Thin Shell Stuff

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Contents

1 Metric Quantities

In the following, we consider a ***thin shell*** of constant thickness $h \in \mathbb{R}$ around an oriented, boundarieless, compact Riemannian 2-manifold (surface) $\mathcal{S}$ defined by

$$\mathcal{S}_h := \mathcal{S} \times \left[-\frac{h}{2}, \frac{h}{2}\right] \subset \mathbb{R}^3 . \tag{1}$$

Constant thickness means, that the orthogonal measurement of the two disjoint boundaries $\Upsilon_h^+ \sqcup \Upsilon_h^- = \Upsilon_h := \partial\mathcal{S}_h$ is h at all boundary points. Thereby, be h small enough, so that $\mathcal{S}_h \subset \mathbb{R}^3$ contains no overlaps, i. e., it exists a surjection $\mathcal{S}_h \twoheadrightarrow \mathcal{S}$.

1.1 Coordinates

We define the coordinate in normal direction $\boldsymbol{\nu}$ of the surface $\mathcal{S}$ by $\xi \in \left[-\frac{h}{2}, \frac{h}{2}\right]$. If we use any choice of local coordinates $(u, v) \in U$ of the surface, so that the immersion $\mathbf{x} : U \rightarrowtail \mathbb{R}^3$ parameterize $\mathcal{S} = \mathrm{Im}(\mathbf{x})$, then we can define the immersion $\tilde{\mathbf{x}} : U \times \left[-\frac{h}{2}, \frac{h}{2}\right] \rightarrowtail \mathbb{R}^3$ with $\mathcal{S}_h = \mathrm{Im}(\tilde{\mathbf{x}})$ by

$$\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(u, v, \xi) := \mathbf{x}(u, v) + \xi\boldsymbol{\nu}(u, v) = \mathbf{x} + \xi\boldsymbol{\nu} . \tag{2}$$

1.2 Arrangements

Lowercase letters $i, j, k, \dots$ are used as index for u and v, e. g., $\alpha_i dx^i$ is an 1-form in $T^*\mathcal{S}$.

Uppercase letters $I, J, K, \dots$ are used for u, v and ξ, e. g., $\tilde{\alpha}^I \partial_I \tilde{\mathbf{x}} = \tilde{\alpha}^i \partial_i \tilde{\mathbf{x}} + \tilde{\alpha}^\xi \partial_\xi \tilde{\mathbf{x}}$ is a (contravariant) vector in $T\mathcal{S}_h$.

The Tilde are used for quantities and relations in context of $\mathcal{S}_h$, e. g., $\tilde{\boldsymbol{\alpha}} \in T\mathcal{S}_h$ but $\boldsymbol{\alpha} \in T\mathcal{S}$ and we can construct a relation $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \alpha^\xi \boldsymbol{\nu}$.

Full covariant descriptions (lower indices) are always used, unless otherwise is defined, e. g., $\mathbf{B} = \{B_{ij}\}$ is the full covariant shape operator, i. e., the second fundamental form in this representation.

Indexing and collector brackets , [] and {}, are used to switch between components and object representations, e. g., $[\mathbf{t}]_{ij} = t_{ij}$ and $\{t_{ij}\} = \mathbf{t}$.

Sharp and flat operator on tensors are generalisations of the usual flat and sharp operator on vector valued quantities and can be realized by matrix multiplications with the metric tensor $\mathbf{g}$ and its inverse $\mathbf{g}^{-1}$, e. g., ${}^\flat\left\{t^i{}_j\right\}^\sharp = \mathbf{g}\left\{t^i{}_j\right\}\mathbf{g}^{-1} = \left\{t_i{}^j\right\} = \mathbf{t}^\sharp$.

Tensor product means always the contraction of the last component of a tensor with the first of another tensor, e. g., $[\boldsymbol{s}\mathbf{t}]_{ij} = s_i{}^k t_{kj}$, e. g., with an usual matrix product $\cdot$, this implies $\boldsymbol{s}\mathbf{t} = \boldsymbol{s} \cdot \mathbf{g}^{-1} \cdot \mathbf{t}$.

