Formulas for Calculus on Surfaces

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Contents

1 Assumptions, Definitions and Notations

$\mathcal{S}$ . . . . . orientable, boundaryless, Riemannian Surface
$\mathbf{g}$ . . . . . metric tensor in $\mathrm{T}_2\mathcal{S} \cong \mathrm{T}^{(2)}\mathcal{S}$
f, φ, ψ . . . . . scalar quantities in $\mathrm{T}^{(0)}\mathcal{S} = C^\infty(\mathcal{S})$
α, β, γ . . . . . vector quantities in $\mathrm{T}^{(1)}\mathcal{S} \cong \mathrm{T}_1\mathcal{S} = \mathrm{T}\mathcal{S} \cong \mathrm{T}^1\mathcal{S} = T^*\mathcal{S} = \Lambda^1\mathcal{S}$
t, s . . . . . tensor quantities in $\mathrm{T}^{(n)}\mathcal{S}$
ω . . . . . skew symmetric 2-tensor quantities (2-forms) in $\mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S} \cong \Lambda^2\mathcal{S}$
$\mu, \mathbf{E}$ . . . . . volume form in $\mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S} \cong \Lambda^2\mathcal{S}$
q . . . . . Q-Tensor (trace-free and symmetric) in $\mathcal{QS} = \mathrm{T}^{(2)}_{\mathrm{Tr}}\mathcal{S} \cap \mathrm{T}^{(2)}_{\mathrm{Sym}}\mathcal{S}$
$\tilde{\alpha}, \tilde{\beta}, \tilde{t}, \tilde{s}$, etc. . . . . . $\mathbb{R}^3$ extensions of quantities, e.g. in $\mathrm{T}^{(n)}\mathbb{R}^3$

If it matters, all quantities are handled **full covariant**, unless otherwise is defined.

2 Wedge Product $\wedge$

$$f \wedge \psi = \psi \wedge f = f\psi \in \mathrm{T}^{(0)}\mathcal{S}$$
$$f \wedge \alpha = \alpha \wedge f = f\alpha \in \mathrm{T}^{(1)}\mathcal{S}$$
$$f \wedge \omega = \omega \wedge f = f\omega \in \mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S}$$
$$\alpha \wedge \beta = -\beta \wedge \alpha = \frac{1}{\sqrt{|g|}}\left(\alpha_1\beta_2 - \alpha_2\beta_1\right)\mu \in \mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S} \qquad [\alpha \wedge \beta]_{ij} = \alpha^k\beta^l E_{kl}E_{ij} = \alpha_i\beta_j - \alpha_j\beta_j$$

2.1 Conclusions

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu \qquad [\alpha \wedge *\beta]_{ij} = \alpha_k\beta^k E_{ij}$$
$$*\left(\alpha \wedge *\beta\right) = \langle \alpha, \beta \rangle \qquad \langle \alpha, \beta \rangle = \alpha^i\alpha_i$$
$$-*\left(\alpha \wedge \beta\right) = \langle \alpha, *\beta \rangle$$

3 Hodge Star $*$

$$*f = f\mu \qquad [*f]_{ij} = fE_{ij}$$
$$**f = f$$
$$*\alpha = \mathrm{i}_\alpha\mu = \alpha\mathbf{E} = -\mathbf{E}\alpha = *_1\alpha \qquad [*\alpha]_i = -E_{ij}\alpha^j$$
$$**\alpha = -\alpha$$
$$**\omega = \omega$$
$$*_1 t = -\mathbf{E}t \qquad [*t_1]_{i_1\ldots i_n} = -E_{i_1 j}t^j{}_{i_2\ldots i_n}$$
$$*_1 *_1 t = -t$$
$$*_r t \qquad [*_r t]_{i_1\ldots i_n} = -E_{i_r j}t_{i_1\ldots i_{r-1}}{}^j{}_{i_{r+1}\ldots i_n}$$
$$*_n t = t\mathbf{E} \qquad [*_n t]_{i_1\ldots i_n} = -E_{i_n j}t^{i_1\ldots i_{n-1}j}$$
$$*q = *_1 q = *_2 q$$

3.1 Conclusions

$$\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$$
$$\|\alpha\| = \|*\alpha\|$$
$$\langle \alpha, *\alpha \rangle = 0$$
$$\langle \alpha, *\beta \rangle = -\langle *\alpha, \beta \rangle = -*\left(\alpha \wedge \beta\right)$$
$$\langle \alpha, *\beta \rangle^2 = \|\alpha \wedge \beta\|^2 = \|\alpha\|^2\|\beta\|^2 - \langle \alpha, \beta \rangle^2$$
$$(*\alpha) \otimes (*\beta) + \beta \otimes \alpha = \langle \alpha, \beta \rangle \mathbf{g}$$
$$(*\alpha) \otimes (*\alpha) + \alpha \otimes \alpha = \|\alpha\|^2 \mathbf{g}$$
$$\alpha \otimes (*\beta) - (*\beta) \otimes \alpha = \langle \alpha, \beta \rangle \mathbf{E}$$
$$\alpha \otimes (*\alpha) - (*\alpha) \otimes \alpha = \|\alpha\|^2 \mathbf{E}$$
$$*_1 t + *_2 t \in \mathrm{T}^{(2)}_{\mathrm{Sym}}\mathcal{S} \qquad \text{for } t \in \mathrm{T}^{(2)}\mathcal{S}$$

4 Levi-Civita Tensor $\mathbf{E}$

$$\mathbf{E}(\alpha,\beta) = \mu(\alpha,\beta) \qquad E_{ij} = \sqrt{|g|}\epsilon_{ij} \cong E^{ij} = \frac{1}{|\mathbf{g}|}E_{ij} = \frac{1}{\sqrt{|g|}}\epsilon_{ij}$$

$$\langle \mathbf{E}, \mathbf{g}\rangle = \mathbf{E}\mathbf{g} = 0 \qquad E_{ij}g^{ij} = 0$$

$$\mathbf{E}^T = -\mathbf{E} \qquad \left[\mathbf{E}^T\right]_{ij} = E_{ji} = -E_{ij}$$

$$\mathbf{E}\otimes\mathbf{E} = (\mathbf{g}\otimes\mathbf{g})^{T_{2,3}} - (\mathbf{g}\otimes\mathbf{g})^{T_{2,4}} \qquad E_{ij}E_{kl} = g_{ik}g_{jl} - g_{il}g_{jk}$$

