1 Arbitrary s.p.d. metric

1.1 Assumptions

- $Ind(M) = 0$
- $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = g_{11} \left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22} \left(dx^2\right)^2$ (**s.p.d.**)

1.2 General proberties

$\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, $\gamma \in \Omega^r(M)$, $\vec{v} \in \mathcal{V}(M)$

1.2.1 Wedge product $\wedge$

- $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$ (**anti-/commutativ**)
- **associativ** $(\alpha \wedge \beta \wedge \gamma)$
- $(c_1\alpha + c_2\beta) \wedge \gamma = c_1\alpha \wedge \gamma + c_2\beta \wedge \gamma$ (**bilinear**)

1.2.2 Exterior derivative $\mathbf{d} : \Omega^p(M) \to \Omega^{p+1}(M)$

$\alpha \in \Omega^p(M)$

- $\mathbf{d} \circ \mathbf{d} = 0$ (**complex proberty**)
- $\mathbf{d}\left(\alpha \wedge \beta\right) = \mathbf{d}\alpha \wedge \beta + (-1)^p\alpha \wedge \mathbf{d}\beta$ (**product rule, $\wedge$-antiderivation**)

1.2.3 Hodge star $* : \Omega^p(M) \to \Omega^{2-p}(M)$

- $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle \, \mu$
- $*1 = \mu \quad (*\mu = 1)$
- $* * \alpha = (-1)^p\alpha$
- $\langle \alpha, \beta \rangle = \langle *\alpha, *\beta \rangle$

1.2.4 Contraction $\mathbf{i} : \left(\mathcal{V} \times \Omega^p\right)(M) \to \Omega^{p-1}(M)$ (inner product)

- $\mathbf{i}_{\vec{v}}\alpha\left(\vec{t}_1, \ldots \vec{t}_{p-1}\right) = \alpha\left(\vec{v}, \vec{t}_1, \ldots \vec{t}_{p-1}\right)$
- $f\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{f\vec{v}}\alpha = \mathbf{i}_{\vec{v}}f\alpha$ (**bilinear**)
- $\mathbf{i}_{\vec{v}}\left(\alpha \wedge \beta\right) = \left(\mathbf{i}_{\vec{v}}\alpha\right) \wedge \beta + (-1)^p\alpha \wedge \left(\mathbf{i}_{\vec{v}}\beta\right)$ (**$\wedge$-antiderivation**)

1.2.5 Lie-derivative $\mathcal{L} : (\mathcal{V} \times \Omega^p)(M) \to \Omega^p(M)$

- $\mathcal{L}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\vec{v}}\alpha$ (**Cartans magic formular**)
- $\mathcal{L}_{f\vec{v}}\alpha = f\mathcal{L}_{\vec{v}}\alpha + \mathbf{d}f \wedge \mathbf{i}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}(\alpha \wedge \beta) = \mathcal{L}_{\vec{v}}\alpha \wedge \beta + \alpha \wedge \mathcal{L}_{\vec{v}}\beta$
- $\mathcal{L}_{\vec{v}}\mathbf{d}\alpha = \mathbf{d}\mathcal{L}_{\vec{v}}\alpha$
- $\mathcal{L}_{\vec{v}}\mathbf{i}_{\vec{v}}\alpha = \mathbf{i}_{\vec{v}}\mathcal{L}_{\vec{v}}\alpha$
  $\Rightarrow \alpha \in \Omega^1(M):\ \mathcal{L}_{\vec{v}}\left\langle \vec{v}^\flat, \alpha \right\rangle = \left\langle \vec{v}^\flat, \mathcal{L}_{\vec{v}}\alpha \right\rangle$
- $\mathcal{L}_{\vec{v}}\vec{w} = [\vec{v}, \vec{w}] = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}$ (**(Levi-Civita-)Conection ∇ is Torsion-free**)

