1 Arbitrary s.p.d. metric

1.1 Assumptions

- $Ind(M) = 0$
- $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2$ (**s.p.d.**)

1.2 General proberties

$\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $\gamma \in \Omega^r(M)$, $\vec{v} \in \mathcal{V}(M)$

1.2.1 Wedge product $\wedge$

- $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$ (**anti-/commutativ**)
- **associativ** $(\alpha \wedge \beta \wedge \gamma)$
- $(c_1\alpha + c_2\beta) \wedge \gamma = c_1\alpha \wedge \gamma + c_2\beta \wedge \gamma$ (**bilinear**)

1.2.2 Exterior derivative $\mathbf{d} : \Omega^p(M) \to \Omega^{p+1}(M)$

$\alpha \in \Omega^p(M)$

- $\mathbf{d} \circ \mathbf{d} = 0$ (**complex proberty**)
- $\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^p\alpha \wedge \mathbf{d}\beta$ (**product rule, $\wedge$-antiderivation**)

1.2.3 Hodge star $* : \Omega^p(M) \to \Omega^{2-p}(M)$

- $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \mu$
- $*1 = \mu \quad (*\mu = 1)$
- $**\alpha = (-1)^p\alpha$
- $\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$

1.2.4 Contraction $\mathbf{i} : (\mathcal{V} \times \Omega^p)(M) \to \Omega^{p-1}(M)$ (inner product)

- $\mathbf{i}_{\vec{v}}\alpha\left(\vec{t}_1, \ldots \vec{t}_{p-1}\right) = \alpha\left(\vec{v}, \vec{t}_1, \ldots \vec{t}_{p-1}\right)$
- $f\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{f\vec{v}}\alpha = \mathbf{i}_{\vec{v}}f\alpha$ (**bilinear**)
- $\mathbf{i}_{\vec{v}}(\alpha \wedge \beta) = (\mathbf{i}_{\vec{v}}\alpha) \wedge \beta + (-1)^p\alpha \wedge (\mathbf{i}_{\vec{v}}\beta)$ (**$\wedge$-antiderivation**)

1.2.5 Lie-derivative $\mathcal{L} : (\mathcal{V} \times \Omega^p)(M) \to \Omega^p(M)$

- $\mathcal{L}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\vec{v}}\alpha$ (**Cartans magic formular**)
- $\mathcal{L}_{f\vec{v}}\alpha = f\mathcal{L}_{\vec{v}}\alpha + \mathbf{d}f \wedge \mathbf{i}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}(\alpha \wedge \beta) = \mathcal{L}_{\vec{v}}\alpha \wedge \beta + \alpha \wedge \mathcal{L}_{\vec{v}}\beta$
- $\mathcal{L}_{\vec{v}}\mathbf{d}\alpha = \mathbf{d}\mathcal{L}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathcal{L}_{\vec{v}}\alpha$
  $\Rightarrow \alpha \in \Omega^1(M): \ \mathcal{L}_{\vec{v}}\left\langle \vec{v}^\flat, \alpha \right\rangle = \left\langle \vec{v}^\flat, \mathcal{L}_{\vec{v}}\alpha \right\rangle$
- $\mathcal{L}_{\vec{v}}\vec{w} = [\vec{v}, \vec{w}] = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}$ (**(Levi-Civita-)Conection ∇ is Torsion-free**)

1.3 Wedge product $\wedge$

$f \in \Omega^0(M)$, $\tilde{f} \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$, $\beta := b_1dx^1 + b_2dx^2 \in \Omega^1(M)$, $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

- $f\tilde{f} = f \wedge \tilde{f} = \tilde{f} \wedge f \in \Omega^0(M)$
- $f\alpha := f \wedge \alpha = \alpha \wedge f = fa_1dx^1 + fa_2dx^2 \in \Omega^1(M)$
- $\alpha \wedge \beta = -\beta \wedge \alpha = (a_1b_2 - a_2b_1)\,dx^1 \wedge dx^2 \in \Omega^2(M)$
- $f\omega := f \wedge \omega = \omega \wedge f = fw_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

1.4 Exterior derivative $\mathbf{d}$

$f \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$

- $\mathbf{d}f = \partial_1 f dx^1 + \partial_2 f dx^2$
- $(\mathbf{d}f)_\mu = \partial_\mu f$ (**Ricci**)
- $\mathbf{d}\alpha = (\partial_1 a_2 - \partial_2 a_1)\,dx^1 \wedge dx^2$
- $(\mathbf{d}\alpha)_{12} = (-1)^{\mu-1}\partial_\mu a_{\bar{\mu}}$ (**Ricci**)

