Notes On Nonic Surfaces Experiment

February 15, 2016

1 Surface Descriptions

We are starting with the standard parametrization of the unit sphere $\mathbb{S}^2$ with local coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, i.e.,

$$\mathbf{x}_{\mathbb{S}^2}(\theta, \phi) = \sin\theta\cos\phi\boldsymbol{e}^x + \sin\theta\sin\phi\boldsymbol{e}^y + \cos\theta\boldsymbol{e}^z \,. \tag{1}$$

For stretching the unit sphere by a displacement function $f : [-1, 1] \to \mathbb{R}$ in the x-direction depending on the z-positions and pressing to the x-z-plane by a press factor $B \in [0, 1)$, we obtain the surface

$$\mathbf{x}_{f,B}(\theta, \phi) := \mathbf{x}_{\mathbb{S}^2}(\theta, \phi) + f(\cos\theta)\boldsymbol{e}^x - B\sin\theta\sin\phi\boldsymbol{e}^y \,, \tag{2}$$

i. e., specially for $B \nearrow 1$ the surface becomes flat. We choose for the displacement function f a double well function, which should break the symmetry referring to the x-y-plane, so that the north pole ($z = 1$) of the initial sphere is shifting in x-direction by $C > 0$ and the south pole ($z = -1$) by $r \cdot C$ with the proportion factor $0 \le r < 1$. This implies

$$f(z) := f_{C,r}(z) = \frac{1}{4}Cz^2\left[(z+1)^2(4-3z) + r(z-1)^2(4+3z)\right] \tag{3}$$

where the double well conditions $f(1) = C$, $f(-1) = r\cdot C$ and $f'(1) = f'(0) = f'(-1) = 0$ are fulfilled, see for example Figure 1. For the immersion $\mathbf{x}_{B,C,r} := \mathbf{x}_{f,B} : [0, \pi]\times[0, 2\pi) \to \mathbb{R}^3$ the surface family $\mathcal{S}_{B,C,r} := \mathrm{Im}(\mathbf{x}_{B,C,r})$ can also expressed implicitly by the 0-Levelset of the function

$$\varphi_{B,C,r}(x, y, z) := (x - f_{C,r}(z))^2 + \frac{1}{(1-B)^2}y^2 + z^2 - 1 \tag{4}$$

defined in a smooth neighbourhood of the surface. We call $\mathcal{S}_{B,C,r}$ a **Nonic Surface**, because $\varphi_{B,C,r}$ is a polynomial of degree 10. The gradient

$$\nabla\varphi_{B,C,r}(x, y, z) = 2\begin{bmatrix} x - f_{C,r}(z) \\ \frac{y}{(1-B)^2} \\ z - (x - f_{C,r}(z))f'_{C,r}(z) \end{bmatrix} \,, \tag{5}$$

restricted to the surface, points in the direction of the outer surface normals.

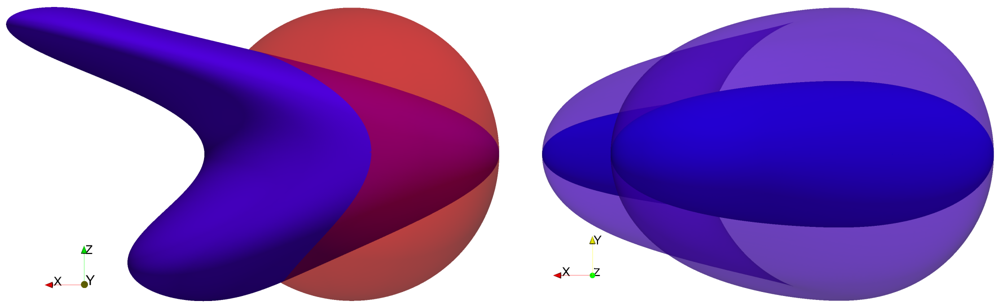

Figure 1: Nonic Surface with parameters $r = 0.5$, $C = 2$ and $B = 0.5$. The left figure shows the stretching of the unit sphere in the x-direction. Hence, by the choice of the parameter, the north pole ($z = 1$) is shifting by $C = 2$ and the south pole ($z = -1$) by $r \cdot C = 1$ units of length to the left. The right figure shows the pressing of the resulting surface to the x-z-plane by the press factor $B = 0.5$.