4.1 Conclusions

$$-\mathbf{E}\alpha = \alpha\mathbf{E} = \mathrm{i}_\alpha\mu = *\alpha \qquad [*\alpha]_i = -E_{ij}\alpha^j$$

$$-\mathbf{E}t = *_1 t \qquad [*_1 t]_{i_1\dots i_n} = -E_{i_1 j}t^j{}_{i_2\dots i_n}$$

$$t\mathbf{E} = *_n t \qquad [*_n t]_{i_1\dots i_n} = -E_{i_n j}t^{i_1\dots i_{n-1}j}$$

$$\mathbf{E}\mathbf{E} = \mathbf{E}^2 = -\mathbf{g} \qquad E_{ik}E^k{}_j = -g_{ij}$$

$$\mathbf{E}^{-1} = -{}^\sharp\mathbf{E}^\sharp \qquad \left[\mathbf{E}^{-1}\right]^{ij} = -E^{ij} = E^{ji}$$

$$\|\mathbf{E}\|^2 = \mathrm{Tr}\left(\mathbf{E}\mathbf{E}^T\right) = 2$$

$$*_1 *_2 t = *_2 *_1 t = -\mathbf{E}t\mathbf{E} = (\mathrm{Tr}t)\,\mathbf{g} - t^T \qquad [*_1 *_2 t]_{ij} = t_k{}^k g_{ij} - t_{ji}$$

$$|t|\,\mathbf{E} = |g|\,t\mathbf{E}t^T \qquad |t|\,E_{ij} = |g|\,E^{kl}t_{ik}t_{jl}$$

$$|t| = |g|\left|t^\sharp\right| = |g|\left|{}^\sharp t\right| = |g|^2\left|{}^\sharp t^\sharp\right|$$

$$= -\frac{|g|}{2}\langle *_1 t, *_2 t\rangle = \frac{|g|}{2}\left((\mathrm{Tr}t)^2 - \mathrm{Tr}t^2\right) \qquad |t| = \frac{|g|}{2}E_{ij}E_{kl}t^{ik}t^{jl} = \frac{|g|}{2}\left(\left(t_k{}^k\right)^2 - t_{kl}t^{lk}\right)$$

$$0 = t^2 - (\mathrm{Tr}t)\,t + \frac{|t|}{|g|}g \qquad [0]_{ij} = t_{ik}t^k{}_j - t_k{}^k t_{ij} + \frac{1}{2}\left(\left(t_k{}^k\right)^2 - t_{kl}t^{lk}\right)g_{ij}$$

$$0 = B^2 - \mathcal{H}B + \mathcal{K}\mathbf{g}$$

$$0 = \|B\|^2 - \mathcal{H}^2 + 2\mathcal{K}$$

5 Christoffel Symbols $\Gamma^{\cdot}_{\cdot\cdot}$

$$\Gamma \qquad \Gamma^k_{ij} = \Gamma^k_{ji} = g^{kl}\Gamma_{lij} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

6 First Order Derivatives $\mathbf{d}$, ∇, div, rot, Rot, $\mathcal{L}_{\gamma^\sharp}$, $\mathcal{D}_\mathcal{Q}$, $\mathcal{D}^*_\mathcal{Q}$

$$\nabla f \cong \mathbf{d}f \qquad [\nabla f]_i = f_{|i} = [\mathbf{d}f]_i = \partial_i f$$

$$\nabla\alpha \qquad [\nabla\alpha]_{i|j} = \alpha_{i|j} = \partial_j\alpha_i - \Gamma^k_{ij}\alpha_k$$

$$\cong \alpha^i{}_{|j} = \partial_j\alpha^i + \Gamma^i_{jk}\alpha^k$$

$$\nabla t \qquad [\nabla t]_{ij|k} = t_{ij|k} = \partial_k t_{ij} - \Gamma^l_{ki}t_{lj} - \Gamma^l_{kj}t_{il}$$

$$\cong t^i{}_{j|k} = \partial_k t^i{}_j + \Gamma^i_{kl}t^l{}_j - \Gamma^l_{kj}t^i{}_l$$

$$\cong t_i{}^j{}_{|k} = \partial_k t_i{}^j - \Gamma^l_{ki}t_l{}^j + \Gamma^j_{kl}t_i{}^l$$

$$\cong t^{ij}{}_k = \partial_k t^{ij} + \Gamma^i_{kl}t^{lj} + \Gamma^j_{kl}t^{il}$$

$$\nabla\mathbf{g} = 0 \qquad g_{ij|k} = 0$$

$$\nabla\mathbf{E} = 0 \qquad E_{ij|k} = 0$$

$$\mathrm{div}\alpha = *\mathbf{d}*\alpha = \langle \nabla\alpha, \mathbf{g}\rangle = \mathrm{Tr}\nabla\alpha \qquad \mathrm{div}\alpha = \alpha^i{}_{|i}$$

$$\mathrm{div}_1 t = \mathbf{g} : \nabla t \qquad [\mathrm{div}_1 t]_i = t^k{}_{i|k}$$

$$\mathrm{div}_2 t = \nabla t : \mathbf{g} = \mathrm{div}_1 t^T \qquad [\mathrm{div}_2 t]_i = t_i{}^k{}_{|k}$$

$$\mathrm{div}_r t \qquad [\mathrm{div}_r t]_{i_1\ldots\widehat{i_r}\ldots i_n} = t_{i_1\ldots i_{r-1}}{}^k{}_{i_{r+1}\ldots i_n|k}$$

$$\mathrm{div} q = \mathrm{div}_1 q = \mathrm{div}_2 q$$

$$\mathrm{rot}\alpha = *\mathbf{d}\alpha = -\langle \nabla\alpha, \mathbf{E}\rangle \qquad \mathrm{rot}\alpha = -E_{ij}\alpha^{i|j} = \frac{1}{\sqrt{|\mathbf{g}|}}\left(\alpha_{2|1} - \alpha_{1|2}\right) = \frac{1}{\sqrt{|\mathbf{g}|}}\left(\partial_1\alpha_2 - \partial_2\alpha_1\right)$$

