1 Arbitrary s.p.d. metric

1.1 Assumptions

- $Ind(M) = 0$
- $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2$ (**s.p.d.**)

1.2 General proberties

$\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $\gamma \in \Omega^r(M)$, $\vec{v} \in \mathcal{V}(M)$

1.2.1 Wedge product $\wedge$

- $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$ (**anti-/commutativ**)
- **associativ** $(\alpha \wedge \beta \wedge \gamma)$
- $(c_1\alpha + c_2\beta) \wedge \gamma = c_1\alpha \wedge \gamma + c_2\beta \wedge \gamma$ (**bilinear**)

1.2.2 Exterior derivative $\mathbf{d} : \Omega^p(M) \to \Omega^{p+1}(M)$

$\alpha \in \Omega^p(M)$

- $\mathbf{d} \circ \mathbf{d} = 0$ (**complex proberty**)
- $\mathbf{d}\left(\alpha \wedge \beta\right) = \mathbf{d}\alpha \wedge \beta + (-1)^p\alpha \wedge \mathbf{d}\beta$ (**product rule, $\wedge$-antiderivation**)

1.2.3 Hodge star $* : \Omega^p(M) \to \Omega^{2-p}(M)$

- $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu$
- $*1 = \mu \quad (*\mu = 1)$
- $* * \alpha = (-1)^p\alpha$
- $\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$

1.2.4 Contraction $\mathbf{i} : \left(\mathcal{V} \times \Omega^p\right)(M) \to \Omega^{p-1}(M)$ (inner product)

- $\mathbf{i}_{\vec{v}}\alpha\left(\vec{t}_1, \dots \vec{t}_{p-1}\right) = \alpha\left(\vec{v}, \vec{t}_1, \dots \vec{t}_{p-1}\right)$
- $f\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{f\vec{v}}\alpha = \mathbf{i}_{\vec{v}}f\alpha$ (**bilinear**)
- $\mathbf{i}_{\vec{v}}\left(\alpha \wedge \beta\right) = \left(\mathbf{i}_{\vec{v}}\alpha\right) \wedge \beta + (-1)^p\alpha \wedge \left(\mathbf{i}_{\vec{v}}\beta\right)$ (**$\wedge$-antiderivation**)

1.2.5 Lie-derivative $\mathcal{L} : (\mathcal{V} \times \Omega^p)(M) \to \Omega^p(M)$

- $\mathcal{L}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\vec{v}}\alpha$ (**Cartans magic formular**)
- $\mathcal{L}_{f\vec{v}}\alpha = f\mathcal{L}_{\vec{v}}\alpha + \mathbf{d}f \wedge \mathbf{i}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}(\alpha \wedge \beta) = \mathcal{L}_{\vec{v}}\alpha \wedge \beta + \alpha \wedge \mathcal{L}_{\vec{v}}\beta$
- $\mathcal{L}_{\vec{v}}\mathbf{d}\alpha = \mathbf{d}\mathcal{L}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathcal{L}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}\vec{w} = [\vec{v}, \vec{w}] = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}$ ((**Levi-Civita-)Conection ∇ is Torsion-free**)

1.3 Wedge product $\wedge$

$f \in \Omega^0(M)$, $\tilde{f} \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\beta := b_1 dx^1 + b_2 dx^2 \in \Omega^1(M)$, $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $f\tilde{f} = f \wedge \tilde{f} = \tilde{f} \wedge f \in \Omega^0(M)$
- $f\alpha := f \wedge \alpha = \alpha \wedge f = f a_1 dx^1 + f a_2 dx^2 \in \Omega^1(M)$
- $\alpha \wedge \beta = -\beta \wedge \alpha = (a_1 b_2 - a_2 b_1)\, dx^1 \wedge dx^2 \in \Omega^2(M)$
- $f\omega := f \wedge \omega = \omega \wedge f = f w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

1.4 Exterior derivative d

$f \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$

- $\mathbf{d}f = \partial_1 f dx^1 + \partial_2 f dx^2$
- $(\mathbf{d}f)_\mu = \partial_\mu f$ (**Ricci**)
- $\mathbf{d}\alpha = (\partial_1 a_2 - \partial_2 a_1)\, dx^1 \wedge dx^2$
- $(\mathbf{d}\alpha)_{12} = (-1)^{\mu-1}\partial_\mu a_{\bar{\mu}}$ (**Ricci**)

1.5 Hodge star $*$

$f \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $*f = f\mu = \sqrt{|g|} f dx^1 \wedge dx^2$
- $*\alpha = \sqrt{|g|}\left(-\left(a_1 g^{12} + a_2 g^{22}\right) dx^1 + \left(a_1 g^{11} + a_2 g^{12}\right) dx^2\right)$
- $(*a)_\mu = (-1)^\mu \sqrt{|g|} g^{\nu\bar{\mu}} a_\nu = (-1)^\mu \sqrt{|g|} a^{\bar{\mu}}$ (**Ricci**)
- $*\omega = \frac{w_{12}}{\sqrt{|g|}}$

