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1 Frank Oseen energy

In \mathbb{R}^3 :

$$E_{\text{OS}} = \frac{1}{2} \int_{\Omega} K_1 (\nabla \cdot \mathbf{p})^2 + K_2 (\mathbf{p} \cdot [\nabla \times \mathbf{p}])^2 + K_3 \|\mathbf{p} \times [\nabla \times \mathbf{p}]\|^2 dV \quad (1)$$

With the Langrange identity for the K_3 -term, we cann rewrite (1) to

$$E_{\text{OS}} = \frac{1}{2} \int_{\Omega} K_1 (\nabla \cdot \mathbf{p})^2 + (K_2 - K_3) (\mathbf{p} \cdot [\nabla \times \mathbf{p}])^2 + K_3 \|\mathbf{p}\|^2 \|\nabla \times \mathbf{p}\|^2 dV \quad (2)$$

If we restrict (2) to a 2-dimensional Manifold $M \subset \Omega$ and postulate that $\mathbf{p} \in T_X M$ is a normalized tangential vector in $X \in M$, we get

$$E_{\text{OS}} = \frac{1}{2} \int_M K_1 (\text{Div} \mathbf{p})^2 + K_3 (\text{Rot} \mathbf{p})^2 dA \quad (3)$$

In terms of exterior calculus with the corresponding 1-form $\mathbf{p}^b \in \Lambda^1(M)$, ,i.e. $(\mathbf{p}^b)^\sharp = \mathbf{p}$, we obtain

$$E_{\text{OS}} = \frac{1}{2} \int_M K_1 \left(\mathbf{d}^* \mathbf{p}^b \right)^2 + K_3 \left(* \mathbf{d} \mathbf{p}^b \right)^2 dA \quad (4)$$

where the exterior coderivative $\mathbf{d}^* := - * \mathbf{d} *$ is the L^2 -orthogonal operator of the exterior derivative \mathbf{d} . (Note $\text{Div} \mathbf{p} = -\mathbf{d}^* \mathbf{p}^b$ and $\text{Rot} \mathbf{p} = * \mathbf{d} \mathbf{p}^b$)

1.1 Functional derivative

With the L^2 -orthogonality of the exterior derivative and coderivative and a arbitrary $\alpha \in \Lambda^1(M)$ we get

$$\int_M \left\langle \frac{\delta E_{OS}}{\delta \mathbf{p}^b}, \alpha \right\rangle dA = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(E_{OS} [\mathbf{p}^b + \epsilon \alpha] - E_{OS} [\mathbf{p}^b] \right) \quad (5)$$

$$= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_M K_1 \left(2\epsilon \left(\mathbf{d}^* \mathbf{p}^b \right) (\mathbf{d}^* \alpha) + \epsilon^2 (\mathbf{d}^* \alpha)^2 \right) \quad (6)$$

$$+ K_3 \left(2\epsilon \left\langle \mathbf{d} \mathbf{p}^b, \mathbf{d} \alpha \right\rangle + \epsilon^2 \|\mathbf{d} \alpha\|^2 \right) dA \quad (7)$$

$$= - \int_M K_1 \left\langle \Delta^{\text{GD}} \mathbf{p}^b, \alpha \right\rangle + K_3 \left\langle \Delta^{\text{RR}} \mathbf{p}^b, \alpha \right\rangle dA \quad (8)$$

$$= \int_M \left\langle - (K_1 \Delta^{\text{GD}} + K_3 \Delta^{\text{RR}}) \mathbf{p}^b, \alpha \right\rangle dA \quad (9)$$

where $\Delta^{\text{RR}} = -\mathbf{d}^* \mathbf{d} = *\mathbf{d} * \mathbf{d}$ is the Vector-Laplace-Beltrami-Operator or Rot-Rot-Laplace and $\Delta^{\text{GD}} = -\mathbf{d} \mathbf{d}^* = \mathbf{d} * \mathbf{d} *$ is the Vector-Laplace-CoBeltrami-Operator or Grad-Div-Laplace. Hence, for a One-Constant-Approximation $K_1 = K_3 =: K_0$, we obtain

$$\int_M \left\langle \frac{\delta E_{OS}}{\delta \mathbf{p}^b}, \alpha \right\rangle dA = \int_M \left\langle K_0 \Delta^{\text{dR}} \mathbf{p}^b, \alpha \right\rangle dA \quad (10)$$

where $\Delta^{\text{dR}} = -\Delta^{\text{RR}} - \Delta^{\text{GD}} = \mathbf{d}^* \mathbf{d} + \mathbf{d} \mathbf{d}^*$ is the Laplace-de Rham operator.