$$\mathrm{rot}_1 t = -\nabla t^T : \mathbf{E} \qquad [\mathrm{rot}_1 t]_i = -E_{jk} t^j{}_i{}^{|k}$$

$$\mathrm{rot}_2 t = -\nabla t : \mathbf{E} = \mathrm{rot}_1 t^T \qquad [\mathrm{rot}_2 t]_i = -E_{jk} t_i{}^{j|k}$$

$$\mathrm{rot}_r t \qquad [\mathrm{rot}_r t]_{i_1\ldots\widehat{i_r}\ldots i_n} = -E_{jk} t_{i_1\ldots i_{r-1}}{}^j{}_{i_{r+1}\ldots i_n}{}^{|k}$$

$$\mathrm{rot} q = \mathrm{rot}_1 q = \mathrm{rot}_2 q$$

$$\mathrm{Rot} f = *\mathbf{d} f = -\mathbf{E}\nabla f \qquad [\mathrm{Rot} f]_i = -E_{ij} f^{|j}$$

$$\mathrm{Rot}\alpha = *_2\nabla\alpha = (\nabla\alpha)\,\mathbf{E} \qquad [\mathrm{Rot}\alpha]_{ij} = -E_{jk}\alpha_i{}^{|k}$$

$$\mathrm{Rot} t = *_n\nabla\alpha = (\nabla\alpha)\,\mathbf{E} \qquad [\mathrm{Rot} t]_{i_1\ldots i_n k} = -E_{kl} t_{i_1\ldots i_n}{}^{|l}$$

$$\mathcal{L}_{\gamma^\sharp} f = \langle \gamma, \nabla f\rangle = \nabla_\gamma f \qquad \mathcal{L}_{\gamma^\sharp} f = \gamma^k\partial_k f$$

$$\mathcal{L}_{\gamma^\sharp}\alpha = \nabla_\gamma\alpha + \alpha\nabla\gamma \qquad \left[\mathcal{L}_{\gamma^\sharp}\alpha\right]_i = \gamma^k\partial_k\alpha_i + \alpha_k\partial_i\gamma^k = \gamma^k\alpha_{i|k} + \alpha^k\gamma_{k|i}$$

$$\mathcal{L}_{\gamma^\sharp}\alpha^\sharp = \nabla_\gamma\alpha - \nabla_\alpha\gamma \qquad \left[\mathcal{L}_{\gamma^\sharp}\alpha^\sharp\right]^i = \gamma^k\partial_k\alpha^i - \alpha^k\partial_k\gamma^i = \gamma^k\alpha^i{}_{|k} - \alpha^k\gamma^i{}_{|k}$$

$$\mathcal{L}_{\gamma^\sharp} t = (\nabla t)\gamma + (\nabla\gamma)^T t + t\nabla\gamma \qquad \left[\mathcal{L}_{\gamma^\sharp} t\right]_{ij} = \gamma^k\partial_k t_{ij} + t_{kj}\partial_i\gamma^k + t_{ik}\partial_j\gamma^k = \gamma^k t_{ij|k} + t_{kj}\gamma^k{}_{|i} + t_{ik}\gamma^k{}_{|j}$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{g} = \nabla\gamma + (\nabla\gamma)^T \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{g}\right]_{ij} = \gamma_{i|j} + \gamma_{j|i}$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{E} = (\nabla*\gamma)^T - (\nabla*\gamma) \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{E}\right]_{ij} = [*\gamma]_{j|i} - [*\gamma]_{i|j} = E_{kj}\gamma^k{}_{|i} + E_{ik}\gamma^k{}_{|j}$$

$$\mathcal{L}_{\gamma^\sharp}{}^\sharp t^\sharp = (\nabla t)\gamma - (\nabla\gamma)\,t - t\,(\nabla\gamma)^T \qquad \left[\mathcal{L}_{\gamma^\sharp}{}^\sharp t^\sharp\right]^{ij} = \gamma^k\partial_k t^{ij} - t^{kj}\partial_k\gamma^i - t^{ik}\partial_k\gamma^i = \gamma^k t^{ij}{}_{|k} - t^{kj}\gamma^i{}_{|k} - t^{ik}\gamma^j{}_{|k}$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{g}^{-1} = \mathcal{L}_{\gamma^\sharp}{}^\sharp\mathbf{g}^\sharp = -\left(\nabla\gamma + (\nabla\gamma)^T\right) \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{g}^{-1}\right]^{ij} = -\left(\gamma^{i|j} + \gamma^{j|i}\right)$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{E}^{-1} = -\mathcal{L}_{\gamma^\sharp}{}^\sharp\mathbf{E}^\sharp = \mathrm{Rot}\gamma - (\mathrm{Rot}\gamma)^T \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{E}^{-1}\right]^{ij} = E^{kj}\gamma^i{}_{|k} + E^{ik}\gamma^j{}_{|k}$$

$$\mathcal{L}_{\gamma^\sharp} t \qquad \begin{aligned}\left[\mathcal{L}_{\gamma^\sharp} t\right]^{i_1\ldots i_r}{}_{j_1\ldots j_s} &= \gamma^k\partial_k t^{i_1\ldots i_r}{}_{j_1\ldots j_s} \\ &\quad - t^{ki_2\ldots i_r}{}_{j_1\ldots j_s}\partial_k\gamma^{i_1} - \ldots - t^{i_1\ldots i_{r-1}k}{}_{j_1\ldots j_s}\partial_k\gamma^{i_r} \text{ (uppers)} \\ &\quad + t^{i_1\ldots i_r}{}_{kj_2\ldots j_s}\partial_{j_1}\gamma^k + \ldots + t^{i_1\ldots i_r}{}_{j_1\ldots j_{s-1}k}\partial_{j_s}\gamma^k \text{ (lowers)} \\ &= \gamma^k t^{i_1\ldots i_r}{}_{j_1\ldots j_s|k} \\ &\quad - t^{ki_2\ldots i_r}{}_{j_1\ldots j_s}\gamma^{i_1}{}_{|k} - \ldots - t^{i_1\ldots i_{r-1}k}{}_{j_1\ldots j_s}\gamma^{i_r}{}_{|k} \text{ (uppers)} \\ &\quad + t^{i_1\ldots i_r}{}_{kj_2\ldots j_s}\gamma^k{}_{|j_1} + \ldots + t^{i_1\ldots i_r}{}_{j_1\ldots j_{s-1}k}\gamma^k{}_{|j_s} \text{ (lowers)}\end{aligned}$$