1.5 Hodge star $*$

$f \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$, $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

- $*f = f\mu = \sqrt{|g|} f dx^1 \wedge dx^2$
- $*\alpha = \sqrt{|g|}\left(-\left(a_1g^{12} + a_2g^{22}\right)dx^1 + \left(a_1g^{11} + a_2g^{12}\right)dx^2\right)$
- $(*a)_\mu = (-1)^\mu \sqrt{|g|} g^{\nu\bar{\mu}} a_\nu = (-1)^\mu \sqrt{|g|} a^{\bar{\mu}}$ (**Ricci**)
- $*\omega = \frac{w_{12}}{\sqrt{|g|}}$

1.6 Rising and lowering indices $\sharp$ / $\flat$

$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\alpha^\sharp = \left(g^{11}a_1 + g^{12}a_2\right)\partial_1 + \left(g^{12}a_1 + g^{22}a_2\right)\partial_2$
- $a^\mu = g^{\mu\nu}a_\nu$ (**Ricci**)
- $\vec{v}^\flat = \left(g_{11}v^1 + g_{12}v^2\right)dx^1 + \left(g_{12}v^1 + g_{22}v^2\right)dx^2$
- $v_\mu = g_{\mu\nu}v^\nu$ (**Ricci**)

1.7 Contraction $\mathbf{i}$

$\alpha := a_1 dx^1 + a_2 dx^2 \in \Omega^1(M)$, $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$ $\vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\mathbf{i}_{\vec{v}}\alpha = \alpha\left(\vec{v}\right) = a_1v^1 + a_2v^2$
- $\mathbf{i}_{\vec{v}}\omega = w_{12}\left(-v^2dx^1 + v^1dx^2\right)$

1.8 Lie-derivative $\mathcal{L}$

$f \in \Omega^0(M), \alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M), \omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M), \vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\mathcal{L}_{\vec{v}}f = v^1\partial_1 f + v^2\partial_2 f$
- $\mathcal{L}_{\vec{v}}\alpha = \sum_{i,k=1,2}\left(v^k\partial_k a_i dx^i + a_i\partial_k v^i dx^k\right)$
- $\mathcal{L}_{\vec{v}}\omega = \left(\partial_1\left(w_{12}v^1\right) + \partial_2\left(w_{12}v^2\right)\right)dx^1 \wedge dx^2$
- $\mathcal{L}_{\vec{v}}\omega = \left(w_{12}\partial_\mu v^\mu + v^\mu\partial_\mu w_{12}\right)dx^1 \wedge dx^2$ (**Ricci**)

1.9 Levi-Civita-Connection (co-/contravariant derivatives)

- $\Gamma^k_{ij} = g^{kl}\Gamma_{ijl} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$ (**Christoffel symbols**)
- $\nabla_j v^i = v^i_{;j} = v^i_{|j} = \partial_j v^i + v^k\Gamma^i_{jk}$
- $\nabla\vec{v} := \left[\nabla_j v^i\right]^i{}_j$
- $\left[\nabla\vec{v}^\flat\right]_{ij} = \left[g\left(\nabla\vec{v}\right)\right]_{ij} = \nabla_j v_i = v_{i;j} = v_{i|j} = \partial_j v_i - v_k\Gamma^k_{ij} = g_{il}\nabla_j v^l$
- $\left[\nabla^\sharp\vec{v}^\flat\right]_i{}^j = \left[g\left(\nabla\vec{v}\right)g^{-1}\right]^j_i = \nabla^j v_i = v_i^{;j} = v_i^{|j} = g^{jk}g_{il}\nabla_k v^l$
- $\nabla_i f = \left[\nabla f\right]_i = \partial_i f$
- $\nabla_{\vec{v}}f = \mathcal{L}_{\vec{v}}f = \langle\vec{v}, \nabla_\Gamma f\rangle = (\mathbf{d}f)(\vec{v}) = v^i\nabla_i f = v^i\partial_i f$