2 Initial Solutions Construction for the Frank-Oseen-Equations

To solve the director field evololutions in paper **NUMERICAL METHODS FOR ORIENTATIONAL ORDER ON SURFACES**, we have to assign initial fields $\mathbf{p}$, $\boldsymbol{\alpha} = \mathbf{p}^\flat$ respectively, with $\|\mathbf{p}\| = \|\boldsymbol{\alpha}\| = 1$ a.e..

2.1 4 Defect Init

The 4 defect configuration, 3 with positive charge at the bulges and 1 with negative charge at the saddle point, is potentially stable depending on the choice of the surface parameter. The proportion factor $r \in [0, 1)$ prevent a metastable solution, because the resulting symmetry break induce different dynamics for 2 defect locations on the bulges. This implies, that the defect on the smaller bulge and the saddle point defect will mutually annihilate, if the 4 defect configuration is not pure stable, see e. g., Figure 2. For the initial solution $\boldsymbol{\alpha}^0$ we can use the x-coordinate potential, i. e.,

$$\boldsymbol{\alpha}^0 = \frac{\mathbf{d}x}{\|\mathbf{d}x\|_\varepsilon}, \tag{6}$$

where

$$\|\mathbf{q}\|_\varepsilon = \begin{cases} \infty & \text{if } \|\mathbf{q}\| < \varepsilon \\ \|\mathbf{q}\| & \text{else} \end{cases} \tag{7}$$

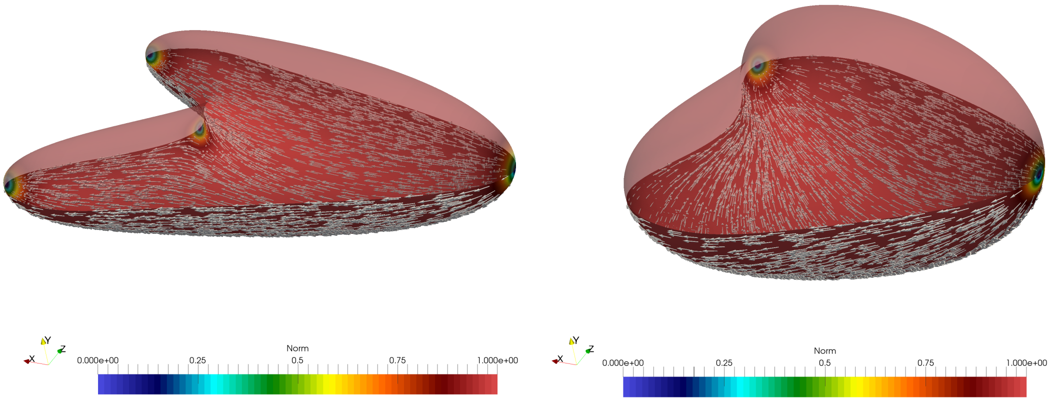

Figure 2: Nonic Surfaces with $r = 0.95$. In the left figure ($B = 0.56$, $C = 1.6$) we see a directional field with stable 4 defect configuration. In the right figure ($B = 0.2625$, $C = 0.75$) the 4 defect initial configuration was not stable, therefor the system was finally gasp to a 2 defect solution.

to prevent ill well-defined in the defect locations. In our experiments ε is mostly chosen by 10^{-10}. Hence, the corresponding contravariant vector field is

$$\mathbf{p}^0 = \frac{\operatorname{grad} x}{\|\operatorname{grad} x\|_\varepsilon} . \tag{8}$$

With the projection map

$$\pi_{\mathcal{S}} = I - \frac{\nabla\varphi}{\|\nabla\varphi\|} \otimes \frac{\nabla\varphi}{\|\nabla\varphi\|} \tag{9}$$

we can use in euclidean coordinates the identity

$$\operatorname{grad} x = \pi_{\mathcal{S}} \nabla x = \pi_{\mathcal{S}} \boldsymbol{e}^x . \tag{10}$$

2.1.1 PD-1-Form Discretization

We can discretize the exact 1-form $\mathbf{d}x$ on an edge $e = [v_1, v_2] \in \mathcal{E}$ by (Stokes theorem)

$$(\mathbf{d}x)_h (e) = v_2^x - v_1^x . \tag{11}$$

If the face $T_1 \succ e$ is right of the edge e and $T_2 \succ e$ located left, so that $\star e = [c(T_1), c(e)] + [c(e), c(T_2)]$ is the dual edge, than we can approximate

$$(*\mathbf{d}x)_h (e) = -\frac{|e|}{|\star e|} (\mathbf{d}x)_h (\star e) = -\frac{|e|}{|\star e|} \left([c(T_2)]^x - [c(T_1)]^x \right) \tag{12}$$