1.3 The Metric Tensor and Shape Operator

With an arbitrary choice of surface coordinates $(u, v) \in U$, we can calculate the canonical basic vectors $\partial_I \tilde{\mathbf{x}} \in T\mathcal{S}_h$ by

$$\partial_i \tilde{\mathbf{x}} = \partial_i \mathbf{x} + \xi \partial_i \boldsymbol{\nu} \tag{3}$$

$$\partial_\xi \tilde{\mathbf{x}} = \boldsymbol{\nu} \,. \tag{4}$$

The ***metric tensor*** (***first fundamental form***) of thin shell is given by its components $\tilde{g}_{IJ} = \langle \partial_I \tilde{\mathbf{x}}, \partial_J \tilde{\mathbf{x}} \rangle_{\mathcal{S}_h}$. Therefore, for the mixed tangential-normal components holds $\tilde{g}_{i\xi} = \tilde{g}_{\xi i} = 0$, because

$$\langle \partial_i \boldsymbol{\nu}, \boldsymbol{\nu} \rangle_{\mathcal{S}_h} = \frac{1}{2} \partial_i \left\| \boldsymbol{\nu} \right\|^2_{\mathcal{S}_h} = 0 \,. \tag{5}$$

For the pure normal component, we obtain $\tilde{g}_{\xi\xi} = \|\boldsymbol{\nu}\|^2_{\mathcal{S}_h} = 1$, i. e., the co- and contravariant normal components of a tensor quantity are equivalently, e. g., (detailed)

$$\tilde{t}^I{}_{\xi J} = \tilde{g}_{\xi K} \tilde{t}^{IK}{}_J = \tilde{g}_{\xi k} \tilde{t}^{Ik}{}_J + \tilde{g}_{\xi\xi} \tilde{t}^{I\xi}{}_J = \tilde{t}^{I\xi}{}_J \,. \tag{6}$$

For the pure tangential components, we get a second degree tensor polynomial in ξ

$$\tilde{g}_{ij} = g_{ij} - 2\xi B_{ij} + \xi^2 \left[\mathbf{B}^2\right]_{ij} = \left[(g - \xi\mathbf{B})^2\right]_{ij} \tag{7}$$

$$= g_{ij} - 2\xi B_{ij} + \xi^2 \left(\mathcal{H} B_{ij} - \mathcal{K} g_{ij}\right) . \tag{8}$$

where the ***covariant shape operator*** (***second fundamental form***) is given by

$$B_{ij} = - \left\langle \partial_i \mathbf{x}, \partial_j \boldsymbol{\nu} \right\rangle_{\mathcal{S}_h} \tag{9}$$

and the ***third fundamental form*** by

$$\left[\mathbf{B}^2\right]_{ij} = \left\langle \partial_i \boldsymbol{\nu}, \partial_j \boldsymbol{\nu} \right\rangle_{\mathcal{S}_h} , \tag{10}$$

see [HW53]. $\mathcal{K} = \left|\mathbf{B}^\sharp\right|$ is the ***Gaussian curvature*** and $\mathcal{H} := \mathrm{Tr}\mathbf{B} = B^i_{\ i}$ the ***mean curvature***. (In a more differential geometrical context on surfaces, this is minus twice the mean curvature.) A more classical representation of the third fundamental form $\mathbf{B}^2$ is

$$\mathbf{B}^2 = \mathcal{H}\mathbf{B} - \mathcal{K}\mathbf{g} . \tag{11}$$

Theorem 1. *For the inverse thin shell metric* $\tilde{\mathbf{g}}^{-1}$ *holds*

$$\tilde{g}^{ij} = \left(g^{ik} + \sum_{\mathfrak{l}=1}^{\infty} \xi^{\mathfrak{l}} \left[\mathbf{B}^{\mathfrak{l}}\right]^{ik} \right) \left(\delta^j_k + \sum_{\mathfrak{k}=1}^{\infty} \xi^{\mathfrak{k}} \left[\mathbf{B}^{\mathfrak{k}}\right]_k^{\ j} \right) \tag{12}$$

$$= \left[\left(\mathbf{g} + \sum_{\mathfrak{k}=1}^{\infty} \xi^{\mathfrak{k}} \mathbf{B}^{\mathfrak{k}} \right)^2 \right]^{ij} , \tag{13}$$

$$\tilde{g}^{\xi\xi} = 1 , \tag{14}$$

$$\tilde{g}^{i\xi} = \tilde{g}^{\xi i} = 0 . \tag{15}$$

Proof. First we define the pure tangential components of the thin shell metric tensor as $\tilde{\mathbf{g}}_t := \{\tilde{g}_{ij}\}$. With $\boldsymbol{\delta} = \left\{\delta^i_j\right\}$ the Kronecker delta, we can write down in usual matrix notation

