Discrete Exterior Calculus (DEC) approximation of curvature on surfaces

Curvature vector

Continuous Problem

- Inclusion map: $\iota : \mathbb{R}^3|_M \hookrightarrow \mathbb{R}^3, \quad \vec{x} \mapsto \vec{x}$
- Laplace-Beltrami-Operator for the inclusion map on a given manifold (componentwise)

$$\Delta_B \iota = (*\mathbf{d} * \mathbf{d})\, \iota = \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^j} \left(g^{ij} \sqrt{|\det g|} \frac{\partial \iota}{\partial x^i} \right)$$

  (g, g^{ij}: metric tensor (e.g. Riemannian metric) resp. its inverse components)

- Curvature Vector, see [Fla63]: $\vec{H} = -\Delta_B \iota$
- Mean curvature: $H = \frac{1}{2} \left\| \vec{H} \right\|$

Discrete Problem

- For a better FEM-like elementwise implementation, the discrete formulation on a vertex v_i is given with respect to the Hodge-/Geometric-Star-Operator:

$$\left\langle *\Delta_B \iota^k, \star v_i \right\rangle = \sum_{\sigma^1 = [v_i, v_j]} \frac{\left|\star\sigma^1\right|}{|\sigma^1|} \left(\iota^k(v_j) - \iota^k(v_i) \right) \,,$$

  ($\iota = \left[\iota^1, \iota^2, \iota^3\right]$ and the global vertex indices i and j)

- DEC-approximated mean curvature:

$$H_d(v_i) = \frac{1}{2\,|\star v_i|} \sqrt{\sum_{k=1}^{3} \left\langle *\Delta_B \iota^k, \star v_i \right\rangle^2} \,.$$

Weingarten map

Continuous problem

- Extended Weingarten map: $\bar{S} := \nabla \vec{\nu} \in \mathbb{R}^{3\times 3} : M \to \mathbb{R}^{3\times 3}$
  (∇: surface gradient)

- The restriction of the extended Weingarten map to the tangential space is the usual Weingarten map S.

- The eigenvalues of S are the principal curvatures κ^1 and κ^2 of the Surface M. The mean curvature and the Gaussian curvature is given by $H = \frac{\kappa^1+\kappa^2}{2}$ resp. $K = \kappa^1 \cdot \kappa^2$.

Discrete problem

- Discrete surface normals $\vec{\nu}$ on a vertex v:
  - Average of element normals $\vec{\nu}^{\sigma^2}$: $\vec{\nu}^{\mathsf{Av}}(v) := \frac{1}{|\star v|} \sum_{\sigma^2 \succ v} \left|\star v \cap \sigma^2\right| \vec{\nu}^{\sigma^2}$
  - From a signed distance function $\varphi : \mathbb{R}^3 \to \mathbb{R}$: $\vec{\nu}(v) = \frac{\nabla_{\mathbb{R}^3}\varphi}{\|\nabla_{\mathbb{R}^3}\varphi\|}$

- Discrete surface Gradient $\nabla^{\overline{pd}}$ as average of the primal-dual-gradient ∇^{pd}, see [Hir03]:

$$\left(\nabla^{\overline{pd}} f\right)(v) = \frac{1}{|\star v|} \sum_{\sigma^2 \succ v} \left|\star v \cap \sigma^2\right| \sum_{\sigma^0 \prec \sigma^2} \left(f(\sigma^0) - f(v)\right) \nabla \Phi^{\sigma^2}_{\sigma^0}$$

  ($\nabla \Phi^{\sigma^2}_{\sigma^0}$: gradient of the linear basis function Φ_{σ^0} on element σ^2)

- Discrete formulation on a vertex v and for components with index $i, j \in \{1, 2, 3\}$:

$$|\star v|\, \bar{S}_{ij}(v) \approx \left\langle * \left[S^{\overline{pd}}\right]_{ij}, \star v \right\rangle := \left\langle * \left[\nabla^{\overline{pd}} \bar{\nu}^i\right]_j, \star v \right\rangle$$

  ($\bar{\nu}^i$: i-th component of $\vec{\nu}$ resp. $\vec{\nu}^{\mathsf{Av}}$)

- Calculation of the eigenvalues of DEC-approximated extended Weingarten map $S^{\overline{pd}}$ on every vertex with QR-Algorithm and cancel out the additional (approx. 0) eigenvalue

References

[Fla63] H. Flanders. *Differential Forms with Applications to the Physical Sciences*. Dover books on advanced mathematics. Dover Publications, 1963.

[Hir03] Anil Nirmal Hirani. *Discrete Exterior Calculus*. PhD thesis, California Institute of Technology, Pasadena, CA, USA, 2003. AAI3086864.