Isoparametric finite element approximation of curvature on hypersurfaces

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Abstract

The discretisation of geometric problems often involves a point-wise approximation of curvature, although the discretised surfaces are globally only of class $C^{0,1}$. We define a discrete second fundamental form by means of an L_2-projection in the context of isoparametric Lagrange-elements of polynomial degree γ. We prove that the order of convergence with respect to the L_2-norm for this discrete second fundamental form is $\gamma - 1$.

1 Introduction

In many interesting mathematical, physical and engineering problems the curvature of surfaces plays an important role. Often these surfaces are free boundaries and are to be determined as one of the major unknowns; some examples are mean-curvature flow, dendrite growth or equilibrium figures of freely rotating drops. The numerical treatment of these problems can be difficult if the curvature enters the governing equations explicitly. Often it is possible to arrive at a weak formulation which does no longer explicitly contain higher derivatives of the free boundary (Dziuk, 1991, for example). When such a weak formulation is not known (Schmidt, 1996; Heine, 2006, for example) one needs a pointwise definition of "curvature" for the discretised surface.

This paper is concerned with the approximation of the second fundamental form of a C^k-hypersurface $\Gamma \subset \mathbb{R}^{n+1}$ in the context of an approximation Γ_h of Γ with isoparametric Lagrange finite elements. Using the coordinates of the ambient space $\mathbb{R}^{n+1}$ we can write

$$II = \nabla_\Gamma \nu_\Gamma \tag{1.1}$$

for the Weingarten map, where ν_Γ denotes the outer unit normal field of Γ and ∇_Γ its tangential gradient. In Section 3 we use a weak formulation of (1.1) to define a discrete second fundamental form II_h for the $C^{0,1}$-surface Γ_h. Section 4 establishes L_2-estimates for II_h for the case $n \leq 3$. Finally Section 5 presents the results of experimental convergence tests with respect to the L_2- and L_∞-norm.

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2 Notation and Fundamental Definitions

2.1 Differential geometry

In the following $\Gamma \subset \mathbb{R}^{n+1}$ is an n-dimensional isometric C^2-immersion into $\mathbb{R}^{n+1}$. We identify the tangent space $T_P\Gamma$ at a point $P \in \Gamma$ with the affine hyperplane of $\mathbb{R}^{n+1}$ which is tangential to Γ at P. We denote the (local) outer unit normal field by $\nu_\Gamma = (\nu_1, \dots, \nu_{n+1}) : \Gamma \to \mathbb{R}^{n+1}$. For $f \in C^1(\Gamma)$ we define the tangential gradient of f on Γ by

$$\nabla_\Gamma f := \nabla f - (\nabla f \cdot \nu_\Gamma)\, \nu_\Gamma \in \mathbb{R}^{n+1},$$

where ∇f denotes the usual gradient of $\mathbb{R}^{n+1}$. Above definition makes sense only if f is extended to a neighbourhood of Γ. However, $\nabla_\Gamma f$ does not depend on the extension but only on the values of f on Γ. Likewise, for $w = (w_i)_{i=1}^{n+1} \in \left(C^1(\Gamma)\right)^{n+1}$ we define its tangential divergence by $\nabla_\Gamma \cdot w := \nabla \cdot w - (\nabla w_i \cdot \nu_\Gamma)\, \nu_i$ and we define the Laplace-Beltrami operator of $g \in C^2(\Gamma)$ by

$$\Delta_\Gamma g := \nabla_\Gamma \cdot \nabla_\Gamma g.$$

The Weingarten-map $I\!I = (I\!I_1, \dots, I\!I_{n+1}) : \Gamma \to \mathbb{R}^{(n+1)\times(n+1)}$ is defined by

$$I\!I_i := \nabla_\Gamma \nu_i, \quad 1 \le i \le n+1.$$

$I\!I$ is the usual second fundamental form but written in the coordinate frame of the ambient space $\mathbb{R}^{n+1}$, with zero-extension to $(T_P\Gamma)^\perp$. For $P \in \Gamma$ the eigenvalues κ_i, $1 \le i \le n$, of the second fundamental form $I\!I(P)|_{T_P\Gamma \times T_P\Gamma}$ are called principal curvatures of Γ at P. Further, we define the mean curvature H_Γ, the curvature vector Y_Γ and the Gauss curvature K_Γ at P by

$$H_\Gamma := \frac{1}{n}\sum_{i=1}^{n} \kappa_i, \quad Y_\Gamma := n\, H_\Gamma\, \nu_\Gamma, \quad K_\Gamma := \prod_{i=1}^{n} \kappa_i. \tag{2.1}$$

Elementary computation yields the formula $-\Delta_\Gamma \mathrm{id}_\Gamma = Y_\Gamma$ for the curvature vector Y_Γ. Finally, for $\partial\Gamma = \emptyset$ the formula for integration by parts on Γ is given by

$$\int_\Gamma \nabla_\Gamma f\, do = -\int_\Gamma f\, Y_\Gamma\, do, \quad f \in C^1(\Gamma). \tag{2.2}$$

2.2 Function spaces

For $\Omega \subset \mathbb{R}^n$ open we denote by $L_p(\Omega)$ and $H_p^m(\Omega)$ the usual Lebesgue- and Sobolev-spaces with the corresponding norms

$$||u||_{L_p(\Omega)} = \Big(\int_\Omega |u|^p\, dx\Big)^{\frac{1}{p}}, \quad ||u||_{H_p^m(\Omega)} = \Big(\sum_{k=0}^{m} ||D^k u||_{L_p(\Omega)}^p\Big)^{\frac{1}{p}},$$

with the usual modification for $p = \infty$. For $p = 2$ we simply write $H^m(\Omega) := H_2^m(\Omega)$. By $C^{m,\alpha}\left(\overline{\Omega}\right)$ and $||\cdot||_{C^{m,\alpha}(\overline{\Omega})}$ we denote the usual Hölder spaces and their associated norms. Using local coordinate-charts it is possible to carry over the definitions of Sobolev- and Hölder-spaces to a differential manifold Γ. The resulting spaces are independent from the special choice of the coordinate system up to norm-equivalence. In particular, Wloka (1982, pp. 92) shows that the spaces $\mathrm{H}^m(\Gamma)$ are well defined for a $\mathrm{C}^{m-1,1}$-manifold Γ for any $m \in \mathbb{N}$, which covers especially the case of the $\mathrm{C}^{0,1}$-manifolds constructed by the finite element approach of Section 2.3 below. For an n-dimensional $\mathrm{C}^{0,1}$-immersion $\Gamma \subset \mathbb{R}^{n+1}$ we choose $|u|_{\mathrm{H}^1(\Gamma)} := \sqrt{\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma u \, do}$ for the H^1 semi-norm.