1.2 Unit vector invariance

If $\mathbf{p} \in T_X M$ is a unit vector on M , we can describe all unit vectors in $X \in M$ as a rotation in the tangential space with angle $\phi \in \mathbb{R}$:

$$\mathbf{q} = \cos \phi \mathbf{p} + \sin \phi (*\mathbf{p}) \quad (11)$$

$*\mathbf{p} = (*\mathbf{p}^b)^\sharp$ is the Hodge dual of \mathbf{p} , i.e. a quarter rotation of \mathbf{p} . For a space independent angle ϕ , i.e. $\mathbf{d}\phi = 0$, straight forward calculations implies

$$\|\text{Rot}(*\mathbf{p})\| = \|\mathbf{d} * \mathbf{p}^b\| = \|\text{Div} \mathbf{p}\| \quad (12)$$

$$\|\text{Div}(*\mathbf{p})\| = \|\mathbf{d} * *\mathbf{p}^b\| = \|\mathbf{d} \mathbf{p}^b\| = \|\text{Rot} \mathbf{p}\| \quad (13)$$

$$\|\text{Rot} \mathbf{q}\|^2 = \|\mathbf{d} \mathbf{q}^b\|^2 = \|\mathbf{d} \mathbf{q}\|^2 \quad (14)$$

$$= \cos^2 \phi \|\text{Rot} \mathbf{p}\|^2 + \sin^2 \phi \|\text{Div} \mathbf{p}\|^2 + 2 \cos \phi \sin \phi \left\langle \mathbf{d} \mathbf{p}^b, \mathbf{d} * \mathbf{p}^b \right\rangle \quad (15)$$

$$\|\text{Div} \mathbf{q}\|^2 = \|\mathbf{d} * \mathbf{q}^b\|^2 = \|\mathbf{d} * \mathbf{q}\|^2 \quad (16)$$

$$= \cos^2 \phi \|\text{Div} \mathbf{p}\|^2 + \sin^2 \phi \|\text{Rot} \mathbf{p}\|^2 - 2 \cos \phi \sin \phi \left\langle \mathbf{d} \mathbf{p}^b, \mathbf{d} * \mathbf{p}^b \right\rangle \quad (17)$$

Finally, we get for the One-Constant-Approximation of the Frank-Oseen-Energy

$$E_{OS}[\mathbf{q}] = E_{OS}[\mathbf{p}] \quad (18)$$

2 Normalizing energy

To constrain \mathbf{p} is normalized, we add

$$E_n = \int_M \frac{K_n}{4} \left(\|\mathbf{p}\|^2 - 1 \right)^2 dA \quad (19)$$

to the Frank Oseen energy. Note that the norm defined by the metric g on the manifold M is invariant regarding lowering or rising the indices, i.e.

$$\|\mathbf{p}\|^2 = p^i g_{ij} p^j = p_i g^{ij} p_j = \|\mathbf{p}^\flat\|^2 \quad (20)$$

2.1 Functional derivative

By varying \mathbf{p}^\flat under the norm with an arbitrary $\alpha \in \Lambda^1(M)$, we obtain

$$\|\mathbf{p}^\flat + \epsilon\alpha\|^2 = \|\mathbf{p}^\flat\|^2 + 2\epsilon \langle \mathbf{p}^\flat, \alpha \rangle + \epsilon^2 \|\alpha\|^2 \quad (21)$$

