Curvature approximation on surfaces – A Discrete Exterior Calculus Approach

I. Nitschke[1], A. Voigt[1,2]

Abstract

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Keywords: Surfaces, Curvature, DEC

1. Introduction

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2. Discrete Exterior Calculus (DEC)

The Discrete Exterior Calculus [7, 4] defines discrete differential p-forms on a triangulated mesh (simplicial complex). For surface meshes, i.e. triangulated orientable 2-manifolds, the degree of the discrete p-forms is 0, 1 or 2 and their are represented by scalars on vertices, edges, triangles or chains of them. Operators for the differential forms, like the exterior derivation $\mathbf{d}$ or the hodge star $\star$,
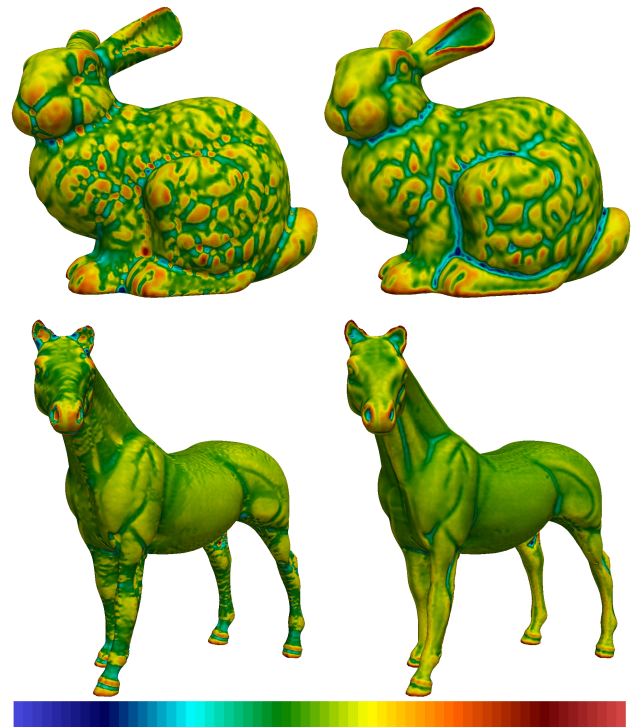

Figure 1: Surface curvature approximations of the Stanford Bunny and a horse. Left: Gaussian curvature K (square root scaled colours). Right: Mean curvature H (linear scaled colours)

Email addresses: `ingo.nitschke@tu-dresden.de` (I. Nitschke), `axel.voigt@tu-dresden.de` (A. Voigt)

[1]Department of Mathematics, TU Dresden, 01062 Dresden, Germany

[2]Center of Advanced Modeling and Simulation, TU Dresden, 01062 Dresden, Germany
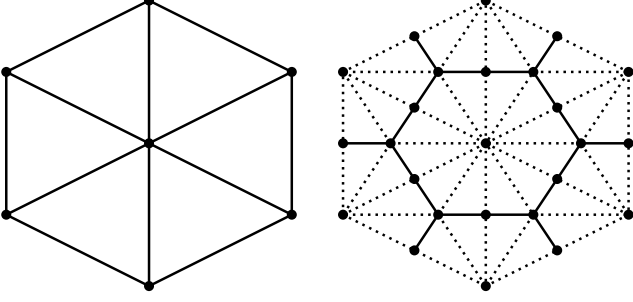
Figure 2: Example of a circumcenter subdivision. Left: A primal mesh K. Right: The resultant dual mesh $\mathrm{csd}K$. All new vertices are the circumcenters of the triangles and edges. The solid lines highlights the edges of the Voronoi mesh $\star K \le \mathrm{csd}K$.

can be approximated by expressions on the discrete geometrical structure. E.g. the integral over a triangle of the exterior derivation $\mathbf{d}$ for a 1-form can be expressed as the integral of this 1-form over the boundaries edges of the triangle. This follows directly from the Stokes Theorem [1, Ch. 7].

