Contents

1 Frank Oseen energy

In $\mathbb{R}^3$:

$$E_{\text{OS}} = \frac{1}{2}\int_{\Omega} K_1 \left(\nabla\cdot\mathbf{p}\right)^2 + K_2 \left(\mathbf{p}\cdot\left[\nabla\times\mathbf{p}\right]\right)^2 + K_3 \left\|\mathbf{p}\times\left[\nabla\times\mathbf{p}\right]\right\|^2 dV \tag{1}$$

With the Langrange identity for the K_3-term, we cann rewrite (1) to

$$E_{\text{OS}} = \frac{1}{2}\int_{\Omega} K_1 \left(\nabla\cdot\mathbf{p}\right)^2 + \left(K_2 - K_3\right)\left(\mathbf{p}\cdot\left[\nabla\times\mathbf{p}\right]\right)^2 + K_3 \left\|\mathbf{p}\right\|^2 \left\|\nabla\times\mathbf{p}\right\|^2 dV \tag{2}$$

If we restrict (2) to a 2-dimensional Manifold $M \subset \Omega$ and postulate that $\mathbf{p} \in T_X M$ is a normalized tangential vector in $X \in M$, we get

$$E_{\text{OS}} = \frac{1}{2}\int_{M} K_1 \left(\text{Div}\mathbf{p}\right)^2 + K_3 \left(\text{Rot}\mathbf{p}\right)^2 dA \tag{3}$$

In terms of exterior calculus with the corresponding 1-form $\mathbf{p}^\flat \in \Lambda^1(M)$, ,i.e. $\left(\mathbf{p}^\flat\right)^\sharp = \mathbf{p}$, we obtain

$$E_{\text{OS}} = \frac{1}{2}\int_{M} K_1 \left(\mathbf{d}^*\mathbf{p}^\flat\right)^2 + K_3 \left(*\mathbf{d}\mathbf{p}^\flat\right)^2 dA \tag{4}$$

where the exterior coderivative $\mathbf{d}^* := -*\mathbf{d}*$ is the L^2-orthogonal operator of the exterior derivative $\mathbf{d}$. (Note $\text{Div}\mathbf{p} = -\mathbf{d}^*\mathbf{p}^\flat$ and $\text{Rot}\mathbf{p} = *\mathbf{d}\mathbf{p}^\flat$)

1.1 Functional derivative

With the L^2-orthogonality of the exterior derivative and coderivative and a arbitrary $\alpha \in \Lambda^1(M)$ we get

$$\int_M \left\langle \frac{\delta E_{\text{OS}}}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(E_{\text{OS}} \left[\mathbf{p}^\flat + \epsilon \alpha \right] - E_{\text{OS}} \left[\mathbf{p}^\flat \right] \right) \tag{5}$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_M K_1 \left(2\epsilon \left(\mathbf{d}^* \mathbf{p}^\flat \right) \left(\mathbf{d}^* \alpha \right) + \epsilon^2 \left(\mathbf{d}^* \alpha \right)^2 \right) \tag{6}$$

$$+ K_3 \left(2\epsilon \left\langle \mathbf{d}\mathbf{p}^\flat, \mathbf{d}\alpha \right\rangle + \epsilon^2 \left\| \mathbf{d}\alpha \right\|^2 \right) dA \tag{7}$$

$$= - \int_M K_1 \left\langle \Delta_{\text{GD}} \mathbf{p}^\flat, \alpha \right\rangle + K_3 \left\langle \Delta_{\text{RR}} \mathbf{p}^\flat, \alpha \right\rangle dA \tag{8}$$

$$= \int_M \left\langle - \left(K_1 \Delta_{\text{GD}} + K_3 \Delta_{\text{RR}} \right) \mathbf{p}^\flat, \alpha \right\rangle dA \tag{9}$$

where $\Delta_{\text{RR}} = -\mathbf{d}^*\mathbf{d} = *\mathbf{d} * \mathbf{d}$ is the Vector-Laplace-Beltrami-Operator or Rot-Rot-Laplace and $\Delta_{\text{GD}} = -\mathbf{d}\mathbf{d}^* = \mathbf{d} * \mathbf{d} *$ is the Vector-Laplace-CoBeltrami-Operator or Grad-Div-Laplace. Hence, for a One-Constant-Approximation $K_1 = K_3 =: K_0$, we obtain

$$\int_M \left\langle \frac{\delta E_{\text{OS}}}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \int_M \left\langle K_0 \Delta_{\text{dR}} \mathbf{p}^\flat, \alpha \right\rangle dA \tag{10}$$

where $\Delta_{\text{dR}} = -\Delta_{\text{RR}} - \Delta_{\text{GD}} = \mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*$ is the Laplace-de Rham operator.

1.2 Unit vector invariance

If $\mathbf{p} \in T_X M$ is a unit vector on M, we can describe all unit vectors in $X \in M$ as a rotation in the tangential space with angle $\phi \in \mathbb{R}$:

$$\mathbf{q} = \cos\phi \mathbf{p} + \sin\phi \left(*\mathbf{p} \right) \tag{11}$$