If we are only interesting in linear terms (in ϵ), this leads to

$$\left(\|\mathbf{p}^\flat + \epsilon\alpha\|^2 - 1 \right)^2 = \left(\|\mathbf{p}^\flat\|^2 - 1 + 2\epsilon \langle \mathbf{p}^\flat, \alpha \rangle + \mathcal{O}(\epsilon^2) \right)^2 \quad (22)$$

$$= \left(\|\mathbf{p}^\flat\|^2 - 1 \right)^2 + 4\epsilon \left(\|\mathbf{p}^\flat\|^2 - 1 \right) \langle \mathbf{p}^\flat, \alpha \rangle + \mathcal{O}(\epsilon^2) \quad (23)$$

Hence, we get for the functional derivative of E_n

$$\int_M \left\langle \frac{\delta E_n}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(E_n[\mathbf{p}^\flat + \epsilon\alpha] - E_n[\mathbf{p}^\flat] \right) \quad (24)$$

$$= \int_M \left\langle K_n \left(\|\mathbf{p}^\flat\|^2 - 1 \right) \mathbf{p}^\flat, \alpha \right\rangle dA \quad (25)$$

3 Model equations

To minimize the energy $E := E_{\text{OS}} + E_n$ we choose a time evolving approach. Hence, with the fundamental lemma of calculus of variations, we will use the time depended differential equation in terms of exterior calculus

$$\partial_t \mathbf{p}^\flat = -\frac{\delta E}{\delta \mathbf{p}^\flat} = -K_0 \Delta^{\text{dR}} \mathbf{p}^\flat - K_n \left(\|\mathbf{p}^\flat\|^2 - 1 \right) \mathbf{p}^\flat \quad (26)$$

or in general, if we don't want to use the One-Constant-Approximation,

$$\partial_t \mathbf{p}^\flat = (K_1 \Delta^{\text{GD}} + K_3 \Delta^{\text{RR}}) \mathbf{p}^\flat - K_n \left(\|\mathbf{p}^\flat\|^2 - 1 \right) \mathbf{p}^\flat \quad (27)$$

Note that if we apply the Hodge operator on the whole equations, we get the Hodge dual equations

$$\partial_t (*\mathbf{p}^\flat) = -K_0 \Delta^{\text{dR}} (*\mathbf{p}^\flat) - K_n \left(\|\mathbf{p}^\flat\|^2 - 1 \right) (*\mathbf{p}^\flat) \quad (28)$$

$$= (K_1 \Delta^{\text{RR}} + K_3 \Delta^{\text{GD}}) (*\mathbf{p}^\flat) - K_n \left(\|\mathbf{p}^\flat\|^2 - 1 \right) (*\mathbf{p}^\flat) \quad (29)$$

which are very useful for the DEC discretization later in context. But this leads to pay attention, because only in the first line (One-Constant-Approximation) we see, that the Hodge dual equations in $*\mathbf{p}^\flat$ is the same as the primal equation in \mathbf{p}^\flat . In the general case (second line), we must "swap" the Laplace operators.

4 Needed DEC stuff

For further information see for example [Whi57, Hir03].

4.1 Surface Mesh

...wellcentered manifoldlike simplicial complex, bla, bla, blub...

4.2 Discrete 1-forms

The main concept to represent a discrete 1-form $\mathbf{p}_h^b \in \Lambda_h^1(K)$ is to approximate the contraction of the continuous 1-Form $\mathbf{p}^b \in \Lambda^1(M)$ on all edges $e \in \mathcal{E}$

$$\mathbf{p}_h^b(e) := \int_{\pi(e)} \mathbf{p}^b \approx \int_0^1 \mathbf{p}_{X_e(\tau)}^b \left(\dot{X}_e(t) \right) dt = \mathbf{p}_{X_e(\tau)}^b(\mathbf{e}) \quad (30)$$

where $\pi : K \rightarrow M$ is the glueing map, who project the elements of the surface mesh to the manifold. $X_e(t) = t\mathbf{v}_2 + (1-t)\mathbf{v}_1$ is the linear barycentric parametrisation of the edge $e = [v_1, v_2]$. The existence of a intermediate value $\tau \in [0, 1]$, so that $\mathbf{e} \in T_{X_e(\tau)}M$, is ensured by the mean value theorem. Other discrete forms of arbitrary degree and theirs hodge duals can be interpreted in a similarly way.

