Formulas for Calculus on Surfaces

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Contents

1 Assumptions, Definitions and Notations

$\mathcal{S}$ . . . orientable, boundaryless, Riemannian Surface
$\mathbf{g}$ . . . metric tensor in $\mathrm{T}_2\mathcal{S} \cong \mathrm{T}^{(2)}\mathcal{S}$
f, φ, ψ . . . scalar quantities in $\mathrm{T}^{(0)}\mathcal{S} = C^\infty(\mathcal{S})$
α, β, γ . . . vector quantities in $\mathrm{T}^{(1)}\mathcal{S} \cong \mathrm{T}_1\mathcal{S} = \mathrm{T}\mathcal{S} \cong \mathrm{T}^1\mathcal{S} = T^*\mathcal{S} = \Lambda^1\mathcal{S}$
t, s . . . tensor quantities in $\mathrm{T}^{(n)}\mathcal{S}$
ω . . . skew symmetric 2-tensor quantities (2-forms) in $\mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S} \cong \Lambda^2\mathcal{S}$
$\mu, \mathbf{E}$ . . . volume form in $\mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S} \cong \Lambda^2\mathcal{S}$
q . . . Q-Tensor (trace-free and symmetric) in $\mathcal{QS} = \mathrm{T}^{(2)}_{\mathrm{Tr}}\mathcal{S} \cap \mathrm{T}^{(2)}_{\mathrm{Sym}}\mathcal{S}$
If it matters, all quantities are handled **full covariant**, unless otherwise is defined.

2 Wedge Product $\wedge$

$$f \wedge \psi = \psi \wedge f = f\psi \in \mathrm{T}^{(0)}\mathcal{S}$$
$$f \wedge \alpha = \alpha \wedge f = f\alpha \in \mathrm{T}^{(1)}\mathcal{S}$$
$$f \wedge \omega = \omega \wedge f = f\omega \in \mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S}$$
$$\alpha \wedge \beta = -\beta \wedge \alpha = \frac{1}{\sqrt{|g|}}\left(\alpha_1\beta_2 - \alpha_2\beta_1\right)\mu \in \mathrm{T}^{(2)}_{\mathrm{Skew}}\mathcal{S} \qquad [\alpha \wedge \beta]_{ij} = \alpha^k\beta^l E_{kl}E_{ij} = \alpha_i\beta_j - \alpha_j\beta_j$$

2.1 Conclusions

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu \qquad [\alpha \wedge *\beta]_{ij} = \alpha_k \beta^k E_{ij}$$

$$*(\alpha \wedge *\beta) = \langle \alpha, \beta \rangle \qquad \langle \alpha, \beta \rangle = \alpha^i \alpha_i$$

$$-*(\alpha \wedge \beta) = \langle \alpha, *\beta \rangle$$

3 Hodge Star $*$

$$*f = f\mu \qquad [*f]_{ij} = fE_{ij}$$

$$**f = f$$

$$*\alpha = \mathrm{i}_\alpha \mu = \alpha \mathbf{E} = -\mathbf{E}\alpha = *_1\alpha \qquad [*\alpha]_i = -E_{ij}\alpha^j$$

$$**\alpha = -\alpha$$

$$**\omega = \omega$$

$$*_1 t = -\mathbf{E}t \qquad [*t_1]_{i_1 \dots i_n} = -E_{i_1 j} t^j{}_{i_2 \dots i_n}$$

$$*_1 *_1 t = -t$$

$$*_r t \qquad [*_r t]_{i_1 \dots i_n} = -E_{i_r j} t_{i_1 \dots i_{r-1}}{}^j{}_{i_{r+1} \dots i_n}$$

$$*_n t = t\mathbf{E} \qquad [*_n t]_{i_1 \dots i_n} = -E_{i_n j} t^{i_1 \dots i_{n-1} j}$$

