Orientation Fields on Closed Surfaces A Discrete Exterior Calculus Primal Dual (DEC-PD) Aproach

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Content

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- vertices, edges, (triangle) faces

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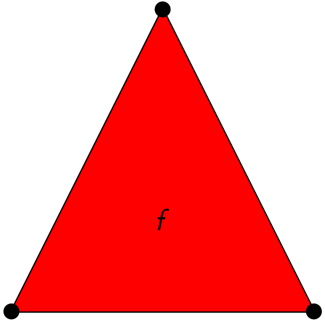

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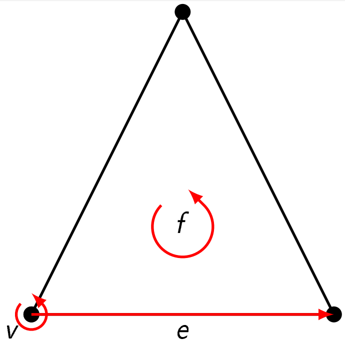

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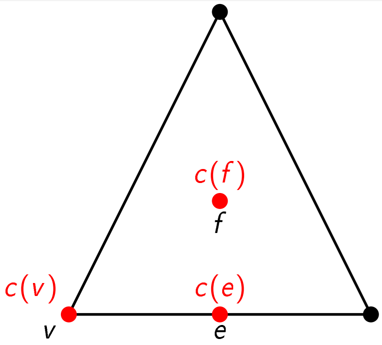
- have circumcenters $c(\sigma) \in Int(\sigma)$ $\Rightarrow$: well-centered)



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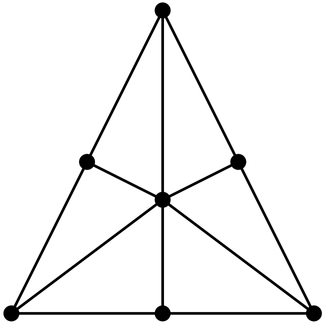
- are refinable (circumcentric subdivision)



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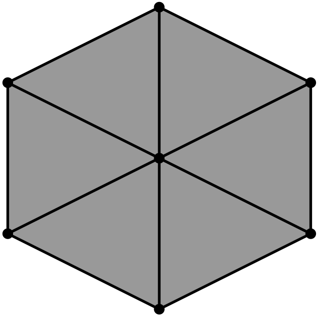

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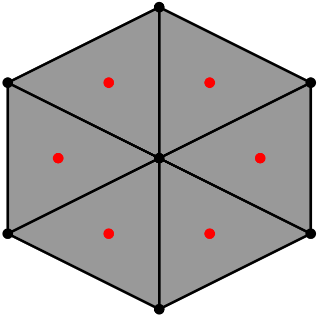

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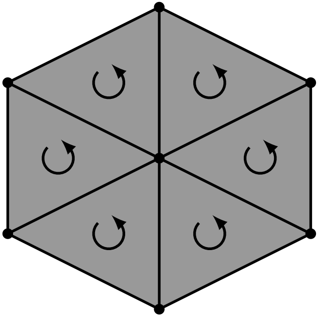

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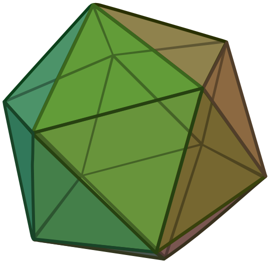
- **manifold-like**: polyhedron $\bigcup_{f\in\mathcal{F}} f$ is a C^0-manifold



https://commons.wikimedia.org/wiki/File:Icosahedron.svg

PDE for orientation fields

$$\partial_t \mathbf{p}^\flat = \left(K_1 \mathbf{\Delta}^{\mathrm{GD}} + K_3 \mathbf{\Delta}^{\mathrm{RR}}\right) \mathbf{p}^\flat - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat$$

We need to discretize

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- the Laplace-Operators $\boldsymbol{\Delta}^{\mathrm{GD}}$ and $\boldsymbol{\Delta}^{\mathrm{RR}}$

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