2.1. Discrete manifolds

In our setting, the surface meshes are linear triangulations of orientable closed 2-manifolds. Such a Triangulation are sets of p-simplices $\{\sigma^p\}$ of the same degree p, e.g. sets of vertices, edges and triangles, and form a simplicial complex of dimension $n = 2$. A simplicial complex K comply two essential rules:

1. Every face of a simplex $\sigma^p \in K$ is in K.
2. The intersection of two simplices in K is either in K or empty.

We shortly write $\sigma^q \prec \sigma^p$ (or $\sigma^p \succ \sigma^q$)), iff $q < p$ and σ^q is a face of σ^p. It is required, that the polytop

$$|K| := \bigcup_{\sigma \in K} \sigma \tag{1}$$

of K is a C^0-manifold and we call K a manifold-like simplicial complex. For a higher consistence to the smooth model, we assume that the simplicial complex is orientable, i.e. all triangles $\sigma^2 \in K$ have the same orientation. The orientation of a triangle is define as one of the two equivalence classes, that arise from the kind of counting the vertices (clockwise or counterclockwise). We write $\sigma^2 = [v_0, v_1, v_2]$ to mark the order of the vertices v_i. In the same manner a edge $\sigma^1 = [v_0, v_1]$ get an declared orientation. Such a manifold-like orientable simplicial complex is called a primal mesh.

For the existence of a dual mesh, we also need, that the primal mesh is well-centered, i.e. for all triangles $\sigma^2 \in K$ the circumcenter $c(\sigma^2)$ lies in the interior of σ^2. A dual mesh $\mathrm{csd}K$ is the circumcenter subdivision of a well-centered primal mesh K. For a explanatory example see figure 2. The dual mesh is also a primal mesh, but it is not well-centered.

2.2. Chains

An important role in the DEC are chains. A p-chain is the formal sum of p-simplices with coefficients in $\mathbb{Z}$. Therewith, we denote the space of all p-chains of K by

$$C_p(K) := \left\{ \sum_{\sigma \in K^{(p)}} a_\sigma \sigma \,\middle|\, a_\sigma \in \mathbb{Z} \right\}. \tag{2}$$

$K^{(p)}$ is the set of all p-simplices of K. For chains we can use the universal property, because $C_p(K)$ is a free abelian group with the generating set $K^{(p)}$. Hence, the diagram

$$
\begin{array}{cc}
C_p(K) & \qquad\qquad (3) \\
\end{array}
$$

commutes for a arbitrary abelian group $\mathfrak{A}$ and homomorphism $op = \widehat{op}|_{K^{(p)}}$ and $\widehat{op}$ is unique determinated by op, i.e. it is quit enough to define operators for chains only on the simplices.

2.3. Geometrical operators

The two momentous operators on chains are the boundary operator ∂ and the star operator $\star$. These are the geometrical tools for the discrete exterior derivation $\mathbf{d}$ resp. the discrete Hodge star $*$, what we will see below.

2.3.1. The star operator

For a general definition of the star operator $\star$ see [7, 4] or [5, Ch. 7]. In our two dimensional well-centered primal mesh case it is enough to define the star operator $\star : C_p(K) \to C_{2-p}(\mathrm{csd}K)$ by

$$\star\sigma^0 := \sum_{\sigma^0 \prec \sigma^1 \prec \sigma^2} s_{\sigma^1\sigma^2} \left[\sigma^0, c(\sigma^1), c(\sigma^2)\right] \tag{4}$$

$$\star\sigma^1 := \sum_{\sigma^1 \prec \sigma^2} s_{\sigma^2} \left[c(\sigma^1), c(\sigma^2)\right] \tag{5}$$

$$\star\sigma^2 := c(\sigma^2) \tag{6}$$

The factors $s_\bullet$ are signs to prevent orientation properties, like the orientability of the dual mesh. (4) describe the Voronoi cell of the primal vertex σ^0, (5) describe the Voronoi edge of the primal edge σ^1 (c.p. figure 2) and (6) describe the Voronoi vertex of a primal triangle σ^2. With the notation $C_{2-p}(\star K)$ for the image of $\star$, the star operator, act on $C_{2-p}(\star K)$ and map to $C_p(K)$, is for $p = 0, 2$ simply the inverse. For $p = 1$ we must swap the orientation, i.e. $\star\star\sigma^1 = -\sigma^1$.