$$\tilde{\mathbf{g}} \cdot \tilde{\mathbf{g}}^{-1} = \begin{bmatrix} \tilde{\mathbf{g}}_t & O \\ O & 1 \end{bmatrix} \cdot \begin{bmatrix} \{\tilde{g}^{ij}\} & \{\tilde{g}^{i\xi}\} \\ \{\tilde{g}^{\xi i}\} & \tilde{g}^{\xi\xi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta} & O \\ O & 1 \end{bmatrix} . \tag{16}$$

Thus, we obtain

$$\tilde{g}^{\xi\xi} = 1 , \tag{17}$$

$$\tilde{g}^{i\xi} = \tilde{g}^{\xi i} = 0 , \tag{18}$$

$$\left\{\tilde{g}^{ij}\right\} = \tilde{\mathbf{g}}_t^{-1} = (\mathbf{g} - \xi\mathbf{B})^{-2} = (\mathbf{g} - \xi\mathbf{B})^{-1} \cdot \left(\boldsymbol{\delta} - \xi\mathbf{B}^\sharp\right)^{-1} . \tag{19}$$

For h small enough, so that $\xi \|\mathbf{B}\| \leq h \|\mathbf{B}\| < 1$ and exponent with a dot indicate matrix (endomorphism) power, we can use the Neumann serie

$$\left(\boldsymbol{\delta} - \xi\mathbf{B}^\sharp\right)^{-1} = \boldsymbol{\delta} + \sum_{\mathfrak{k}=1}^{\infty} \xi^{\mathfrak{k}} \left(\mathbf{B}^\sharp\right)^{\cdot\mathfrak{k}} , \tag{20}$$

and therefore the assertion, because with $\mathbf{B}^{\mathfrak{k}} = \left(\mathbf{B}\cdot\mathbf{g}^{-1}\right)^{\cdot\mathfrak{k}} \cdot \mathbf{g}$ we get

$$\left(\mathbf{B}^\sharp\right)^{\cdot\mathfrak{k}} = \left(\mathbf{B}\cdot\mathbf{g}^{-1}\right)^{\cdot\mathfrak{k}} = \mathbf{B}^{\mathfrak{k}}\cdot\mathbf{g}^{-1} = \left(\mathbf{B}^{\mathfrak{k}}\right)^\sharp \tag{21}$$

and

$$\left(\mathbf{g} - \xi\mathbf{B}\right)^{-1} = \left(\left(\boldsymbol{\delta} - \xi\mathbf{B}^\sharp\right)\cdot\mathbf{g}\right)^{-1} = \mathbf{g}^{-1}\cdot\left(\boldsymbol{\delta} - \xi\mathbf{B}^\sharp\right)^{-1} = {}^\sharp\!\left(\boldsymbol{\delta} - \xi\mathbf{B}^\sharp\right)^{-1} . \tag{22}$$

□

Therefore, we get for $\tilde{\mathbf{g}}_t^{-1}$ a polynomial in $\xi\mathbf{B}$ and with successively applying (11),i. e., $\mathbf{B}^{\mathfrak{k}} = \mathcal{H}\mathbf{B}^{\mathfrak{k}-1} - \mathcal{K}\mathbf{B}^{\mathfrak{k}-2}$, we can always find polynomials p and q in $\mathcal{K}$, $\mathcal{H}$ and ξ, so that holds $\tilde{\mathbf{g}}_t^{-1} = p(\mathcal{K},\mathcal{H},\xi)\mathbf{g}^{-1} + q(\mathcal{K},\mathcal{H},\xi)^\sharp\mathbf{B}^\sharp$. We will not carry this out in full generality, but let us mention some developments in ξ.

