

# Thin Shell Stuff

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## 1 Metric Quantities

In the following, we consider a *thin shell* of constant thickness  $h \in \mathbb{R}$  around an oriented, boundarieless, compact Riemannian 2-manifold (surface)  $\mathcal{S}$  defined by

$$\mathcal{S}_h := \mathcal{S} \times \left[ -\frac{h}{2}, \frac{h}{2} \right] \subset \mathbb{R}^3. \quad (1)$$

Constant thickness means, that the orthogonal measurement of the two disjoint boundaries  $\partial^+ \mathcal{S}_h \sqcup \partial^- \mathcal{S}_h = \partial \mathcal{S}_h$  is  $h$  at all boundary points. Thereby, be  $h$  small enough, so that  $\mathcal{S}_h \subset \mathbb{R}^3$  contains no overlaps, i. e., it exists a surjection  $\mathcal{S}_h \rightarrow \mathcal{S}$ .

### 1.1 Coordinates

We define the coordinate in normal direction  $\boldsymbol{\nu}$  of the surface  $\mathcal{S}$  by  $\xi \in \left[ -\frac{h}{2}, \frac{h}{2} \right]$ . If we use any choice of local coordinates  $(u, v) \in U$  of the surface, so that the immersion  $\mathbf{x} : U \rightarrow \mathbb{R}^3$  parameterize  $\mathcal{S} = \text{Im}(\mathbf{x})$ , then we can define the immersion  $\tilde{\mathbf{x}} : U \times \left[ -\frac{h}{2}, \frac{h}{2} \right] \rightarrow \mathbb{R}^3$  with  $\mathcal{S}_h = \text{Im}(\tilde{\mathbf{x}})$  by

$$\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(u, v, \xi) := \mathbf{x}(u, v) + \xi \boldsymbol{\nu}(u, v) = \mathbf{x} + \xi \boldsymbol{\nu}. \quad (2)$$

## 1.2 Arrangements

**Lowercase letters**  $i, j, k, \dots$  are used as index for  $u$  and  $v$ , e. g.,  $\alpha_i dx^i$  is an 1-form in  $T^*\mathcal{S}$ .

**Uppercase letters**  $I, J, K, \dots$  are used for  $u, v$  and  $\xi$ , e. g.,  $\tilde{\alpha}^I \partial_I \tilde{\mathbf{x}} = \tilde{\alpha}^i \partial_i \tilde{\mathbf{x}} + \tilde{\alpha}^\xi \partial_\xi \tilde{\mathbf{x}}$  is a (contravariant) vector in  $T\mathcal{S}_h$ .

**The Tilde** are used for quantities and relations in context of  $\mathcal{S}_h$ , e. g.,  $\tilde{\alpha} \in T\mathcal{S}_h$  but  $\alpha \in T\mathcal{S}$  and we can construct a relation  $\tilde{\alpha} = \alpha + \alpha^\xi \nu$ .

**Full covariant descriptions** (lower indices) are always used, unless otherwise is defined, e. g.,  $\mathbf{B} = \{B_{ij}\}$  is the full covariant shape operator, i. e., the second fundamental form in this representation.

**Indexing and collector brackets**  $\square$  and  $\{\}$ , are used to switch between components and object representations, e. g.,  $[\mathbf{t}]_{ij} = t_{ij}$  and  $\{t_{ij}\} = \mathbf{t}$ .

**Sharp and flat operator on tensors** are generalisations of the usual flat and sharp operator on vector valued quantities and can be realized by matrix multiplications with the metric tensor  $\mathbf{g}$  and its inverse  $\mathbf{g}^{-1}$ , e. g.,  $\flat \left\{ t^i_j \right\}^\sharp = \mathbf{g} \left\{ t^i_j \right\} \mathbf{g}^{-1} = \left\{ t^i_j \right\} = \mathbf{t}^\sharp$ .

**Tensor product** means always the contraction of the last component of a tensor with the first of another tensor, e. g.,  $[\mathbf{st}]_{ij} = s_i^k t_{kj}$ , e. g., with an usual matrix product  $\cdot$ , this implies  $\mathbf{st} = \mathbf{s} \cdot \mathbf{g}^{-1} \cdot \mathbf{t}$ .

