

1 Arbitrary s.p.d. metric

1.1 Assumptions

- $Ind(M) = 0$
- $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = g_{11} (dx^1)^2 + 2g_{12} dx^1 dx^2 + g_{22} (dx^2)^2$ (s.p.d.)

1.2 General properties

$$\alpha \in \Omega^p(M), \beta \in \Omega^q(M), \gamma \in \Omega^r(M), \vec{v} \in \mathcal{V}(M)$$

1.2.1 Wedge product \wedge

- $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ (**anti-/commutativ**)
- **associativ** $(\alpha \wedge \beta \wedge \gamma)$
- $(c_1 \alpha + c_2 \beta) \wedge \gamma = c_1 \alpha \wedge \gamma + c_2 \beta \wedge \gamma$ (**bilinear**)

1.2.2 Exterior derivative $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$

$$\alpha \in \Omega^p(M)$$

- $d \circ d = 0$ (**complex property**)
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ (**product rule, \wedge -antiderivation**)

1.2.3 Hodge star $*$: $\Omega^p(M) \rightarrow \Omega^{2-p}(M)$

- $\alpha \wedge * \beta = \beta \wedge * \alpha = \langle \alpha, \beta \rangle \mu$
- $*1 = \mu$ ($*\mu = 1$)
- $**\alpha = (-1)^p \alpha$
- $\langle \alpha, \beta \rangle = \langle * \alpha, * \beta \rangle$

1.2.4 Contraction $i : (\mathcal{V} \times \Omega^p)(M) \rightarrow \Omega^{p-1}(M)$ (**inner product**)

- $i_{\vec{v}} \alpha(\vec{t}_1, \dots, \vec{t}_{p-1}) = \alpha(\vec{v}, \vec{t}_1, \dots, \vec{t}_{p-1})$
- $f i_{\vec{v}} \alpha = i_{f\vec{v}} \alpha = i_{\vec{v}} f \alpha$ (**bilinear**)
- $i_{\vec{v}}(\alpha \wedge \beta) = (i_{\vec{v}} \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_{\vec{v}} \beta)$ (**\wedge -antiderivation**)

1.2.5 Lie-derivative $\mathcal{L} : (\mathcal{V} \times \Omega^p)(M) \rightarrow \Omega^p(M)$

- $\mathcal{L}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\vec{v}}\alpha$ (**Cartans magic formular**)
- $\mathcal{L}_{f\vec{v}}\alpha = f\mathcal{L}_{\vec{v}}\alpha + \mathbf{d}f \wedge \mathbf{i}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}(\alpha \wedge \beta) = \mathcal{L}_{\vec{v}}\alpha \wedge \beta + \alpha \wedge \mathcal{L}_{\vec{v}}\beta$
- $\mathcal{L}_{\vec{v}}\mathbf{d}\alpha = \mathbf{d}\mathcal{L}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathcal{L}_{\vec{v}}\alpha$
 $\Rightarrow \alpha \in \Omega^1(M) : \mathcal{L}_{\vec{v}}\langle \vec{v}^b, \alpha \rangle = \langle \vec{v}^b, \mathcal{L}_{\vec{v}}\alpha \rangle$
- $\mathcal{L}_{\vec{v}}\vec{w} = [\vec{v}, \vec{w}] = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}$ (**(Levi-Civita-)Conection ∇ is Torsion-free**)

1.3 Wedge product \wedge

$f \in \Omega^0(M)$, $\tilde{f} \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\beta := b_1 dx^1 + b_2 dx^2 \in \Omega^1(M)$,
 $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $f\tilde{f} = f \wedge \tilde{f} = \tilde{f} \wedge f \in \Omega^0(M)$
- $f\alpha := f \wedge \alpha = \alpha \wedge f = f a_1 dx^1 + f a_2 dx^2 \in \Omega^1(M)$
- $\alpha \wedge \beta = -\beta \wedge \alpha = (a_1 b_2 - a_2 b_1) dx^1 \wedge dx^2 \in \Omega^2(M)$
- $f\omega := f \wedge \omega = \omega \wedge f = f w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