Conclusion 1. *The developments of $\tilde{\mathbf{g}}_t^{-1}$ up to second degree in ξ are*

$$\tilde{\mathbf{g}}_t^{-1} = \mathbf{g}^{-1} + \mathcal{O}(\xi) , \tag{23}$$

$$\tilde{\mathbf{g}}_t^{-1} = \mathbf{g}^{-1} + 2\xi^\sharp\mathbf{B}^\sharp + \mathcal{O}(\xi^2) , \tag{24}$$

$$\begin{aligned}\tilde{\mathbf{g}}_t^{-1} &= \mathbf{g}^{-1} + 2\xi^\sharp\mathbf{B}^\sharp + 3\xi^2\, {}^\sharp\!\left(\mathbf{B}^2\right)^\sharp + \mathcal{O}(\xi^3) &(25)\\ &= \left(1 - 3\xi^2\mathcal{K}\right)\mathbf{g}^{-1} + \xi\left(2 + 3\xi\mathcal{H}\right){}^\sharp\mathbf{B}^\sharp + \mathcal{O}(\xi^3) . &(26)\end{aligned}$$

1.3.1 The Volume Element

To develop the thin shell volume element $\tilde{\mu}$ in normal direction at the surface volume element μ, we need a development of the determinant of the metric tensor $\tilde{\mathbf{g}}$.

Theorem 2. *For the determinant of the thin shell metric tensor $|\tilde{\mathbf{g}}|$ holds*

$$|\tilde{\mathbf{g}}| = \left(1 - \xi\mathcal{H} + \xi^2\mathcal{K}\right)^2 |\mathbf{g}| , \tag{27}$$

Proof. The mixed components are zero, so we get

$$|\tilde{\mathbf{g}}| = \tilde{g}_{\xi\xi}\, |\tilde{\mathbf{g}}_t| = |\tilde{\mathbf{g}}_t| . \tag{28}$$

Now, we define $\sqrt{\tilde{\mathbf{g}}_t^\sharp} := \left(\mathbf{g} - \xi\mathbf{B}\right)^\sharp$ as a square root of $\tilde{\mathbf{g}}_t^\sharp$, because

$$\tilde{\mathbf{g}}_t^\sharp = \left(\left(\mathbf{g} - \xi\mathbf{B}\right)^2\right)^\sharp = \left(\left(\mathbf{g} - \xi\mathbf{B}\right)^\sharp\left(\mathbf{g} - \xi\mathbf{B}\right)\right)^\sharp = \left(\mathbf{g} - \xi\mathbf{B}\right)^\sharp\left(\mathbf{g} - \xi\mathbf{B}\right)^\sharp = \left(\sqrt{\tilde{\mathbf{g}}_t^\sharp}\right)^2 . \tag{29}$$

Hence, we can calculate

$$|\tilde{\mathbf{g}}| = |\tilde{\mathbf{g}}_t| = \left|\tilde{\mathbf{g}}_t^\sharp \mathbf{g}\right| = \left|\tilde{\mathbf{g}}_t^\sharp\right| |\mathbf{g}| = \left|\sqrt{\tilde{\mathbf{g}}_t^\sharp}\right|^2 |\mathbf{g}| \ . \tag{30}$$

For the determinant of $\sqrt{\tilde{\mathbf{g}}_t^\sharp}$, we regard that $\mathbf{g}^\sharp$ is the Kronecker delta, so we obtain

$$\left|\sqrt{\tilde{\mathbf{g}}_t^\sharp}\right| = \left|\mathbf{g}^\sharp - \xi \mathbf{B}^\sharp\right| = (1 - \xi {B_u}^u)(1 - \xi {B_v}^v) - \xi^2 {B_u}^v {B_v}^u \tag{31}$$
$$= 1 - \xi({B_u}^u + {B_v}^v) + \xi^2({B_u}^u {B_v}^v - {B_u}^v {B_v}^u) = \left(1 - \xi\mathcal{H} + \xi^2\mathcal{K}\right) . \tag{32}$$

□

Therefore a representation of the ***thin shell volume element*** $\tilde{\mu}$, depending on the surface volume element μ, is

$$\tilde{\mu} = \sqrt{|\tilde{\mathbf{g}}|}du \wedge dv \wedge d\xi = \left(1 - \xi\mathcal{H} + \xi^2\mathcal{K}\right)\mu \wedge d\xi \tag{33}$$
$$= \left(1 - \xi\mathcal{H} + \xi^2\mathcal{K}\right) d\xi \wedge \mu \, . \tag{34}$$

1.3.2 The Levi-Civita Tensor and Hodge-Dualism

1.4 Christoffel Symbols

The ***Christoffel symbols*** are needed to define an unique metric compatible derivation (Levi-Civita connection). With a choice of coordinates, the christoffel symbols (of second kind) on the thin shell are

$$\widetilde{\Gamma}_{IJ}^K = \frac{1}{2}\tilde{g}^{KL}\left(\partial_I \tilde{g}_{JL} + \partial_J \tilde{g}_{IL} - \partial_L \tilde{g}_{IJ}\right) . \tag{35}$$

On the surface $\mathcal{S}$, they are equal defined, just omit the tilde and use lowercase letters for indexing.