## 1.3 The Metric Tensor and Shape Operator

With an arbitrary choice of surface coordinates  $(u, v) \in U$ , we can calculate the canonical basic vectors  $\partial_I \tilde{\mathbf{x}} \in T\mathcal{S}_h$  by

$$\partial_i \tilde{\mathbf{x}} = \partial_i \mathbf{x} + \xi \partial_i \nu \quad (3)$$

$$\partial_\xi \tilde{\mathbf{x}} = \nu. \quad (4)$$

The *metric tensor (first fundamental form)* of thin shell is given by its components  $\tilde{g}_{IJ} = \langle \partial_I \tilde{\mathbf{x}}, \partial_J \tilde{\mathbf{x}} \rangle_{\mathcal{S}_h}$ . Therefore, for the mixed tangential-normal components holds  $\tilde{g}_{i\xi} = \tilde{g}_{\xi i} = 0$ , because

$$\langle \partial_i \nu, \nu \rangle_{\mathcal{S}_h} = \frac{1}{2} \partial_i \|\nu\|_{\mathcal{S}_h}^2 = 0. \quad (5)$$

For the pure normal component, we obtain  $\tilde{g}_{\xi\xi} = \|\nu\|_{\mathcal{S}_h}^2 = 1$ , i. e., the co- and contravariant normal components of a tensor quantity are equivalently, e. g., (detailed)

$$\tilde{t}^I_{\xi J} = \tilde{g}_{\xi K} \tilde{t}^{IK}_J = \tilde{g}_{\xi k} \tilde{t}^{Ik}_J + \tilde{g}_{\xi\xi} \tilde{t}^{I\xi}_J = \tilde{t}^{I\xi}_J. \quad (6)$$

For the pure tangential components, we get a second degree tensor polynomial in  $\xi$

$$\tilde{g}_{ij} = g_{ij} - 2\xi B_{ij} + \xi^2 [\mathbf{B}^2]_{ij} = \left[ (g - \xi \mathbf{B})^2 \right]_{ij} \quad (7)$$

$$= g_{ij} - 2\xi B_{ij} + \xi^2 (\mathcal{H}B_{ij} - \mathcal{K}g_{ij}) . \quad (8)$$

where the *covariant shape operator (second fundamental form)* is given by

$$B_{ij} = -\langle \partial_i \mathbf{x}, \partial_j \boldsymbol{\nu} \rangle_{\mathcal{S}_h} \quad (9)$$

and the *third fundamental form* by

$$[\mathbf{B}^2]_{ij} = \langle \partial_i \boldsymbol{\nu}, \partial_j \boldsymbol{\nu} \rangle_{\mathcal{S}_h} , \quad (10)$$

see [HW53].  $\mathcal{K} = |\mathbf{B}^\#|$  is the *Gaussian curvature* and  $\mathcal{H} := \text{Tr} \mathbf{B} = B^i_i$  the *mean curvature*. (In a more differential geometrical context on surfaces, this is minus twice the mean curvature.) A more classical representation of the third fundamental form  $\mathbf{B}^2$  is

$$\mathbf{B}^2 = \mathcal{H} \mathbf{B} - \mathcal{K} \mathbf{g} . \quad (11)$$

**Theorem 1.** *For the inverse thin shell metric  $\tilde{\mathbf{g}}^{-1}$  holds*

$$\tilde{g}^{ij} = \left( g^{ik} + \sum_{\mathfrak{l}=1}^{\infty} \xi^{\mathfrak{l}} [\mathbf{B}^{\mathfrak{l}}]^{ik} \right) \left( \delta_k^j + \sum_{\mathfrak{k}=1}^{\infty} \xi^{\mathfrak{k}} [\mathbf{B}^{\mathfrak{k}}]_k^j \right) \quad (12)$$

$$= \left[ \left( \mathbf{g} + \sum_{\mathfrak{k}=1}^{\infty} \xi^{\mathfrak{k}} \mathbf{B}^{\mathfrak{k}} \right)^2 \right]^{ij} , \quad (13)$$

$$\tilde{g}^{\xi\xi} = 1 , \quad (14)$$

$$\tilde{g}^{i\xi} = \tilde{g}^{\xi i} = 0 . \quad (15)$$

*Proof.* First we define the pure tangential components of the thin shell metric tensor as  $\tilde{\mathbf{g}}_t := \{\tilde{g}_{ij}\}$ . With  $\boldsymbol{\delta} = \{\delta_j^i\}$  the Kronecker delta, we can write down in usual matrix notation

$$\tilde{\mathbf{g}} \cdot \tilde{\mathbf{g}}^{-1} = \begin{bmatrix} \tilde{\mathbf{g}}_t & O \\ O & 1 \end{bmatrix} \cdot \begin{bmatrix} \{\tilde{g}^{ij}\} & \{\tilde{g}^{i\xi}\} \\ \{\tilde{g}^{\xi i}\} & \tilde{g}^{\xi\xi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta} & O \\ O & 1 \end{bmatrix} . \quad (16)$$