$$*q = *_1 q = *_2 q$$

3.1 Conclusions

$$\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$$

$$\|\alpha\| = \|*\alpha\|$$

$$\langle \alpha, *\alpha \rangle = 0$$

$$\langle \alpha, *\beta \rangle = -\langle *\alpha, \beta \rangle = -*(\alpha \wedge \beta)$$

$$\langle \alpha, *\beta \rangle^2 = \|\alpha \wedge \beta\|^2 = \|\alpha\|^2 \|\beta\|^2 - \langle \alpha, \beta \rangle^2$$

$$(*\alpha) \otimes (*\beta) + \beta \otimes \alpha = \langle \alpha, \beta \rangle \mathbf{g}$$

$$(*\alpha) \otimes (*\alpha) + \alpha \otimes \alpha = \|\alpha\|^2 \mathbf{g}$$

$$\alpha \otimes (*\beta) - (*\beta) \otimes \alpha = \langle \alpha, \beta \rangle \mathbf{E}$$

$$\alpha \otimes (*\alpha) - (*\alpha) \otimes \alpha = \|\alpha\|^2 \mathbf{E}$$

$$*_1 t + *_2 t \in \mathrm{T}^{(2)}_{\mathrm{Sym}} \mathcal{S} \qquad \text{for } t \in \mathrm{T}^{(2)} \mathcal{S}$$

4 Levi-Civita Tensor $\mathbf{E}$

$$\mathbf{E}(\alpha, \beta) = \mu(\alpha, \beta) \qquad E_{ij} = \sqrt{|g|}\epsilon_{ij} \cong E^{ij} = \frac{1}{|\mathbf{g}|} E_{ij} = \frac{1}{\sqrt{|g|}} \epsilon_{ij}$$

$$\langle \mathbf{E}, \mathbf{g} \rangle = \mathbf{E}\mathbf{g} = 0 \qquad E_{ij} g^{ij} = 0$$

$$\mathbf{E}^T = -\mathbf{E} \qquad \left[\mathbf{E}^T\right]_{ij} = E_{ji} = -E_{ij}$$

$$\mathbf{E} \otimes \mathbf{E} = (\mathbf{g} \otimes \mathbf{g})^{T_{2,3}} - (\mathbf{g} \otimes \mathbf{g})^{T_{2,4}} \qquad E_{ij} E_{kl} = g_{ik} g_{jl} - g_{il} g_{jk}$$

4.1 Conclusions

$$-\mathbf{E}\alpha = \alpha\mathbf{E} = \mathrm{i}_\alpha\mu = *\alpha \qquad [*\alpha]_i = -E_{ij}\alpha^j$$

$$-\mathbf{E}t = *_1 t \qquad [*_1 t]_{i_1\ldots i_n} = -E_{i_1 j}t^j{}_{i_2\ldots i_n}$$

$$t\mathbf{E} = *_n t \qquad [*_n t]_{i_1\ldots i_n} = -E_{i_n j}t^{i_1\ldots i_{n-1}j}$$

$$\mathbf{E}\mathbf{E} = \mathbf{E}^2 = -\mathbf{g} \qquad E_{ik}E^k{}_j = -g_{ij}$$

$$\mathbf{E}^{-1} = -{}^\sharp\mathbf{E}^\sharp \qquad \left[\mathbf{E}^{-1}\right]^{ij} = -E^{ij} = E^{ji}$$

$$\|\mathbf{E}\|^2 = \mathrm{Tr}\left(\mathbf{E}\mathbf{E}^T\right) = 2$$

$$*_1 *_2 t = *_2 *_1 t = -\mathbf{E}t\mathbf{E} = (\mathrm{Tr}t)\,\mathbf{g} - t^T \qquad [*_1 *_2 t]_{ij} = t_k{}^k g_{ij} - t_{ji}$$

$$|t|\,\mathbf{E} = |g|\,t\mathbf{E}t^T \qquad |t|\,E_{ij} = |g|\,E^{kl}t_{ik}t_{jl}$$