4.3 Discrete Laplace operators

In the discrete exterior calculus discrete Operators are defined by successively interpretation of the basic operations on the forms, like the Hodge operator $*$ or the exterior derivative \mathbf{d} , as geometric operators on the simplices, like the Voronoi dual operator \star or the boundary operator ∂ (see [Hir03]). This results for example to a discrete definition of Δ^{RR} for a discrete 1-form $\mathbf{p}_h^b \in \Lambda_h^1(M)$ on a edge $e \in \mathcal{E}$

$$\Delta_h^{\text{RR}} \mathbf{p}_h^b(e) := \left(* \mathbf{d} * \mathbf{d} \mathbf{p}_h^b \right) (e) = - \frac{|e|}{|\star e|} \left(\mathbf{d} * \mathbf{d} \mathbf{p}_h^b \right) (\star e) \quad (31)$$

$$= - \frac{|e|}{|\star e|} \left(* \mathbf{d} \mathbf{p}_h^b \right) (\partial \star e) = - \frac{|e|}{|\star e|} \sum_{f \succ e} s_{f,e} \left(* \mathbf{d} \mathbf{p}_h^b \right) (\star f) \quad (32)$$

$$= - \frac{|e|}{|\star e|} \sum_{f \succ e} \frac{s_{f,e}}{|f|} \left(\mathbf{d} \mathbf{p}_h^b \right) (f) = - \frac{|e|}{|\star e|} \sum_{f \succ e} \frac{s_{f,e}}{|f|} \mathbf{p}_h^b(\partial f) \quad (33)$$

$$= - \frac{|e|}{|\star e|} \sum_{f \succ e} \frac{s_{f,e}}{|f|} \sum_{\tilde{e} \prec f} s_{f,\tilde{e}} \mathbf{p}_h^b(\tilde{e}) \quad (34)$$

or for Δ^{GD}

$$\Delta_h^{\text{GD}} \mathbf{p}_h^b(e) := \left(\mathbf{d} * \mathbf{d} * \mathbf{p}_h^b \right) (e) = \left(* \mathbf{d} * \mathbf{p}_h^b \right) (\partial e) \quad (35)$$

$$= \sum_{v \prec e} s_{v,e} \left(* \mathbf{d} * \mathbf{p}_h^b \right) (v) = \sum_{v \prec e} \frac{s_{v,e}}{|\star v|} \left(\mathbf{d} * \mathbf{p}_h^b \right) (\star v) \quad (36)$$

$$= \sum_{v \prec e} \frac{s_{v,e}}{|\star v|} \left(* \mathbf{p}_h^b \right) (\partial \star v) = - \sum_{v \prec e} \frac{s_{v,e}}{|\star v|} \sum_{\tilde{e} \succ v} s_{v,\tilde{e}} \left(* \mathbf{p}_h^b \right) (\star \tilde{e}) \quad (37)$$

$$= - \sum_{v \prec e} \frac{s_{v,e}}{|\star v|} \sum_{\tilde{e} \succ v} s_{v,\tilde{e}} \frac{|\star \tilde{e}|}{|\tilde{e}|} \mathbf{p}_h^b(\tilde{e}) \quad (38)$$

where the sign $s_{f,e}$ is +1 if the face $f \succ e$ is in the left of the edge e (-1 otherwise) and $s_{v,e}$ is +1 if the edge $e \succ v$ points to the vertex v and -1 if e points away from v . We see Δ_h^{RR} , Δ_h^{GD} and also $\Delta_h^{\text{dR}} := -\Delta_h^{\text{RR}} - \Delta_h^{\text{GD}}$ are linear operators in $\mathbf{p}_h^b(\tilde{e})$ and therefor results in sparse matrices if the $\mathbf{p}_h^b(\tilde{e})$ are our degree of freedoms. ...explainable figures...

