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Discrete Exterior Calculus (DEC) approximation of curvature on surfaces

Motivation

The Discrete Exterior Calculus (DEC) gives the advantage to discretize differential p-forms in $\Omega^p(M)$ its Operators, e.g the exterior derivative $\mathbf{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ or the Hodge-Star-Operator $* : \Omega^p(M) \to \Omega^{2-p}(M)$, on a surface M. Such discrete formulations can be obtained on vertices, edges or higher order simplices, which approximate the surface linear.

In many mathematical, physical and engineering problems the curvature of surfaces plays a important role. With the DEC it is possible to approximate the curvature vector and the Weingarten map to get the mean or the Gaussian curvature on the vertices of the C^0-manifold.

Curvature vector

Continuous Problem

- Inclusion map: $\iota : \mathbb{R}^3|_M \hookrightarrow \mathbb{R}^3, \quad \vec{x} \mapsto \vec{x}$
- Laplace-Beltrami-Operator for the inclusion map on a given manifold (componentwise)

$$\Delta_B \iota = (*\mathbf{d} * \mathbf{d})\, \iota = \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^j} \left(g^{ij} \sqrt{|\det g|} \frac{\partial \iota}{\partial x^i} \right)$$

(g, g^{ij}: metric tensor (e.g. Riemannian metric) resp. its inverse components)

- Curvature Vector, see [Fla63]: $\vec{H} = -\Delta_B \iota$
- Mean curvature: $H = \frac{1}{2} \left\| \vec{H} \right\|$

Discrete Problem

- For a better FEM-like elementwise implementation, the discrete formulation on a vertex v_i is given with respect to the Hodge-/Geometric-Star-Operator:

$$\left\langle *\Delta_B \iota^k, \star v_i \right\rangle = \sum_{\sigma^1 = [v_i, v_j]} \frac{|\star\sigma^1|}{|\sigma^1|} \left(\iota^k(v_j) - \iota^k(v_i) \right),$$

($\iota = \left[\iota^1, \iota^2, \iota^3\right]$ and the global vertex indices i and j)

- DEC-approximated mean curvature:

$$H_d(v_i) = \frac{1}{2\,|\star v_i|} \sqrt{\sum_{k=1}^{3} \left\langle *\Delta_B \iota^k, \star v_i \right\rangle^2}.$$

Weingarten map

Continuous problem

- Extended Weingarten map: $\bar{S} := \nabla \vec{\nu} \in \mathbb{R}^{3\times 3} : M \to \mathbb{R}^{3\times 3}$ (∇: surface gradient)
- The restriction of the extended Weingarten map to the tangential space is the usual Weingarten map S.
- The eigenvalues of S are the principal curvatures κ^1 and κ^2 of the Surface M. The mean curvature and the Gaussian curvature is given by $H = \frac{\kappa^1 + \kappa^2}{2}$ resp. $K = \kappa^1 \cdot \kappa^2$.

Discrete problem

- Discrete surface normals $\vec{\nu}$ on a vertex v:
  - Average of element normals $\vec{\nu}^{\sigma^2}$: $\quad \vec{\nu}^{\mathsf{Av}}(v) := \frac{1}{|\star v|} \sum_{\sigma^2 \succ v} \left|\star v \cap \sigma^2\right| \vec{\nu}^{\sigma^2}$
  - From a signed distance function $\varphi : \mathbb{R}^3 \to \mathbb{R}$: $\quad \vec{\nu}(v) = \frac{\nabla_{\mathbb{R}^3}\varphi}{\left\|\nabla_{\mathbb{R}^3}\varphi\right\|}$
- Discrete surface Gradient $\nabla^{\overline{pd}}$ as average of the primal-dual-gradient ∇^{pd}, see [Hir03]:

$$\left(\nabla^{\overline{pd}} f\right)(v) = \frac{1}{|\star v|} \sum_{\sigma^2 \succ v} \left|\star v \cap \sigma^2\right| \sum_{\sigma^0 \prec \sigma^2} \left(f(\sigma^0) - f(v)\right) \nabla\Phi_{\sigma^0}^{\sigma^2}$$

($\nabla\Phi_{\sigma^0}^{\sigma^2}$: gradient of the linear basis function Φ_{σ^0} on element σ^2)

- Discrete formulation on a vertex v and for components with index $i, j \in \{1, 2, 3\}$:

$$|\star v|\, \bar{S}_{ij}(v) \approx \left\langle * \left[S^{\overline{pd}}\right]_{ij}, \star v \right\rangle := \left\langle * \left[\nabla^{\overline{pd}} \bar{\nu}^i\right]_j, \star v \right\rangle$$

($\bar{\nu}^i$: i-th component of $\vec{\nu}$ resp. $\vec{\nu}^{\mathsf{Av}}$)

- Calculation of the eigenvalues of DEC-approximated extended Weingarten map $S^{\overline{pd}}$ on every vertex with QR-Algorithm and cancel out the additional (approx. 0) eigenvalue

Results and Conclusion

All DEC-Operators, which were needed for curvature calculations, were able to implemented as element operators in the FEM-Toolbox AMDiS. Hence, the FEM-Part to provide the element matrices was replaced by a DEC-formulations, which holds locally on the triangles. The global matrix assembly and solving the linear system can be done by AMDiS.