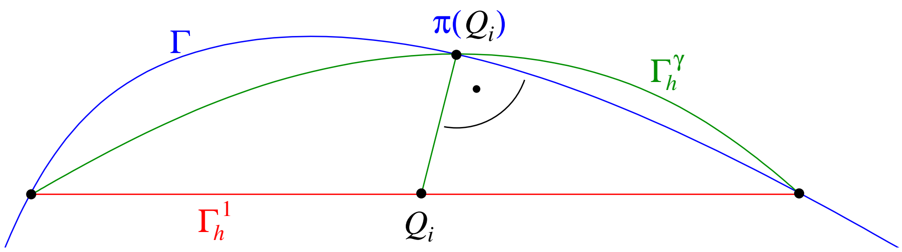
2.3 Isoparametric finite elements

For simplicity we consider a compact closed $\mathrm{C}^{k,1}$-hypersurface $\Gamma \subset \mathbb{R}^{n+1}$ where $k \geq 1$. We note that the results could be extended to immersions but this would involve some notational complications. Because the sectional curvature of Γ is bounded there is $\delta > 0$ such that the decomposition

$$x = \pi(x) + d(x)\,\nu(x), \quad \text{with } \pi : U \to \Gamma,$$

is unique for $x \in U := \{x \in \mathbb{R}^{n+1} |\, \mathrm{dist}(x, \Gamma) < \delta\}$, where $d(x) \in \mathrm{C}^k(U)$ is a signed distance function and $\nu(x) := \nabla d|_{\pi(x)}$. It follows that $\nu(x)$ is normal to Γ at $\pi(x)$ with $|\nu(x)| \equiv 1$.

Figure 2.1 Construction of Γ_h^γ



To construct an isoparametric finite element approximation Γ_h^γ of Γ of given polynomial degree γ we first approximate Γ by a simplicial polytope Γ_h^1 with associated triangulation $\mathcal{T}_h^1 = \{T_h^1 | T_h^1 \text{ is a face of } \Gamma_h^1\}$. The polytope Γ_h^1 must be contained in the δ-strip U where the projection π is well-defined. Then, for any simplex $T_h^1 \in \mathcal{T}_h^1$, we take the images $\pi(Q_i)$ of the canonical Lagrange-nodes Q_i of T_h^1 and define an isoparametric simplex T_h^γ by applying Lagrange-interpolation of degree γ to the coordinates of the projected Lagrange-nodes $\pi(Q_i)$, see Figure 2.1. The result is a discrete surface Γ_h^γ of class $\mathrm{C}^{0,1}$ and associated isoparametric triangulation $\mathcal{T}_h^\gamma$ where $T_h^\gamma := \Phi_{T_h^1}^\gamma(\widehat{T}) \in \mathcal{T}_h^\gamma$ is an image of a reference simplex $\widehat{T} \subset \mathbb{R}^n$ under a polynomial parametrisation $\Phi_{T_h^1}^\gamma : \widehat{T} \to T_h^\gamma$ of degree γ. By construction Γ_h^1 interpolates Γ_h^γ between its vertex Lagrange-nodes. This will be a helpful feature in Section 4 below.

On Γ_h^γ we define parametric finite elements which again are piece-wise polynomial of degree γ or over the reference simplex $\widehat{T}$:

$$W_h^\gamma(\Gamma_h^\gamma) := \{v_h \in \mathrm{C}^{0,1}(\Gamma_h^\gamma) \mid \quad \Phi^\gamma_{T_h^1} * v_h := (v_h \circ \Phi^\gamma_{T_h^1}) \in \mathbb{P}_\gamma(\widehat{T}) \quad \forall T_h^\gamma \in \mathcal{T}_h^\gamma\}. \tag{2.3}$$

The finite element space $W_h^\gamma(\Gamma_h^\gamma)$ is virtually *iso*parametric as the local parametrisation of Γ_h^γ over the reference simplex is piecewise a polynomial of degree γ; so we have in particular $\mathrm{id}_{\Gamma_h^\gamma} \in (W_h^\gamma(\Gamma_h^\gamma))^{n+1}$.

Because $\pi|_{\Gamma_h^\gamma}$ is by construction a bijection we can lift $v_h \in W_h^\gamma(\Gamma_h^\gamma)$ to $\overline{v_h} \in \mathrm{C}^{0,1}(\Gamma)$ by

$$\overline{v_h}(y) := (\pi^{-1} * v_h)(y) = v_h(\pi^{-1}(y)), \quad x \in \Gamma_h^\gamma, \quad y = \pi(x).$$

We call a family of isoparametric triangulations $\{\mathcal{T}_h^\gamma\}_{h>0}$ quasi-uniform if there is $c > 0$ independent from h with

$$h \le c\,\rho(T) \quad \forall T \in \mathcal{T}_h^\gamma \quad \text{with} \quad h = \max_{T\in\mathcal{T}_h^\gamma} h(T),$$

where for $T \in \mathcal{T}_h^\gamma$ we denote by $h(T)$ the diameter of T and by $\rho(T)$ the radius of the largest ball contained in T.

3 Definition of the discrete curvature

In the following Γ_h always denotes a $\mathrm{C}^{0,1}$-approximation of a closed compact $\mathrm{C}^{1\,1}$-immersion $\Gamma \subset \mathbb{R}^{n+1}$ of dimension n with $W_h \le H^1(\Gamma_h)$ such that $\mathrm{id}_{\Gamma_h} \in (W_h)^{n+1}$.

Definition 3.1 (Discrete Curvature Vector) *Under the given assumptions the* discrete curvature vector $Y_h \in (W_h)^{n+1}$ *is defined as the solution of*

$$(Y_h,\ \psi_h)_{\mathrm{L}_2(\Gamma_h)} = (\nabla_{\Gamma_h}\mathrm{id}_{\Gamma_h},\ \nabla_{\Gamma_h}\psi_h)_{\mathrm{L}_2(\Gamma_h)} \quad \forall\,\psi_h \in (W_h)^{n+1}. \tag{3.1}$$

This definition follows Schmidt (1993) and is motivated by the identity

$$\int_\Gamma Y_\Gamma \cdot \psi\, do = -\int_\Gamma \Delta_\Gamma \mathrm{id}_\Gamma \cdot \psi\, do = \int_\Gamma \nabla_\Gamma \mathrm{id}_\Gamma \cdot \nabla_\Gamma \psi\, do \quad \forall \psi \in (H^1(\Gamma))^{n+1},$$

with $Y_\Gamma := n\,H\,\nu$ and $\Gamma \subset \mathbb{R}^{n+1}$ at least $\mathrm{C}^{1,\,1}$, see Section 2.1. The experimental convergence tests for the case of a sphere presented in Schmidt (1993) show approximate linear convergence in the L^2 and L^∞ norms for quadratic Lagrange elements and no convergence for linear elements.

For the definition of the scalar mean and Gaussian curvature we have a look at the Weingarten map $I\!I$, see Section 2.1. To derive a weak formulation for $I\!I = (I\!I_1, \ldots, I\!I_{n+1}) \in \mathbb{R}^{(n+1)\times(n+1)}$ we multiply the identity $I\!I_j = \nabla_\Gamma(\nu_\Gamma)_j$ by a test-function $\psi \in \mathrm{H}^1(\Gamma)$, integrate over Γ and take the integration by parts formula (2.2) on page 2 into account. Carried over to Γ_h and W_h this yields:

DEFINITION 3.2 (DISCRETE WEINGARTEN MAP) *The* discrete Weingarten map $II_h = (II_h^1, \dots, II_h^{n+1}) \in (W_h)^{(n+1)\times(n+1)}$ *is defined as the solution of*

$$\big(II_h^j,\, \psi_h\big)_{L_2(\Gamma_h)} = -\big(\nu_h^j,\, \nabla_{\Gamma_h} \cdot \psi_h\big)_{L_2(\Gamma_h)} - \big(\nu_h^j,\, Y_h \cdot \psi_h\big)_{L_2(\Gamma_h)} \quad \forall \psi_h \in (W_h)^{n+1}, \tag{3.2}$$

where Y_h is the discrete curvature vector defined in (3.1) *and $\nu_h = (\nu_h^1, \dots, \nu_h^{n+1})$ is the (discontinuous) outer normal field of Γ_h.*

We use II_h to define the discrete mean curvature H_h and discrete Gaussian curvature K_h in the obvious way, see Section 2.1.