2.3.2. The boundary operator

Let K be a primal mesh (not necessary well-centered). The boundary operator $\partial : C_p(K) \to C_{p-1}(K)$ is defined

by

$$\partial \sigma^p := \begin{cases} \sum\limits_{i=0}^{p} (-1)^i \left[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_p\right] & \text{for } p = 1, 2 \\ 0 & \text{for } p = 0 \end{cases}$$

(7)

($\hat{v}_i$ is omitted) and comply the property $\partial \circ \partial = 0$. This is the authorization to call the sequence $(C_p(K), \partial)$ (and also $(C_p(\star K), \partial))$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_2(K) & \xrightarrow{\partial} & C_1(K) & \xrightarrow{\partial} & C_0(K) & \longrightarrow & 0 \\
& & \updownarrow{\star} & & \updownarrow{\star} & & \updownarrow{\star} & & \\
0 & \longleftarrow & C_0(\star K) & \xleftarrow{\partial} & C_1(\star K) & \xleftarrow{\partial} & C_2(\star K) & \longleftarrow & 0
\end{array}
$$

(8)

a chain complex.

2.4. Discrete differential forms

A discrete p-form α is a homomorphism from the chain group $C_p(K)$ to the additive group $\mathbb{R}$. Thus, the space of discrete p-form is the space of cochains $\mathrm{Hom}\,(C_p(K), \mathbb{R})$, denoted as $C^p(K)$ or, under attention the analogy to smooth differential forms, $\Omega_d^p(K)$. The crucial map, that define such homomorphisms and give the relation to the smooth forms, is the de Rham map

$$
\begin{aligned}
\psi : \Omega^p(M) &\to C^p(L) = \Omega_d^p(L) \\
\alpha &\mapsto \left(\sigma^p \mapsto \int_{\sigma^p} \alpha =: \langle \psi(\alpha), \sigma^p \rangle \right).
\end{aligned}
$$

(9)

The right hand side is the pairing notation, which we want to use. At this, L is the abstract simplicial complex of K and arise if all simplices from K are projected to the underlying manifold M, i.e. $|L| = M$ and $L = \pi(K)$ with a projection map $\pi : |K| \to M$. But the projection and in the most cases the manifold M is not exactly known, so for interest of simplification we can approximate the integral in (9) with a linear quadrature I on the vertices of a given simplex $\sigma^p \in K$:

$$\langle \psi(\alpha), \pi(\sigma^p) \rangle \approx I_{\sigma^p}(\alpha).$$

(10)

For $p = 0$ and a function (0-form) $f : M \to \mathbb{R}$ and a vertex (0-simplex) is this exact, i.e.

$$\langle \psi(f), \pi(v) \rangle = \langle \psi(f), v \rangle = f(v).$$

(11)

For $p > 1$ the question about evaluating the integral in (9) is pure formal in this paper, because in all of our computation, we can reduce the problems to scalar valued formulations. Henceforth, the projection π is omitted in the pairing notation, i.e. we write $\langle \psi(\alpha), \sigma^p \rangle$ instead of $\langle \psi(\alpha), \pi(\sigma^p) \rangle$.

2.5. DEC operators

For the definition of a discrete version of the Hodge star $*$ or the exterior derivative $\mathbf{d}$ we can fall back to the geometric operators.

2.5.1. The discrete Hodge star operator

The discrete Hodge star $* : \Omega_d^p(K) \to \Omega_d^{2-p}(*K)$ is nothing more than scale the evaluation of a discrete p-form by the ratio of the dual (Voronoi) and the primal volumes.

$$\langle *\alpha, \star\sigma^p \rangle := \frac{|\star \sigma^p|}{|\sigma^p|} \langle \alpha, \sigma^p \rangle$$

(12)

for a simplex $\sigma^p \in K$ and a $\alpha \in \Omega_d^p(K)$. Note that the intrinsic volume of a 0-simplex is 1. (In [9] we can see that (12) is consistent with $\mathcal{O}(h^{3-p})$ for the maximum circumscribed diameter h.) The discrete Hodge star on a discrete dual form $\hat{\alpha} \in \Omega_d^p(*K)$ can be obtained implicitly by the rule $* * \hat{\alpha} = (-1)^p \hat{\alpha}$, wich holds on a even-dimensional simplicial complex as well as for smooth forms on a even-dimensional manifold.