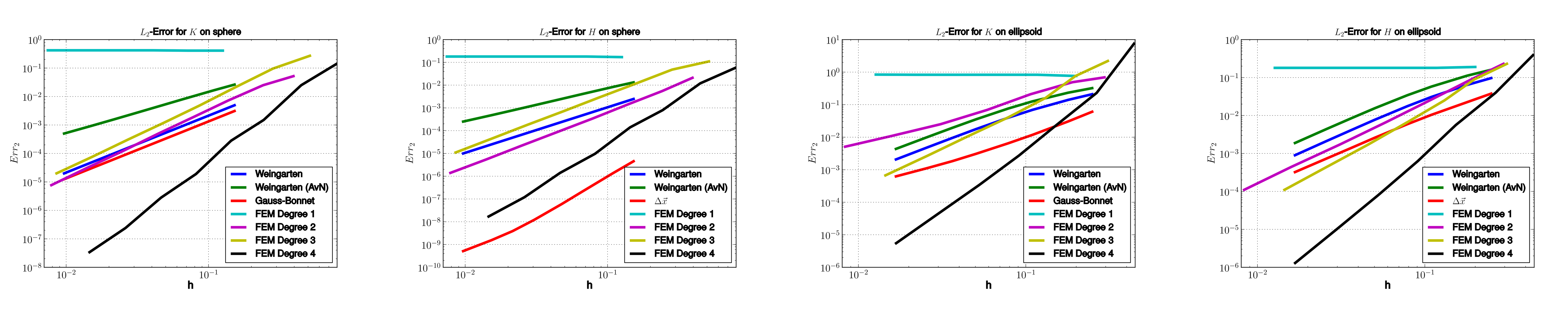

Figure 1: Log-Log-Plot of the discrete relative L_2-Error of the Gaussian curvature K and the mean Curvature H on a sphere and a ellipsoid given by the signed distance function $\varphi(x,y,z) := (3x)^2 + (6y)^2 + (2z)^2 - 9$. (AvN) means the additional computation of the average element normals. $\Delta\vec{x}$ is the calculation of the curvature vector with the DEC-discretized Laplace-Beltrami-Operator. The results were tested against a Gauss-Bonnet-Approximation and a isoparametric FEM of different degrees, see [Hei04]. All computational costs are approximative lower than the costs for the FE-Method of degree 1 (or equal for Weingarten(AvN)).

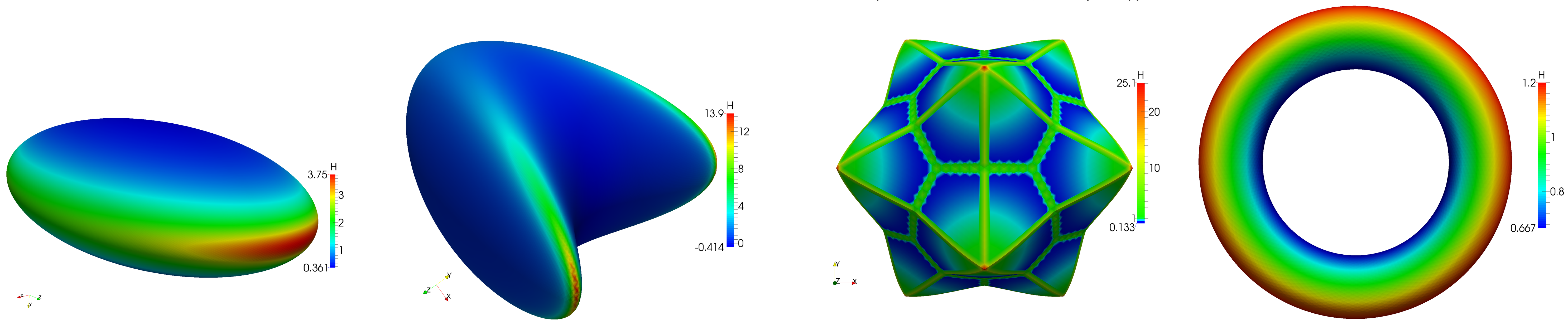

Figure 2: Mean curvature of a ellipsoid, a quartic surface ($\varphi(x,y,z) := (x - z^2)^2 + (y - z^2)^2 + z^2 - 1$), a handmade surface (merge of a sphere and a icosahedron) and a torus.

References

[Fla63] H. Flanders. *Differential Forms with Applications to the Physical Sciences.* Dover books on advanced mathematics. Dover Publications, 1963.

[Hei04] C.-J. Heine. Isoparametric finite element approximation of curvature on hypersurfaces. *Preprint Fak. f. Math. Phys. Univ. Freiburg*, (26), 2004.

[Hir03] Anil Nirmal Hirani. *Discrete Exterior Calculus.* PhD thesis, California Institute of Technology, Pasadena, CA, USA, 2003. AAI3086864.