Curvature approximation on surfaces – A Discrete Exterior Calculus Approach

I. Nitschke[1], A. Voigt[1,2]

Abstract

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Keywords: Surfaces, Curvature, DEC

1. Introduction

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2. Discrete Exterior Calculus (DEC)

The Discrete Exterior Calculus [5, 2] defines discrete differential p-forms on a triangulated mesh (simplicial complex). For surface meshes, i.e. triangulated orientable 2-manifolds, the degree of the discrete p-forms is 0, 1 or 2 and their are represented by scalars on vertices, edges, triangles or chains of them. Operators for the differential forms, like the exterior derivation $\mathbf{d}$ or the hodge star $\star$, can be approximated by expressions on the discrete geometrical structure. E.g. the integral over a triangle of the exterior derivation $\mathbf{d}$ for a 1-form can be expressed as the integral of the 1-form over the boundaries edges of the triangle. This follows directly from the Stokes Theorem [1, Ch. 7].

2.1. Discrete manifolds

In our setting, the surface meshes are linear triangulations of orientable closed 2-manifolds. Such a Triangulation are sets of p-simplices $\{\sigma^p\}$ of the same degree p, e.g. sets of vertices, edges and triangles, and form a simplicial complex of dimension $n = 2$. A simplicial complex K comply two essential rules:

1. Every face of a simplex $\sigma^p \in K$ is in K.
2. The intersection of two simplices in K is either in K or empty.

We shortly write $\sigma^q \prec \sigma^p$ (or $\sigma^p \succ \sigma^q$)), iff $q < p$ and σ^q is a face of σ^p. It is required, that the polytop

$$|K| := \bigcup_{\sigma \in K} \sigma \qquad (1)$$

of K is a C^0-manifold and we call K a manifold-like simplicial complex. For a higher consistence to the smooth model, we assume that the simplicial complex is orientable, i.e. all triangles $\sigma^2 \in K$ have the same orientation. The orientation of a triangle is define as one of the two equivalence classes, that arise from the kind of counting the vertices (clockwise or counterclockwise). We write $\sigma^2 = [v_0, v_1, v_2]$ to mark the order of the vertices v_i. In the same manner a edge $\sigma^1 = [v_0, v_1]$ get an declared orientation. Such a manifold-like orientable simplicial complex is called a primal mesh.

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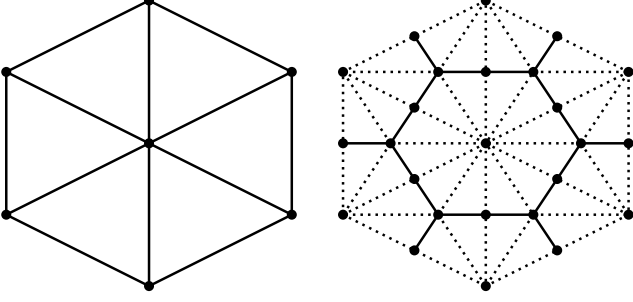
Figure 1: Example of a circumcenter subdivision. Left: A primal mesh K. Right: The resultant dual mesh csd$K$. All new vertices are the circumcenters of the triangles and edges. The solid lines highlights the edges of the Voronoi mesh $\star K \leq$ csd$K$.

For the existence of a dual mesh, we also need, that the primal mesh is well-centered, i.e. for all triangles $\sigma^2 \in K$ the circumcenter $c(\sigma^2)$ lies in the interior of σ^2. A dual mesh csd$K$ is the circumcenter subdivision of a well-centered primal mesh K. For a explanatory example see figure 1. The dual mesh is also a primal mesh, but it is not well-centered.

2.2. Chains

An important role in the DEC are chains. A p-chain is the formal sum of p-simplices with coefficients in $\mathbb{Z}$. Therewith, we denote the space of all p-chains of K by

$$C_p(K) := \left\{ \sum_{\sigma \in K^{(p)}} a_\sigma \sigma \,\middle|\, a_\sigma \in \mathbb{Z} \right\} . \qquad (2)$$

$K^{(p)}$ is the set of all p-simplices of K. For chains we can use the universal property, because $C_p(K)$ is a free abelian group with the generating set $K^{(p)}$. Hence, the diagram

$$
\begin{array}{cc}
C_p(K) & \qquad (3) \\
\uparrow \quad \searrow^{\widehat{op}} & \\
K^{(p)} \xrightarrow{op} \mathfrak{A} &
\end{array}
$$

commutes for a arbitrary abelian group $\mathfrak{A}$ and homomorphism $op = \widehat{op}|_{K^{(p)}}$ and $\widehat{op}$ is unique determinated by op, i.e. it is quit enough to define operators for chains only on the simplices.

2.3. Geometrical operators

The two momentous operators on chains are the boundary operator ∂ and the star operator $\star$. These are the geometrical tools for the discrete exterior derivation d resp. the discrete Hodge star $*$, what we will see below.