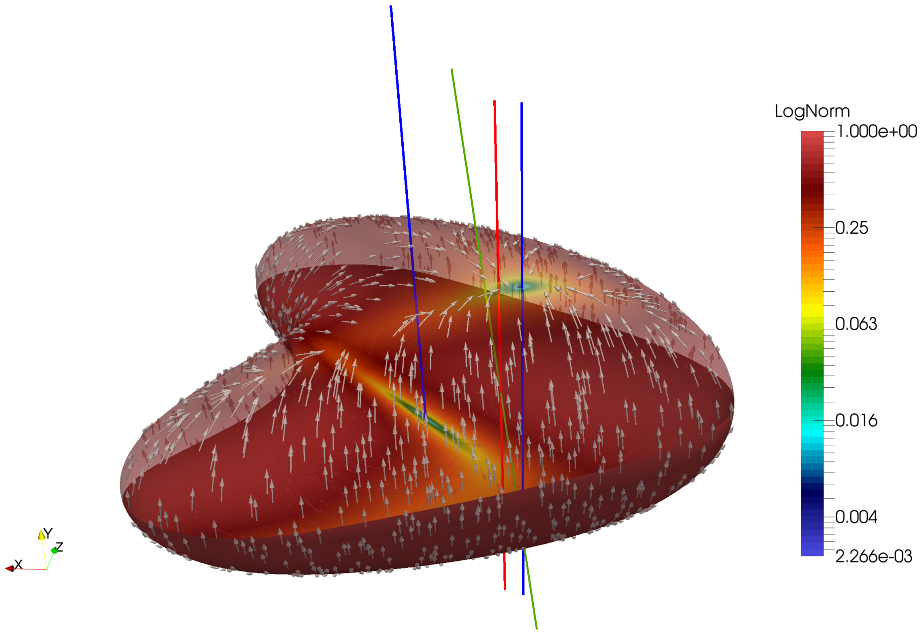

Figure 3: Nonic Surface with $r = 0.95$, $B = 0.35$ and $C = 1$. The green line is the y-axis and the red line is the rotated y-axis throw the origin. This is a rotation by a radian of $\gamma = 1.5$ in the normal plane of the vector $[-1, 0, 1]^T$. The defect locations are at the points, where the rotated y-axis is orthogonal to the surface (see blue lines). The colouring is the logarithm of the norm of the resulting unnormalized vector field $\check{\mathbf{p}}^0$. The arrows show the normalized vector field $\mathbf{p}^0$.

With the discrete norm (**??**) of PD-1-forms, we obtain the discrete initial PD-1-form on $e \in \mathcal{E}$ by

$$\underline{\boldsymbol{\alpha}}^0(e) = \frac{\left(v_2^x - v_1^x, -\frac{|e|}{|\star e|}\left([c(T_2)]^x - [c(T_1)]^x\right)\right)}{\sqrt{\frac{1}{|e|^2}\left(v_2^x - v_1^x\right)^2 + \frac{1}{|\star e|^2}\left([c(T_2)]^x - [c(T_1)]^x\right)^2}}\,, \tag{13}$$

if $\sqrt{\frac{1}{|e|^2}\left(v_2^x - v_1^x\right)^2 + \frac{1}{|\star e|^2}\left([c(T_2)]^x - [c(T_1)]^x\right)^2} \geq \varepsilon$, else we set $\underline{\boldsymbol{\alpha}}^0(e) = (0, 0)$.