4.4 Discrete norm

Approximating the norm $\|\mathbf{p}^b\|$ on a edge $e \in \mathcal{E}$ is not so easy like the development of discrete linear operators. We only know how \mathbf{p}_h^b "act" on a single edge, so \mathbf{p}_h^b gives us only one dimensional informations. In other words, we only know the proportion $\mathbf{p}_h \cdot \mathbf{e} = \mathbf{p}_h^b(e)$ of the discrete contravariant vector field $\mathbf{p}_h = (\mathbf{p}_h^b)^\sharp$ in the \mathbf{e} direction, but we don't know the length of \mathbf{p}_h defined on this edge.

...maybe some averaging techniques, which I tried and why these sucks...

One way out is to rise the dimension of the discrete 1-forms, therefor we introduce some bases at the intersection $c(e) = e \cap (\star e)$ of a edge and its dual edge to describe discrete contra- and covariant vector fields in a local (piecewise flat) coordinate system.

The basis for contravariant vectors at $c(e)$ is composed of

$$\partial_e X := \mathbf{e} = \mathbf{v}_2 - \mathbf{v}_1 \quad (39)$$

$$\partial_{\star e} X := \star \mathbf{e} = c(f_2) - c(f_1) \quad (40)$$

if $e = [v_1, v_2]$ and the face f_1 lay right and f_2 left of e (...figure...). This definitions are consistent with the canonical basis, if the position X is a barycentric parametrisation of the edge e resp. its dual, i.e. for

$$X_e(t_e) = t_e \mathbf{v}_2 + (1 - t_e) \mathbf{v}_1 \text{ for } t \in [0, 1] \quad (41)$$

$$X_{\star e}(t_{\star e}) = \begin{cases} 2t_{\star e}c(e) + (1 - 2t_{\star e})c(f_1) & \text{if } t_{\star e} \in [0, \frac{1}{2}] \\ (1 - t_{\star e})c(e) + (2t_{\star e} - 1)c(f_1) & \text{if } t_{\star e} \in [\frac{1}{2}, 1] \end{cases} \quad (42)$$

holds $\partial_e X = \partial_{t_e} X_e$ and $\partial_{\star e} X = \frac{1}{2} \left(\partial_{t_{\star e}} X_{\star e}|_{[0, \frac{1}{2}]} + \partial_{t_{\star e}} X_{\star e}|_{[\frac{1}{2}, 1]} \right)$. Therefore we get the local metric tensor

$$\mathbf{g}_h(e) = |e|^2 (dx^e)^2 + |\star e|^2 (dx^{\star e})^2 \quad (43)$$

where $\{dx^e, dx^{\star e}\}$ are the dual base of $\{\partial_e X, \partial_{\star e} X\}$, i.e. $dx^i(\partial_j X) = \delta_j^i$ for $i, j \in \{e, \star e\}$. This gives us the great possibility to define

$$\underline{\mathbf{p}}_h^b(e) := \left[\begin{matrix} \mathbf{p}_h^b \\ \star \mathbf{p}_h^b \end{matrix} \right]_{\text{PD}}(e) := \mathbf{p}_h^b(e) dx^e + \mathbf{p}_h^b(\star e) dx^{\star e} \quad (44)$$

$$= \mathbf{p}_h^b(e) dx^e - \frac{|\star e|}{|e|} \left(\star \mathbf{p}_h^b \right)(e) dx^{\star e} \quad (45)$$

$$\underline{\mathbf{p}}_h(e) := \left(\underline{\mathbf{p}}_h^b(e) \right)^\sharp = \frac{1}{|e|^2} \mathbf{p}_h^b(e) \partial_e X + \frac{1}{|\star e|^2} \mathbf{p}_h^b(\star e) \partial_{\star e} X \quad (46)$$

$$= \frac{1}{|e|} \left(\frac{1}{|e|} \mathbf{p}_h^b(e) \partial_e X - \frac{1}{|\star e|} \left(\star \mathbf{p}_h^b \right)(e) \partial_{\star e} X \right) \quad (47)$$