1.4 Exterior derivative \mathbf{d}

$f \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$

- $\mathbf{d}f = \partial_1 f dx^1 + \partial_2 f dx^2$
- $(\mathbf{d}f)_\mu = \partial_\mu f$ (**Ricci**)
- $\mathbf{d}\alpha = (\partial_1 a_2 - \partial_2 a_1) dx^1 \wedge dx^2$
- $(\mathbf{d}\alpha)_{12} = (-1)^{\mu-1} \partial_\mu a_{\bar{\mu}}$ (**Ricci**)

1.5 Hodge star $*$

$f \in \Omega^0(M)$, $\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M)$

- $*f = f\mu = \sqrt{|g|} f dx^1 \wedge dx^2$
- $*\alpha = \sqrt{|g|} (- (a_1 g^{12} + a_2 g^{22}) dx^1 + (a_1 g^{11} + a_2 g^{12}) dx^2)$
- $(*a)_\mu = (-1)^\mu \sqrt{|g|} g^{\nu\bar{\mu}} a_\nu = (-1)^\mu \sqrt{|g|} a^{\bar{\mu}}$ (**Ricci**)
- $*\omega = \frac{w_{12}}{\sqrt{|g|}}$

1.6 Rising and lowering indices \sharp / \flat

$$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\alpha^\sharp = (g^{11}a_1 + g^{12}a_2) \partial_1 + (g^{12}a_1 + g^{22}a_2) \partial_2$
- $a^\mu = g^{\mu\nu} a_\nu$ (**Ricci**)
- $\vec{v}^\flat = (g_{11}v^1 + g_{12}v^2) dx^1 + (g_{12}v^1 + g_{22}v^2) dx^2$
- $v_\mu = g_{\mu\nu} v^\nu$ (**Ricci**)

1.7 Contraction \mathbf{i}

$$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\mathbf{i}_{\vec{v}}\alpha = \alpha(\vec{v}) = a_1 v^1 + a_2 v^2$
- $\mathbf{i}_{\vec{v}}\omega = w_{12} (-v^2 dx^1 + v^1 dx^2)$

1.8 Lie-derivative \mathcal{L}

$$f \in \Omega^0(M), \alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M), \omega := w_{12} dx^1 \wedge dx^2 \in \Omega^2(M), \vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\mathcal{L}_{\vec{v}}f = v^1 \partial_1 f + v^2 \partial_2 f$
- $\mathcal{L}_{\vec{v}}\alpha = \sum_{i,k=1,2} (v^k \partial_k a_i dx^i + a_i \partial_k v^i dx^k)$
- $\mathcal{L}_{\vec{v}}\omega = (\partial_1 (w_{12} v^1) + \partial_2 (w_{12} v^2)) dx^1 \wedge dx^2$
- $\mathcal{L}_{\vec{v}}\omega = (w_{12} \partial_\mu v^\mu + v^\mu \partial_\mu w_{12}) dx^1 \wedge dx^2$ (**Ricci**)

1.9 Levi-Civita-Connection (co-/contravariant derivatives)

- $\Gamma_{ij}^k = g^{kl} \Gamma_{ijl} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ (**Christoffel symbols**)
- $\nabla_j v^i = v^i_{;j} = v^i_{|j} = \partial_j v^i + v^k \Gamma_{jk}^i$
- $\nabla \vec{v} := [\nabla_j v^i]_j^i$
- $[\nabla \vec{v}^\flat]_{ji} = [(\nabla \vec{v}) g]_{ji} = \nabla_j v_i = v_{i;j} = v_{i|j} = \partial_j v_i - v_k \Gamma_{ij}^k = g_{il} \nabla_j v^l$
- $[\nabla^\sharp \vec{v}^\flat]_i^j = [g^{-1} (\nabla \vec{v}) g]_i^j = \nabla^j v_i = v_i^{;j} = v_i^{|j} = g^{jk} g_{il} \nabla_k v^l$
- $\nabla_i f = [\nabla f]_i = \partial_i f$
- $\nabla_{\vec{v}} f = \mathcal{L}_{\vec{v}} f = \langle \vec{v}, \nabla_\Gamma f \rangle = (\mathbf{d}f)(\vec{v}) = v^i \nabla_i f = v^i \partial_i f$