$$\begin{aligned} |t| &= |g|\left|t^\sharp\right| = |g|\left|{}^\sharp t\right| = |g|^2\left|{}^\sharp t^\sharp\right| \\ &= -\frac{|g|}{2}\langle *_1 t, *_2 t\rangle = \frac{|g|}{2}\left((\mathrm{Tr}t)^2 - \mathrm{Tr}t^2\right) \end{aligned} \qquad |t| = \frac{|g|}{2}E_{ij}E_{kl}t^{ik}t^{jl} = \frac{|g|}{2}\left(\left(t_k{}^k\right)^2 - t_{kl}t^{lk}\right)$$

$$0 = t^2 - (\mathrm{Tr}t)\,t + \frac{|t|}{|g|}g \qquad [0]_{ij} = t_{ik}t^k{}_j - t_k{}^k t_{ij} + \frac{1}{2}\left(\left(t_k{}^k\right)^2 - t_{kl}t^{lk}\right)g_{ij}$$

5 Christoffel Symbols $\Gamma^{\cdot}_{\cdot\cdot}$

$$\Gamma \qquad \Gamma^k_{ij} = \Gamma^k_{ji} = g^{kl}\Gamma_{lij} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

6 First Order Derivatives $\mathbf{d}$, ∇, div, rot, Rot, $\mathcal{L}_{\gamma^\sharp}$, $\mathcal{D}_{\mathcal{Q}}$, $\mathcal{D}^*_{\mathcal{Q}}$

$$\nabla f \cong \mathbf{d}f \qquad [\nabla f]_i = f_{|i} = [\mathbf{d}f]_i = \partial_i f$$

$$\nabla\alpha \qquad \begin{aligned} [\nabla\alpha]_{i|j} &= \alpha_{i|j} = \partial_j\alpha_i - \Gamma^k_{ij}\alpha_k \\ &\cong \alpha^i{}_{|j} = \partial_j\alpha^i + \Gamma^i_{jk}\alpha^k \end{aligned}$$

$$\nabla t \qquad \begin{aligned} [\nabla t]_{ij|k} &= t_{ij|k} = \partial_k t_{ij} - \Gamma^l_{ki}t_{lj} - \Gamma^l_{kj}t_{il} \\ &\cong t^i{}_{j|k} = \partial_k t^i{}_j + \Gamma^i_{kl}t^l{}_j - \Gamma^l_{kj}t^i{}_l \\ &\cong t_i{}^j{}_{|k} = \partial_k t_i^j - \Gamma^l_{ki}t_l^j + \Gamma^j_{kl}t_i{}^l \\ &\cong t^{ij}{}_k = \partial_k t^{ij} + \Gamma^i_{kl}t^{lj} + \Gamma^j_{kl}t^{il} \end{aligned}$$

$$\nabla\mathbf{g} = 0 \qquad g_{ij|k} = 0$$

$$\nabla\mathbf{E} = 0 \qquad E_{ij|k} = 0$$

$$\mathrm{div}\alpha = *\mathbf{d}*\alpha = \langle\nabla\alpha, \mathbf{g}\rangle = \mathrm{Tr}\nabla\alpha \qquad \mathrm{div}\alpha = \alpha^i{}_{|i}$$

$$\mathrm{div}_1 t = \mathbf{g} : \nabla t \qquad [\mathrm{div}_1 t]_i = t^k{}_{i|k}$$

$$\mathrm{div}_2 t = \nabla t : \mathbf{g} = \mathrm{div}_1 t^T \qquad [\mathrm{div}_2 t]_i = t_i{}^k{}_{|k}$$

$$\mathrm{div}_r t \qquad [\mathrm{div}_r t]_{i_1\ldots\widehat{i_r}\ldots i_n} = t_{i_1\ldots i_{r-1}}{}^k{}_{i_{r+1}\ldots i_n|k}$$