1.10 Shape-Operator S, etc

- **Second fundamental form**:
  $[II]_{ij} = [S^\flat]_{ij} = h_{ij} = -\partial_i \vec{N} \cdot \partial_j \vec{X} = -\left[\nabla \vec{N}\right]_{ij} = \vec{n} \cdot \partial_i \partial_j \vec{X}$
- **Shape operator (Weingarten map)**:
  $[S]^i_j = g^{ik} h_{kj} = -\left[\nabla_\Gamma \vec{N}\right]^i \cdot \partial_j \vec{X} = -\left[\nabla_\Gamma \vec{N}\right]^i_j$
- **Inverse of second fundamental form**:
  $b^{ij} = \left[II^{-1}\right]^{ij} = \frac{1}{|g|K}\left[II^{\mathrm{Adj}}\right]^{ij}$
- $[S(\vec{v})]_i = -\left[\nabla_{\vec{v}} \vec{N}\right]_i = v^j h_{ij}$
- $S^T\alpha = \alpha S = S(\alpha^\sharp)$

1.11 Conclusions

$\vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\mathrm{Grad} f = \nabla_\Gamma f = \nabla^\sharp f = (\mathbf{d}f)^\sharp$
  $[\mathrm{Grad} f]^i = \nabla^i f = g^{ij}\nabla_j f = g^{ij}\partial_j f$
- $$\begin{aligned}\mathrm{Div}\vec{v} &= -\delta\vec{v}^\flat = *\mathbf{d}*\vec{v}^\flat = \nabla_i v^i = \partial_i v^i + v^k\Gamma^i_{ik} = \partial_i v^i + v^k\partial_k \log\sqrt{|g|}\\ &= \sum_{i=1,2}\frac{1}{\sqrt{|g|}}\partial_i\sqrt{|g|}v^i = \sum_{i=1,2}\frac{v^i}{\sqrt{|g|}}\partial_i\sqrt{|g|} + \partial_i v^i\end{aligned}$$
- $\delta(f\alpha) = f\delta\alpha - \langle \mathbf{d}f, \alpha\rangle$
- $\mathrm{Div}(f\vec{v}) = f\mathrm{Div}\vec{v} + \nabla_{\vec{v}} f = f\nabla_i v^i + v^i\nabla_i f$
- **Laplace-Beltrami operator**:
  $\Delta_B f = -\delta\mathbf{d}f = *\mathbf{d}*\mathbf{d}f = \mathrm{DivGrad} f = \nabla_i\nabla^i f = \frac{1}{\sqrt{|g|}}\partial_j\left(g^{ij}\sqrt{|g|}\partial_i f\right)$
- **Laplace-de Rham operator**:
  $\Delta_{dR}\alpha = (\delta\mathbf{d} + \mathbf{d}\delta)\alpha =: -(\Delta_B + \Delta_{CB})\alpha$
  $\Delta_{dR}\vec{v} = \left(\Delta_{dR}\vec{v}^\flat\right)^\sharp$
- **Giaquinta-Hildebrandt operator**:
  $\Delta_{GH} f = \Box f = \mathrm{Div}\left(KII^{-1}\mathbf{d}f\right) = -\delta\left(KS^{-T}\mathbf{d}f\right) = \frac{1}{\sqrt{|g|}}\partial_j\left(\sqrt{|g|}Kb^{ij}\partial_i f\right)$
- $-\delta\left(S^T\alpha\right) = -H\mathrm{Div}\alpha^\sharp - \mathrm{Div}\left(KII^{-1}\alpha\right) - \nabla_{\alpha^\sharp}H = H\delta\alpha + \delta\left(KS^{-T}\alpha\right) - \langle\alpha, \mathbf{d}H\rangle$
  $-\delta\left(S^T\mathbf{d}f\right) = -H\Delta_B f - \Delta_{GH} f - \langle\mathbf{d}H, \mathbf{d}f\rangle$

1.12 Moving Surfaces $M(t)$

$\vec{V} := \vec{v} + v_n\vec{N} = \partial_t\vec{X}$ (surface velocity), $\vec{X} : M \to E^3$ (parametrization)