1.10 Shape-Operator S , etc

- **Second fundamental form:**
 $[II]_{ij} = [S^b]_{ij} = h_{ij} = -\partial_i \vec{N} \cdot \partial_j \vec{X} = -\left[\nabla \vec{N}\right]_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{X}$
- **Shape operator (Weingarten map):**
 $[S]_j^i = g^{ik} h_{kj} = -\left[\nabla_\Gamma \vec{N}\right]^i \cdot \partial_j \vec{X} = -\left[\nabla_\Gamma \vec{N}\right]_j^i$
- **Inverse of second fundamental form:**
 $b^{ij} = [II^{-1}]^{ij} = \frac{1}{|g|K} [II^{\text{Adj}}]^{ij}$
- $[S(\vec{v})]_i = -\left[\nabla_{\vec{v}} \vec{N}\right]_i = v^j h_{ij}$
- $S^T \alpha = \alpha S = S(\alpha^\sharp)$

1.11 Conclusions

$$\vec{v} := v^1 \partial_1 + v^2 \partial_2 \in \mathcal{V}(M)$$

- $\text{Grad} f = \nabla_\Gamma f = \nabla^\sharp f = (\mathbf{d}f)^\sharp$
 $[\text{Grad} f]^i = \nabla^i f = g^{ij} \nabla_j f = g^{ij} \partial_j f$
- $\text{Div} \vec{v} = -\delta \vec{v}^\flat = * \mathbf{d} * \vec{v}^\flat = \nabla_i v^i = \partial_i v^i + v^k \Gamma_{ik}^i = \partial_i v^i + v^k \partial_k \log \sqrt{|g|}$

$$= \sum_{i=1,2} \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} v^i = \sum_{i=1,2} \frac{v^i}{\sqrt{|g|}} \partial_i \sqrt{|g|} + \partial_i v^i$$
- $\delta(f\alpha) = f\delta\alpha - \langle \mathbf{d}f, \alpha \rangle$
- $\text{Div}(f\vec{v}) = f\text{Div} \vec{v} + \nabla_{\vec{v}} f = f\nabla_i v^i + v^i \nabla_i f$
- **Laplace-Beltrami operator:**
 $\Delta_B f = -\delta \mathbf{d}f = * \mathbf{d} * \mathbf{d}f = \text{Div Grad} f = \nabla_i \nabla^i f = \frac{1}{\sqrt{|g|}} \partial_j \left(g^{ij} \sqrt{|g|} \partial_i f \right)$
- **Laplace-de Rham operator:**
 $\Delta_{dR} \alpha = (\delta \mathbf{d} + \mathbf{d} \delta) \alpha =: -(\Delta_B + \Delta_{CB}) \alpha$
 $\Delta_{dR} \vec{v} = (\Delta_{dR} \vec{v}^\flat)^\sharp$
- **Giaquinta-Hildebrandt operator:**
 $\Delta_{GH} f = \square f = \text{Div} (K II^{-1} \mathbf{d}f) = -\delta (K S^{-T} \mathbf{d}f) = \frac{1}{\sqrt{|g|}} \partial_j \left(\sqrt{|g|} K b^{ij} \partial_i f \right)$
- $-\delta(S^T \alpha) = -H \text{Div} \alpha^\sharp - \text{Div} (K II^{-1} \alpha) - \nabla_{\alpha^\sharp} H = H \delta \alpha + \delta(K S^{-T} \alpha) - \langle \alpha, \mathbf{d}H \rangle$
 $-\delta(S^T \mathbf{d}f) = -H \Delta_B f - \Delta_{GH} f - \langle \mathbf{d}H, \mathbf{d}f \rangle$

1.12 Moving Surfaces $M(t)$

$\vec{V} := \vec{v} + v_n \vec{N} = \partial_t \vec{X}$ (surface velocity), $\vec{X} : M \rightarrow E^3$ (parametrization)