4 Error estimates

THEOREM 4.1 *Let $\Gamma \subset \mathbb{R}^{n+1}$ $(n \le 3)$ be a closed compact embedded hypersurface of class $C^{m,1}$ with $m \ge 1$ and $1 \le \gamma \le m$. Let $\{\mathcal{T}_h^\gamma\}_{h>0}$ be a quasi-uniform family of isoparametric triangulations of Γ of polynomial degree γ. Then it holds for the discrete curvature vector and the discrete Weingarten map Y_h, $II_h \in W_h^\gamma(\Gamma_h^\gamma)$ that*

$$||Y - \overline{Y_h}||_{L_2(\Gamma)} \le c\, h^{\gamma-1}, \tag{4.1}$$

$$||II - \overline{II_h}||_{L_2(\Gamma)} \le c\, h^{\gamma-1}, \tag{4.2}$$

where $c = c(\sigma, \gamma, \Gamma)$ does not depend on h.

REMARK 4.2 The regularity assumption for Γ ensures that Y, $II \in H^{m-1,\infty}(\Gamma)$.

REMARK 4.3 Our proof of Theorem 4.1 is based on Dziuk (1988) who proves error estimates for the discretisation of the Poisson equation on arbitrary surfaces using linear finite elements. The numerical experiments performed by Schmidt (1996) and our own numerical experiments (see Section 5) are in accordance with this proposition. We continue with some intermediate results before we give a proof of Theorem 4.1.

LEMMA 4.4 *For $u_h \in W_h^\gamma(\Gamma_h^\gamma)$ it holds:*

$$\begin{aligned} \frac{1}{c}||u_h||_{L_2(\Gamma_h)} \le ||\overline{u_h}||_{L_2(\Gamma)} \le c\,||u_h||_{L_2(\Gamma_h)}, \\ \frac{1}{c}||\nabla_{\Gamma_h} u_h||_{L_2(\Gamma_h)} \le ||\nabla_\Gamma \overline{u_h}||_{L_2(\Gamma)} \le c\,||\nabla_{\Gamma_h} u_h||_{L_2(\Gamma_h)}. \end{aligned} \tag{4.3}$$

Proof . See Dziuk (1988). □

LEMMA 4.5 (INVERSE INEQUALITY) *Let $T \in \mathcal{T}_h^\gamma$. Then it holds*

$$||\nabla_{\Gamma_h} u_h||_{L_2(T)} \le c\frac{\sigma(T)^{n/2+1}}{h(T)}||u_h||_{L_2(T_h^\gamma)} \quad \forall u_h \in W_h^\gamma(\Gamma_h^\gamma). \tag{4.4}$$

Proof . The proof is a one-to-one copy of the proof for non-parametric finite elements. □

Following Dziuk (1988) we introduce the following abbreviations:

$$P_h \nabla u_h := \nabla_{\Gamma} u_h \equiv \nabla u_h - \nu_{\Gamma_h} \cdot \nabla u_h \, \nu_{\Gamma_h},$$
$$P_{ik} := \delta_{ik} - \nu_i \, \nu_k, \quad H_{ik} := d_{x_i x_k} = \nu_{i,x_k} = \nu_{k,x_i}.$$

We have that $\nabla u_h(x) = (P(x) - d(x)\, H(x)) \nabla \overline{u_h}|_{x - d(x)\,\nu(x)}$ and $P\,H = H\,P = H$. The defining equations (3.1) for Y_h can be rewritten using integrals over Γ instead of Γ_h:

$$\int_{\Gamma} \overline{Y_h} \cdot \overline{\varphi_h} \frac{1}{\mu_h} \, do = \int_{\Gamma} (P_h(I - d\,H) \nabla_{\Gamma} \overline{\mathrm{id}_{\Gamma_h}}) : (P_h(I - d\,H) \nabla_{\Gamma} \overline{\varphi_h}) \frac{1}{\mu_h} \, do, \tag{4.5}$$

with $\mu_h := \frac{do}{do_h}$. Obviously we have $\frac{1}{c} \le \mu_h \le c < \infty$. By defining the abbreviation $A_h := \frac{1}{\mu_h} P(I - d\,H) P_h \,(I - d\,H) P$ we arrive at

$$(4.5) \iff \int_{\Gamma} \overline{Y_h} \cdot \overline{\varphi_h} \frac{1}{\mu_h} \, do = \int_{\Gamma} \left(A_h \cdot \nabla_{\Gamma} \overline{\mathrm{id}_{\Gamma_h}} \right) : \nabla_{\Gamma} \overline{\varphi_h} \, do. \tag{4.6}$$

Alternatively, we can use the following formulation:

$$\int_{\Gamma} \overline{Y_h} \cdot \overline{\varphi_h} \frac{1}{\mu_h} \, do = \int_{\Gamma} (P_h(I - d\,H) \nabla_{\Gamma}) \cdot \overline{\varphi_h} \frac{1}{\mu_h} \, do = \int_{\Gamma} B_h : \nabla_{\Gamma} \overline{\varphi_h} \, do, \tag{4.7}$$

with $B_h := \frac{1}{\mu_h} P_h(I - d\,H) P$. To cope with the error introduced by lifting $W_h^{\gamma}(\Gamma_h^{\gamma})$ to Γ we need the following estimates:

LEMMA 4.6 *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a closed compact embedded hypersurface of class $\mathrm{C}^{m,1}$ with $m \ge 1$. Let $\mathcal{T}_h^{\gamma}$ be a quasi-uniform isoparametric triangulation of Γ of polynomial degree $1 \le \gamma \le m$. Then*

$$|d| \le c\, h^{\gamma+1}, \tag{4.8}$$
$$|1 - \mu_h| \le c\, h^{\gamma+1}, \tag{4.9}$$
$$|(A_h - I)\, P| \le c\, h^{\gamma+1}, \tag{4.10}$$
$$|(B_h - I)\, P| \le c\, h^{\gamma}, \tag{4.11}$$

where $c = c(\sigma, \gamma, \Gamma)$.

(4.8) and (4.9) are already mentioned without proof in Nédélec (1976) for surfaces in $\mathbb{R}^3$. Dziuk (1988) proves (4.10) for the linear case ($\gamma = 1$) for surfaces in $\mathbb{R}^3$. We nevertheless give a detailed proof of (4.9) because we think that the result is not obvious, though the calculations are elementary.

