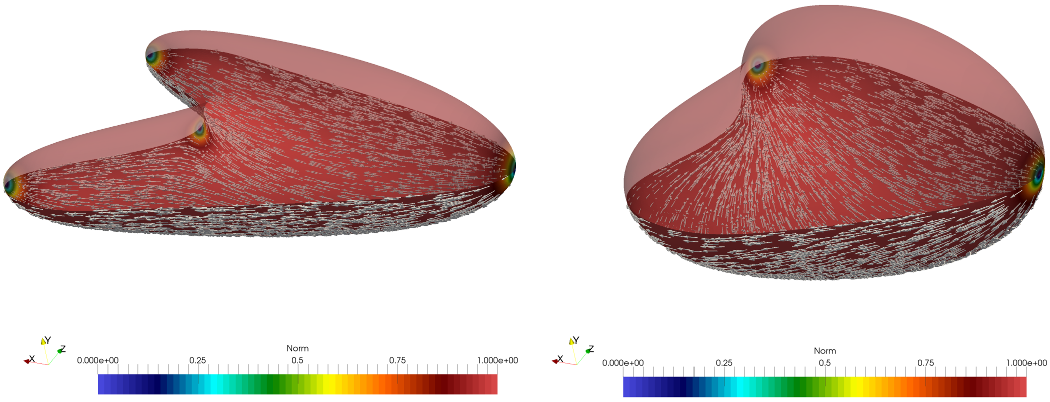

Figure 2: Nonic Surfaces with $r = 0.95$. In the left figure ($B = 0.56$, $C = 1.6$) we see a directional field with stable 4 defect configuration. In the right figure ($B = 0.2625$, $C = 0.75$) the 4 defect initial configuration was not stable, therefor the system was finally gasp to a 2 defect solution.

to prevent ill well-defined in the defect locations. In our experiments ε is mostly chosen by 10^{-10}. Hence, the corresponding contravariant vector field is

$$\mathbf{p}^0 = \frac{\operatorname{grad} x}{\|\operatorname{grad} x\|_\varepsilon} . \tag{8}$$

With the projection map

$$\pi_{\mathcal{S}} = I - \frac{\nabla\varphi}{\|\nabla\varphi\|} \otimes \frac{\nabla\varphi}{\|\nabla\varphi\|} \tag{9}$$

we can use in euclidean coordinates the identity

$$\operatorname{grad} x = \pi_{\mathcal{S}} \nabla x = \pi_{\mathcal{S}} \boldsymbol{e}^x . \tag{10}$$

2.1.1 PD-1-Form Discretization

We can discretize the exact 1-form $\mathbf{d}x$ on an edge $e = [v_1, v_2] \in \mathcal{E}$ by (Stokes theorem)

$$\left(\mathbf{d}x\right)_h(e) = v_2^x - v_1^x . \tag{11}$$

If the face $T_1 \succ e$ is right of the edge e and $T_2 \succ e$ located left, so that $\star e = [c(T_1), c(T_2)]$ is the dual edge, than we can approximate

$$\left(*\mathbf{d}x\right)_h(e) = -\frac{|e|}{|\star e|}\left(\mathbf{d}x\right)_h(\star e) = -\frac{|e|}{|\star e|}\left([c(T_2)]^x - [c(T_1)]^x\right) \tag{12}$$