Thus, we obtain

$$\tilde{g}^{\xi\xi} = 1 , \quad (17)$$

$$\tilde{g}^{i\xi} = \tilde{g}^{\xi i} = 0 , \quad (18)$$

$$\{\tilde{g}^{ij}\} = \tilde{\mathbf{g}}_t^{-1} = (\mathbf{g} - \xi \mathbf{B})^{-2} = (\mathbf{g} - \xi \mathbf{B})^{-1} \cdot (\boldsymbol{\delta} - \xi \mathbf{B}^\#)^{-1} . \quad (19)$$

For  $h$  small enough, so that  $\xi \|\mathbf{B}\| \leq h \|\mathbf{B}\| < 1$  and exponent with a dot indicate matrix (endomorphism) power, we can use the Neumann serie

$$\left(\delta - \xi \mathbf{B}^\sharp\right)^{-1} = \delta + \sum_{\mathfrak{k}=1}^{\infty} \xi^{\mathfrak{k}} \left(\mathbf{B}^\sharp\right)^{\cdot \mathfrak{k}}, \quad (20)$$

and therefore the assertion, because with  $\mathbf{B}^{\mathfrak{k}} = (\mathbf{B} \cdot \mathbf{g}^{-1})^{\cdot \mathfrak{k}} \cdot \mathbf{g}$  we get

$$\left(\mathbf{B}^\sharp\right)^{\cdot \mathfrak{k}} = (\mathbf{B} \cdot \mathbf{g}^{-1})^{\cdot \mathfrak{k}} = \mathbf{B}^{\mathfrak{k}} \cdot \mathbf{g}^{-1} = \left(\mathbf{B}^{\mathfrak{k}}\right)^\sharp \quad (21)$$

and

$$(\mathbf{g} - \xi \mathbf{B})^{-1} = \left((\delta - \xi \mathbf{B}^\sharp) \cdot \mathbf{g}\right)^{-1} = \mathbf{g}^{-1} \cdot (\delta - \xi \mathbf{B}^\sharp)^{-1} = {}^\sharp (\delta - \xi \mathbf{B}^\sharp)^{-1}. \quad (22)$$

□

Therefore, we get for  $\tilde{\mathbf{g}}_t^{-1}$  a polynomial in  $\xi \mathbf{B}$  and with successively applying (11), i. e.,  $\mathbf{B}^{\mathfrak{k}} = \mathcal{H} \mathbf{B}^{\mathfrak{k}-1} - \mathcal{K} \mathbf{B}^{\mathfrak{k}-2}$ , we can always find polynomials  $p$  and  $q$  in  $\mathcal{K}$ ,  $\mathcal{H}$  and  $\xi$ , so that holds  $\tilde{\mathbf{g}}_t^{-1} = p(\mathcal{K}, \mathcal{H}, \xi) \mathbf{g}^{-1} + q(\mathcal{K}, \mathcal{H}, \xi) {}^\sharp \mathbf{B}^\sharp$ . We will not carry this out in full generality, but let us mention some developments in  $\xi$ .

**Conclusion 1.** *The developments of  $\tilde{\mathbf{g}}_t^{-1}$  up to second degree in  $\xi$  are*

$$\tilde{\mathbf{g}}_t^{-1} = \mathbf{g}^{-1} + \mathcal{O}(\xi), \quad (23)$$

$$\tilde{\mathbf{g}}_t^{-1} = \mathbf{g}^{-1} + 2\xi {}^\sharp \mathbf{B}^\sharp + \mathcal{O}(\xi^2), \quad (24)$$

$$\tilde{\mathbf{g}}_t^{-1} = \mathbf{g}^{-1} + 2\xi {}^\sharp \mathbf{B}^\sharp + 3\xi^2 {}^\sharp (\mathbf{B}^2)^\sharp + \mathcal{O}(\xi^3) \quad (25)$$

$$= (1 - 3\xi^2 \mathcal{K}) \mathbf{g}^{-1} + \xi (2 + 3\xi \mathcal{H}) {}^\sharp \mathbf{B}^\sharp + \mathcal{O}(\xi^3). \quad (26)$$

### 1.3.1 The Volume Element

To develop the thin shell volume element  $\tilde{\mu}$  in normal direction at the surface volume element  $\mu$ , we need a development of the determinant of the metric tensor  $\tilde{\mathbf{g}}$ .