$$\mathrm{div}q = \mathrm{div}_1 q = \mathrm{div}_2 q$$

$$\mathrm{rot}\alpha = *\mathbf{d}\alpha = -\langle \nabla\alpha, \mathbf{E}\rangle \qquad \mathrm{rot}\alpha = -E_{ij}\alpha^{i|j} = \frac{1}{\sqrt{|\mathbf{g}|}}\left(\alpha_{2|1} - \alpha_{1|2}\right) = \frac{1}{\sqrt{|\mathbf{g}|}}\left(\partial_1\alpha_2 - \partial_2\alpha_1\right)$$

$$\mathrm{rot}_1 t = -\nabla t^T : \mathbf{E} \qquad [\mathrm{rot}_1 t]_i = -E_{jk}t^j{}_i{}^{|k}$$

$$\mathrm{rot}_2 t = -\nabla t : \mathbf{E} = \mathrm{rot}_1 t^T \qquad [\mathrm{rot}_2 t]_i = -E_{jk}t_i{}^{j|k}$$

$$\mathrm{rot}_r t \qquad [\mathrm{rot}_r t]_{i_1\ldots\widehat{i_r}\ldots i_n} = -E_{jk}t_{i_1\ldots i_{r-1}}{}^{j}{}_{i_{r+1}\ldots i_n}{}^{|k}$$

$$\mathrm{rot}q = \mathrm{rot}_1 q = \mathrm{rot}_2 q$$

$$\mathrm{Rot}f = *\mathbf{d}f = -\mathbf{E}\nabla f \qquad [\mathrm{Rot}f]_i = -E_{ij}f^{|j}$$

$$\mathrm{Rot}\alpha = *_2\nabla\alpha = (\nabla\alpha)\,\mathbf{E} \qquad [\mathrm{Rot}\alpha]_{ij} = -E_{jk}\alpha_i{}^{|k}$$

$$\mathrm{Rot}t = *_n\nabla\alpha = (\nabla\alpha)\,\mathbf{E} \qquad [\mathrm{Rot}t]_{i_1\ldots i_n k} = -E_{kl}t_{i_1\ldots i_n}{}^{|l}$$

$$\mathcal{L}_{\gamma^\sharp} f = \langle\gamma, \nabla f\rangle = \nabla_\gamma f \qquad \mathcal{L}_{\gamma^\sharp} f = \gamma^k\partial_k f$$

$$\mathcal{L}_{\gamma^\sharp}\alpha = \nabla_\gamma\alpha + \alpha\nabla\gamma \qquad \left[\mathcal{L}_{\gamma^\sharp}\alpha\right]_i = \gamma^k\partial_k\alpha_i + \alpha_k\partial_i\gamma^k = \gamma^k\alpha_{i|k} + \alpha^k\gamma_{k|i}$$

$$\mathcal{L}_{\gamma^\sharp}\alpha^\sharp = \nabla_\gamma\alpha - \nabla_\alpha\gamma \qquad \left[\mathcal{L}_{\gamma^\sharp}\alpha^\sharp\right]^i = \gamma^k\partial_k\alpha^i - \alpha^k\partial_k\gamma^i = \gamma^k\alpha^i{}_{|k} - \alpha^k\gamma^i{}_{|k}$$

$$\mathcal{L}_{\gamma^\sharp} t = (\nabla t)\gamma + (\nabla\gamma)^T t + t\nabla\gamma \qquad \left[\mathcal{L}_{\gamma^\sharp} t\right]_{ij} = \gamma^k\partial_k t_{ij} + t_{kj}\partial_i\gamma^k + t_{ik}\partial_j\gamma^k = \gamma^k t_{ij|k} + t_{kj}\gamma^k{}_{|i} + t_{ik}\gamma^k{}_{|j}$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{g} = \nabla\gamma + (\nabla\gamma)^T \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{g}\right]_{ij} = \gamma_{i|j} + \gamma_{j|i}$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{E} = (\nabla * \gamma)^T - (\nabla * \gamma) \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{E}\right]_{ij} = [*\gamma]_{j|i} - [*\gamma]_{i|j} = E_{kj}\gamma^k{}_{|i} + E_{ik}\gamma^k{}_{|j}$$