Theorem 3. *With ${\beta_{ij}}^k := {B_i}^k{}_{|j} + {B_j}^k{}_{|i} - {B_{ij}}^{|k}$ the second order expansions in normal direction of Christoffel symbols are*

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k - \xi {\beta_{ij}}^k + \mathcal{O}(\xi^2) \tag{36}$$
$$\widetilde{\Gamma}_{ij}^\xi = B_{ij} - \xi\left[\mathbf{B}^2\right]_{ij} \qquad = (1 - \xi\mathcal{H}) B_{ij} + \xi\mathcal{K} g_{ij} \tag{37}$$
$$\widetilde{\Gamma}_{i\xi}^k = \widetilde{\Gamma}_{\xi i}^k = -{B_i}^k - \xi\left[\mathbf{B}^2\right]_i{}^k + \mathcal{O}(\xi^2) \qquad = -(1 + \xi\mathcal{H}){B_i}^k + \xi\mathcal{K}\delta_i^k + \mathcal{O}(\xi^2) \tag{38}$$
$$\widetilde{\Gamma}_{\xi\xi}^K = 0 \tag{39}$$
$$\widetilde{\Gamma}_{I\xi}^\xi = \widetilde{\Gamma}_{\xi I}^\xi = 0 \, . \tag{40}$$

Proof. Properties of the thin shell metric $\tilde{\mathbf{g}}$ are the mixed tangential-normal components are zero (the same holds for the inverse metric) and the pure normal component is constant. Hence, we get

$$\widetilde{\Gamma}^K_{\xi\xi} = \frac{1}{2}\tilde{g}^{KL}\left(\partial_\xi \tilde{g}_{\xi L} + \partial_\xi \tilde{g}_{\xi L} - \partial_L \tilde{g}_{\xi\xi}\right) = 0\,, \tag{41}$$

$$\widetilde{\Gamma}^\xi_{I\xi} = \frac{1}{2}\tilde{g}^{\xi\xi}\left(\partial_I \tilde{g}_{\xi\xi} + \partial_\xi \tilde{g}_{\xi L} - \partial_\xi \tilde{g}_{I\xi}\right)\,. \tag{42}$$

The partial derivative in normal direction of the tangential part of thin shell metric is

$$\partial_\xi \tilde{g}_{ij} = 2\left(-B_{ij} + \xi\left[\mathbf{B}^2\right]_{ij}\right)\,. \tag{43}$$

Therefore, we obtain

$$\widetilde{\Gamma}^\xi_{ij} = \frac{1}{2}\tilde{g}^{\xi\xi}\left(\partial_i \tilde{g}_{j\xi} + \partial_j \tilde{g}_{i\xi} - \partial_\xi \tilde{g}_{ij}\right) = B_{ij} - \xi\left[\mathbf{B}^2\right]_{ij}\,, \tag{44}$$

$$\widetilde{\Gamma}^k_{i\xi} = \frac{1}{2}\tilde{g}^{kl}\left(\partial_i \tilde{g}_{\xi l} + \partial_\xi \tilde{g}_{il} - \partial_l \tilde{g}_{i\xi}\right) = \left(g^{kl} + 2\xi B^{kl} + \mathcal{O}(\xi^2)\right)\left(-B_{il} + \xi\left[\mathbf{B}^2\right]_{il}\right) \tag{45}$$

$$= -B_i^{\ k} - \xi\left[\mathbf{B}^2\right]_i^{\ k} + \mathcal{O}(\xi^2) \tag{46}$$

and with the substitution (11) the remaining statements of these two terms. For the pure tangential thin shell Christoffel symbols, we first determine ${\beta_{ij}}^k$ at the surface in terms of partial derivatives and take advantage of the symmetry of the shape operator,i. e.,