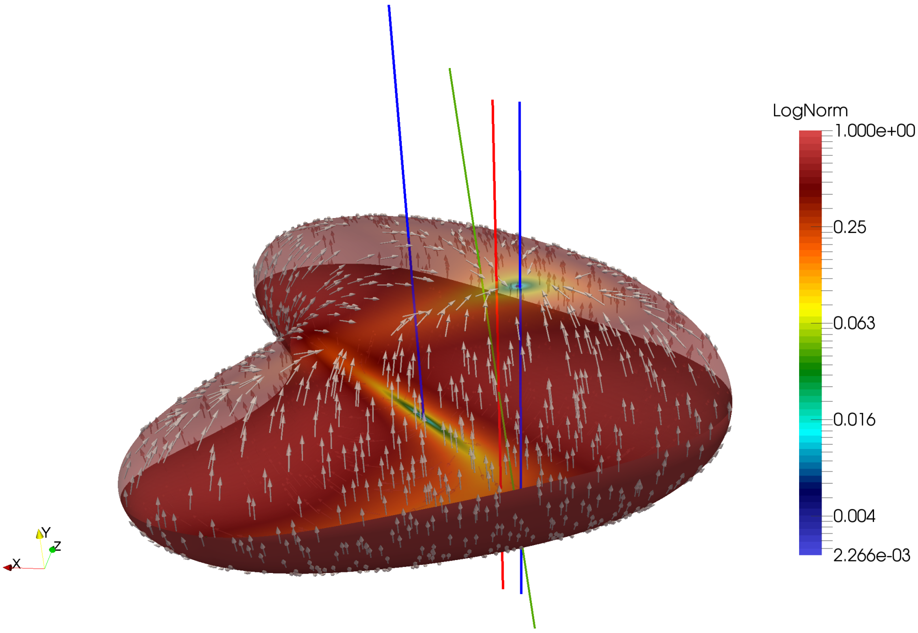

Figure 3: Nonic Surface with $r = 0.95$, $B = 0.35$ and $C = 1$. The green line is the y-axis and the red line is the rotated y-axis throw the origin. This is a rotation by a radian of $\gamma = 1.5$ in the normal plane of the vector $[-1, 0, 1]^T$. The defect locations are at the points, where the rotated y-axis is orthogonal to the surface (see blue lines). The colouring is the logarithm of the norm of the resulting unnormalized vector field $\check{\mathbf{p}}^0$. The arrows show the normalized vector field $\mathbf{p}^0$.

With the discrete norm (**??**) of PD-1-forms, we obtain the discrete initial PD-1-form on $e \in \mathcal{E}$ by

$$\underline{\boldsymbol{\alpha}}^0(e) = \frac{\begin{bmatrix} v_2^x - v_1^x \\ -\frac{|e|}{|\star e|}\left([c(T_2)]^x - [c(T_1)]^x\right) \end{bmatrix}}{\sqrt{\frac{1}{|e|^2}\left(v_2^x - v_1^x\right)^2 + \frac{1}{|\star e|^2}\left([c(T_2)]^x - [c(T_1)]^x\right)^2}}\,, \tag{13}$$

if $\sqrt{\frac{1}{|e|^2}\left(v_2^x - v_1^x\right)^2 + \frac{1}{|\star e|^2}\left([c(T_2)]^x - [c(T_1)]^x\right)^2} \geq \varepsilon$, else we set $\underline{\boldsymbol{\alpha}}^0(e) = [0, 0]^T$.

2.2 2 Defect Init

To provoke a 2 defect solution in the equilibrium, like in Figure 2 (right), we use a normalized projected slightly rotated $\boldsymbol{e}^y$ Field, see e. g., Figure 3. With the symmetry of the surface, $\pi_{\mathcal{S}}\boldsymbol{e}^y$ would be result in a metastable state. To disturb this, we define a rotation R_γ by an angle γ in the normal plane of the vector $[-1,0,1]^T$, i. e.,

$$R_\gamma := \begin{bmatrix} \frac{1+\cos\gamma}{2} & -\frac{\sin\gamma}{\sqrt{2}} & \frac{-1+\cos\gamma}{2} \\ \frac{\sin\gamma}{\sqrt{2}} & \cos\gamma & \frac{\sin\gamma}{\sqrt{2}} \\ \frac{-1+\cos\gamma}{2} & -\frac{\sin\gamma}{\sqrt{2}} & \frac{1+\cos\gamma}{2} \end{bmatrix}. \tag{14}$$

Hence, we get the unnormalized vector field $\check{\mathbf{p}}^0 := \pi_{\mathcal{S}} R_\gamma \boldsymbol{e}^y$. The advantage of $\check{\mathbf{p}}^0$ is that one of the two defects is closer on the larger bulge. so that the defect move to them in the evolution and not to the smaller bulge.

3 Appendix

3.1 Some Reverse Transformations

$$\cos\theta = z \tag{15}$$

$$\sin\theta = \sqrt{1-z^2} \tag{16}$$

$$\cot\theta = \frac{z}{\sqrt{1-z^2}} \tag{17}$$

$$\csc\theta = \frac{1}{\sqrt{1-z^2}} \tag{18}$$

$$\cos\phi = \frac{x-f(z)}{\sqrt{1-z^2}} \tag{19}$$

$$\sin\phi = \frac{y}{(1-B)\sqrt{1-z^2}} \tag{20}$$

$$\tag{21}$$