$$\mathcal{D}_\mathcal{Q}\alpha = \mathcal{L}_{\alpha^\sharp}\mathbf{g} - (\mathrm{div}\alpha)\,\mathbf{g} = \nabla\alpha + (\nabla\alpha)^T - (\mathrm{div}\alpha)\,\mathbf{g} = 2\Pi_\mathcal{Q}(\nabla\alpha) \in \mathcal{QS} \qquad [\mathcal{D}_\mathcal{Q}\alpha]_{ij} = \alpha_{i|j} + \alpha_{j|i} - \alpha^k{}_{|k} g_{ij}$$

$$\mathcal{D}^*_\mathcal{Q} q = -2\mathrm{div} q = -2*\mathrm{rot} q = -2\mathrm{rot}*q \qquad \int_\mathcal{S}\left\langle \mathcal{D}^*_\mathcal{Q} q, \alpha\right\rangle\mu = \int_\mathcal{S}\langle q, \mathcal{D}_\mathcal{Q}\alpha\rangle\,\mu$$

6.1 Conclusions

$$\mathrm{rot} * \alpha = *\mathbf{d} * \alpha = \mathrm{div}\alpha$$

$$\mathrm{Rot} * \alpha = *_2 \nabla * \alpha = *_1 *_2 \nabla\alpha = (\mathrm{div}\alpha)\,\mathbf{g} - (\nabla\alpha)^T \qquad [\mathrm{Rot} * \alpha]_{ij} = \alpha^k{}_{|k} g_{ij} - \alpha_{j|i}$$

$$\mathrm{rot}_1 *_1 t = \mathrm{div}_1 t \qquad [\mathrm{rot}_1 *_1 t]_i = t^k{}_{i|k}$$

$$\mathrm{rot}_2 *_2 t = \mathrm{div}_2 t \qquad [\mathrm{rot}_2 *_2 t]_i = t_i{}^k{}_{|k}$$

$$*\mathrm{rot}_1 t = \mathrm{rot}_1 *_2 t = \mathrm{div}_2 t - \nabla \mathrm{Tr} t \qquad [\mathrm{rot}_1 *_2 t]_i = t_i{}^k{}_{|k} - t^k{}_{k|i}$$

$$*\mathrm{rot}_2 t = \mathrm{rot}_2 *_1 t = \mathrm{div}_1 t - \nabla \mathrm{Tr} t \qquad [\mathrm{rot}_2 *_1 t]_i = t^k{}_{i|k} - t^k{}_{k|i}$$

$$\nabla(\psi f) = \psi\nabla f + f\nabla\psi \qquad [\nabla(\psi f)]_i = \psi f_{|i} + f\psi_{|i}$$

$$\nabla\langle\alpha,\beta\rangle = \alpha\nabla\beta + \beta\nabla\alpha \qquad [\nabla\langle\alpha,\beta\rangle]_i = \alpha^k\beta_{k|i} + \beta^k\alpha_{k|i}$$

$$\nabla(f\alpha) = f\nabla\alpha + \alpha\otimes\nabla f \qquad [\nabla(f\alpha)]_{ij} = f\alpha_{i|j} + \alpha_i f_{|j}$$

$$\nabla(ft) = f\nabla t + t\otimes\nabla f \qquad [\nabla(ft)]_{ijk} = ft_{ij|k} + t_{ij}f_{|k}$$

$$\nabla\|t\|^2 = 2t : \nabla t \qquad \left(t^{jk}t_{jk}\right)_{|i} = 2t^{jk}t_{jk|i}$$

$$\nabla\left\langle t, t^T\right\rangle = 2t^T : \nabla t \qquad \left(t^{jk}t_{kj}\right)_{|i} = 2t^{kj}t_{jk|i}$$

$$\mathrm{rot}(f\alpha) = f\mathrm{rot}\alpha + \langle\alpha, \mathrm{Rot} f\rangle \qquad \mathrm{rot}(f\alpha) = -fE_{ij}\alpha^{i|j} - \alpha^i E_{ij} f^{|j}$$

$$\mathrm{div}(f\alpha) = f\mathrm{div}\alpha + \langle\alpha,\nabla f\rangle \qquad \mathrm{div}(f\alpha) = f\alpha^k{}_{|k} + \alpha^k f_{|k}$$

$$\mathrm{div}_1(ft) = f\mathrm{div}_1 t + (\nabla f)t \qquad [\mathrm{div}_1(ft)]_i = ft^k{}_{i|k} + f^{|k}t_{ki}$$

$$\mathrm{div}_2(ft) = f\mathrm{div}_2 t + t\nabla f \qquad [\mathrm{div}_2(ft)]_i = ft_i{}^k{}_{|k} + f^{|k}t_{ik}$$

$$\mathrm{div}(f\mathbf{g}) = \nabla f$$

$$\mathrm{div}\mathcal{L}_{\alpha^\sharp}\alpha = \mathrm{div}(\nabla_\alpha\alpha + \alpha\nabla\alpha) = \Delta\|\alpha\|^2 - (\mathrm{rot}\alpha)^2 - \left\langle\Delta^{\mathrm{Rr}}\alpha, \alpha\right\rangle$$

$$\nabla_\alpha\alpha = \frac{1}{2}\mathbf{d}\|\alpha\|^2 + (\mathrm{rot}\alpha)(*\alpha) \qquad [\nabla_\alpha\alpha]_i = \alpha^j\alpha_{i|j} = \frac{1}{2}\left(\alpha^j\alpha_j\right)_{|i} + (\mathrm{rot}\alpha)(*\alpha)_i$$