$${\beta_{ij}}^k = g^{kl}\left(B_{il|j} + B_{jl|i} - B_{ij|l}\right) \tag{47}$$

$$= g^{kl}\left(\partial_j B_{il} - \Gamma^m_{ij}B_{ml} - \Gamma^m_{jl}B_{im} + \partial_i B_{jl} - \Gamma^m_{ij}B_{ml} - \Gamma^m_{il}B_{jm} - \partial_l B_{ij} + \Gamma^m_{il}B_{mj} + \Gamma^m_{jl}B_{im}\right) \tag{48}$$

$$= g^{kl}\left(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij} - 2\Gamma^m_{ij}B_{ml}\right) \tag{49}$$

$$= g^{kl}\left(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij}\right) - 2\Gamma_{ijl}B^{kl} \tag{50}$$

$$= g^{kl}\left(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij}\right) - B^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right)\,. \tag{51}$$

Hence, we get

$$\widetilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}\left(\partial_i \tilde{g}_{jl} + \partial_j \tilde{g}_{il} - \partial_l \tilde{g}_{ij}\right) \tag{52}$$

$$= \frac{1}{2}\left(g^{kl} + 2\xi B^{kl} + \mathcal{O}(\xi^2)\right)\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij} - 2\xi\left(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij}\right)\right) \tag{53}$$

$$= \Gamma^k_{ij} + \xi\left(B^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right) - g^{kl}\left(\partial_j B_{il} + \partial_i B_{jl} - \partial_l B_{ij}\right)\right) + \mathcal{O}(\xi^2) \tag{54}$$

$$= \Gamma^k_{ij} - \xi{\beta_{ij}}^k + \mathcal{O}(\xi^2)\,. \tag{55}$$

□

2 Boundary Conditions

If we consider the limit case $h \to 0$ for an differential expression or especially an operator L_h, then we may get remaining partial normal derivation ∂_ξ^p with arbitrary order $p \geq 0$.

Those expressions may be undetermined on the surface $\mathcal{S}$. One way out is to set an additional condition on the whole thin shell, e. g.***parallel transportation*** of quantities, e. g.for a tensor $\tilde{\mathbf{t}}$, we set $\nabla_\xi \tilde{\mathbf{t}} \equiv 0$. Another possibility is to take boundary conditions of the thin shell into account by expanding in normal directions.

2.1 No-Penetration Condition (NPC) for vector quantities

We consider the ***No-Penetration Condition***

$$\boldsymbol{\nu} \cdot \tilde{\boldsymbol{\alpha}} = \tilde{\alpha}_\xi = 0 \text{ on } \Upsilon_h\,. \tag{NPC}$$

Taylor expansion at the surface $\mathcal{S}$ results in

$$0 = \tilde{\alpha}_\xi\big|_{\Upsilon_h^\pm} = \tilde{\alpha}_\xi\big|_{\mathcal{S}} \pm \frac{h}{2}\partial_\xi\tilde{\alpha}_\xi\big|_{\mathcal{S}} + \frac{h^2}{8}\partial_\xi^2\tilde{\alpha}_\xi\big|_{\mathcal{S}} + \mathcal{O}(h^3) \tag{56}$$

$$0 = \tilde{\alpha}_\xi\big|_{\Upsilon_h^+} + \tilde{\alpha}_\xi\big|_{\Upsilon_h^-} = 2\tilde{\alpha}_\xi\big|_{\mathcal{S}} + \mathcal{O}(h^2) \qquad \Rightarrow \boxed{\tilde{\alpha}_\xi\big|_{\mathcal{S}} = \mathcal{O}(h^2)} \tag{57}$$

$$0 = \tilde{\alpha}_\xi\big|_{\Upsilon_h^+} - \tilde{\alpha}_\xi\big|_{\Upsilon_h^-} = h\partial_\xi\tilde{\alpha}_\xi\big|_{\mathcal{S}} + \mathcal{O}(h^3) \qquad \Rightarrow \boxed{\partial_\xi\tilde{\alpha}_\xi\big|_{\mathcal{S}} = \mathcal{O}(h^2)}\,. \tag{58}$$

2.2 Neumann Condition (NC) for vector quantities

We consider the ***Neumann Condition***

$$\tilde{\nabla}_{\boldsymbol{\nu}}\tilde{\boldsymbol{\alpha}} = \left\{\tilde{\nabla}_\xi\tilde{\alpha}_I\right\} = \left\{\tilde{\nabla}_\xi\tilde{\alpha}^I\right\} = 0 \text{ on } \Upsilon_h\,. \tag{NPC}$$