- $\partial_i \vec{V} \cdot \partial_j \vec{X} = g_{jk} \nabla_i v^k - v_n h_{ij} = \left[(\nabla \vec{v} - v_n S)^b \right]_{ij}$
- (rate-of-deformation tensor \mathbf{d})

$$\frac{d}{dt} g = \left(\nabla \vec{v}^b \right) + \left(\nabla \vec{v}^b \right)^T - 2v_n H = \mathcal{L}_{\vec{V}} g = 2\mathbf{d}$$

$$\frac{d}{dt} g_{ij} = g_{ik} \nabla_j v^k + g_{jk} \nabla_i v^k - 2v_n h_{ij} = 2d_{ij}$$
- $\left\langle \frac{d}{dt} \vec{p}^b, \alpha \right\rangle = \dot{\vec{p}} \alpha + 2\vec{p} \mathbf{d} \alpha^\sharp = \dot{\vec{p}} \alpha + \vec{p} \left((\nabla \vec{v})^T + \left(\nabla^\sharp \vec{v}^b \right) - 2 * v_n S^T \right) \alpha$

$$= \dot{p}^i \alpha_i + p^j \left(\nabla_j v^i + \nabla^i v_j - 2v_n h_j^i \right) \alpha_i$$
- $\frac{d}{dt} \alpha^\sharp = \left[\dot{\alpha} + \left(2v_n S^T - (\nabla \vec{v})^T - \left(\nabla^\sharp \vec{v}^b \right) \right) \alpha \right]^\sharp$
- $\frac{d}{dt} * \omega = * [\dot{\omega} - (\text{Div} \vec{v} + v_n H) \omega]$
- $\frac{d}{dt} * \vec{p}^b = * \left[\dot{\vec{p}} + (\text{Div} \vec{v} + v_n H) \vec{p} \right]^b$
- $\frac{d}{dt} * \alpha = * \left[\dot{\alpha} + \left(2v_n S^T - (\nabla \vec{v})^T - \left(\nabla^\sharp \vec{v}^b \right) \right) \alpha + (\text{Div} \vec{v} + v_n H) \alpha \right]$
- $\frac{1}{2} \frac{d}{dt} \|\alpha\|^2 = \langle \dot{\alpha} + v_n S^T \alpha - (\nabla \vec{v})^T \alpha, \alpha \rangle = \dot{\alpha} \alpha^\sharp + \alpha (v_n S - \nabla \vec{v}) \alpha^\sharp$

$$= \dot{\alpha}_i \alpha^i + v_n \alpha_i h_j^i \alpha^j - \alpha_i (\nabla_j v^i) \alpha^j$$
- $\frac{1}{2} \frac{d}{dt} \|\omega\|^2 = \frac{1}{2} \frac{d}{dt} (*\omega)^2 = \langle \dot{\omega}, \omega \rangle - (\text{Div} \vec{v} + v_n H) \|\omega\|^2$
- $\frac{1}{2} \frac{d}{dt} \|\delta \alpha\|^2 = \left\langle \delta \left[\dot{\alpha} + \left(2v_n S^T - (\nabla \vec{v})^T - \left(\nabla^\sharp \vec{v}^b \right) \right) \alpha + (\text{Div} \vec{v} + v_n H) \alpha \right], \delta \alpha \right\rangle - (\text{Div} \vec{v} + v_n H) \|\delta \alpha\|^2$
- $\frac{1}{2} \frac{d}{dt} \|\delta \vec{p}^b\|^2 = \left\langle \delta \left[\dot{\vec{p}} + (\text{Div} \vec{v} + v_n H) \vec{p} \right]^b, \delta \vec{p}^b \right\rangle - (\text{Div} \vec{v} + v_n H) \|\delta \vec{p}^b\|^2$
- $\frac{d}{dt} \int_{M(t)} f \mu = \int_{M(t)} \dot{f} + f (\text{Div} \vec{v} + v_n H) \mu$
- $\frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\alpha\|^2 \mu = \int_{M(t)} \langle \dot{\alpha}, \alpha \rangle + v_n \left\langle S^T \alpha + \frac{1}{2} H \alpha, \alpha \right\rangle + \left\langle \frac{1}{2} (\text{Div} \vec{v}) \alpha - (\nabla \vec{v})^T \alpha, \alpha \right\rangle \mu$
- $\frac{d}{dt} \int_{M(t)} \frac{1}{2} \|\omega\|^2 \mu = \int_{M(t)} \langle \dot{\omega}, \omega \rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \|\omega\|^2 \mu$