- $\partial_i\vec{V}\cdot\partial_j\vec{X} = g_{jk}\nabla_i v^k - v_n h_{ij} = \left[(\nabla\vec{v} - v_n S)^\flat\right]_{ij}$
- (rate-of-deformation tensor $\boldsymbol{d}$)
  $\dfrac{d}{dt}g = \left(\nabla\vec{v}^\flat\right) + \left(\nabla\vec{v}^\flat\right)^T - 2v_n II = \mathcal{L}_{\vec{V}}g = 2\boldsymbol{d}$
  $\dfrac{d}{dt}g_{ij} = g_{ik}\nabla_j v^k + g_{jk}\nabla_i v^k - 2v_n h_{ij} = 2d_{ij}$
- $$\left\langle \frac{d}{dt}\vec{p}^\flat, \alpha \right\rangle = \dot{\vec{p}}\alpha + 2\vec{p}\boldsymbol{d}\alpha^\sharp = \dot{\vec{p}}\alpha + \vec{p}\left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2 * v_n S^T\right)\alpha$$
  $$= \dot{p}^i\alpha_i + p^j\left(\nabla_j v^i + \nabla^i v_j - 2v_n h^i_j\right)\alpha_i$$
- $\dfrac{d}{dt}\alpha^\sharp = \left[\dot{\alpha} + \left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\alpha\right]^\sharp$
- $\dfrac{d}{dt} * \omega = * \left[\dot{\omega} - (\mathrm{Div}\vec{v} + v_n H)\,\omega\right]$
- $\dfrac{d}{dt} * \vec{p}^\flat = * \left[\dot{\vec{p}} + (\mathrm{Div}\vec{v} + v_n H)\,\vec{p}\right]^\flat$
- $\dfrac{d}{dt} * \alpha = * \left[\dot{\alpha} + \left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\alpha + (\mathrm{Div}\vec{v} + v_n H)\,\alpha\right]$
- $$\frac{1}{2}\frac{d}{dt}\|\alpha\|^2 = \left\langle \dot{\alpha} + v_n S^T\alpha - (\nabla\vec{v})^T\alpha, \alpha\right\rangle = \dot{\alpha}\alpha^\sharp + \alpha\left(v_n S - \nabla\vec{v}\right)\alpha^\sharp$$
  $$= \dot{\alpha}_i\alpha^i + v_n\alpha_i h^i_j\alpha^j - \alpha_i\left(\nabla_j v^i\right)\alpha^j$$
- $\dfrac{1}{2}\dfrac{d}{dt}\|\omega\|^2 = \dfrac{1}{2}\dfrac{d}{dt}(*\omega)^2 = \langle\dot{\omega}, \omega\rangle - (\mathrm{Div}\vec{v} + v_n H)\,\|\omega\|^2$
- $\dfrac{1}{2}\dfrac{d}{dt}\|\delta\alpha\|^2 = \left\langle \delta\left[\dot{\alpha} + \left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\alpha + (\mathrm{Div}\vec{v} + v_n H)\,\alpha\right], \delta\alpha\right\rangle - (\mathrm{Div}\vec{v} + v_n H)\,\|\delta\alpha\|^2$
- $\dfrac{1}{2}\dfrac{d}{dt}\left\|\delta\vec{p}^\flat\right\|^2 = \left\langle \delta\left[\dot{\vec{p}} + (\mathrm{Div}\vec{v} + v_n H)\,\vec{p}\right]^\flat, \delta\vec{p}^\flat\right\rangle - (\mathrm{Div}\vec{v} + v_n H)\left\|\delta\vec{p}^\flat\right\|^2$
- $\dfrac{d}{dt}\displaystyle\int_{M(t)} f\mu = \int_{M(t)} \dot{f} + f\,(\mathrm{Div}\vec{v} + v_n H)\,\mu$
- $\dfrac{d}{dt}\displaystyle\int_{M(t)} \frac{1}{2}\|\alpha\|^2\mu = \int_{M(t)} \langle\dot{\alpha}, \alpha\rangle + v_n\left\langle S^T\alpha + \frac{1}{2}H\alpha, \alpha\right\rangle + \left\langle \frac{1}{2}(\mathrm{Div}\vec{v})\,\alpha - (\nabla\vec{v})^T\alpha, \alpha\right\rangle\mu$
- $\dfrac{d}{dt}\displaystyle\int_{M(t)} \frac{1}{2}\|\omega\|^2\mu = \int_{M(t)} \langle\dot{\omega}, \omega\rangle - \frac{1}{2}(\mathrm{Div}\vec{v} + v_n H)\,\|\omega\|^2\mu$

- $$\frac{d}{dt}\int_{M(t)} \frac{1}{2}\left\|\mathbf{d}\vec{p}^\flat\right\|^2\mu = \int_{M(t)} \left\langle \frac{d}{dt}\vec{p}^\flat, \delta\mathbf{d}\vec{p}^\flat \right\rangle - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\mathbf{d}\vec{p}^\flat\right\|^2\mu$$
$$= \int_{M(t)} \dot{\vec{p}}\delta\mathbf{d}\vec{p}^\flat + \vec{p}\left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2v_n S^T\right)\delta\mathbf{d}\vec{p}^\flat$$
$$- \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\mathbf{d}\vec{p}^\flat\right\|^2\mu$$