2.5.2. The discrete exterior derivative

The main advantage in the definition of discrete differential forms as integral evaluation of smooth forms on (abstract) simplices is that we are able to use the Stokes theorem (14) to describe a discrete exterior derivative $\mathbf{d} : \Omega_d^p(K) \to \Omega_d^{p+1}(K)$:

$$\langle \mathbf{d}\psi(\alpha), \sigma^p \rangle := \langle \psi(\mathbf{d}\alpha), \sigma^p \rangle$$

(13)

$$= \langle \psi(\alpha), \partial\sigma^p \rangle$$

(14)

(this applies on $\star K$ also) and there is no approximation error accept the interpretation of the integral expression. Hence, the discrete exterior derivative inherit the complex property of the boundary operator, i.e. $\mathbf{d} \circ \mathbf{d} = 0$. That's why the sequence of discrete differential forms (cochains) $(\Omega_d^p(K), \mathbf{d})$ (and $(\Omega_d^p(\star K), \mathbf{d}))$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_d^0(K) & \xrightarrow{\mathbf{d}} & \Omega_d^1(K) & \xrightarrow{\mathbf{d}} & \Omega_d^2(K) & \longrightarrow & 0 \\
& & \updownarrow{\star} & & \updownarrow{\star} & & \updownarrow{\star} & & \\
0 & \longleftarrow & \Omega_d^2(\star K) & \xleftarrow{\mathbf{d}} & \Omega_d^1(\star K) & \xleftarrow{\mathbf{d}} & \Omega_d^0(\star K) & \longleftarrow & 0
\end{array}
$$

(15)

is a cochain complex.

2.5.3. The discrete Laplace-Beltrami operator

The Laplace-Beltrami operator $\Delta_B = *\mathbf{d}*\mathbf{d}$ is the special case of the negative Laplace-de Rham operator $-\Delta_{dR} = *\mathbf{d}*\mathbf{d} + \mathbf{d}*\mathbf{d}*$ for 0-forms on a 2-manifold or in general on a even-dimensional manifold. To discretize Δ_B all of work is done above, because we can use the definitions (12) and (13) step by step in the same manner
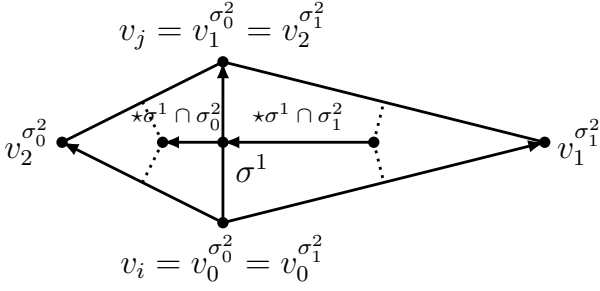
Figure 3: Two adjacent (triangle) elements share a common edge $\sigma^1 = [v_i, v_j]$. The vertices are local indexed on these elements.

like the smooth exterior calculus. I.e. for a 0-form f on a vertex v it results in

$$\langle \Delta_B f, v \rangle = \frac{1}{|\star v|} \sum_{\sigma^1 = [v, v_i] \in K} \frac{|\star \sigma^1|}{|\sigma^1|} \left(f(v_i) - f(v) \right) . \quad (16)$$

For greater details see [7] or [4]. (16), equivalent to the cotan-formular described in [8], is a approximation of order 2 (see [12]) and was already earlier published in [10] as a non-DEC consequence.

Note on Implementation. In section 3.1 we need the discrete Laplace-Beltrami operator to approximate the mean curvature. One way to do that is to represent the discrete linear operators $\ast$ and $\mathbf{d}$ as matrices act on a vector of all vertices, edges or triangles. So the discrete Laplace-Beltrami is nothing more than the product of these matrices (see [6], [2] or [3]).

A more "FEM-like" manner is to decompose (16) in parts, which can be computed on all (triangle) elements without knowledge of their neighbours. For this we formulate the operator on the Voronoi cells to cancel the scale factor $|\star v|^{-1}$ in (16), split the sum and use a local vertex indexing on all elements:

$$\langle \ast \Delta_B f, \star v_i \rangle = \sum_{\sigma^1 = [v_i, v_j]} \frac{|\star \sigma^1|}{|\sigma^1|} \left(f(v_j) - f(v_i) \right) \quad (17)$$

$$= \sum_{\substack{\sigma^2 = \left[v_0^{\sigma^2}, v_1^{\sigma^2}, v_2^{\sigma^2} \right] \\ v_0^{\sigma^2} = v_i}} \sum_{l=1,2} C_{0,l}^{\sigma^2} \left(f_l^{\sigma^2} - f_0^{\sigma^2} \right) . \quad (18)$$