2.2 2 Defect Init

To provoke a 2 defect solution in the equilibrium, like in Figure 2 (right), we use a normalized projected slightly rotated $\boldsymbol{e}^y$ Field, see e. g., Figure 3. With the symmetry of the surface, $\pi_{\mathcal{S}}\boldsymbol{e}^y$ would be result in a metastable state. To disturb this, we define a rotation R_γ by an angle γ in the normal plane of the vector $[-1, 0, 1]^T$, i. e.,

$$R_\gamma := \begin{bmatrix} \frac{1+\cos\gamma}{2} & -\frac{\sin\gamma}{\sqrt{2}} & \frac{-1+\cos\gamma}{2} \\ \frac{\sin\gamma}{\sqrt{2}} & \cos\gamma & \frac{\sin\gamma}{\sqrt{2}} \\ \frac{-1+\cos\gamma}{2} & -\frac{\sin\gamma}{\sqrt{2}} & \frac{1+\cos\gamma}{2} \end{bmatrix} . \tag{14}$$

Hence, we get the unnormalized vector field $\check{\mathbf{p}}^0 := \pi_{\mathcal{S}} R_\gamma \boldsymbol{e}^y$. The advantage of $\check{\mathbf{p}}^0$ is that one of the two defects is closer on the larger bulge, so that the defect move to them in the evolution and not to the smaller bulge. By normalizing we get the initial director field $\mathbf{p}^0 = \frac{\check{\mathbf{p}}^0}{\|\check{\mathbf{p}}^0\|_\varepsilon}$. We use $\gamma = 0.05$ in our experiment, because a small radian is enough to influence the dynamic in the way we want and a larger γ would be result in long "defect lines" at the beginning, which can particular split up to 2 defects, for large stretch factors C.

2.2.1 PD-1-Form Discretization

By definition of the dual basis, we can usually flat a vector field $\mathbf{p}$ to 1-form in the continuum, with testing the vector field with its basis. The evaluation of a vector field with the dual edge vector $\boldsymbol{e}_\star$ on a edge e, i. e., at the intersection $e \cap \star e = c(e)$, is ambiguous, hence we define canonical to the definition of the dual 1-chain $\star e = \star e|_{T_1} + \star e|_{T_2}$

$$\mathbf{p}(c(e)) \cdot \boldsymbol{e}_\star := \mathbf{p}(c(e)) \cdot (\boldsymbol{e}_\star|_{T_1} + \boldsymbol{e}_\star|_{T_2}) = \mathbf{p}(c(e)) \cdot (c(T_2) - c(T_1)) \ , \tag{15}$$

where the face $T_1 \succ e$ is right of the edge e and $T_2 \succ e$ is located left. Hence, we get for the initial discrete PD-1-form

$$\underline{\boldsymbol{\alpha}}^0(e) = \frac{\left(\check{\mathbf{p}}^0(c(e)) \cdot \boldsymbol{e}, -\frac{|e|}{|\star e|}\check{\mathbf{p}}^0(c(e)) \cdot \boldsymbol{e}_\star\right)}{\sqrt{\frac{1}{|e|^2}\left(\check{\mathbf{p}}^0(c(e)) \cdot \boldsymbol{e}\right)^2 + \frac{1}{|\star e|^2}\left(\check{\mathbf{p}}^0(c(e)) \cdot \boldsymbol{e}_\star\right)^2}} \ , \tag{16}$$

if $\sqrt{\frac{1}{|e|^2}\left(\check{\mathbf{p}}^0(c(e)) \cdot \boldsymbol{e}\right)^2 + \frac{1}{|\star e|^2}\left(\check{\mathbf{p}}^0(c(e)) \cdot \boldsymbol{e}_\star\right)^2} \geq \varepsilon$, else we set $\underline{\boldsymbol{\alpha}}^0(e) = (0, 0)$.

3 Experiments

To study the energy and the stability of the defects behavior, we use a sequence of nonic surfaces, starting with the unit sphere, i. e., $B = C = 0$, up to $B = 0.7$ and $C = 2.0$, see Figure 4. We always chose the press factor B and the stretch factor C in the same ratio, i. e., $C : B = 20 : 7$. The proportion factor is set to $r = 0.95$ and the radian for the 2 defect initial solution is $\gamma = 0.05$, see Figure 5 results. For the parameter discretization see Appendix.