We call $\underline{\mathbf{p}}_h^b(e)$ the discrete Primal-Dual-1-form (short PD-1-form) and $\underline{\mathbf{p}}_h$ the discrete Primal-Dual-vector field (short PD-vector field). Hence, we get for the square of the norm on $c(e)$ by contract the PD-1-Form with its corresponding PD-Vector

$$\left\| \underline{\mathbf{p}}_h^b \right\|_h^2(e) := \left(\underline{\mathbf{p}}_h^b \left(\underline{\mathbf{p}}_h \right) \right)(e) = \frac{1}{|e|^2} \left(\left[\mathbf{p}_h^b(e) \right]^2 + \left[\left(\star \mathbf{p}_h^b \right)(e) \right]^2 \right) \quad (48)$$

In general, we also get a local discrete inner product between $\underline{\mathbf{p}}_h^b$ and another PD-1-form $\underline{\mathbf{q}}_h^b$ by

$$\left\langle \underline{\mathbf{q}}_h^b, \underline{\mathbf{p}}_h^b \right\rangle_h(e) := \left(\underline{\mathbf{q}}_h^b \left(\underline{\mathbf{p}}_h \right) \right)(e) = \frac{1}{|e|^2} \left(\left[\mathbf{p}_h^b \mathbf{q}_h^b \right](e) + \left[\left(\star \mathbf{p}_h^b \right) \left(\star \mathbf{q}_h^b \right) \right](e) \right) \quad (49)$$

5 Discrete PD-Problem, a DEC approach

To use a PD-1-form solution, we must also determine the dual part $\star \mathbf{p}_h^b$, therefore we discretize the Hodge dual equation (28) (resp. (29)) simultaneous to the primal equation (26) (resp. (27)). This leads to the DEC-discretized Primal-Dual-problem (DEC-PD-problem)

$$\left[\partial_t + K_0 \Delta_h^{\text{dR}} + K_n \left(\left\| \underline{\mathbf{p}}_h^b \right\|_h^2 - 1 \right) \right] \underline{\mathbf{p}}_h^b = 0 \quad (50)$$

respective, without the One-Constant-Approximation,

$$\left[\partial_t - \begin{bmatrix} K_1 \\ K_3 \end{bmatrix} \Delta_h^{\text{GD}} - \begin{bmatrix} K_3 \\ K_1 \end{bmatrix} \Delta_h^{\text{RR}} + K_n \left(\left\| \underline{\mathbf{p}}_h^b \right\|_h^2 - 1 \right) \right] \underline{\mathbf{p}}_h^b = 0 \quad (51)$$

5.1 Time discretization

The simplest way to discretize the DEC-PD-problem in time is to use an implicit Euler scheme, where we handle the norm $\|\underline{\mathbf{p}}_h^b\|_h^2$ explicit. For one Euler step, we have to solve

$$\left[\frac{1}{\tau} + K_0 \Delta_h^{\text{dR}} + K_n \left(\|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2(e) - 1 \right) \right] \underline{\mathbf{p}}_h^b(e) = \frac{1}{\tau} \widehat{\underline{\mathbf{p}}}_h^b(e) \quad (52)$$

resp.

$$\left[\frac{1}{\tau} - \begin{bmatrix} K_1 \\ K_3 \end{bmatrix} \Delta_h^{\text{GD}} - \begin{bmatrix} K_3 \\ K_1 \end{bmatrix} \Delta_h^{\text{RR}} + K_n \left(\|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2(e) - 1 \right) \right] \underline{\mathbf{p}}_h^b(e) = \frac{1}{\tau} \widehat{\underline{\mathbf{p}}}_h^b(e) \quad (53)$$

for all $e \in \mathcal{E}$. $\widehat{\underline{\mathbf{p}}}_h^b$ is the solution of the last time step or the initial condition, if this is the first Euler step. $\tau = t - \hat{t}$ is the time step wide.