First we investigate the tangential parts

$$\begin{aligned}
\tilde{\nabla}_\xi\tilde{\alpha}_i &= \partial_\xi\tilde{\alpha}_i - \widetilde{\Gamma}_{\xi i}^J\tilde{\alpha}_J = \partial_\xi\tilde{\alpha}_i - \widetilde{\Gamma}_{\xi i}^j\tilde{\alpha}_j &(59)\\
&= \partial_\xi\tilde{\alpha}_i + B_i^{\ j}\tilde{\alpha}_j + \xi\left[\mathbf{B}^2\right]_i^{\ j}\tilde{\alpha}_j + \mathcal{O}(\xi^2) &(60)\\
\tilde{\nabla}_\xi\tilde{\alpha}_i\big|_{\Upsilon_h^\pm} &= \partial_\xi\tilde{\alpha}_i\big|_{\Upsilon_h^\pm} + B_i^{\ j}\tilde{\alpha}_j\big|_{\Upsilon_h^\pm} \pm \frac{h}{2}\left[\mathbf{B}^2\right]_i^{\ j}\tilde{\alpha}_j\big|_{\Upsilon_h^\pm} + \mathcal{O}(h^2)\,. &(61)
\end{aligned}$$

Taylor expansion at the surface $\mathcal{S}$ for $p \geq 0$ results in

$$\partial_\xi^p\tilde{\boldsymbol{\alpha}}\big|_{\Upsilon_h^+} + \partial_\xi^p\tilde{\boldsymbol{\alpha}}\big|_{\Upsilon_h^-} = 2\partial_\xi^p\tilde{\boldsymbol{\alpha}}\big|_{\mathcal{S}} + \mathcal{O}(h^2) \tag{62}$$

$$\partial_\xi^p\tilde{\boldsymbol{\alpha}}\big|_{\Upsilon_h^+} - \partial_\xi^p\tilde{\boldsymbol{\alpha}}\big|_{\Upsilon_h^-} = h\partial_\xi^{p+1}\tilde{\boldsymbol{\alpha}}\big|_{\mathcal{S}} + \mathcal{O}(h^3)\,. \tag{63}$$

Therefor by making up the sum and the difference of (61) one obtain

$$\begin{aligned}
0 = \tilde{\nabla}_\xi\tilde{\alpha}_i\big|_{\Upsilon_h^+} + \tilde{\nabla}_\xi\tilde{\alpha}_i\big|_{\Upsilon_h^-} &= 2\partial_\xi\tilde{\alpha}_i\big|_{\mathcal{S}} + 2B_i^{\ j}\alpha_j + \mathcal{O}(h^2) &(64)\\
&\Longrightarrow \boxed{\partial_\xi\tilde{\alpha}_i\big|_{\mathcal{S}} = -B_i^{\ j}\alpha_j + \mathcal{O}(h^2)}\,, &(65)\\
0 = \tilde{\nabla}_\xi\tilde{\alpha}_i\big|_{\Upsilon_h^+} - \tilde{\nabla}_\xi\tilde{\alpha}_i\big|_{\Upsilon_h^-} &= h\partial_\xi^2\tilde{\alpha}_i\big|_{\mathcal{S}} + hB_i^{\ j}\partial_\xi\tilde{\alpha}_j\big|_{\mathcal{S}} + h\left[\mathbf{B}^2\right]_i^{\ j}\alpha_j + \mathcal{O}(h^3) &(66)\\
&= h\partial_\xi^2\tilde{\alpha}_i\big|_{\mathcal{S}} + \mathcal{O}(h^3) &(67)\\
&\Longrightarrow \boxed{\partial_\xi^2\tilde{\alpha}_i\big|_{\mathcal{S}} = \mathcal{O}(h^3)}\,. &(68)
\end{aligned}$$