- $$\frac{d}{dt}\int_{M(t)} \frac{1}{2}\left\|\delta\vec{p}^\flat\right\|^2\mu = \int_{M(t)} \left\langle \frac{d}{dt}\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat \right\rangle + \left\langle \left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat \right\rangle$$
$$+ \left\langle \left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat \right\rangle - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\delta\vec{p}^\flat\right\|^2\mu$$
$$= \int_{M(t)} \dot{\vec{p}}\mathbf{d}\delta\vec{p}^\flat + \left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}\mathbf{d}\delta\vec{p}^\flat - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\delta\vec{p}^\flat\right\|^2\mu$$

- $$\frac{d}{dt}\int_{M(t)} \frac{1}{2}\left(\left\|\mathbf{d}\vec{p}^\flat\right\|^2 + \left\|\delta\vec{p}^\flat\right\|^2\right)\mu = \frac{d}{dt}\int_{M(t)} \frac{1}{2}\left(\|\mathrm{Rot}\vec{p}\|^2 + \|\mathrm{Div}\vec{p}\|^2\right)\mu$$
$$= \int_{M(t)} \left\langle \frac{d}{dt}\vec{p}^\flat, \Delta_{dR}\vec{p}^\flat \right\rangle - \frac{\mathrm{Div}\vec{v} + v_n H}{2}\left(\left\|\mathbf{d}\vec{p}^\flat\right\|^2 + \left\|\delta\vec{p}^\flat\right\|^2\right)$$
$$+ \left\langle \left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat \right\rangle$$
$$- \left\langle \left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2v_n S^T\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat \right\rangle\mu$$
$$= \int_{M(t)} \left\langle \dot{\vec{p}}, \Delta_{dR}\vec{p} \right\rangle - \frac{\mathrm{Div}\vec{v} + v_n H}{2}\left(\|\mathrm{Rot}\vec{p}\|^2 + \|\mathrm{Div}\vec{p}\|^2\right)$$
$$+ \left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}\mathbf{d}\delta\vec{p}^\flat$$
$$+ \vec{p}\left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2v_n S^T\right)\delta\mathbf{d}\vec{p}^\flat\mu$$

- $$\frac{d}{dt}\int_{M(t)} \frac{1}{4}\left(\|\vec{p}\|^2 - 1\right)^2\mu = \int_{M(t)} \left(\left\|\vec{p}^\flat\right\|^2 - 1\right)\left[\left\langle \frac{d}{dt}\vec{p}^\flat, \vec{p}^\flat \right\rangle + \left\langle \left(v_n S^T - (\nabla\vec{v})^T\right)\vec{p}^\flat, \vec{p}^\flat \right\rangle\right.$$
$$\left. + \frac{1}{4}\left(\left\|\vec{p}^\flat\right\|^2 - 1\right)\left(\mathrm{Div}\vec{v} + v_n H\right)\right]\mu$$
$$= \int_{M(t)} \left(\left\|\vec{p}^\flat\right\|^2 - 1\right)\left[\left\langle \dot{\vec{p}}, \vec{p} \right\rangle + \left\langle \left(\left(\nabla^\sharp\vec{v}^\flat\right)^T - v_n S\right)\vec{p}, \vec{p} \right\rangle\right.$$
$$\left. + \frac{1}{4}\left(\|\vec{p}\|^2 - 1\right)\left(\mathrm{Div}\vec{v} + v_n H\right)\right]\mu$$

- $$\int_{M(t)} \left\langle \left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat \right\rangle - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\delta\vec{p}^\flat\right\|^2\mu$$
$$= \int_{M(t)} \mathcal{L}_{\vec{p}}\left\langle \vec{p}^\flat, \mathbf{d}\delta\vec{v}^\flat + \mathbf{d}\left(v_v H\right) \right\rangle + \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\delta\vec{p}^\flat\right\|^2\mu$$

2 Tensors

2.1 Flat / Sharp

- $t := t^i{}_j \partial_i \otimes dx^j$
- ${}^\flat t = gt = g_{ik} t^k{}_j dx^i \otimes dx^j = t_{ij} dx^i \otimes dx^j$
- $t^\sharp = tg^{-1} = t^i{}_k g^{kj} \partial_i \otimes \partial_j = t^{ij} \partial_i \otimes \partial_j$
- ${}^\flat t^\sharp = gtg^{-1} = g_{ik} t^k{}_l g^{lj} dx^i \otimes \partial_j = t_i{}^j dx^i \otimes \partial_j$