$$\mathcal{L}_{\gamma^\sharp}{}^\sharp t^\sharp = (\nabla t)\gamma - (\nabla\gamma)\,t - t\,(\nabla\gamma)^T \qquad \left[\mathcal{L}_{\gamma^\sharp}{}^\sharp t^\sharp\right]^{ij} = \gamma^k\partial_k t^{ij} - t^{kj}\partial_k\gamma^i - t^{ik}\partial_k\gamma^i = \gamma^k t^{ij}{}_{|k} - t^{kj}\gamma^i{}_{|k} - t^{ik}\gamma^j{}_{|k}$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{g}^{-1} = \mathcal{L}_{\gamma^\sharp}{}^\sharp\mathbf{g}^\sharp = -\left(\nabla\gamma + (\nabla\gamma)^T\right) \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{g}^{-1}\right]^{ij} = -\left(\gamma^{i|j} + \gamma^{j|i}\right)$$

$$\mathcal{L}_{\gamma^\sharp}\mathbf{E}^{-1} = -\mathcal{L}_{\gamma^\sharp}{}^\sharp\mathbf{E}^\sharp = \mathrm{Rot}\gamma - (\mathrm{Rot}\gamma)^T \qquad \left[\mathcal{L}_{\gamma^\sharp}\mathbf{E}^{-1}\right]^{ij} = E^{kj}\gamma^i{}_{|k} + E^{ik}\gamma^j{}_{|k}$$

$$\mathcal{L}_{\gamma^\sharp} t \qquad \begin{aligned}\left[\mathcal{L}_{\gamma^\sharp} t\right]^{i_1\ldots i_r}{}_{j_1\ldots j_s} &= \gamma^k\partial_k t^{i_1\ldots i_r}{}_{j_1\ldots j_s} \\ &\quad - t^{ki_2\ldots i_r}{}_{j_1\ldots j_s}\partial_k\gamma^{i_1} - \ldots - t^{i_1\ldots i_{r-1}k}{}_{j_1\ldots j_s}\partial_k\gamma^{i_r} \text{ (uppers)} \\ &\quad + t^{i_1\ldots i_r}{}_{kj_2\ldots j_s}\partial_{j_1}\gamma^k + \ldots + t^{i_1\ldots i_r}{}_{j_1\ldots j_{s-1}k}\partial_{j_s}\gamma^k \text{ (lowers)} \\ &= \gamma^k t^{i_1\ldots i_r}{}_{j_1\ldots j_s|k} \\ &\quad - t^{ki_2\ldots i_r}{}_{j_1\ldots j_s}\gamma^{i_1}{}_{|k} - \ldots - t^{i_1\ldots i_{r-1}k}{}_{j_1\ldots j_s}\gamma^{i_r}{}_{|k} \text{ (uppers)} \\ &\quad + t^{i_1\ldots i_r}{}_{kj_2\ldots j_s}\gamma^k{}_{|j_1} + \ldots + t^{i_1\ldots i_r}{}_{j_1\ldots j_{s-1}k}\gamma^k{}_{|j_s} \text{ (lowers)}\end{aligned}$$

$$\mathcal{D}_{\mathcal{Q}}\alpha = \mathcal{L}_{\alpha^\sharp}\mathbf{g} - (\mathrm{div}\alpha)\,\mathbf{g} = \nabla\alpha + (\nabla\alpha)^T - (\mathrm{div}\alpha)\,\mathbf{g} = 2\Pi_{\mathcal{Q}}(\nabla\alpha) \in \mathcal{QS} \qquad [\mathcal{D}_{\mathcal{Q}}\alpha]_{ij} = \alpha_{i|j} + \alpha_{j|i} - \alpha^k{}_{|k}g_{ij}$$