1.3 Wedge product $\wedge$

$f \in \Omega^0(M),\ \tilde{f} \in \Omega^0(M),\ \alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M),\ \beta := b_1dx^1 + b_2dx^2 \in \Omega^1(M),$
$\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

- $f\tilde{f} = f \wedge \tilde{f} = \tilde{f} \wedge f \in \Omega^0(M)$
- $f\alpha := f \wedge \alpha = \alpha \wedge f = fa_1dx^1 + fa_2dx^2 \in \Omega^1(M)$
- $\alpha \wedge \beta = -\beta \wedge \alpha = (a_1b_2 - a_2b_1)\,dx^1 \wedge dx^2 \in \Omega^2(M)$
- $f\omega := f \wedge \omega = \omega \wedge f = fw_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

1.4 Exterior derivative $\mathbf{d}$

$f \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$

- $\mathbf{d}f = \partial_1 f dx^1 + \partial_2 f dx^2$
- $(\mathbf{d}f)_\mu = \partial_\mu f$ (**Ricci**)
- $\mathbf{d}\alpha = (\partial_1 a_2 - \partial_2 a_1)\,dx^1 \wedge dx^2$
- $(\mathbf{d}\alpha)_{12} = (-1)^{\mu-1}\partial_\mu a_{\bar{\mu}}$ (**Ricci**)

1.5 Hodge star $*$

$f \in \Omega^0(M)$, $\alpha := a_1dx^1 + a_2dx^2 \in \Omega^1(M)$, $\omega := w_{12}dx^1 \wedge dx^2 \in \Omega^2(M)$

- $*f = f\mu = \sqrt{|g|}f dx^1 \wedge dx^2$
- $*\alpha = \sqrt{|g|}\left(-\left(a_1g^{12} + a_2g^{22}\right)dx^1 + \left(a_1g^{11} + a_2g^{12}\right)dx^2\right)$
- $(*a)_\mu = (-1)^\mu \sqrt{|g|}g^{\nu\bar{\mu}}a_\nu = (-1)^\mu \sqrt{|g|}a^{\bar{\mu}}$ (**Ricci**)
- $*\omega = \frac{w_{12}}{\sqrt{|g|}}$

- $1-\frac{\langle\alpha,\beta\rangle^2}{\|\alpha\|^2\|\beta\|^2}=\frac{\|\alpha\wedge\beta\|^2}{\|\alpha\|^2\|\beta\|^2}=\frac{\langle\alpha,*\beta\rangle^2}{\|\alpha\|^2\|\beta\|^2}$ $(\sim 1-\cos^2\phi=\sin^2\phi)$
- $\langle\alpha,\beta\rangle^2+\langle\alpha,*\beta\rangle^2=\|\alpha\|^2\|\beta\|^2$
- $\langle *\alpha,\beta\rangle=-\langle\alpha,*\beta\rangle$

1.6 Rising and lowering indices $\sharp$ / $\flat$

$\alpha:=a_1dx^1+a_2dx^2\in\Omega^1(M)$, $\vec{v}:=v^1\partial_1+v^2\partial_2\in\mathcal{V}(M)$

- $\alpha^\sharp=\left(g^{11}a_1+g^{12}a_2\right)\partial_1+\left(g^{12}a_1+g^{22}a_2\right)\partial_2$
- $a^\mu=g^{\mu\nu}a_\nu$ (**Ricci**)
- $\vec{v}^\flat=\left(g_{11}v^1+g_{12}v^2\right)dx^1+\left(g_{12}v^1+g_{22}v^2\right)dx^2$
- $v_\mu=g_{\mu\nu}v^\nu$ (**Ricci**)

1.7 Contraction $\mathbf{i}$

$\alpha:=a_1dx^1+a_2dx^2\in\Omega^1(M)$, $\omega:=w_{12}dx^1\wedge dx^2\in\Omega^2(M)$ $\vec{v}:=v^1\partial_1+v^2\partial_2\in\mathcal{V}(M)$