Proof Lemma 4.6. For any isoparametric surface Γ_h^{γ} we also have naturally a simplicial polytope Γ_h^1 which interpolates the $\mathrm{C}^{m,1}$-surface Γ as well as Γ_h^{γ} linearly in the vertex nodes of the triangulation $\mathcal{T}_h^{\gamma}$. Let $T_h^{\gamma} \in \mathcal{T}_h^{\gamma}$ be an isoparametric element, $T = \pi(T_h^\gamma)$ its associated lifted element and T_h^1 the corresponding triangle which interpolates T_h^γ and T in the vertex Lagrange-nodes (Figure 4.1). Without loss of generality we can assume that $T_h^1 \subset \mathbb{R}^n \times \{0\}$. We use the following notations for the coordinates on $\mathbb{R}^{n+1}$, T_h^1, T_h^γ and T:

$\mathbb{R}^{n+1}$: $x = (x_1, \ldots, x_{n+1})$.

T_h^1: $\eta = (\eta_1, \ldots, \eta_n)$. Without loss of generality we can assume $\eta_i = x_i$, $1 \le i \le n$. We parametrise T_h^γ and T over T_h^1.

T_h^γ: $\xi = (\xi_1(\eta), \ldots, \xi_{n+1}(\eta))$. The coordinate functions $\xi_i(\eta)$ of T_h^γ are polynomials of degree γ over T_h^1.

T: $\pi(\xi(\eta)) = (\pi_1 \circ \xi(\eta), \ldots, \pi_{n+1} \circ \xi(\eta)) = \xi(\eta) - d(\xi(\eta))\, \nu(\xi(\eta))$.

We remark that the coordinate functions $\xi_i(\eta)$ of Γ_h^γ are "ordinary" (i.e. non-parametric) finite element interpolations of polynomial degree γ of the functions $\pi_i \circ \xi(\eta) \in \mathrm{H}^{m+1,\infty}(T_h^1)$. This means that the usual estimates for Lagrange-interpolation apply with respect to the interpolation of π_i by ξ_i. Equation (4.8) is then clear because

$$c\, h^{\gamma+1} \ge |\pi_i(\xi(\eta)) - \xi_i(\eta)| = |d(\xi(\eta))\, \nu_i(\xi(\eta))|, \quad 1 \le i \le (n+1).$$

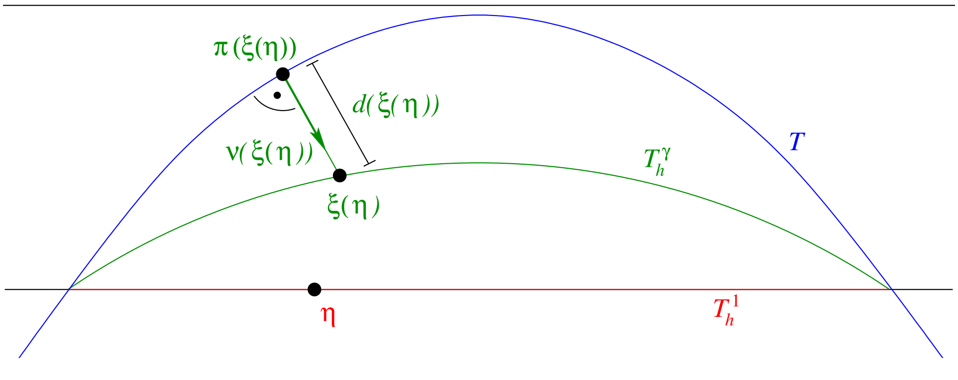
The estimate (4.9) for the surface element is more complicated. For any $(n+1)$-tuple $(a_1, \ldots, a_{n+1})$ we denote by $a_{\widehat{i}}$ the n-tuple where the i-th component has been omitted (e.g. $a_{\widehat{1}} = (a_2, \ldots, a_{n+1})$). In this notation the surface elements do_h on T_h^γ and do on T are

$$do = \sqrt{\left|\frac{\partial \pi_{\widehat{n+1}}}{\partial \eta}\right|^2 + \sum_{k=1}^{n} \left|\frac{\partial \pi_{\widehat{k}}}{\partial \eta}\right|^2}\, d\eta, \quad do_h = \sqrt{\left|\frac{\partial x_{\widehat{n+1}}}{\partial \eta}\right|^2 + \sum_{k=1}^{n} \left|\frac{\partial \xi_{\widehat{k}}}{\partial \eta}\right|^2}\, d\eta.$$

So we have

$$|1 - \frac{do}{do_h}| \le \left| \left|\frac{\partial x_{\widehat{n+1}}}{\partial \eta}\right|^2 - \left|\frac{\partial \pi_{\widehat{n+1}}}{\partial \eta}\right|^2 \right| + \sum_{k=1}^{n} \left| \left|\frac{\partial \xi_{\widehat{k}}}{\partial \eta}\right|^2 - \left|\frac{\partial \pi_{\widehat{k}}}{\partial \eta}\right|^2 \right|. \tag{4.12}$$

Figure 4.1 Parametrisation of Γ over T_h^γ over T_h^1



For estimating the individual determinants we employ Leibniz' formula, using the notation $k_i := i$ for $i < k$, $k_i := i+1$ for $i \geq k$:

$$\left|\frac{\partial \xi_{\widehat{k}}}{\partial \eta}\right| = \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha) \prod_{i=1}^{n} \frac{\partial \xi_{k_i}}{\partial \eta_{\alpha(i)}}, \quad \left|\frac{\partial \pi_{\widehat{k}}}{\partial \eta}\right| = \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha) \prod_{i=1}^{n} \frac{\partial \pi_{k_i}}{\partial \eta_{\alpha(i)}}. \tag{4.13}$$

First we note that T_h^1 interpolates T_h^γ as well as T linearly between their vertices. So we have the following interpolation estimates for the derivatives of the coordinate functions ($1 \leq j \leq n$):

$$c\,h \geq \left|\frac{\partial \xi_i}{\partial \eta_j} - \frac{\partial \eta_i}{\partial \eta_j}\right| = \left|\frac{\partial \xi_i}{\partial \eta_j} - \delta_{ij}\right|, \quad c\,h \geq \left|\frac{\partial \pi_i}{\partial \eta_j} - \delta_{ij}\right| \quad (1 \leq i \leq n), \tag{4.14}$$

$$c\,h \geq \left|\frac{\partial \xi_{n+1}}{\partial \eta_j}\right|, \quad c\,h \geq \left|\frac{\partial \pi_{n+1}}{\partial \eta_j}\right|. \tag{4.15}$$

It follows that

$$\left|\frac{\partial \xi_{\widehat{i}}}{\partial \eta}\right| \leq c\,h, \quad \left|\frac{\partial \pi_{\widehat{i}}}{\partial \eta}\right| \leq c\,h \quad (1 \leq i \leq n), \tag{4.16}$$

because each of the summands in Leibniz' formula (4.13) contains a factor $\frac{\partial \xi_{n+1}}{\partial \eta_j}$ respectively $\frac{\partial \pi_{n+1}}{\partial \eta_j}$. Further, the polynomials $\xi_i(\eta)$ interpolate $\pi_i(\eta)$, so we have $|\frac{\partial \pi_k}{\partial \eta_i} - \frac{\partial \xi_k}{\partial \eta_i}| \leq c\,h^\gamma$. Together with (4.16) this yields

$$\left|\left|\frac{\partial \xi_{\widehat{k}}}{\partial \eta}\right|^2 - \left|\frac{\partial \pi_{\widehat{k}}}{\partial \eta}\right|^2\right| = \Bigg|\underbrace{\left|\frac{\partial \xi_{\widehat{k}}}{\partial \eta}\right| - \left|\frac{\partial \pi_{\widehat{k}}}{\partial \eta}\right|}_{=O(h^\gamma)}\Bigg|\,\Bigg|\underbrace{\left|\frac{\partial \xi_{\widehat{k}}}{\partial \eta}\right| + \left|\frac{\partial \pi_{\widehat{k}}}{\partial \eta}\right|}_{=O(h)}\Bigg| \leq c\,h^{\gamma+1} \quad (1 \leq k \leq n). \tag{4.17}$$