The drawback of this semi-implicit Euler scheme is, that we need very small τ . Therefore, it is better to use a Taylor-linearisation for $\|\underline{\mathbf{p}}_h^b\|_h^2 \underline{\mathbf{p}}_h^b$. First we calculate the partially component derivative on an edge $e \in \mathcal{E}$ of

$$\Phi(\underline{\mathbf{p}}_h^b) = \Phi(\underline{\mathbf{p}}_h^b, * \underline{\mathbf{p}}_h^b) := \|\underline{\mathbf{p}}_h^b\|_h^2 \underline{\mathbf{p}}_h^b = \frac{1}{|e|^2} \left((\underline{\mathbf{p}}_h^b)^2 + (* \underline{\mathbf{p}}_h^b)^2 \right) \begin{bmatrix} \underline{\mathbf{p}}_h^b \\ * \underline{\mathbf{p}}_h^b \end{bmatrix} \quad (54)$$

(Henceforward, for a better readability, the arguments e are omitted). Hence, we obtain

$$\partial_{\underline{\mathbf{p}}_h^b} \Phi = \frac{1}{|e|^2} \begin{bmatrix} 3 (\underline{\mathbf{p}}_h^b)^2 + (* \underline{\mathbf{p}}_h^b)^2 \\ 2 \underline{\mathbf{p}}_h^b (* \underline{\mathbf{p}}_h^b) \end{bmatrix} \quad (55)$$

$$\partial_{* \underline{\mathbf{p}}_h^b} \Phi = \frac{1}{|e|^2} \begin{bmatrix} 2 \underline{\mathbf{p}}_h^b (* \underline{\mathbf{p}}_h^b) \\ (\underline{\mathbf{p}}_h^b)^2 + 3 (* \underline{\mathbf{p}}_h^b)^2 \end{bmatrix} \quad (56)$$

For one step Taylor at $\widehat{\underline{\mathbf{p}}}_h^b$, we get

$$\Phi(\underline{\mathbf{p}}_h^b) \approx \Phi(\widehat{\underline{\mathbf{p}}}_h^b) + (\underline{\mathbf{p}}_h^b - \widehat{\underline{\mathbf{p}}}_h^b) \partial_{\underline{\mathbf{p}}_h^b} \Phi(\widehat{\underline{\mathbf{p}}}_h^b) + (* \underline{\mathbf{p}}_h^b - * \widehat{\underline{\mathbf{p}}}_h^b) \partial_{* \underline{\mathbf{p}}_h^b} \Phi(\widehat{\underline{\mathbf{p}}}_h^b) \quad (57)$$

$$= \|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2 \widehat{\underline{\mathbf{p}}}_h^b + \frac{1}{|e|^2} \begin{bmatrix} (\underline{\mathbf{p}}_h^b - \widehat{\underline{\mathbf{p}}}_h^b) \left(3 (\underline{\mathbf{p}}_h^b)^2 + (* \underline{\mathbf{p}}_h^b)^2 \right) + 2 (* \underline{\mathbf{p}}_h^b - * \widehat{\underline{\mathbf{p}}}_h^b) \underline{\mathbf{p}}_h^b (* \underline{\mathbf{p}}_h^b) \\ 2 (\underline{\mathbf{p}}_h^b - \widehat{\underline{\mathbf{p}}}_h^b) \underline{\mathbf{p}}_h^b (* \underline{\mathbf{p}}_h^b) + (* \underline{\mathbf{p}}_h^b - * \widehat{\underline{\mathbf{p}}}_h^b) \left(3 (\underline{\mathbf{p}}_h^b)^2 + (* \underline{\mathbf{p}}_h^b)^2 \right) \end{bmatrix} \quad (58)$$

$$= \|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2 \widehat{\underline{\mathbf{p}}}_h^b - \frac{3}{|e|^2} \left((\widehat{\underline{\mathbf{p}}}_h^b)^2 + (* \widehat{\underline{\mathbf{p}}}_h^b)^2 \right) \widehat{\underline{\mathbf{p}}}_h^b + \frac{2}{|e|^2} \begin{bmatrix} (* \widehat{\underline{\mathbf{p}}}_h^b)^2 \\ (\widehat{\underline{\mathbf{p}}}_h^b)^2 \end{bmatrix} \widehat{\underline{\mathbf{p}}}_h^b \quad (59)$$