2.1.1 Conclusions

$\alpha = \vec{v}^\flat = {}^\flat\vec{v}$, $\vec{w} = \beta^\sharp = {}^\sharp\beta$, $s = \vec{v} \otimes \beta$:

- t symmetric ($t_{12} = t_{21}$ resp. $t^{12} = t^{21}$): ${}^\flat t^\sharp = t^T$
- $\alpha t \vec{w} = \vec{v}^\flat t \vec{w} = \alpha t^\sharp \beta = \vec{v}^\flat t^\sharp \beta$ (Associativity referring to arguments)

$\Rightarrow$ $t(\alpha, \vec{w}) = {}^\flat t(\vec{v}, \vec{w}) = t^\sharp(\alpha, \beta) = {}^\flat t^\sharp(\vec{v}, \beta)$

- $s = \vec{v} \otimes \beta$: ${}^\flat s = \alpha \otimes \beta$, $s^\sharp = \vec{v} \otimes \vec{w}$, ${}^\flat s^\sharp = \alpha \otimes \vec{w}$ (Associativity referring to factors, tensor product is metric compatible)
- $t^T := (({}^\flat t)^T)^\sharp = {}^\flat((t^\sharp)^T) = t_j{}^i dx^j \otimes \partial_i$

2.2 Covariant Derivative $\nabla_\bullet = g_{\bullet i} \nabla^\bullet$

- $\nabla_k f = \partial_k f$
- $\nabla_k v^i = \partial_k v^i + \Gamma_{kl}{}^i v^l$
- $\nabla_k v_i = \partial_k v_i - \Gamma_{ki}{}^l v_l$
- $\nabla_k t^i{}_j = \partial_k t^i{}_j + \Gamma_{kl}{}^i t^l{}_j - \Gamma_{kj}{}^l t^i{}_l$
- $\nabla_k t_i{}^j = \partial_k t_i{}^j - \Gamma_{ki}{}^l t_l{}^j + \Gamma_{kl}{}^j t_i{}^l$
- $\nabla_k t^{ij} = \partial_k t^{ij} + \Gamma_{kl}{}^i t^{lj} + \Gamma_{kl}{}^j t^{il}$
- $\nabla_k t_{ij} = \partial_k t_{ij} - \Gamma_{ki}{}^l t_{lj} - \Gamma_{kj}{}^l t_{il}$
- $\mathrm{Div}(v) = \nabla_i v^i = \nabla^i v_i = \mathrm{Tr}(\nabla \vec{v}) = \mathrm{Tr}({}^\flat \nabla^\sharp \alpha)$
- $\mathrm{Rot}(v) = (\sqrt{|g|})^{-1} \epsilon^{ki} \nabla_k v_i = (\sqrt{|g|})^{-1} \mathrm{Tr}((\nabla \alpha) \epsilon^\sharp)$
- $\mathrm{Div}(t) = \nabla^j t^i{}_j \partial_i = \nabla_j t_i{}^j dx^i = \nabla_j t^{ij} \partial_i = \nabla^j t_{ij} dx^i$

2.3 Rotation $R = R^{-T} : T_pM \to (T_pM)'$

- $R = R_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} = R^i{}_j \partial_i \otimes dx^j$

2.3.1 Pull-back $R^* = R_*^T$

Hint: Use rotated metric $g' = R_* g = RgR^T$ in rotated coords. system!

- $R^* v' = R^{-1} v' = R^T v' = v \in T_pM \quad \rightsquigarrow [R^* v']^i = [R^T]^i{}_j v'^j$
- $R^* \alpha' = \alpha'(R\bullet) = \alpha' R = \alpha \in T_p^*M \quad \rightsquigarrow [R^*\alpha']_i = R^j{}_i v'_j$
- $R^* \, {}^\bullet t'^\bullet = R^T \, {}^\bullet t'^\bullet R = {}^\bullet t^\bullet$ (Rotation is metric compatible)
- $R^*(\bullet \otimes \bullet) = (R^*\bullet) \otimes (R^*\bullet)$

2.3.2 Push-forward $R_* = R^{*T}$

- $R_* v = Rv = v'$
- $R_* \alpha = \alpha R^T = \alpha'$
- $R^* \, {}^\bullet t^\bullet = R \, {}^\bullet t^\bullet R^T = {}^\bullet t'^\bullet$
- $R_*(\bullet \otimes \bullet) = (R_*\bullet) \otimes (R_*\bullet)$