$f_l^{\sigma^2} \in \mathbb{R}$ are the evaluations of f on the local vertices $v_l^{\sigma^2}$ of the element σ^2. Figure 3 clarify this setting. The coefficients $C_{0,l}^{\sigma^2} \in \mathbb{R}$ are generally defined by

$$C_{k,l}^{\sigma^2} := C_{l,k}^{\sigma^2} := \frac{\left| \star \left[v_k^{\sigma^2}, v_l^{\sigma^2} \right] \cap \sigma^2 \right|}{\left| \left[v_k^{\sigma^2}, v_l^{\sigma^2} \right] \right|} . \quad (19)$$

Hence we can extract from (18) a local coefficients matrix

$$A^{\sigma^2} := \begin{bmatrix} -\left(C_{01}^{\sigma^2} + C_{02}^{\sigma^2} \right) & C_{01}^{\sigma^2} & C_{02}^{\sigma^2} \\ C_{01}^{\sigma^2} & -\left(C_{01}^{\sigma^2} + C_{12}^{\sigma^2} \right) & C_{12}^{\sigma^2} \\ C_{02}^{\sigma^2} & C_{12}^{\sigma^2} & -\left(C_{02}^{\sigma^2} + C_{12}^{\sigma^2} \right) \end{bmatrix} \quad (20)$$

which holds all computation parts for the local vertices. All A^{σ^2} can be assembly in a familiar way to get a sparse global matrix $A \in \mathbb{R}^{N_{\sigma^0} \times N_{\sigma^0}}$, where N_{σ^0} are the number of vertices in K. At the end, for $f_h \in \mathbb{R}^{N_{\sigma^0}}$, the vector of evaluations for f (resp. vector of DOFs[3]), $A f_h$ is the result of (17) on all global vertices.

The great advantage of this proceeding is the compatibility to existing Finite Element code like AMDiS[4] [11]. Only the element operator A^{σ^2} must be implemented under use of linear test functions for the interpolation. All other works, like mesh management, problem formulation on the user level, assembling and solving of the linear System, can be done by the FE-toolbox.

2.5.4. The discrete gradient

The gradient of a scalar valued function f can be obtained by rising the indices of its exterior derivative, in other words $\nabla f = (\mathbf{d}f)^\sharp$. Unfortunately, the sharp operator $\sharp$ depends on the (unknown) metric of the manifold. In [7] it is suggested to use the following discrete sharp operator for such a exact 1-form $\mathbf{d}f$ to get a discrete gradient:

$$\left\langle (\mathbf{d}f)^\sharp, \star \sigma^2 \right\rangle := \sum_{\sigma^0 \prec \sigma^2} \left(f(\sigma^0) - f(w) \right) \nabla_{\mathbb{R}^3} \Phi_{\sigma_0}^{\sigma^2} , \quad (21)$$

with a arbitrary vertex $w \prec \sigma^2$, $\Phi_{\sigma_0}^{\sigma^2}$ is the linear test function (hat function) for the vertex σ^0 restricted to σ^2 and $\nabla_{\mathbb{R}^3}$ is the usual $\mathbb{R}^3$ gradient in the ambient space. The formulation for the $\mathbb{R}^3$ vector $(\mathbf{d}f)^\sharp$ is to read component wise as a $\mathbb{R}^3$-vectorized scalar formulation. (21) gives us a tangential vector of the polytop $|K|$ on the circumcenter of the triangle σ^2. But in our setting we want to use a gradient on a vertex of the primal mesh, resp. interpreted on the associated Voronoi cell. A simple way to get such a discrete gradient, is to average (21) over all dual vertices of the Voronoi cell related to the primal vertex v:

$$\left\langle \ast \overline{\nabla} f, \star v \right\rangle = \sum_{\sigma^2 \succ v} |\star v \cap \sigma^2| \sum_{\sigma^0 \prec \sigma^2} \left(f(\sigma^0) - f(v) \right) \nabla_{\mathbb{R}^3} \Phi_{\sigma_0}^{\sigma^2} \quad (22)$$

The drawback of the discrete average gradient, which we will use in section 3.2, is that $|K|$ has no tangential space on its vertices, because the polytop is not continuous differentiable in v. This makes the situation for a numerical analysis slightly difficult. But comparing calculations for several manifolds and functions and the results in section 4 indicate the consistence of (22) to the smooth gradient, if the triangulation is fine enough.