$$\alpha\nabla\alpha = \frac{1}{2}\mathbf{d}\|\alpha\|^2 \qquad [\alpha\nabla_\alpha]_i = \alpha^j\alpha_{j|i} = \frac{1}{2}\left(\alpha^j\alpha_j\right)_{|i}$$

$$\nabla_\beta\alpha - \beta\nabla\alpha = (\mathrm{rot}\alpha)(*\beta)$$

$$\|\nabla\alpha\|^2 = \mathrm{div}(\alpha\nabla\alpha) - \left\langle\Delta^{\mathrm{dG}}\alpha,\alpha\right\rangle \qquad \alpha^{i|j}\alpha_{i|j} = \left(\alpha_j\alpha^{j|i}\right)_{|i} - \alpha^j\alpha_j{}^{|i}{}_{|i}$$

$$\left\langle\nabla\alpha, (\nabla\alpha)^T\right\rangle = \mathrm{div}(\nabla_\alpha\alpha) - \langle\mathrm{div}_1\nabla\alpha, \alpha\rangle \qquad \alpha^{i|j}\alpha_{j|i} = \left(\alpha_j\alpha^{i|j}\right)_{|i} - \alpha^j\alpha^i{}_{|j|i}$$

$$= \mathrm{div}(\nabla_\alpha\alpha) + \left\langle\Delta^{\mathrm{Rr}}\alpha - \Delta^{\mathrm{dG}}\alpha, \alpha\right\rangle$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{d}f = \mathbf{d}\mathcal{L}_{\gamma^\sharp}f$$

$$\mathcal{L}_{\gamma^\sharp}\alpha = \nabla\langle\gamma,\alpha\rangle + (\mathrm{rot}\alpha)(*\gamma)$$

$$\left(\mathcal{L}_{\gamma^\sharp}\alpha^\sharp\right)^\flat = -\left(\mathcal{L}_{\alpha^\sharp}\gamma^\sharp\right)^\flat = (\mathrm{div}\alpha)\,\gamma - (\mathrm{div}\gamma)\,\alpha - \mathrm{Rot}\langle\gamma, *\alpha\rangle$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{g} = -{}^\flat\!\left(\mathcal{L}_{\gamma^\sharp}\mathbf{g}^{-1}\right)^\flat$$

$$*_1\mathcal{L}_{*\gamma^\sharp}\mathbf{g} = (\mathrm{div}\gamma)\,\mathbf{g} - 2\nabla\gamma \in \mathrm{T}^{(2)}_{\mathrm{Tr}}\mathcal{S} \qquad \left[*_1\mathcal{L}_{*\gamma^\sharp}\mathbf{g}\right]_{ij} = \gamma^k{}_{|k}g_{ij} - 2\gamma_{i|j}$$

$$*_2\mathcal{L}_{*\gamma^\sharp}\mathbf{g} = (\mathrm{div}\gamma)\,\mathbf{g} - 2(\nabla\gamma)^T \in \mathrm{T}^{(2)}_{\mathrm{Tr}}\mathcal{S} \qquad \left[*_2\mathcal{L}_{*\gamma^\sharp}\mathbf{g}\right]_{ij} = \gamma^k{}_{|k}g_{ij} - 2\gamma_{j|i}$$

$$\mathcal{L}_{*\gamma^\sharp}\mathbf{g} = \nabla * \gamma + (\nabla * \gamma)^T = (\mathrm{div}\gamma)\,\mathbf{E} + 2\nabla * \gamma \in \mathrm{T}^{(2)}_{\mathrm{Sym}}\mathcal{S} \qquad \left[\mathcal{L}_{*\gamma^\sharp}\mathbf{g}\right]_{ij} = E_{ki}\gamma^k{}_{|j} + E_{kj}\gamma^k{}_{|i} = \gamma^k{}_{|k}E_{ij} + 2E_{ki}\gamma^k{}_{|j}$$

$$\left\|\mathcal{L}_{\gamma^\sharp}\mathbf{g}\right\|^2 = 2\left\langle\nabla\gamma, \nabla\gamma + (\nabla\gamma)^T\right\rangle \qquad \left\|\mathcal{L}_{\gamma^\sharp}\mathbf{g}\right\|^2 = 2\gamma^{i|j}\left(\gamma_{i|j} + \gamma_{j|i}\right)$$

$$\left(\mathcal{L}_{\alpha^\sharp}\mathbf{g}\right)\beta = \beta\left(\mathcal{L}_{\alpha^\sharp}\mathbf{g}\right) = \mathcal{L}_{\alpha^\sharp}\beta - \left(\mathcal{L}_{\alpha^\sharp}\beta^\sharp\right)^\flat$$