**Theorem 2.** *For the determinant of the thin shell metric tensor  $|\tilde{\mathbf{g}}|$  holds*

$$|\tilde{\mathbf{g}}| = (1 - \xi \mathcal{H} + \xi^2 \mathcal{K})^2 |\mathbf{g}|, \quad (27)$$

*Proof.* The mixed components are zero, so we get

$$|\tilde{\mathbf{g}}| = \tilde{g}_{\xi\xi} |\tilde{\mathbf{g}}_t| = |\tilde{\mathbf{g}}_t|. \quad (28)$$

Now, we define  $\sqrt{{}^\sharp \tilde{\mathbf{g}}_t^\sharp} := (\mathbf{g} - \xi \mathbf{B})^\sharp$  as a square root of  $\tilde{\mathbf{g}}_t^\sharp$ , because

$$\tilde{\mathbf{g}}_t^\sharp = \left((\mathbf{g} - \xi \mathbf{B})^2\right)^\sharp = \left((\mathbf{g} - \xi \mathbf{B})^\sharp (\mathbf{g} - \xi \mathbf{B})\right)^\sharp = (\mathbf{g} - \xi \mathbf{B})^\sharp (\mathbf{g} - \xi \mathbf{B})^\sharp = \left(\sqrt{{}^\sharp \tilde{\mathbf{g}}_t^\sharp}\right)^2. \quad (29)$$

Hence, we can calculate

$$|\tilde{\mathbf{g}}| = |\tilde{\mathbf{g}}_t| = |\tilde{\mathbf{g}}_t^\# \mathbf{g}| = |\tilde{\mathbf{g}}_t^\#| |\mathbf{g}| = \left| \sqrt{\tilde{\mathbf{g}}_t^\#} \right|^2 |\mathbf{g}|. \quad (30)$$

For the determinant of  $\sqrt{\tilde{\mathbf{g}}_t^\#}$ , we regard that  $\mathbf{g}^\#$  is the Kronecker delta, so we obtain

$$\left| \sqrt{\tilde{\mathbf{g}}_t^\#} \right| = \left| \mathbf{g}^\# - \xi \mathbf{B}^\# \right| = (1 - \xi B_u^u) (1 - \xi B_v^v) - \xi^2 B_u^v B_v^u \quad (31)$$

$$= 1 - \xi (B_u^u + B_v^v) + \xi^2 (B_u^u B_v^v - B_u^v B_v^u) = (1 - \xi \mathcal{H} + \xi^2 \mathcal{K}). \quad (32)$$

□

Therefore a representation of the *thin shell volume element*  $\tilde{\mu}$ , depending on the surface volume element  $\mu$ , is

$$\tilde{\mu} = \sqrt{|\tilde{\mathbf{g}}|} du \wedge dv \wedge d\xi = (1 - \xi \mathcal{H} + \xi^2 \mathcal{K}) \mu \wedge d\xi \quad (33)$$

$$= (1 - \xi \mathcal{H} + \xi^2 \mathcal{K}) d\xi \wedge \mu. \quad (34)$$

### 1.3.2 The Levi-Civita Tensor and Hodge-Dualism

### 1.4 Christoffel Symbols

The *Christoffel symbols* are needed to define an unique metric compatible derivation (Levi-Civita connection). With a choice of coordinates, the christoffel symbols (of second kind) on the thin shell are

$$\tilde{\Gamma}_{IJ}^K = \frac{1}{2} \tilde{g}^{KL} (\partial_I \tilde{g}_{JL} + \partial_J \tilde{g}_{IL} - \partial_L \tilde{g}_{IJ}). \quad (35)$$

On the surface  $\mathcal{S}$ , they are equal defined, just omit the tilde and use lowercase letters for indexing.

**Theorem 3.** With  $\beta_{ij}^k := B_i^k|_j + B_j^k|_i - B_{ij}^{|k}$  the second order expansions in normal direction of Christoffel symbols are