- $$\begin{aligned}
\frac{d}{dt} \int_{M(t)} \frac{1}{2} \left\| \mathbf{d}\vec{p}^\flat \right\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^\flat, \delta \mathbf{d}\vec{p}^\flat \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \mathbf{d}\vec{p}^\flat \right\|^2 \mu \\
&= \int_{M(t)} \dot{\vec{p}} \delta \mathbf{d}\vec{p}^\flat + \vec{p} \left((\nabla \vec{v})^T + \left(\nabla^\sharp \vec{v}^\flat \right) - 2v_n S^T \right) \delta \mathbf{d}\vec{p}^\flat \\
&\quad - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \mathbf{d}\vec{p}^\flat \right\|^2 \mu
\end{aligned}$$
- $$\begin{aligned}
\frac{d}{dt} \int_{M(t)} \frac{1}{2} \left\| \delta \vec{p}^\flat \right\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle + \left\langle \left(2v_n S^T - (\nabla \vec{v})^T - \left(\nabla^\sharp \vec{v}^\flat \right) \right) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle \\
&\quad + \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu \\
&= \int_{M(t)} \dot{\vec{p}} \mathbf{d} \delta \vec{p}^\flat + (\text{Div} \vec{v} + v_n H) \vec{p}^\flat \mathbf{d} \delta \vec{p}^\flat - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu
\end{aligned}$$
- $$\begin{aligned}
\frac{d}{dt} \int_{M(t)} \frac{1}{2} \left(\left\| \mathbf{d}\vec{p}^\flat \right\|^2 + \left\| \delta \vec{p}^\flat \right\|^2 \right) \mu &= \frac{d}{dt} \int_{M(t)} \frac{1}{2} \left(\left\| \text{Rot} \vec{p} \right\|^2 + \left\| \text{Div} \vec{p} \right\|^2 \right) \mu \\
&= \int_{M(t)} \left\langle \frac{d}{dt} \vec{p}^\flat, \Delta_{dR} \vec{p}^\flat \right\rangle - \frac{\text{Div} \vec{v} + v_n H}{2} \left(\left\| \mathbf{d}\vec{p}^\flat \right\|^2 + \left\| \delta \vec{p}^\flat \right\|^2 \right) \\
&\quad + \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle \\
&\quad - \left\langle \left((\nabla \vec{v})^T + \left(\nabla^\sharp \vec{v}^\flat \right) - 2v_n S^T \right) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle \mu \\
&= \int_{M(t)} \left\langle \dot{\vec{p}}, \Delta_{dR} \vec{p} \right\rangle - \frac{\text{Div} \vec{v} + v_n H}{2} \left(\left\| \text{Rot} \vec{p} \right\|^2 + \left\| \text{Div} \vec{p} \right\|^2 \right) \\
&\quad + (\text{Div} \vec{v} + v_n H) \vec{p}^\flat \mathbf{d} \delta \vec{p}^\flat \\
&\quad + \vec{p} \left((\nabla \vec{v})^T + \left(\nabla^\sharp \vec{v}^\flat \right) - 2v_n S^T \right) \delta \mathbf{d}\vec{p}^\flat \mu
\end{aligned}$$
- $$\begin{aligned}
&\int_{M(t)} \left\langle (\text{Div} \vec{v} + v_n H) \vec{p}^\flat, \mathbf{d} \delta \vec{p}^\flat \right\rangle - \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu \\
&= \int_{M(t)} \mathcal{L}_{\vec{p}} \left\langle \vec{p}^\flat, \mathbf{d} \delta \vec{v}^\flat + \mathbf{d} (v_n H) \right\rangle + \frac{1}{2} (\text{Div} \vec{v} + v_n H) \left\| \delta \vec{p}^\flat \right\|^2 \mu
\end{aligned}$$