It remains to estimate

$$\left|\left|\frac{\partial \xi_{\widehat{n+1}}}{\partial \eta}\right|^2 - \left|\frac{\partial \pi_{\widehat{n+1}}}{\partial \eta}\right|^2\right| \overset{(4.14)}{=} (2 + O(h))\left|\left|\frac{\partial \xi_{\widehat{n+1}}}{\partial \eta}\right| - \left|\frac{\partial \pi_{\widehat{n+1}}}{\partial \eta}\right|\right|. \tag{4.18}$$

To find an estimate we have to consider the differences $|\frac{\partial \xi_i}{\partial \eta_j} - \frac{\partial \pi_i}{\partial \eta_j}| = |\frac{\partial d}{\partial \eta_j}\,\nu_i - d\,\frac{\partial \nu_i}{\partial \eta_j}|$ for $1 \leq i,\, j \leq n$. From the approximation properties of $\xi_i(\eta)$ we first get

$$\begin{aligned} &c\,h^\gamma \geq \left|\frac{\partial \xi_k}{\partial \eta_j} - \frac{\partial \pi_k}{\partial \eta_j}\right| = \left|\frac{\partial d}{\partial \eta_j}\,\nu_k + d\,\frac{\partial \nu_k}{\partial \eta_j}\right| \quad (1 \leq k \leq n+1) \\ \Longrightarrow \quad &\left|\frac{\partial d}{\partial \eta_j}\right| \leq c\,h^\gamma \text{ a.e., because } d = O(h^{\gamma+1}) \text{ and } \frac{\partial \nu_k}{\partial \eta_j} \text{ is bounded as } \Gamma \text{ is } \mathrm{C}^{1,1}. \end{aligned} \tag{4.19}$$

Further, we have $\nu_i = O(h)$ for $1 \leq i \leq n$. To see this we apply the chain rule to

$$\frac{\partial d(\xi(\eta))}{\partial \eta_i} = \frac{\partial d}{\partial x_{n+1}}\Big|_{\xi(\eta)} \frac{\partial \xi_{n+1}}{\partial \eta_i} + \sum_{j=1}^{n} \frac{\partial d}{\partial x_j}\Big|_{\xi(\eta)} \frac{\partial \xi_j}{\partial \eta_i}$$

So we get $\nu_i = \frac{\partial d}{\partial x_i} = O(h)$, $1 \leq i \leq n$, by estimating each term according to (4.14), (4.15) and (4.19). Therefore it holds

$$|\frac{\partial \xi_i}{\partial \eta_j} - \frac{\partial \pi_i}{\partial \eta_j}| = |\frac{\partial d}{\partial \eta_j}\, \nu_i - d\, \frac{\partial \nu_i}{\partial \eta_j}| = O(h^{\gamma+1}), \quad 1 \leq i \leq n, \tag{4.20}$$

because $\frac{\partial \nu_i}{\partial \eta_j}$ is bounded as Γ is at least $\mathrm{C}^{1,1}$. Together with (4.13) and (4.18) this yields

$$|\left|\frac{\partial \xi_{\widehat{n+1}}}{\partial \eta}\right|^2 - \left|\frac{\partial \pi_{\widehat{n+1}}}{\partial \eta}\right|^2| = c\,|\left|\frac{\partial x_{\widehat{n+1}}}{\partial \eta}\right| - \left|\frac{\partial \pi_{\widehat{n+1}}}{\partial \eta}\right|| \leq c\, h^{\gamma+1},$$

which completes the proof of (4.9). We quote the proof for (4.10) from Dziuk (1988) where the result is proved for the linear case ($\gamma = 1$). The generalisation for $\gamma > 1$ is just a consequence of (4.9):

$$\begin{aligned}
(A_h - I)\, P &= \frac{1}{\mu_h} P\,((I - d\,H)\, P_h\, (I - d\,H)\, P - I)\, P \\
&\overset{(4.9)}{=} P\,(I - d\,H)\, P_h\, (I - d\,H)\, P - I)\, P + O(h^{\gamma+1}) \\
&\overset{(4.8)}{=} P\, P_h\, P - P + O(h^{\gamma+1}) \\
&= -P\, \nu_h\, \nu_h^t\, P + O(h^{\gamma+1}).
\end{aligned}$$

It follows from (4.17) and (4.20) that $\nu_h = \nu + O(h^\gamma)$. So we have

$$|(A_h - I)\, P| \leq |P\, \nu_h|^2 + c\, h^{\gamma+1} \leq \tilde{c}\, h^{\gamma+1},$$

For (4.11) we similarly get

$$\begin{aligned}
(B_h - I)\, P &= \frac{1}{\mu_h}\, P_h\, ((I - d\,H)\, P - I)\, P \\
&\overset{(4.9)}{=} P_h\, (I - d\,H)\, P - I)\, P + O(h^{\gamma+1}) \\
&\overset{(4.8)}{=} P_h\, P - P + O(h^{\gamma+1}) = -\nu_h\, \nu_h^t\, P + O(h^{\gamma+1}) = O(h^\gamma).
\end{aligned}$$

□

Proof Theorem 4.1. Let $\varphi_h \in W_h^\gamma(\Gamma_h^\gamma)$ be an arbitrary test-function. For brevity we set $||\cdot||_2 := ||\cdot||_{\mathrm{L}_2(\Gamma)}$ and $||\cdot||_\infty := ||\cdot||_{\mathrm{L}_\infty(\Gamma)}$. We have

$$||Y - \overline{Y_h}||_2^2 \leq \underbrace{|\int_\Gamma (Y - \overline{Y_h}) \cdot (\overline{\varphi_h} - \overline{Y_h})\, do|}_{:=I_1} + \underbrace{|\int_\Gamma (Y - \overline{Y_h}) \cdot (Y - \overline{\varphi_h})\, do|}_{:=I_2}. \tag{4.21}$$