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[3]Degree Of Freedom
[4]Adaptive MultiDimensional Simulation toolbox, developed at Institute for Scientific Computing, TU-Dresden
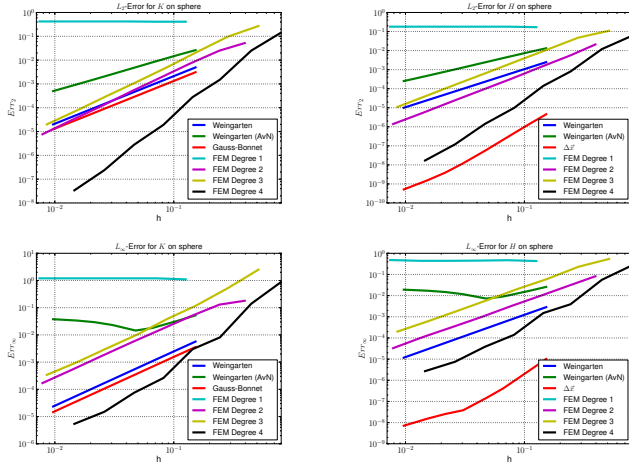
Figure 4: Sphere (25): Relative L_2 (top) resp. L_∞ (bottom) error for K (left) and H (right) for different h in a log-log-plot.

Note on Implementation. Like in section 2.5.3, we can deploy a coefficients matrix for the q-th component of (22) for a triangle σ^2:

$$\begin{bmatrix} -\left(C_{01q}^{\sigma^2}+C_{02q}^{\sigma^2}\right) & C_{01q}^{\sigma^2} & C_{02q}^{\sigma^2} \\ C_{10q}^{\sigma^2} & -\left(C_{10q}^{\sigma^2}+C_{12q}^{\sigma^2}\right) & C_{12q}^{\sigma^2} \\ C_{20q}^{\sigma^2} & C_{21q}^{\sigma^2} & -\left(C_{20q}^{\sigma^2}+C_{21q}^{\sigma^2}\right) \end{bmatrix} \tag{23}$$

with

$$C_{klq}^{\sigma^2} := \left| \star v_k^{\sigma^2} \cap \sigma^2 \right| \frac{\partial}{\partial x^q} \Phi_{v_l^{\sigma^2}}^{\sigma^2} , \tag{24}$$

for $k,l,q \in \{0,1,2\}$ and $\left(x^0, x^1, x^2\right)$ are the ordinary $\mathbb{R}^3$ coordinates.

3. Curvature approximations

3.1. Curvature vector
3.2. Weingarten map
4. Results

$$\varphi\left(x,y,z\right) := x^2 + y^2 + z^2 - 1 \tag{25}$$

$$\varphi\left(x,y,z\right) := \left(3x\right)^2 + \left(6y\right)^2 + \left(2z\right)^2 - 9 \tag{26}$$

$$\varphi\left(x,y,z\right) := \left(x-z^2\right)^2 + \left(y-z^2\right)^2 + z^2 - 1 \tag{27}$$

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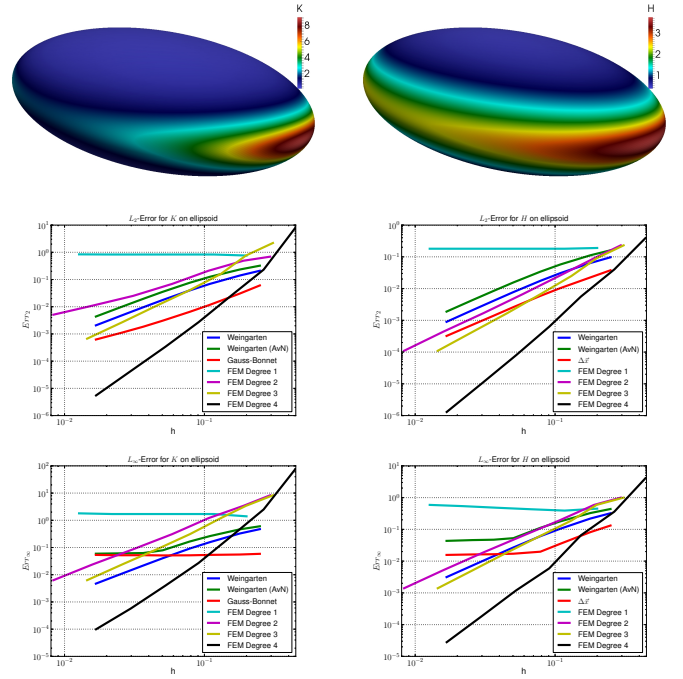URL http://doi.acm.org/10.1145/2382585.2382588