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k - \xi \beta_{ij}^k + \mathcal{O}(\xi^2) \quad (36)$$

$$\tilde{\Gamma}_{ij}^\xi = B_{ij} - \xi [\mathbf{B}^2]_{ij} = (1 - \xi \mathcal{H}) B_{ij} + \xi \mathcal{K} g_{ij} \quad (37)$$

$$\tilde{\Gamma}_{i\xi}^k = \tilde{\Gamma}_{\xi i}^k = -B_i^k - \xi [\mathbf{B}^2]_i^k + \mathcal{O}(\xi^2) = -(1 + \xi \mathcal{H}) B_i^k + \xi \mathcal{K} \delta_i^k + \mathcal{O}(\xi^2) \quad (38)$$

$$\tilde{\Gamma}_{\xi\xi}^K = 0 \quad (39)$$

$$\tilde{\Gamma}_{I\xi}^\xi = \tilde{\Gamma}_{\xi I}^\xi = 0. \quad (40)$$

*Proof.* Properties of the thin shell metric  $\tilde{\mathbf{g}}$  are the mixed tangential-normal components are zero (the same holds for the inverse metric) and the pure normal component is

constant. Hence, we get

$$\tilde{\Gamma}_{\xi\xi}^K = \frac{1}{2}\tilde{g}^{KL}(\partial_\xi\tilde{g}_{\xi L} + \partial_\xi\tilde{g}_{\xi L} - \partial_L\tilde{g}_{\xi\xi}) = 0, \quad (41)$$

$$\tilde{\Gamma}_{I\xi}^\xi = \frac{1}{2}\tilde{g}^{\xi\xi}(\partial_I\tilde{g}_{\xi\xi} + \partial_\xi\tilde{g}_{\xi I} - \partial_\xi\tilde{g}_{I\xi}). \quad (42)$$

The partial derivative in normal direction of the tangential part of thin shell metric is

$$\partial_\xi\tilde{g}_{ij} = 2\left(-B_{ij} + \xi[\mathbf{B}^2]_{ij}\right). \quad (43)$$

Therefore, we obtain

$$\tilde{\Gamma}_{ij}^\xi = \frac{1}{2}\tilde{g}^{\xi\xi}(\partial_i\tilde{g}_{j\xi} + \partial_j\tilde{g}_{i\xi} - \partial_\xi\tilde{g}_{ij}) = B_{ij} - \xi[\mathbf{B}^2]_{ij}, \quad (44)$$

$$\tilde{\Gamma}_{i\xi}^k = \frac{1}{2}\tilde{g}^{kl}(\partial_i\tilde{g}_{\xi l} + \partial_\xi\tilde{g}_{il} - \partial_l\tilde{g}_{i\xi}) = \left(g^{kl} + 2\xi B^{kl} + \mathcal{O}(\xi^2)\right)(-B_{il} + \xi[\mathbf{B}^2]_{il}) \quad (45)$$

$$= -B_i{}^k - \xi[\mathbf{B}^2]_i{}^k + \mathcal{O}(\xi^2) \quad (46)$$

and with the substitution (11) the remaining statements of these two terms. For the pure tangential thin shell Christoffel symbols, we first determine  $\beta_{ij}{}^k$  at the surface in terms of partial derivatives and take advantage of the symmetry of the shape operator, i.e.,

$$\beta_{ij}{}^k = g^{kl}(B_{il|j} + B_{jl|i} - B_{ij|l}) \quad (47)$$

$$= g^{kl}(\partial_j B_{il} - \Gamma_{ij}^m B_{ml} - \Gamma_{jl}^m B_{im} + \partial_i B_{jl} - \Gamma_{ij}^m B_{ml} - \Gamma_{il}^m B_{jm} - \partial_l B_{ij} + \Gamma_{il}^m B_{mj} + \Gamma_{jl}^m B_{im}) \quad (48)$$

$$= g^{kl}(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij} - 2\Gamma_{ij}^m B_{ml}) \quad (49)$$

$$= g^{kl}(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij}) - 2\Gamma_{ijl} B^{kl} \quad (50)$$

$$= g^{kl}(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij}) - B^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}). \quad (51)$$

Hence, we get

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2}\tilde{g}^{kl}(\partial_i\tilde{g}_{jl} + \partial_j\tilde{g}_{il} - \partial_l\tilde{g}_{ij}) \quad (52)$$

$$= \frac{1}{2}\left(g^{kl} + 2\xi B^{kl} + \mathcal{O}(\xi^2)\right)(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij} - 2\xi(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij})) \quad (53)$$

$$= \Gamma_{ij}^k + \xi\left(B^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) - g^{kl}(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij})\right) + \mathcal{O}(\xi^2) \quad (54)$$

$$= \Gamma_{ij}^k - \xi\beta_{ij}{}^k + \mathcal{O}(\xi^2). \quad (55)$$

□

## References

- [HW53] Philip Hartman and Aurel Wintner. On the third fundamental form of a surface. *American Journal of Mathematics*, 75(2):298–334, 1953.