To use the weak formulation (4.6) with I_1 we have to compensate for the factor $\frac{1}{\mu_h}$ introduced by projecting the finite element functions to the smooth surface Γ:

$$
\begin{aligned}
I_1 &= |\int_\Gamma (Y + \mu_h\, Y - \mu_h\, Y - \overline{Y_h}) \cdot (\overline{\varphi_h} - \overline{Y_h})\, do| \\
&\le |\int_\Gamma (Y - \mu_h\, Y) \cdot (\overline{\varphi_h} - \overline{Y_h})\, do + ||\mu_h||_\infty \int_\Gamma (Y - \frac{1}{\mu_h}\overline{Y_h}) \cdot (\overline{\varphi_h} - \overline{Y_h})\, do| \\
&\overset{(4.7)}{\le} ||1-\mu_h||_\infty\, ||Y||_2\, ||\overline{\varphi_h} - \overline{Y_h}||_2 + ||\mu_h||_\infty \int_\Gamma (I - B_h)\, P : \nabla_\Gamma(\overline{\varphi_h} - \overline{Y_h})\, do \\
&\overset{\text{Lem. 4.6}}{\le} c\, h^{\gamma+1}\, ||\overline{\varphi_h} - \overline{Y_h}||_2 + c\, h^\gamma\, ||\nabla_\Gamma(\overline{\varphi_h} - \overline{Y_h})||_1 \\
&\overset{\text{Lem. 4.5}}{\le} c\, h^{\gamma-1}\, ||\overline{\varphi_h} - \overline{Y_h}||_2.
\end{aligned}
$$

So we finally get from (4.21):

$$
\begin{aligned}
&||Y - \overline{Y_h}||_2^2 \le ||Y - \overline{Y_h}||_2\, ||Y - \overline{\varphi_h}||_2 + c\, h^{\gamma-1}\, ||\overline{\varphi_h} - \overline{Y_h}||_2 \quad \forall \varphi \in W_h^\gamma(\Gamma_h^\gamma). \\
\Longrightarrow\ &||Y - \overline{Y_h}||_2 \le\ c\, h^{\gamma-1},\ \text{by choosing } \varphi_h = I_h^{\gamma-1} Y,\ \text{using that } Y \in \mathrm{H}^{m-1,\infty}(\Gamma).
\end{aligned}
$$

This proves equation (4.1) of Theorem 4.1 on page 5.

The proof of the estimate (4.2) for the discrete second fundamental form $I\!I_h$ works very similar. As for the proof of (4.1) we start by transforming the integrals in the defining equation (3.2) for $I\!I_h$ into surface integrals over Γ; we get

$$
\int_\Gamma \overline{I\!I_h}^j \cdot \overline{\varphi_h} \frac{1}{\mu_h}\, do = -\int_\Gamma \overline{\nu_h}^j \big(A_h \nabla_\Gamma(\overline{\mathrm{id}_{\Gamma_h}})\big) : \nabla_\Gamma \overline{\varphi_h}\, do - \int_\Gamma \overline{\nu_h}^j\, \overline{Y_h} \cdot \overline{\varphi_h}\, \frac{1}{\mu_h}\, do. \tag{4.22}
$$

Therefore

$$
\begin{aligned}
||I\!I_j - \overline{I\!I_h}^j||_2^2 &\le |\int_\Gamma (I\!I_j - \overline{I\!I_h}^j) \cdot (I\!I_j - \overline{\varphi_h})\, do| + |\int_\Gamma (I\!I_j - \overline{I\!I_h}^j) \cdot (\overline{\varphi_h} - \overline{I\!I_h}^j)\, do| \\
&\le ||I\!I_j - \overline{I\!I_h}^j||_2\, ||I\!I_j - \overline{\varphi_h}||_2 + ||1-\mu_h||_\infty\, ||I\!I_j||_2\, ||\overline{\varphi_h} - \overline{I\!I_h}^j||_2 \\
&\quad + ||\mu_h||_\infty\, \underbrace{|\int_\Gamma (I\!I_j - \frac{1}{\mu_h}\overline{I\!I_h}^j) \cdot (\overline{\varphi_h} - \overline{I\!I_h}^j)|}_{:=I_3}.
\end{aligned} \tag{4.23}
$$

We transform I_3 by employing the weak formulation (4.22):

$$
\begin{aligned}
I_3 \le {} & |\int_\Gamma (\nu_j \nabla_\Gamma \mathrm{id}_\Gamma - \overline{\nu_h}^j A_h \nabla_\Gamma \overline{\mathrm{id}_{\Gamma_h}}) : (\nabla_\Gamma \overline{\varphi_h} - \nabla_\Gamma \overline{\mathit{II}_h}^j)\, do| \\
& + |\int_\Gamma (\nu_j Y - \frac{\overline{\nu_h}^j}{\mu_h} \overline{Y_h}) \cdot (\overline{\varphi_h} - \overline{\mathit{II}_h}^j)\, do| \\
\le {} & \big(||(A_h - I)\, P||_\infty \, ||\nabla_\Gamma \overline{\mathrm{id}_{\Gamma_h}}||_2 + ||\nabla_\Gamma (\mathrm{id}_\Gamma - \overline{\mathrm{id}_{\Gamma_h}})||_2 \big) \, ||\nabla_\Gamma (\overline{\varphi_h} - \overline{\mathit{II}_h}^j)||_2 \\
& \qquad + ||\nu_j - \overline{\nu_h}^j||_\infty \, ||\nabla_\Gamma \mathrm{id}_\Gamma||_2 \, ||\nabla_\Gamma (\overline{\varphi_h} - \overline{\mathit{II}_h}^j)||_2 \\
& + ||\mu_h^{-1}||_\infty \, ||Y - \overline{Y_h}||_2 \, ||\overline{\mathit{II}_h}^j - \overline{\varphi_h}||_2 \\
& + \big(||\mu_h^{-1}||_\infty \, ||\nu_j - \overline{\nu_h}^j||_\infty \, ||Y||_2 + ||1 - \mu_h^{-1}||_\infty \, ||Y||_2 \big) \, ||\overline{\mathit{II}_h}^j - \overline{\varphi_h}||_2 \\
\le {} & c\, h^{\gamma - 1} \, ||\overline{\mathit{II}_h}^j - \overline{\varphi_h}||_2 \quad \text{(see (4.1), (4.9), (4.10), Lemma 4.5).}
\end{aligned}
$$

The remaining terms in (4.23) are of higher order so that we finally arrive at $||\mathit{II}_j - \overline{\mathit{II}_h}^j||_2 \le c\, h^{\gamma-1}$, which completes the proof of Theorem 4.1 on page 5. □

5 Numerical experiments

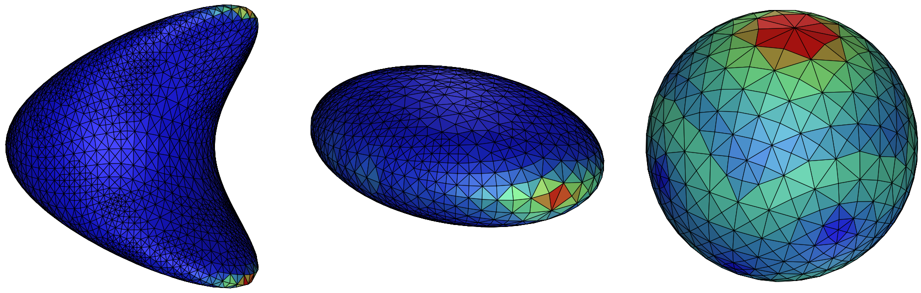
We have implemented our definition of the discrete curvature (see Section 3) for isoparametric finite elements in the context of the finite element toolbox ALBERT (Schmidt and Siebert, 1998). Table 5.1 to 5.4 on pages 13–16 show the results of our experimental convergence tests with piecewise polynomial discretisations of degree 1, 2, 3 and 4 for the following model problems (see Figure 5.1 on the next page):

$$(x - z^2)^2 + y^2 + z^2 - 1 = 0, \tag{5.1a}$$
$$(x - z^2)^2 + (y - z^2)^2 + z^2 - 1 = 0, \tag{5.1b}$$
$$x^2 + (2\, y)^2 + (\frac{2}{3}\, z)^2 - 1 = 0, \tag{5.1c}$$
$$x^2 + y^2 + z^2 - 1 = 0. \tag{5.1d}$$

Table 5.2 on page 14 corresponding to surface (5.1b) has already been published in Heine (2006).