But we are carefully about the meaning of the results relating to rising the indices, i. e., $\partial_\xi \tilde{\alpha}^i\big|_{\mathcal{S}} \neq g^{ij}\partial_\xi \tilde{\alpha}_j\big|_{\mathcal{S}}$ generally, because $\partial_\xi \tilde{\mathbf{g}} \neq 0$ neither at $\mathcal{S}$ nor in whole $\mathcal{S}_h$. But with

$$
\begin{aligned}
\partial_\xi \tilde{g}^{ij} &= 2B^{ij} + \mathcal{O}(\xi)\,, && (69)\\
\partial_\xi^2 \tilde{g}^{ij} &= 6\left[\mathbf{B}^2\right]^{ij} + \mathcal{O}(\xi)\,, && (70)\\
\partial_\xi \tilde{\alpha}^i &= \partial_\xi\left(\tilde{g}^{ij}\tilde{\alpha}_j\right) = \tilde{g}^{ij}\partial_\xi\tilde{\alpha}_j + \tilde{\alpha}_j\partial_\xi\tilde{g}^{ij} && (71)\\
&= \tilde{g}^{ij}\partial_\xi\tilde{\alpha}_j + 2B^{ij}\tilde{\alpha}_j + \mathcal{O}(\xi)\,, && (72)\\
\partial_\xi^2 \tilde{\alpha}^i &= \partial_\xi^2\left(\tilde{g}^{ij}\tilde{\alpha}_j\right) = \tilde{g}^{ij}\partial_\xi^2\tilde{\alpha}_j + \tilde{\alpha}_j\partial_\xi^2\tilde{g}^{ij} + 2\left(\partial_\xi\tilde{g}^{ij}\right)\left(\partial_\xi\tilde{\alpha}_j\right) && (73)\\
&= \tilde{g}^{ij}\partial_\xi^2\tilde{\alpha}_j + 4B^{ij}\partial_\xi\tilde{\alpha}_j + 6\left[\mathbf{B}^2\right]^{ij}\tilde{\alpha}_j + \mathcal{O}(\xi)\,, && (74)
\end{aligned}
$$

(65), and (68), we get for the restriction to the surface

$$\boxed{\partial_\xi \tilde{\alpha}^i\big|_{\mathcal{S}} = B^i{}_j\alpha^j + \mathcal{O}(h^2)}\,, \tag{75}$$

$$\boxed{\partial_\xi^2 \tilde{\alpha}^i\big|_{\mathcal{S}} = 2\left[\mathbf{B}^2\right]^i{}_j\alpha^j + \mathcal{O}(h^2)}\,. \tag{76}$$

The former is also consistent to the covariant normal derivation formulation

$$\boxed{\tilde{\nabla}_\xi \tilde{\alpha}^i\big|_{\mathcal{S}} = g^{ij}\tilde{\nabla}_\xi\tilde{\alpha}_i\big|_{\mathcal{S}} = \mathcal{O}(h^2)}\,. \tag{77}$$

For the boundary condition in normal direction, namely

$$\tilde{\nabla}_\xi\tilde{\alpha}_\xi\big|_{\Upsilon_h^\pm} = \partial_\xi\tilde{\alpha}_\xi\big|_{\Upsilon_h^\pm} = 0\,, \tag{78}$$

we have

$$
\begin{aligned}
0 &= \tilde{\nabla}_\xi\tilde{\alpha}_\xi\big|_{\Upsilon_h^+} + \tilde{\nabla}_\xi\tilde{\alpha}_\xi\big|_{\Upsilon_h^-} = 2\partial_\xi\tilde{\alpha}_\xi\big|_{\mathcal{S}} + \mathcal{O}(h^2) \quad \Rightarrow \boxed{\partial_\xi\tilde{\alpha}_\xi\big|_{\mathcal{S}} = \partial_\xi\tilde{\alpha}^\xi\big|_{\mathcal{S}} = \mathcal{O}(h^2)}\,, && (79)\\
0 &= \tilde{\nabla}_\xi\tilde{\alpha}_\xi\big|_{\Upsilon_h^+} - \tilde{\nabla}_\xi\tilde{\alpha}_\xi\big|_{\Upsilon_h^-} = h\partial_\xi^2\tilde{\alpha}_\xi\big|_{\mathcal{S}} + \mathcal{O}(h^3) \quad \Rightarrow \boxed{\partial_\xi^2\tilde{\alpha}_\xi\big|_{\mathcal{S}} = \partial_\xi^2\tilde{\alpha}^\xi\big|_{\mathcal{S}} = \mathcal{O}(h^2)}\,. && (80)
\end{aligned}
$$

References

[HW53] Philip Hartman and Aurel Wintner. On the third fundamental form of a surface. *American Journal of Mathematics*, 75(2):298–334, 1953.