$$
\begin{aligned}
*\mathcal{D}_{\mathcal{Q}}\alpha &= *_1\mathcal{D}_{\mathcal{Q}}\alpha = *_2\mathcal{D}_{\mathcal{Q}}\alpha = \mathcal{D}_{\mathcal{Q}} * \alpha \in \mathcal{QS} \\
*\mathcal{D}^*_{\mathcal{Q}}q &= \mathcal{D}^*_{\mathcal{Q}} *_1 q = \mathcal{D}^*_{\mathcal{Q}} *_2 q = \mathcal{D}^*_{\mathcal{Q}} * q \\
\mathcal{D}_{\mathcal{Q}}\alpha &= -\frac{1}{2}\left(*_1 + *_2\right)\mathcal{L}_{*\alpha^\sharp}\mathbf{g} = - * \mathcal{D}_{\mathcal{Q}} * \alpha \\
\mathcal{D}_{\mathcal{Q}}\mathbf{d}f &= - * \mathcal{L}_{\mathrm{Rot}f}\mathbf{g} = - * \mathcal{D}_{\mathcal{Q}}\mathrm{Rot}f \\
\mathcal{D}_{\mathcal{Q}}\mathrm{Rot}f &= \mathcal{L}_{\mathrm{Rot}f}\mathbf{g} = *\mathcal{D}_{\mathcal{Q}}\mathbf{d}f \\
\mathcal{D}_{\mathcal{Q}}\left(\mathrm{Rot}\phi + \mathbf{d}\psi + \gamma\right) &= \mathcal{L}_{\mathrm{Rot}\phi}\mathbf{g} - *\mathcal{L}_{\mathrm{Rot}\psi}\mathbf{g} + \mathcal{L}_{\gamma^\sharp}\mathbf{g} && \text{for } \mathrm{div}\gamma = \mathrm{rot}\gamma = 0 \\
\left\|\mathcal{D}_{\mathcal{Q}}\alpha\right\|^2 &= \left\|*\mathcal{D}_{\mathcal{Q}}\alpha\right\|^2 = \left\|\mathcal{L}_{\alpha^\sharp}\mathbf{g}\right\|^2 - 2\left(\mathrm{div}\alpha\right)^2 \\
&= 2\left(\Delta\left\|\alpha\right\|^2 - 2\left\langle\Delta^{\mathrm{dG}}\alpha,\alpha\right\rangle - \left(\mathrm{div}\alpha\right)^2 - \left(\mathrm{rot}\alpha\right)^2\right) \\
&= 2\left(\Delta\left\|\alpha\right\|^2 - 2\mathcal{K}\left\|\alpha\right\|^2 - 2\left\langle\Delta\alpha,\alpha\right\rangle - \left(\mathrm{div}\alpha\right)^2 - \left(\mathrm{rot}\alpha\right)^2\right) \\
\beta\left(\mathcal{D}_{\mathcal{Q}}\alpha\right)\gamma &= -\sum_{\sigma\in\{\mathrm{id},(\alpha,\beta),(\alpha,\gamma)\}} \mathrm{sgn}(\sigma)\left\langle\sigma(\alpha),\mathbf{d}\left\langle\sigma(\beta),\sigma(\gamma)\right\rangle\right\rangle \\
&\quad - \sum_{\sigma\in\{\mathrm{id},(*\alpha,\beta),(*\alpha,\gamma)\}} \mathrm{sgn}(\sigma)\mathrm{rot}\sigma(*\alpha)\left\langle\sigma(\beta),\sigma(\gamma)\right\rangle \\
&= \left\langle\gamma,\mathbf{d}\left\langle\alpha,\beta\right\rangle\right\rangle + \left\langle\beta,\mathbf{d}\left\langle\alpha,\gamma\right\rangle\right\rangle - \left\langle\alpha,\mathbf{d}\left\langle\beta,\gamma\right\rangle\right\rangle \\
&\quad + \left(\mathrm{rot}\gamma\right)\left\langle*\alpha,\beta\right\rangle + \left(\mathrm{rot}\beta\right)\left\langle*\alpha,\gamma\right\rangle - \left(\mathrm{div}\alpha\right)\left\langle\beta,\gamma\right\rangle \\
\beta\mathcal{D}_{\mathcal{Q}}\alpha &= \left(\mathcal{D}_{\mathcal{Q}}\alpha\right)\beta = \mathcal{L}_{\alpha^\sharp}\beta - \left(\mathcal{L}_{\alpha^\sharp}\beta^\sharp\right)^\flat - \left(\mathrm{div}\alpha\right)\beta \\
&= \nabla\left\langle\alpha,\beta\right\rangle + \mathrm{Rot}\left\langle\alpha,*\beta\right\rangle - \left(\mathrm{div}\beta\right)\alpha - \left(\mathrm{div}\left(*\beta\right)\right)\left(*\alpha\right) \\
\alpha\mathcal{D}_{\mathcal{Q}}\alpha &= \left(\mathcal{D}_{\mathcal{Q}}\alpha\right)\alpha = \mathcal{L}_{\alpha^\sharp}\alpha - \left(\mathrm{div}\alpha\right)\alpha \\
&= \left(\mathrm{rot}\alpha\right)\left(*\alpha\right) - \left(\mathrm{div}\alpha\right)\alpha + \mathbf{d}\left\|\alpha\right\|^2 \\
&= \mathbf{d}\left\|\alpha\right\|^2 - \left(\mathrm{div}\alpha\right)\alpha - \left(\mathrm{div}\left(*\alpha\right)\right)\left(*\alpha\right)
\end{aligned}
$$

$$
EW(\mathcal{D}_{\mathcal{Q}}\alpha) = \left\{\pm\frac{\left\|\mathcal{D}_{\mathcal{Q}}\alpha\right\|}{\sqrt{2}}\right\} = \left\{\pm\sqrt{\Delta\left\|\alpha\right\|^2 - 2\left\langle\Delta^{\mathrm{dG}}\alpha,\alpha\right\rangle - \left(\mathrm{div}\alpha\right)^2 - \left(\mathrm{rot}\alpha\right)^2}\right\}
$$