$$\mathcal{D}^*_{\mathcal{Q}}q = -2\mathrm{div}q = -2 * \mathrm{rot}q = -2\mathrm{rot} * q \qquad \int_{\mathcal{S}}\left\langle\mathcal{D}^*_{\mathcal{Q}}q, \alpha\right\rangle\mu = \int_{\mathcal{S}}\langle q, \mathcal{D}_{\mathcal{Q}}\alpha\rangle\,\mu$$

6.1 Conclusions

$$
\begin{aligned}
\mathrm{rot} * \alpha &= *\mathbf{d} * \alpha = \mathrm{div}\alpha \\
\mathrm{Rot} * \alpha &= *_2 \nabla * \alpha = *_1 *_2 \nabla\alpha = (\mathrm{div}\alpha)\,\mathbf{g} - (\nabla\alpha)^T & [\mathrm{Rot} * \alpha]_{ij} &= \alpha^k{}_{|k} g_{ij} - \alpha_{j|i} \\
\mathrm{rot}_1 *_1 t &= \mathrm{div}_1 t & [\mathrm{rot}_1 *_1 t]_i &= t^k{}_{i|k} \\
\mathrm{rot}_2 *_2 t &= \mathrm{div}_2 t & [\mathrm{rot}_2 *_2 t]_i &= t_i{}^k{}_{|k} \\
*\mathrm{rot}_1 t &= \mathrm{rot}_1 *_2 t = \mathrm{div}_2 t - \nabla \mathrm{Tr} t & [\mathrm{rot}_1 *_2 t]_i &= t_i{}^k{}_{|k} - t^k{}_{k|i} \\
*\mathrm{rot}_2 t &= \mathrm{rot}_2 *_1 t = \mathrm{div}_1 t - \nabla \mathrm{Tr} t & [\mathrm{rot}_2 *_1 t]_i &= t^k{}_{i|k} - t^k{}_{k|i} \\
\nabla(\psi f) &= \psi\nabla f + f\nabla\psi & [\nabla(\psi f)]_i &= \psi f_{|i} + f\psi_{|i} \\
\nabla\langle\alpha,\beta\rangle &= \alpha\nabla\beta + \beta\nabla\alpha & [\nabla\langle\alpha,\beta\rangle]_i &= \alpha^k\beta_{k|i} + \beta^k\alpha_{k|i} \\
\nabla(f\alpha) &= f\nabla\alpha + \alpha\otimes\nabla f & [\nabla(f\alpha)]_{ij} &= f\alpha_{i|j} + \alpha_i f_{|j} \\
\nabla(ft) &= f\nabla t + t\otimes\nabla f & [\nabla(ft)]_{ijk} &= f t_{ij|k} + t_{ij} f_{|k} \\
\mathrm{rot}(f\alpha) &= f\mathrm{rot}\alpha + \langle\alpha, \mathrm{Rot} f\rangle & \mathrm{rot}(f\alpha) &= -fE_{ij}\alpha^{i|j} - \alpha^i E_{ij} f^{|j} \\
\mathrm{div}(f\alpha) &= f\mathrm{div}\alpha + \langle\alpha,\nabla f\rangle & \mathrm{div}(f\alpha) &= f\alpha^k{}_{|k} + \alpha^k f_{|k} \\
\mathrm{div}_1(ft) &= f\mathrm{div}_1 t + (\nabla f)t & [\mathrm{div}_1(ft)]_i &= f t^k{}_{i|k} + f^{|k} t_{ki} \\
\mathrm{div}_2(ft) &= f\mathrm{div}_2 t + t\nabla f & [\mathrm{div}_2(ft)]_i &= f t_i{}^k{}_{|k} + f^{|k} t_{ik} \\
\mathrm{div}(f\mathbf{g}) &= \nabla f
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\gamma^\sharp}\mathbf{g} &= -{}^\flat\left(\mathcal{L}_{\gamma^\sharp}\mathbf{g}^{-1}\right)^\flat \\
\mathcal{L}_{\gamma^\sharp}\mathbf{d}f &= \mathbf{d}\mathcal{L}_{\gamma^\sharp} f \\
*_1\mathcal{L}_{*\gamma^\sharp}\mathbf{g} &= (\mathrm{div}\gamma)\,\mathbf{g} - 2\nabla\gamma \in \mathrm{T}^{(2)}_{\mathrm{Tr}}\mathcal{S} & \left[*_1\mathcal{L}_{*\gamma^\sharp}\mathbf{g}\right]_{ij} &= \gamma^k{}_{|k} g_{ij} - 2\gamma_{i|j} \\
*_2\mathcal{L}_{*\gamma^\sharp}\mathbf{g} &= (\mathrm{div}\gamma)\,\mathbf{g} - 2(\nabla\gamma)^T \in \mathrm{T}^{(2)}_{\mathrm{Tr}}\mathcal{S} & \left[*_2\mathcal{L}_{*\gamma^\sharp}\mathbf{g}\right]_{ij} &= \gamma^k{}_{|k} g_{ij} - 2\gamma_{j|i} \\
\mathcal{L}_{*\gamma^\sharp}\mathbf{g} &= \nabla * \gamma + (\nabla * \gamma)^T = (\mathrm{div}\gamma)\,\mathbf{E} + 2\nabla * \gamma \in \mathrm{T}^{(2)}_{\mathrm{Sym}}\mathcal{S} & \left[\mathcal{L}_{*\gamma^\sharp}\mathbf{g}\right]_{ij} &= E_{ki}\gamma^k{}_{|j} + E_{kj}\gamma^k{}_{|i} = \gamma^k{}_{|k} E_{ij} + 2E_{ki}\gamma^k{}_{|j}
\end{aligned}
$$