$$+ \frac{1}{|e|^2} \left((\widehat{\underline{\mathbf{p}}}_h^b)^2 + (* \widehat{\underline{\mathbf{p}}}_h^b)^2 \right) \underline{\mathbf{p}}_h^b + \frac{2}{|e|^2} \begin{bmatrix} \underline{\mathbf{p}}_h^b \widehat{\underline{\mathbf{p}}}_h^b \\ (* \underline{\mathbf{p}}_h^b) (* \widehat{\underline{\mathbf{p}}}_h^b) \end{bmatrix} \widehat{\underline{\mathbf{p}}}_h^b \\ + \frac{2}{|e|^2} \begin{bmatrix} (* \underline{\mathbf{p}}_h^b - * \widehat{\underline{\mathbf{p}}}_h^b) (* \underline{\mathbf{p}}_h^b) \\ (\underline{\mathbf{p}}_h^b - \widehat{\underline{\mathbf{p}}}_h^b) \underline{\mathbf{p}}_h^b \end{bmatrix} \widehat{\underline{\mathbf{p}}}_h^b$$

$$= -2 \|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2 \widehat{\underline{\mathbf{p}}}_h^b + \|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2 \underline{\mathbf{p}}_h^b + 2 \langle \underline{\mathbf{p}}_h^b, \widehat{\underline{\mathbf{p}}}_h^b \rangle_h \widehat{\underline{\mathbf{p}}}_h^b \quad (60)$$

The inner product term can be also expressed as matrix-vector multiplication:

$$\langle \underline{\mathbf{p}}_h^b, \widehat{\underline{\mathbf{p}}}_h^b \rangle_h \widehat{\underline{\mathbf{p}}}_h^b = \frac{1}{|e|^2} \begin{bmatrix} (* \widehat{\underline{\mathbf{p}}}_h^b)^2 & \widehat{\underline{\mathbf{p}}}_h^b (* \widehat{\underline{\mathbf{p}}}_h^b) \\ \widehat{\underline{\mathbf{p}}}_h^b (* \widehat{\underline{\mathbf{p}}}_h^b) & (\widehat{\underline{\mathbf{p}}}_h^b)^2 \end{bmatrix} \cdot \underline{\mathbf{p}}_h^b =: (\underline{\widehat{\mathbf{p}}}_h^b \otimes_h \widehat{\underline{\mathbf{p}}}_h^b)^{\sharp, \cdot} \cdot \underline{\mathbf{p}}_h^b \quad (61)$$

with a formally defined local outer product \otimes_h for PD-1-forms, which respect the discrete local metric.

With the Taylor linearization and the implicit Euler scheme, we have to solve in every time steps and all $e \in \mathcal{E}$

$$\left[\frac{1}{\tau} + K_0 \Delta_h^{\text{dR}} + K_n \left(\|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2 - 1 \right) \right] \underline{\mathbf{p}}_h^b + 2 K_n (\underline{\widehat{\mathbf{p}}}_h^b \otimes_h \widehat{\underline{\mathbf{p}}}_h^b) \cdot \underline{\mathbf{p}}_h^b = \left[\frac{1}{\tau} + 2 K_n \|\widehat{\underline{\mathbf{p}}}_h^b\|_h^2 \right] \widehat{\underline{\mathbf{p}}}_h^b \quad (62)$$

resp.

$$\left[\frac{1}{\tau} - \begin{bmatrix} K_1 \\ K_3 \end{bmatrix} \Delta_h^{\text{GD}} - \begin{bmatrix} K_3 \\ K_1 \end{bmatrix} \Delta_h^{\text{RR}} + K_n \left(\left\| \hat{\underline{\mathbf{p}}}_h^{\flat} \right\|_h^2 - 1 \right) \right] \underline{\mathbf{p}}_h^{\flat} + 2K_n \left(\hat{\underline{\mathbf{p}}}_h^{\flat} \otimes_h \hat{\underline{\mathbf{p}}}_h^{\flat} \right) \cdot \underline{\mathbf{p}}_h^{\flat} = \left[\frac{1}{\tau} + 2K_n \left\| \hat{\underline{\mathbf{p}}}_h^{\flat} \right\|_h^2 \right] \hat{\underline{\mathbf{p}}}_h^{\flat} \quad (63)$$

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