Figure 5: Ellipsoid (26): Top row: Gaussian curvature K (left) and mean curvature H (right). Middle and bottom row: Relative L_2 resp. L_∞ error for K (left) and H (right) for different h in a log-log-plot.
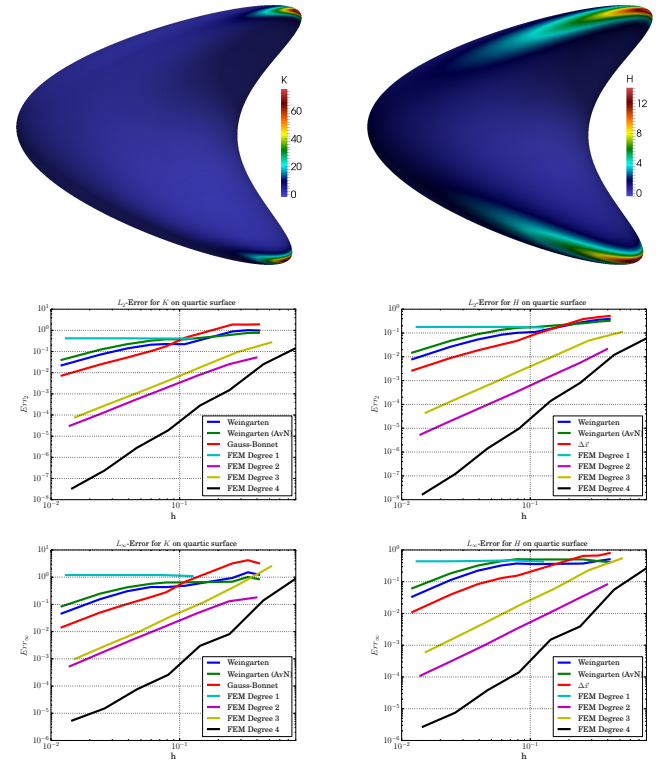

Figure 6: Quartic Surface (27): Top row: Gaussian curvature K (left) and mean curvature H (right). Middle and bottom row: Relative L_2 resp. L_∞ error for K (left) and H (right) for different h in a log-log-plot.
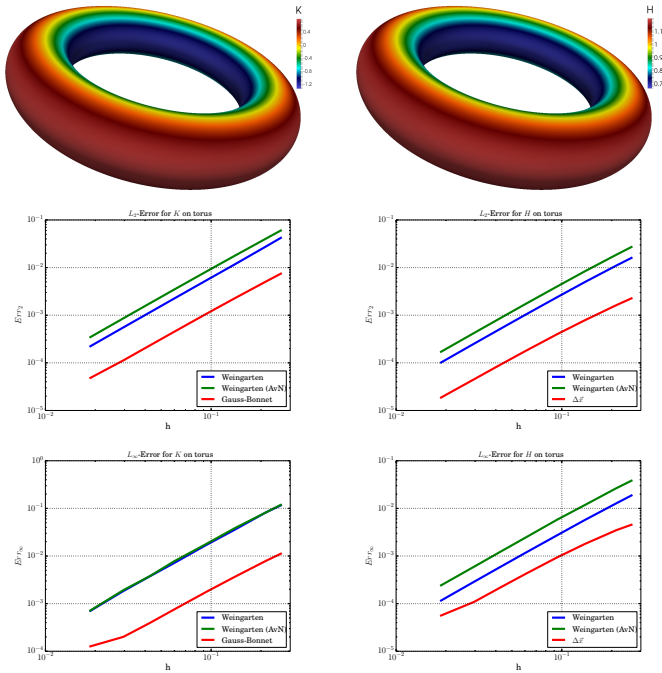
Figure 7: Torus: Top row: Gaussian curvature K (left) and mean curvature H (right). Middle and bottom row: Relative L_2 resp. L_∞ error for K (left) and H (right) for different h in a log-log-plot.

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