$*\mathbf{p} = \left(*\mathbf{p}^\flat \right)^\sharp$ is the Hodge dual of $\mathbf{p}$, i.e. a quarter rotation of $\mathbf{p}$. For a space independent angle ϕ, i.e. $\mathbf{d}\phi = 0$, straight forward calculations implies

$$\left\| \text{Rot}(*\mathbf{p}) \right\| = \left\| *\mathbf{d} * \mathbf{p}^\flat \right\| = \left\| \text{Div}\mathbf{p} \right\| \tag{12}$$

$$\left\| \text{Div}(*\mathbf{p}) \right\| = \left\| *\mathbf{d} * *\mathbf{p}^\flat \right\| = \left\| *\mathbf{d}\mathbf{p}^\flat \right\| = \left\| \text{Rot}\mathbf{p} \right\| \tag{13}$$

$$\left\| \text{Rot}\mathbf{q} \right\|^2 = \left\| *\mathbf{d}\mathbf{q}^\flat \right\|^2 = \left\| \mathbf{d}\mathbf{q}^\flat \right\|^2 \tag{14}$$

$$= \cos^2\phi \left\| \text{Rot}\mathbf{p} \right\|^2 + \sin^2\phi \left\| \text{Div}\mathbf{p} \right\|^2 + 2\cos\phi\sin\phi \left\langle \mathbf{d}\mathbf{p}^\flat, \mathbf{d} * \mathbf{p}^\flat \right\rangle \tag{15}$$

$$\left\| \text{Div}\mathbf{q} \right\|^2 = \left\| *\mathbf{d} * \mathbf{q}^\flat \right\|^2 = \left\| \mathbf{d} * \mathbf{q}^\flat \right\|^2 \tag{16}$$

$$= \cos^2\phi \left\| \text{Div}\mathbf{p} \right\|^2 + \sin^2\phi \left\| \text{Rot}\mathbf{p} \right\|^2 - 2\cos\phi\sin\phi \left\langle \mathbf{d}\mathbf{p}^\flat, \mathbf{d} * \mathbf{p}^\flat \right\rangle \tag{17}$$

Finally, we get for the One-Constant-Approximation of the Frank-Oseen-Energy

$$E_{\mathrm{OS}}[\mathbf{q}] = E_{\mathrm{OS}}[\mathbf{p}] \tag{18}$$

2 Normalizing energy

To constrain $\mathbf{p}$ is normalized, we add

$$E_n = \int_M \frac{K_n}{4}\left(\|\mathbf{p}\|^2 - 1\right)^2 dA \tag{19}$$

to the Frank Oseen energy. Note that the norm defined by the metric g on the manifold M is invariant regarding lowering or rising the indices, i.e.

$$\|\mathbf{p}\|^2 = p^i g_{ij} p^j = p_i g^{ij} p_j = \left\|\mathbf{p}^\flat\right\|^2 \tag{20}$$

2.1 Functional derivative

By variating $\mathbf{p}^\flat$ under the norm with an arbitrary $\alpha \in \Lambda^1(M)$, we obtain

$$\left\|\mathbf{p}^\flat + \epsilon\alpha\right\|^2 = \left\|\mathbf{p}^\flat\right\|^2 + 2\epsilon\left\langle \mathbf{p}^\flat, \alpha\right\rangle + \epsilon^2\|\alpha\|^2 \tag{21}$$

If we are only interesting in linear terms (in ϵ), this leads to

$$\left(\left\|\mathbf{p}^\flat + \epsilon\alpha\right\|^2 - 1\right)^2 = \left(\left\|\mathbf{p}^\flat\right\|^2 - 1 + 2\epsilon\left\langle \mathbf{p}^\flat, \alpha\right\rangle + \mathcal{O}\left(\epsilon^2\right)\right)^2 \tag{22}$$

$$= \left(\left\|\mathbf{p}^\flat\right\|^2 - 1\right)^2 + 4\epsilon\left(\left\|\mathbf{p}^\flat\right\|^2 - 1\right)\left\langle \mathbf{p}^\flat, \alpha\right\rangle + \mathcal{O}\left(\epsilon^2\right) \tag{23}$$

Hence, we get for the functional derivative of E_n

$$\int_M \left\langle \frac{\delta E_n}{\delta \mathbf{p}^\flat}, \alpha\right\rangle dA = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(E_n\left[\mathbf{p}^\flat + \epsilon\alpha\right] - E_n\left[\mathbf{p}^\flat\right]\right) \tag{24}$$

$$= \int_M \left\langle K_n\left(\left\|\mathbf{p}^\flat\right\|^2 - 1\right)\mathbf{p}^\flat, \alpha\right\rangle dA \tag{25}$$

3 Model equations

To minimize the energy $E := E_{\mathrm{OS}} + E_n$ we choose a time evolving approach. Hence, with the fundamental lemma of calculus of variations, we will use the time depended differential equation in terms of exterior calculus

$$\partial_t \mathbf{p}^\flat = -\frac{\delta E}{\delta \mathbf{p}^\flat} = -K_0\Delta_{\mathrm{dR}}\mathbf{p}^\flat - K_n\left(\left\|\mathbf{p}^\flat\right\|^2 - 1\right)\mathbf{p}^\flat \tag{26}$$

or in general, if we don't want to use the One-Constant-Approximation,

$$\partial_t \mathbf{p}^\flat = \left(K_1 \Delta_{\mathrm{GD}} + K_3 \Delta_{\mathrm{RR}}\right) \mathbf{p}^\flat - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat \tag{27}$$

Note that if we apply the Hodge operator on the whole equations, we get the Hodge dual equations

$$\partial_t (*\mathbf{p}^\flat) = -K_0 \Delta_{\mathrm{dR}} (*\mathbf{p}^\flat) - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) (*\mathbf{p}^\flat) \tag{28}$$

$$= \left(K_1 \Delta_{\mathrm{RR}} + K_3 \Delta_{\mathrm{GD}}\right) (*\mathbf{p}^\flat) - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) (*\mathbf{p}^\flat) \tag{29}$$

which are very useful for the DEC discretization later in context. But this leads to pay attention, because only in the first line (One-Constant-Approximation) we see, that the Hodge dual equations in $*\mathbf{p}^\flat$ is the same as the primal equation in $\mathbf{p}^\flat$. In the general case (second line), we must "swap" the Laplace operators.