The underlying macro triangulation for the discretisation of all model-problems is an icosahedron. The discrete surface Γ_h is just the interpolation of the exact shape Γ with Lagrange-polynomials. At first the mesh was refined adaptively until the L_2 error with respect to the exact surface was below 0.005. Afterwards the meshes were refined globally by doubly bisecting the elements in each refinement step. The tables 5.1 to 5.4 on pages 13–16 show the maximal mesh width, the relative L_2 and L_∞ errors and the experimental order of convergence (*EOC*) of the discrete mean and Gaussian curvatures H_h and K_h derived from the discrete second fundamental form as defined in Definition 3.2 on page 5.

Figure 5.1 Surface meshes for testing our definition of the discrete curvature. The colour indicates the magnitude of the discrete Gaussian curvature. The defining equations for the surfaces are (5.1b)-(5.1d). A picture for (5.1a) has been omitted as it is of the same type as the surface shown in figure (a).



(a) Equation (5.1b) (b) Equation (5.1c) (c) Equation (5.1d)

For the normalisation of the relative errors we use the curvature of the exact surfaces defined by (5.1a) to (5.1d):

$$\mathcal{E}_{C,r} := \frac{||C - C_h||_{\mathrm{L}_r(\Gamma_h)}}{||C||_{\mathrm{L}_r(\Gamma_h)}}, \quad \text{where } C \in \{K,\ H\}, \quad r \in \{2,\ \infty\}.$$

In order to compare functions defined on Γ with those defined on Γ_h we were using an approximate orthogonal lift from Γ onto Γ_h with respect to an averaged normal field on Γ_h. For a sequence of isoparametric triangulations $\mathcal{T}_{h_j}$ the *experimental order of convergence* (*eoc*) is defined by $EOC_j := (\log \frac{\mathcal{E}_j}{\mathcal{E}_{j+1}})/(\log \frac{h_j}{h_{j+1}})$, where $\mathcal{E}_j := ||C - C_{h_j}||$ denotes the error between the discrete solution C_{h_j} and the exact solution C in the appropriate norm.

While it is not surprising that the discrete curvatures originating from Definition 3.2 on page 5 do not converge for linearly interpolated surfaces, the experimental order of convergence for higher order parametrisations is surprising. The tables 5.1 to 5.4 on pages 13–16 suggest that for Lagrange-interpolates of polynomial degree k the experimental order of convergence for the discrete curvature is greater than $k - 1$, provided that $k > 1$. We would have expected values for the *EOC* of at most $k - 1$, but not greater.

However, one notes that the results for the less complicated surfaces (5.1c) – an ellipsoid – and (5.1d) – a sphere – are not as suggestive as the results for the more complicated surfaces: for them table 5.3 on page 15 and 5.4 on page 16 show a *decreasing* *EOC* as the number of unknowns increases. This might indicate that the values for the *EOC* as shown in the tables 5.1 on the next page and 5.2 on page 14 just did not stabilise yet.

Table 5.1 Discrete curvature, surface (5.1a), relative errors

Polynomial degree: 1

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.135	0.79	.	1.4	.	0.19	.	0.36	.
0.0689	0.87	−0.16	2.2	−0.61	0.19	0.055	0.36	−0.028
0.0361	0.9	−0.041	2.1	0.015	0.19	−0.0011	0.4	−0.15
0.0199	0.91	−0.022	2.3	−0.12	0.19	−0.0065	0.42	−0.081
0.0114	0.91	−0.0072	2.3	−0.02	0.19	−0.0042	0.43	−0.024
0.00654	0.91	−0.0023	2.3	−0.0059	0.19	−0.0021	0.43	−0.0092

Polynomial degree: 2

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.372	1.7	.	2.5	.	0.15	.	0.48	.
0.206	1.	0.85	2.7	−0.14	0.085	0.92	0.31	0.75
0.118	0.54	1.2	1.5	1.	0.044	1.2	0.15	1.3
0.0681	0.2	1.8	0.69	1.4	0.02	1.4	0.067	1.5
0.0382	0.082	1.6	0.44	0.78	0.0099	1.3	0.044	0.74
0.0219	0.032	1.7	0.13	2.2	0.0047	1.3	0.018	1.6
0.0123	0.015	1.4	0.044	1.9	0.0023	1.2	0.0088	1.2

Polynomial degree: 3

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.488	1.5	.	3.6	.	0.13	.	0.49	.
0.32	2.1	−0.83	5.4	−1.	0.15	−0.26	0.58	−0.4
0.191	1.1	1.1	6.3	−0.3	0.087	1.	0.62	−0.12
0.11	0.17	3.5	1.3	2.9	0.022	2.5	0.17	2.3
0.064	0.054	2.1	0.46	1.9	0.0068	2.1	0.083	1.4
0.0359	0.014	2.3	0.13	2.2	0.0018	2.3	0.026	2.
0.0202	0.0036	2.4	0.034	2.3	0.00046	2.4	0.0069	2.3

Polynomial degree: 4

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.549	4.5	.	13.	.	0.3	.	1.4	.
0.324	1.6	2.	8.6	0.77	0.12	1.8	0.75	1.2
0.196	0.34	3.1	1.5	3.5	0.035	2.4	0.24	2.3
0.121	0.04	4.4	0.5	2.2	0.0048	4.1	0.055	3.
0.0697	0.0047	3.9	0.046	4.3	0.00074	3.4	0.0092	3.2
0.0402	0.00061	3.7	0.0062	3.6	9.1e–05	3.8	0.0012	3.7
0.0232	7.4e–05	3.8	0.00068	4.	1.1e–05	3.8	0.00015	3.8

Table 5.2 Discrete curvature, surface (5.1b), relative errors

Polynomial degree: 1

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.205	1.7	.	1.9	.	0.39	.	0.69	.
0.0957	1.4	0.22	1.7	0.14	0.24	0.64	0.59	0.21
0.0508	1.5	−0.11	2.3	−0.44	0.2	0.29	0.37	0.75
0.0263	1.7	−0.19	3.3	−0.56	0.19	0.057	0.37	−0.025
0.0151	1.8	−0.091	3.7	−0.21	0.19	−0.0025	0.41	−0.17
0.00864	1.8	−0.033	3.7	−0.037	0.2	−0.0041	0.43	−0.085

Polynomial degree: 2

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.368	2.7	.	5.2	.	0.25	.	0.58	.
0.204	4.4	−0.83	11.	−1.3	0.15	0.9	0.42	0.52
0.116	2.4	1.1	6.4	1.	0.073	1.2	0.31	0.53
0.0673	0.96	1.7	3.8	0.95	0.039	1.2	0.21	0.71
0.0379	0.34	1.8	1.6	1.5	0.018	1.3	0.087	1.6
0.0214	0.11	2.	0.47	2.2	0.0085	1.4	0.039	1.4