$$
\begin{aligned}
*\mathcal{D}_\mathcal{Q}\alpha &= *_1\mathcal{D}_\mathcal{Q}\alpha = *_2\mathcal{D}_\mathcal{Q}\alpha = \mathcal{D}_\mathcal{Q} * \alpha \in \mathcal{Q}\mathcal{S} \\
*\mathcal{D}^*_\mathcal{Q} q &= \mathcal{D}^*_\mathcal{Q} *_1 q = \mathcal{D}^*_\mathcal{Q} *_2 q = \mathcal{D}^*_\mathcal{Q} * q \\
\mathcal{D}_\mathcal{Q}\alpha &= -\frac{1}{2}(*_1 + *_2)\,\mathcal{L}_{*\alpha^\sharp}\mathbf{g} = - * \mathcal{D}_\mathcal{Q} * \alpha \\
\mathcal{D}_\mathcal{Q}\mathbf{d}f &= - * \mathcal{L}_{\mathrm{Rot} f}\mathbf{g} = - * \mathcal{D}_\mathcal{Q}\mathrm{Rot} f \\
\mathcal{D}_\mathcal{Q}\mathrm{Rot} f &= \mathcal{L}_{\mathrm{Rot} f}\mathbf{g} = *\mathcal{D}_\mathcal{Q}\mathbf{d}f \\
\mathcal{D}_\mathcal{Q}(\mathrm{Rot}\phi + \mathbf{d}\psi + \gamma) &= \mathcal{L}_{\mathrm{Rot}\phi}\mathbf{g} - *\mathcal{L}_{\mathrm{Rot}\psi}\mathbf{g} + \mathcal{L}_{\gamma^\sharp}\mathbf{g} & &\text{for } \mathrm{div}\gamma = \mathrm{rot}\gamma = 0
\end{aligned}
$$