7 Laplace-like Derivatives

$$
\begin{aligned}
\Delta f &= \Delta^{\mathrm{B}}f = \Delta^{\mathrm{dG}}f = -\Delta^{\mathrm{DeR}}f && \Delta f = f^{|i}{}_{|i} \\
&= *\mathbf{d} * \mathbf{d}f = \mathrm{Tr}\mathcal{H}f \\
\Delta\alpha &= -\Delta^{\mathrm{DeR}}\alpha = \left(\Delta^{\mathrm{Gd}} + \Delta^{\mathrm{Rr}}\right)\alpha && [\Delta\alpha]_i = \alpha_i{}^{|k}{}_{|k} + \alpha^k{}_{|k|i} - \alpha^k{}_{|i|k} \\
&= \left(\mathbf{d} * \mathbf{d} * + * \mathbf{d} * \mathbf{d}\right)\alpha = \Delta^{\mathrm{dG}}\alpha - \mathcal{K}\alpha = \Delta^{\mathcal{Q}}\alpha - 2\mathcal{K}\alpha \\
\Delta^{\mathrm{Gd}}\alpha &= \nabla\mathrm{div}\alpha && \left[\Delta^{\mathrm{Gd}}\alpha\right]_i = \alpha^k{}_{|k|i} \\
\Delta^{\mathrm{Rr}}\alpha &= \mathrm{Rotrot}\alpha && \left[\Delta^{\mathrm{Rr}}\alpha\right]_i = \alpha_i{}^{|k}{}_{|k} - \alpha^k{}_{|i|k} \\
\Delta^{\mathrm{dG}}\alpha &= \mathrm{div}_2\nabla\alpha = \Delta\alpha + \mathcal{K}\alpha = \Delta^{\mathcal{Q}}\alpha - \mathcal{K}\alpha && \left[\Delta^{\mathrm{dG}}\alpha\right]_i = \alpha_i{}^{|k}{}_{|k} \\
\Delta^{\mathcal{Q}}\alpha &= -\frac{1}{2}\mathcal{D}^*_{\mathcal{Q}}\mathcal{D}_{\mathcal{Q}}\alpha = \mathrm{div}\mathcal{D}_{\mathcal{Q}}\alpha = 2\mathrm{div}\Pi_{\mathcal{Q}}\nabla\alpha && \left[\Delta^{\mathcal{Q}}\alpha\right]_i = \alpha_i{}^{|k}{}_{|k} - \alpha^k{}_{|k|i} + \alpha^k{}_{|i|k} \\
&= \Delta^{\mathrm{dG}}\alpha + \mathcal{K}\alpha = \Delta\alpha + 2\mathcal{K}\alpha \\
\Delta q &= \Delta^{\mathrm{Rr}}q + \Delta^{\mathrm{Gd}}q = \Delta^{\mathrm{dG}}q - 2\mathcal{K}q && [\Delta q]_{ij} = q_{ij}{}^{|k}{}_{|k} + q_i{}^k{}_{|k|j} - q_i{}^k{}_{|j|k} \\
&= \Delta^{\mathcal{Q}}q = -\frac{1}{2}\mathcal{D}_{\mathcal{Q}}\mathcal{D}^*_{\mathcal{Q}}q = \mathcal{D}_{\mathcal{Q}}\mathrm{div}q = 2\Pi_{\mathcal{Q}}\nabla\mathrm{div}q \\
&= 2\Pi_{\mathcal{Q}}\Delta^{\mathrm{Gd}}q = 2\Pi_{\mathcal{Q}}\Delta^{\mathrm{Rr}}q \\
\Delta^{\mathrm{Gd}}q &= \nabla\mathrm{div}q && \left[\Delta^{\mathrm{Gd}}q\right]_{ij} = q_i{}^k{}_{|k|j} \\
\Delta^{\mathrm{Rr}}q &= \mathrm{Rotrot}q && \left[\Delta^{\mathrm{Rr}}q\right]_{ij} = q_{ij}{}^{|k}{}_{|k} - q_i{}^k{}_{|j|k} \\
\Delta^{\mathrm{dG}}q &= \mathrm{div}\nabla q = \Delta q + 2\mathcal{K}q && \left[\Delta^{\mathrm{dG}}q\right]_{ij} = q_{ij}{}^{|k}{}_{|k}
\end{aligned}
$$

8 $\mathbb{R}^3$ Representations

$$\Pi = \mathrm{Id}_{\mathbb{R}^3} - \nu \otimes \nu = \mathrm{Id}_{\mathcal{S}} \qquad \Pi^I{}_J = \delta^I{}_J - \nu^I \nu_J$$

$$\Pi\left[\tilde{t}\right] = t \in \mathrm{T}^{(n)}\mathcal{S} \qquad t^{i_1\ldots i_n} \cong t^{I_1\ldots I_n} = \Pi^{I_1}{}_{J_1}\ldots\Pi^{I_n}{}_{J_n}\tilde{t}^{J_1\ldots J_n}$$

$$D = \Pi\left[\partial\right] = \Pi\cdot\partial \qquad D_I =_{:I}= \Pi^J{}_I\partial_J$$

$$B = \mathrm{Gram}\left(\Pi\left[\partial\right]\nu, \Pi\left[\partial\right]X\right) = \nu\cdot\left(\Pi\left[\partial\right]\otimes\Pi\left[\partial\right]\right)X \qquad B_{ij} = -\partial_i\nu\cdot\partial_j X = \nu\cdot\partial_i\partial_j X$$

$$B^2 = \mathrm{Gram}\left(\Pi\left[\partial\right]\nu, \Pi\left[\partial\right]\nu\right) \qquad B_i{}^k B_{kj} = \partial_i\nu\cdot\partial_j\nu$$

$$\mathcal{H} = \mathrm{Tr}B \qquad \mathcal{H} = B^i{}_i = B^I{}_I$$

$$0 = B^2 - \mathcal{H}B + \mathcal{K}\pi \qquad \left[0\right]^I{}_J = B^I{}_K B^K{}_J - \mathcal{H}B^I{}_J + \mathcal{K}\Pi^I{}_J$$

8.1 Thin Shell Metric Quantities

$$\tilde{X} = \tilde{X}\left(\left\{x^i\right\},\xi\right) = X\left(\left\{x^i\right\}\right) + \xi\nu\left(\left\{x^i\right\}\right) = X + \xi\nu \qquad \tilde{X}_I = X_I + \xi\nu_I$$

$$\Pi\left[\partial\right]\tilde{X} = \Pi\left[\partial\right]X + \xi\Pi\left[\partial\right]\nu \qquad \partial_i\tilde{X}_J = \partial_i X_J + \xi\partial_i\nu_J$$

$$\partial_\xi\tilde{X} = \nu \qquad \partial_\xi\tilde{X}_I = \nu_I$$

$$\Pi\left[\tilde{\mathbf{g}}\right] = \left(\mathbf{g} - \xi B\right)^2 = \mathbf{g} - 2\xi B + \xi^2 B^2 \qquad \tilde{g}_{ij} = g_{ij} - 2\xi B_{ij} + \xi^2 B_i{}^k B_{kj}$$

$$\Pi\cdot\tilde{\mathbf{g}}\cdot\nu = \nu\cdot\tilde{\mathbf{g}}\cdot\Pi = 0 \qquad \tilde{g}_{i\xi} = \tilde{g}_{\xi i} = 0$$