Polynomial degree: 3

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.468	12.	.	19.	.	0.45	.	0.84	.
0.322	13.	−0.15	29.	−1.1	0.34	0.73	1.1	−0.78
0.19	10.	0.39	50.	−1.1	0.19	1.2	1.4	−0.4
0.113	1.3	4.	8.	3.5	0.055	2.3	0.43	2.3
0.0653	0.36	2.3	3.6	1.4	0.022	1.7	0.27	0.82
0.0373	0.086	2.6	0.88	2.5	0.0062	2.2	0.1	1.8

Polynomial degree: 4

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.477	15.	.	39.	.	0.45	.	1.2	.
0.304	8.	1.4	58.	−0.87	0.22	1.6	1.3	−0.16
0.196	1.2	4.3	9.2	4.2	0.055	3.2	0.45	2.5
0.115	0.22	3.3	3.1	2.	0.013	2.7	0.14	2.2
0.0655	0.022	4.1	0.34	3.9	0.0019	3.5	0.026	3.
0.0374	0.0024	3.9	0.025	4.7	0.00023	3.8	0.0032	3.7

Table 5.3 Discrete curvature, surface (5.1c), relative errors

Polynomial degree: 1

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.2	0.76	.	1.4	.	0.19	.	0.45	.
0.117	0.83	−0.14	1.7	−0.39	0.18	0.056	0.39	0.25
0.0674	0.83	−0.0022	1.7	0.028	0.18	0.019	0.43	−0.2
0.0384	0.83	−0.0086	1.7	−0.098	0.18	−0.0021	0.48	−0.17
0.0218	0.83	−0.0067	1.7	0.0033	0.18	−0.0034	0.54	−0.2
0.0127	0.84	−0.0037	1.8	−0.012	0.18	−0.0019	0.59	−0.19

Polynomial degree: 2

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.295	0.69	.	1.6	.	0.086	.	0.27	.
0.19	0.49	0.77	1.7	−0.23	0.06	0.81	0.23	0.41
0.107	0.21	1.5	1.1	0.85	0.026	1.4	0.12	1.1
0.0588	0.069	1.8	0.47	1.4	0.011	1.5	0.063	1.1
0.0309	0.025	1.6	0.17	1.6	0.0049	1.2	0.031	1.1
0.0159	0.011	1.3	0.056	1.7	0.0024	1.1	0.013	1.3
0.00833	0.0051	1.2	0.026	1.2	0.0012	1.1	0.0064	1.1

Polynomial degree: 3

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.309	2.2	.	8.3	.	0.23	.	1.	.
0.198	0.76	2.3	3.4	2.	0.09	2.1	0.59	1.2
0.131	0.16	3.8	1.2	2.6	0.025	3.1	0.19	2.8
0.0788	0.041	2.7	0.31	2.6	0.0067	2.6	0.063	2.1
0.0457	0.011	2.5	0.086	2.4	0.0017	2.5	0.019	2.2
0.0259	0.0027	2.4	0.024	2.2	0.00044	2.4	0.0052	2.3
0.0145	0.00067	2.4	0.0063	2.3	0.00011	2.4	0.0014	2.3

Polynomial degree: 4

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.443	7.5	.	72.	.	0.39	.	4.1	.
0.265	0.23	6.8	2.5	6.5	0.04	4.4	0.36	4.7
0.153	0.024	4.1	0.24	4.3	0.0056	3.6	0.065	3.1
0.0908	0.0027	4.2	0.027	4.2	0.00063	4.2	0.0059	4.6
0.0527	0.00034	3.8	0.004	3.5	8.0e–05	3.8	0.0012	2.9
0.0297	4.3e–05	3.6	0.00057	3.4	1.0e–05	3.6	0.00018	3.4
0.0168	5.5e–06	3.6	9.9e–05	3.1	1.3e–06	3.6	2.8e–05	3.2

Table 5.4 Discrete curvature, surface (5.1d), relative errors

Polynomial degree: 1

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.126	0.41	.	1.1	.	0.17	.	0.43	.
0.0717	0.41	−0.015	1.2	−0.19	0.18	−0.014	0.47	−0.13
0.0407	0.42	−0.0048	1.2	0.088	0.18	−0.0062	0.45	0.085
0.0231	0.42	−0.0038	1.2	0.012	0.18	−0.0041	0.44	0.012
0.0131	0.42	−0.0022	1.2	0.0027	0.18	−0.0023	0.44	0.0022
0.00743	0.42	−0.0012	1.2	0.0011	0.18	−0.0012	0.48	−0.13

Polynomial degree: 2

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.395	0.052	.	0.18	.	0.021	.	0.082	.
0.242	0.025	1.5	0.13	0.74	0.0055	2.7	0.031	2.
0.138	0.0073	2.2	0.048	1.7	0.0014	2.5	0.01	2.
0.0784	0.0019	2.4	0.016	1.9	0.00034	2.4	0.0033	2.
0.0441	0.00049	2.4	0.0053	2.	8.6e–05	2.4	0.001	2.
0.0249	0.00012	2.4	0.0017	2.	2.2e–05	2.4	0.00033	2.
0.014	3.1e–05	2.4	0.00053	2.	5.4e–06	2.4	0.00011	2.
0.00787	7.7e–06	2.4	0.00017	2.	1.4e–06	2.4	3.3e–05	2.

Polynomial degree: 3

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.515	0.27	.	2.5	.	0.11	.	0.54	.
0.284	0.096	1.7	0.52	2.6	0.048	1.5	0.23	1.4
0.154	0.021	2.5	0.12	2.4	0.011	2.4	0.06	2.2
0.0863	0.0048	2.5	0.037	2.	0.0028	2.4	0.02	1.9
0.0486	0.0012	2.4	0.01	2.2	0.0007	2.4	0.0061	2.1
0.0273	0.00031	2.4	0.0032	2.	0.00018	2.4	0.0019	2.
0.0154	7.8e–05	2.4	0.001	2.	4.5e–05	2.4	0.00061	2.
0.00856	2.e–05	2.4	0.00034	1.9	1.1e–05	2.4	0.0002	1.9

Polynomial degree: 4

h_{max}	$\mathcal{E}_{K,2}$	EOC	$\mathcal{E}_{K,\infty}$	EOC	$\mathcal{E}_{H,2}$	EOC	$\mathcal{E}_{H,\infty}$	EOC
0.793	0.14	.	0.86	.	0.058	.	0.26	.
0.448	0.025	2.9	0.14	3.2	0.012	2.7	0.056	2.7
0.244	0.0015	4.7	0.0081	4.7	0.0008	4.5	0.0039	4.4
0.144	0.00028	3.2	0.003	1.9	0.00014	3.4	0.0015	1.8
0.0817	1.9e–05	4.7	0.00026	4.3	9.7e–06	4.6	0.00014	4.2
0.0464	2.8e–06	3.4	7.5e–05	2.2	1.4e–06	3.4	3.8e–05	2.3
0.0261	2.4e–07	4.2	1.5e–05	2.8	1.2e–07	4.2	7.5e–06	2.8
0.0146	3.4e–08	3.4	5.4e–06	1.8	1.7e–08	3.4	2.7e–06	1.8

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