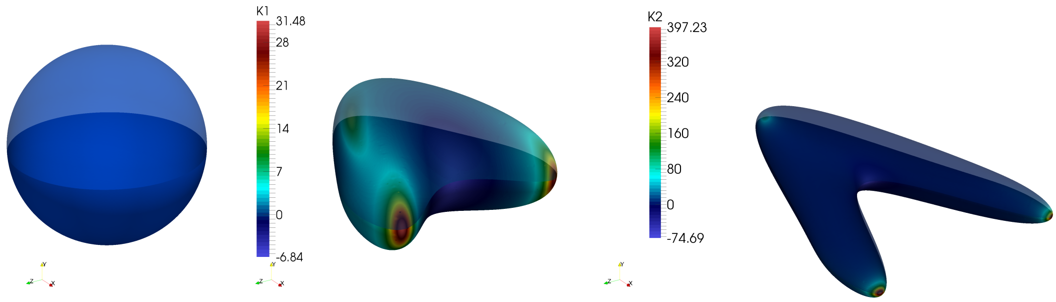

Figure 4: Nonic Surface with $r = 0.95$. Form left to right:$B = C = 0$, $B = 0.35$ and $C = 1$, $B = 0.7$ and $C = 2$. The colouring represent the Gaussian curvature (on unit Sphere $K = 1$).

4 Appendix

4.1 Parameter Discretization

We use a ID of the form $XXXX\dots$ with $X \in \{1, \dots, 9\}$ to identify the nonic surfaces. The longer the ID the finer is the choice of the parameter B and C. Furthermore, every surface get a name, which contains the parameter r, B and C, of the form `nonic`$[\![r]\!]$`r`$[\![C]\!]$`c`$[\![B]\!]$`b`, where $[\![x]\!]$ is a point-free representation of a floating number $x \in [0, 10)$, e. g., $[\![0.51]\!] =$ `051` or $[\![4.2]\!] =$ `42`. The 9 base surfaces, which discretized the parameter space are

ID	B	C	Name
1	0.0	0.0	`nonic095r0c0b`
2	0.0875	0.25	`nonic095r025c00875b`
3	0.175	0.5	`nonic095r05c0175b`
4	0.2625	0.75	`nonic095r075c02625b`
5	0.35	1.0	`nonic095r1c035b`
6	0.4375	1.25	`nonic095r125c04375b`
7	0.525	1.5	`nonic095r15c0525b`
8	0.6125	1.75	`nonic095r175c06125b`
9	0.7	2.0	`nonic095r2c07b`

(17)