$$\nu\cdot\tilde{\mathbf{g}}\cdot\nu = 1 \qquad \tilde{g}_{\xi\xi} = 1$$

$$\Pi\left[\tilde{\mathbf{g}}^{-1}\right] = \mathbf{g}^{-1} + \mathcal{O}\left(\xi\right) \qquad \tilde{g}^{ij} = g^{ij} + \mathcal{O}\left(\xi\right)^{ij} = \frac{g^{ij} - 2\xi\mathcal{K}\left[B^{-1}\right]^{ij} + \xi^2\mathcal{K}^2\left[B^{-2}\right]^{ij}}{\left(1 + \xi\mathcal{H} + \xi^2\mathcal{K}\right)^2}$$

$$\Pi\cdot\tilde{\mathbf{g}}^{-1}\cdot\nu = \nu\cdot\tilde{\mathbf{g}}^{-1}\cdot\Pi = 0 \qquad \tilde{g}^{i\xi} = \tilde{g}^{\xi i} = 0$$

$$\nu\cdot\tilde{\mathbf{g}}^{-1}\cdot\nu = 1 \qquad \tilde{g}^{\xi\xi} = 1$$

$$\sqrt{\left|\tilde{\mathbf{g}}\right|} = \left(1 + \xi\mathcal{H} + \xi^2\mathcal{K}\right)\sqrt{\left|\mathbf{g}\right|}$$

$$\tilde{\Gamma} \qquad \tilde{\Gamma}^K_{IJ} = \frac{1}{2}\tilde{g}^{KL}\left(\partial_I\tilde{g}_{JL} + \partial_J\tilde{g}_{IL} - \partial_L\tilde{g}_{IJ}\right)$$

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \mathcal{O}\left(\xi\right)^k_{ij}$$

$$\tilde{\Gamma}^\xi_{ij} = B_{ij} + \mathcal{O}\left(\xi\right)_{ij}$$

$$\tilde{\Gamma}^k_{i\xi} = \tilde{\Gamma}^k_{\xi i} = -B_i{}^k + \mathcal{O}\left(\xi\right)^k_i = -B^k{}_i + \mathcal{O}\left(\xi\right)^k_i$$

$$\tilde{\Gamma}^K_{\xi\xi} = \tilde{\Gamma}^\xi_{I\xi} = \tilde{\Gamma}^\xi_{\xi I} = 0$$

$$\tilde{\mathbf{E}} = \sqrt{\left|\tilde{\mathbf{g}}\right|}\varepsilon_{\mathbb{R}^3} = \sqrt{\left|\mathbf{g}\right|}\varepsilon_{\mathbb{R}^3} + \mathcal{O}\left(\xi\right) \qquad \tilde{E}_{IJK} = \sqrt{\left|\mathbf{g}\right|}\varepsilon_{IJK} + \mathcal{O}\left(\xi\right)_{IJK}$$

$$\tilde{E}_{\xi ij} = -\tilde{E}_{i\xi j} = \tilde{E}_{ij\xi} = E_{ij} + \mathcal{O}\left(\xi\right)_{ij}$$

8.2 First Order Derivatives on Surfaces ($\xi = 0$)

$$\nabla\tilde{\alpha} = \Pi\left[\nabla_{\mathbb{R}^3}\tilde{\alpha}\right] + \left(\tilde{\alpha}\cdot\nu\right)B \qquad \tilde{\alpha}^I{}_{|J} = \Pi^I{}_K\Pi^L{}_J\partial_L\tilde{\alpha}^K + \nu_K\tilde{\alpha}^K B^I{}_J$$

$$= \Pi\cdot D\alpha + \left(\tilde{\alpha}\cdot\nu\right)B \qquad = \Pi^I{}_K\tilde{\alpha}^K{}_{:J} + \nu_K\tilde{\alpha}^K B^I{}_J$$

$$\nabla\tilde{t} = \Pi\left[\nabla_{\mathbb{R}^3}\tilde{t}\right] + \left(\left(\nu\cdot\tilde{t}\cdot\Pi\right)\otimes B\right)^{T_{1,2}} + \left(\Pi\cdot\tilde{t}\cdot\nu\right)\otimes B \qquad \tilde{t}^{IJ}{}_K = \Pi^I{}_{\hat{I}}\Pi^J{}_{\hat{J}}\Pi^{\hat{K}}{}_K\partial_{\hat{K}}\tilde{t}^{\hat{I}\hat{J}} + \nu_L\Pi^J{}_{\hat{J}}\tilde{t}^{L\hat{J}}B^I{}_K + \nu_L\Pi^I{}_{\hat{I}}\tilde{t}^{\hat{I}L}B^J{}_K$$

Not verifiable with `Mathematica`. (Complexity).

$$\mathrm{div}\tilde{\alpha} = \mathrm{Tr}D\tilde{\alpha} + \mathcal{H}\tilde{\alpha}\cdot\nu \qquad \mathrm{div}\tilde{\alpha} = \tilde{\alpha}^I{}_{:I} + \mathcal{H}\tilde{\alpha}^I\nu_I$$

8.3 Weak Formulations

$$
\begin{aligned}
\int_S \left\langle \Delta^{\mathrm{dG}} \tilde{\alpha}, \tilde{\gamma} \right\rangle \mu &= -\int_S \left(\Pi \cdot D\tilde{\alpha}\right) : D\tilde{\gamma} + \left(B : D\tilde{\alpha}\right)\left(\nu \cdot \tilde{\gamma}\right) + \left(\nu \cdot \tilde{\alpha}\right)\left(B : D\tilde{\gamma}\right) + \|B\|^2 \left(\nu \cdot \tilde{\alpha}\right)\left(\nu \cdot \tilde{\gamma}\right) \mu \\
&= -\int_S \Pi^I{}_J \tilde{\alpha}_{I:K} \tilde{\gamma}^{J:K} + \nu_J B^K{}_I \tilde{\alpha}^I{}_{:K} \tilde{\gamma}^J + \nu_I B^K{}_J \tilde{\alpha}^I \tilde{\gamma}^J{}_{:K} + \nu_I \nu_J B^K{}_L B^L{}_K \tilde{\alpha}^I \tilde{\gamma}^J \mu
\end{aligned}
$$