1.6 Rising and lowering indices $\sharp$ / $\flat$

$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\alpha^\sharp = \left(g^{11}a_1 + g^{12}a_2\right)\partial_1 + \left(g^{12}a_1 + g^{22}a_2\right)\partial_2$
- $a^\mu = g^{\mu\nu}a_\nu$ (**Ricci**)
- $\vec{v}^\flat = \left(g_{11}v^1 + g_{12}v^2\right)dx^1 + \left(g_{12}v^1 + g_{22}v^2\right)dx^2$
- $v_\mu = g_{\mu\nu}v^\nu$ (**Ricci**)

1.7 Contraction $\mathbf{i}$

$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$ $\vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\mathbf{i}_{\vec{v}}\alpha = \alpha\left(\vec{v}\right) = a_1v^1 + a_2v^2$
- $\mathbf{i}_{\vec{v}}\omega = w_{12}\left(-v^2dx^1 + v^1dx^2\right)$

1.8 Lie-derivative $\mathcal{L}$

$f \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$, $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$, $\vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\mathcal{L}_{\vec{v}}f = v^1\partial_1 f + v^2\partial_2 f$
- $\mathcal{L}_{\vec{v}}\alpha = \sum_{i,k=1,2}\left(v^k\partial_k a_i dx^i + a_i\partial_k v^i dx^k\right)$
- $\mathcal{L}_{\vec{v}}\omega = \left(\partial_1\left(w_{12}v^1\right) + \partial_2\left(w_{12}v^2\right)\right)dx^1 \wedge dx^2$
- $\mathcal{L}_{\vec{v}}\omega = \left(w_{12}\partial_\mu v^\mu + v^\mu\partial_\mu w_{12}\right)dx^1 \wedge dx^2$ (**Ricci**)

1.9 Levi-Civita-Connection (Contravariant Derivative)

- $\Gamma^k_{ij} = g^{kl}\Gamma_{ijl} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$ (**Christoffel symbols**)
- $\nabla_j v^i = v^i_{;j} = v^i_{|j} = \partial_j v^i + v^k\Gamma^i_{jk}$
- $\nabla\vec{v} := \left[\nabla_j v^i\right]^i_j$
- $\nabla_i f = \left[\nabla f\right]_i = \partial_i f$
- $\nabla_{\vec{v}}f = \mathcal{L}_{\vec{v}}f = \langle\vec{v}, \nabla_\Gamma f\rangle = (\mathbf{d}f)(\vec{v}) = v^i\nabla_i f = v^i\partial_i f$

1.10 Shape-Operator S, etc

- **Second fundamental form**:
  $[II]_{ij} = [S^\flat]_{ij} = h_{ij} = -\partial_i \vec{N} \cdot \partial_j \vec{X} = -\left[\nabla \vec{N}\right]_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{X}$
- **Shape operator (Weingarten map)**:
  $[S]^i_j = g^{ik} h_{kj} = -\left[\nabla_\Gamma \vec{N}\right]^i \cdot \partial_j \vec{X} = -\left[\nabla_\Gamma \vec{N}\right]^i_j$
- $[S(\vec{v})]_i = -\left[\nabla_{\vec{v}} \vec{N}\right]_i = v^j h_{ij}$
- $S^T \alpha = \alpha S = S(\alpha^\sharp)$

1.11 Conclusions

$\vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$

- $\mathrm{Grad} f = \nabla_\Gamma f = \nabla^\sharp f = (\mathbf{d} f)^\sharp$
  $[\mathrm{Grad} f]^i = \nabla^i f = g^{ij} \nabla_j f = g^{ij} \partial_j f$
- $$\mathrm{Div} \vec{v} = -\delta \vec{v}^\flat = *\mathbf{d} * \vec{v}^\flat = \nabla_i v^i = \partial_i v^i + v^k \Gamma^i_{ik} = \partial_i v^i + v^k \partial_k \log \sqrt{|g|}$$
  $$= \sum_{i=1,2} \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} v^i = \sum_{i=1,2} \frac{v^i}{\sqrt{|g|}} \partial_i \sqrt{|g|} + \partial_i v^i$$
- $\mathrm{Div}(f\vec{v}) = f \mathrm{Div} \vec{v} + \nabla_{\vec{v}} f = f \nabla_i v^i + v^i \nabla_i f$
- $\Delta_B f = -\delta \mathbf{d} f = *\mathbf{d} * \mathbf{d} f = \mathrm{Div}\mathrm{Grad} f = \nabla_i \nabla^i f = \frac{1}{\sqrt{|g|}} \partial_j \left(g^{ij} \sqrt{|g|} \partial_i f \right)$

1.12 Moving Surfaces

$\vec{V} := \vec{v} + v_n \vec{N}$ (surface velocity), $\vec{X} : M \to E^3$ (parametrization)

- $\partial_i \vec{V} \cdot \partial_j \vec{X} = g_{jk} \nabla_i v^k - v_n h_{ij} = \left[(\nabla \vec{v} - v_n S)^\flat \right]_{ij}$
- $$\frac{1}{2} \frac{d}{dt} \|\alpha\|^2 = \left\langle \dot{\alpha} + v_n S^T \alpha - (\nabla \vec{v})^T \alpha, \alpha \right\rangle = \dot{\alpha} \alpha^\sharp + \alpha \left(v_n S - \nabla \vec{v} \right) \alpha^\sharp$$
  $$= \dot{\alpha}_i \alpha^i + v_n \alpha_i h^i_j \alpha^j - \alpha_i \left(\nabla_j v^i \right) \alpha^j$$