2.3.1. The star operator

For a general definition of the star operator $\star$ see [5, 2] or [3, Ch. 7]. In our two dimensional well-centered

primal mesh case it is enough to define the star operator $\star : C_p(K) \to C_{2-p}(\text{csd}K)$ by

$$\star\sigma^0 := \sum_{\sigma^0 \prec \sigma^1 \prec \sigma^2} s_{\sigma^1\sigma^2} \left[\sigma^0, c(\sigma^1), c(\sigma^2) \right] \qquad (4)$$

$$\star\sigma^1 := \sum_{\sigma^1 \prec \sigma^2} s_{\sigma^2} \left[c(\sigma^1), c(\sigma^2) \right] \qquad (5)$$

$$\star\sigma^2 := c(\sigma^2) \qquad (6)$$

The factors $s_\bullet$ are signs to prevent orientation properties, like the orientability of the dual mesh. (4) describe the Voronoi cell of the primal vertex σ^0, (5) describe the Voronoi edge of the primal edge σ^1 (c.p. figure 1) and (6) describe the Voronoi vertex of a primal triangle σ^2. With the notation $C_{2-p}(\star K)$ for the image of $\star$, the star operator act on $C_{2-p}(\star K)$ and map to $C_p(K)$ is for $p = 0, 2$ simply the inverse. For $p = 1$ we must swap the orientation, i.e. $\star \star \sigma^1 = -\sigma^1$.

2.3.2. The boundary operator

Let K be a primal mesh (not necessary well-centered). The boundary operator $\partial : C_p(K) \to C_{p-1}(K)$ is defined by

$$\partial\sigma^p := \begin{cases} \sum_{i=0}^{p} (-1)^i \left[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_p \right] & \text{for } p = 1, 2 \\ 0 & \text{for } p = 0 \end{cases} \qquad (7)$$

($\hat{v}_i$ is omitted) and comply the property $\partial \circ \partial = 0$. This is the authorization to call the sequence $(C_p(K), \partial)$ (and also $(C_p(\star K), \partial)$)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_2(K) & \xrightarrow{\partial} & C_1(K) & \xrightarrow{\partial} & C_0(K) & \longrightarrow & 0 \\
& & \uparrow\downarrow{\scriptstyle\star} & & \uparrow\downarrow{\scriptstyle\star} & & \uparrow\downarrow{\scriptstyle\star} & & \\
0 & \longleftarrow & C_0(\star K) & \xleftarrow{\partial} & C_1(\star K) & \xleftarrow{\partial} & C_2(\star K) & \longleftarrow & 0
\end{array}
$$
$$(8)$$

a chain complex.

2.4. Discrete differential forms

A discrete p-form α is a homomorphism from the chain group $C_p(K)$ to the additive group $\mathbb{R}$. Thus, the space of discrete p-form is the space of cochains Hom $(C_p(K), \mathbb{R})$, denoted as $C^p(K)$ or, under attention the analogy to smooth differential forms, $\Omega_d^p(K)$. The crucial map, that define such homomorphisms and give the relation to the smooth forms, is the de Rham map

$$\psi : \Omega^p(M) \to C^p(L) = \Omega_d^p(L)$$
$$\alpha \mapsto \left(\sigma^p \mapsto \int_{\sigma^p} \alpha =: \langle \psi(\alpha), \sigma^p \rangle \right) . \qquad (9)$$

The right hand side is the pairing notation, which we want to use. At this, L is the abstract simplicial complex of

K and arise if all simplices from K are projected to the underlying manifold M, i.e. $|L| = M$ and $L = \pi(K)$ with a projection map $\pi : |K| \to M$. But the projection and in the most cases the manifold M is not exactly known, so for interest of simplification we can approximate the integral in (9) with a linear quadrature I on the vertices of a given simplex $\sigma^p \in K$:

$$\langle \psi(\alpha), \pi(\sigma^p) \rangle \approx I_{\sigma^p}(\alpha). \qquad (10)$$

For $p = 0$ and a function (0-form) $f : M \to \mathbb{R}$ and a vertex (0-simplex) is this exact, i.e.

$$\langle \psi(f), \pi(v) \rangle = \langle \psi(f), v \rangle = f(v). \qquad (11)$$

For $p > 1$ the question about evaluating the integral in (9) is pure formal in this paper, because in all of our computation, we can reduce the problems to scalar valued formulations. Henceforth, the projection π is omitted in the pairing notation, i.e. we write $\langle \psi(\alpha), \sigma^p \rangle$ instead of $\langle \psi(\alpha), \pi(\sigma^p) \rangle$.

2.5. DEC operators

For the definition of a discrete version of the Hodge star $*$ or the exterior derivative d we can fall back to the geometric operators.

2.5.1. The discrete Hodge star operator

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