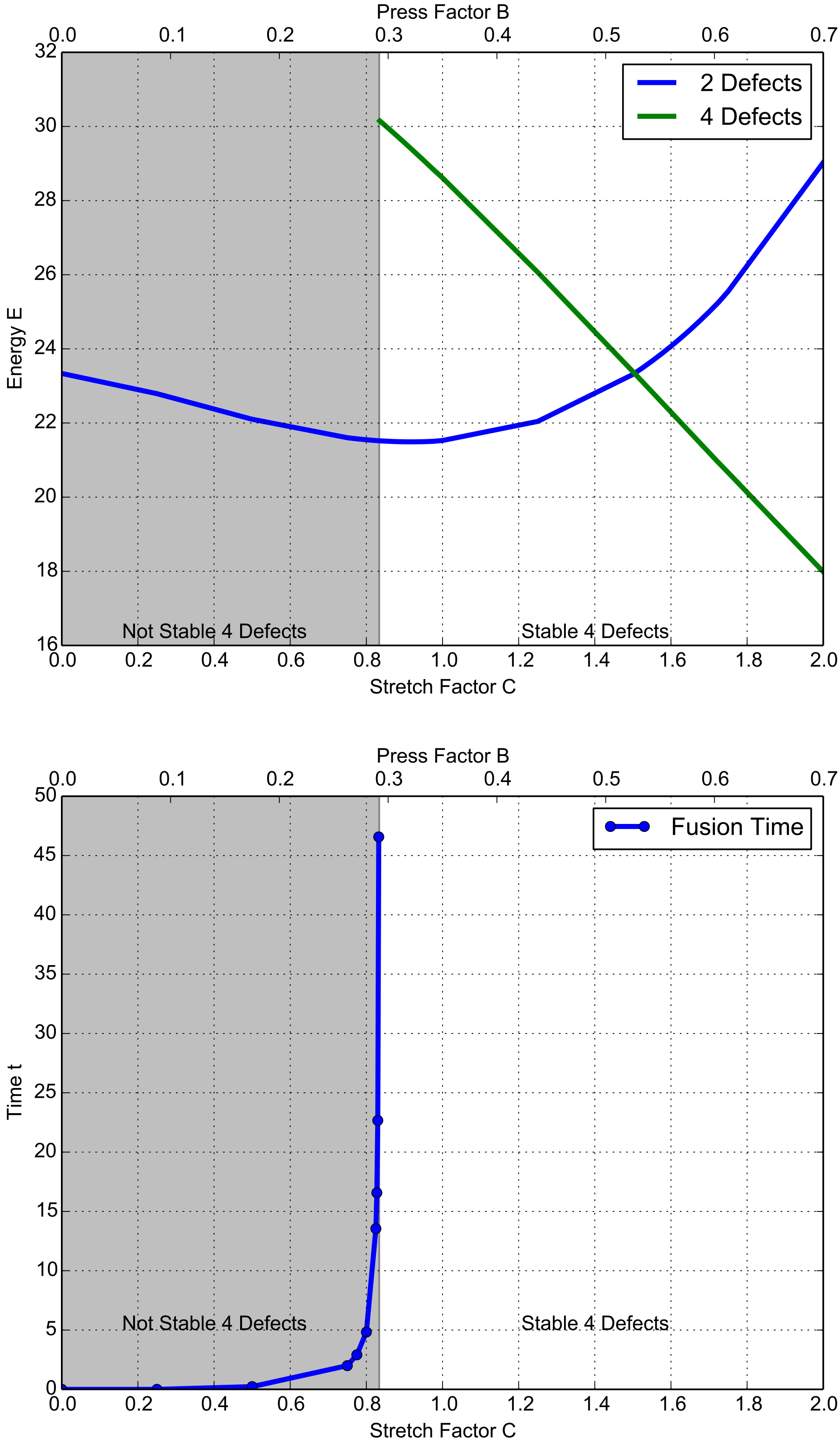

Figure 5: Experiment on Nonic Surface with $r = 0.95$.

For a better resolution of the parameter space, where the 4 defect solution becomes stable, we refine between the ID 4 and 5, i. e.,

ID	B	C	Name
41	0.27125	0.775	`nonic095r0775c027125b`
42	0.28	0.8	`nonic095r08c028b`
43	0.28875	0.825	`nonic095r0825c028875b`
44	0.2975	0.85	`nonic095r085c02975b`
45	0.30625	0.875	`nonic095r0875c030625b`
46	0.315	0.9	`nonic095r09c0315b`
47	0.32375	0.925	`nonic095r0925c032375b`
48	0.3325	0.95	`nonic095r095c03325b`
49	0.34125	0.975	`nonic095r0975c034125b`

(18)

and again between 43 and 44, i. e.,

ID	B	C	Name
431	0.289625	0.8275	`nonic095r08275c0289625b`
432	0.2905	0.83	`nonic095r083c02905b`
433	0.291375	0.8325	`nonic095r08325c0291375b`
434	0.29225	0.835	`nonic095r0835c029225b`
435	0.293125	0.8375	`nonic095r08375c0293125b`
436	0.294	0.84	`nonic095r084c0294b`
437	0.294875	0.8425	`nonic095r08425c0294875b`
438	0.29575	0.845	`nonic095r0845c029575b`
439	0.296625	0.8475	`nonic095r08475c0296625b`

. (19)

For the parameter region, where the 4 defect solution becomes cheaper than the 2 defect solution, we refine between ID 7 and 8, i. e.,

ID	B	C	Name
71	0.53375	1.525	`nonic095r1525c053375b`
72	0.5425	1.55	`nonic095r155c05425b`
73	0.55125	1.575	`nonic095r1575c055125b`
74	0.56	1.6	`nonic095r16c056b`
75	0.56875	1.625	`nonic095r1625c056875b`
76	0.5775	1.65	`nonic095r165c05775b`
77	0.58625	1.675	`nonic095r1675c058625b`
78	0.595	1.7	`nonic095r17c0595b`
79	0.60375	1.725	`nonic095r1725c060375b`

. (20)

4.2 Some Reverse Transformations

$$\cos\theta = z \tag{21}$$

$$\sin\theta = \sqrt{1-z^2} \tag{22}$$

$$\cot\theta = \frac{z}{\sqrt{1-z^2}} \tag{23}$$

$$\csc\theta = \frac{1}{\sqrt{1-z^2}} \tag{24}$$

$$\cos\phi = \frac{x-f(z)}{\sqrt{1-z^2}} \tag{25}$$

$$\sin\phi = \frac{y}{(1-B)\sqrt{1-z^2}} \tag{26}$$

$$\tag{27}$$