- $\mathbf{i}_{\vec{v}}\alpha=\alpha\left(\vec{v}\right)=a_1v^1+a_2v^2$
- $\mathbf{i}_{\vec{v}}\omega=w_{12}\left(-v^2dx^1+v^1dx^2\right)$

1.8 Lie-derivative $\mathcal{L}$

$f\in\Omega^0(M)$, $\alpha:=a_1dx^1+a_2dx^2\in\Omega^1(M)$, $\omega:=w_{12}dx^1\wedge dx^2\in\Omega^2(M)$, $\vec{v}:=v^1\partial_1+v^2\partial_2\in\mathcal{V}(M)$

- $\mathcal{L}_{\vec{v}}f=v^1\partial_1f+v^2\partial_2f$
- $\mathcal{L}_{\vec{v}}\alpha=\sum_{i,k=1,2}\left(v^k\partial_ka_idx^i+a_i\partial_kv^idx^k\right)$
- $\mathcal{L}_{\vec{v}}\omega=\left(\partial_1\left(w_{12}v^1\right)+\partial_2\left(w_{12}v^2\right)\right)dx^1\wedge dx^2$
- $\mathcal{L}_{\vec{v}}\omega=\left(w_{12}\partial_\mu v^\mu+v^\mu\partial_\mu w_{12}\right)dx^1\wedge dx^2$ (**Ricci**)

1.9 Levi-Civita-Connection (co-/contravariant derivatives)

- $\Gamma^k_{ij}=g^{kl}\Gamma_{ijl}=\frac{1}{2}g^{kl}\left(\partial_ig_{jl}+\partial_jg_{il}-\partial_lg_{ij}\right)$ (**Christoffel symbols**)
- $\nabla_jv^i=v^i_{;j}=v^i_{|j}=\partial_jv^i+v^k\Gamma^i_{jk}$
- $\nabla\vec{v}:=\left[\nabla_jv^i\right]^i{}_j$
- $\left[\nabla\vec{v}^\flat\right]_{ij}=\left[g\left(\nabla\vec{v}\right)\right]_{ij}=\nabla_jv_i=v_{i;j}=v_{i|j}=\partial_jv_i-v_k\Gamma^k_{ij}=g_{il}\nabla_jv^l$

- $\left[\nabla^\sharp \vec{v}^\flat\right]_i^{\ j} = \left[g\left(\nabla\vec{v}\right)g^{-1}\right]_i^j = \nabla^j v_i = v_i^{;j} = v_i^{|j} = g^{jk}g_{il}\nabla_k v^l$
- $\nabla_i f = [\nabla f]_i = \partial_i f$
- $\nabla_{\vec{v}} f = \mathcal{L}_{\vec{v}} f = \langle \vec{v}, \nabla_\Gamma f\rangle = (\mathbf{d}f)(\vec{v}) = v^i \nabla_i f = v^i \partial_i f$

1.10 Shape-Operator S, etc

- **Second fundamental form**:
  $[II]_{ij} = [S^\flat]_{ij} = h_{ij} = -\partial_i \vec{N}\cdot\partial_j\vec{X} = -\left[\nabla\vec{N}\right]_{ij} = \vec{n}\cdot\partial_i\partial_j\vec{X}$
- **Shape operator (Weingarten map)**:
  $[S]_j^i = g^{ik}h_{kj} = -\left[\nabla_\Gamma\vec{N}\right]^i\cdot\partial_j\vec{X} = -\left[\nabla_\Gamma\vec{N}\right]_j^i$
- **Inverse of second fundamental form**:
  $b^{ij} = \left[II^{-1}\right]^{ij} = \frac{1}{|g|K}\left[II^{\mathrm{Adj}}\right]^{ij}$
- $[S(\vec{v})]_i = -\left[\nabla_{\vec{v}}\vec{N}\right]_i = v^j h_{ij}$
- $S^T\alpha = \alpha S = S(\alpha^\sharp)$

1.11 Conclusions

$\vec{v} := v^1\partial_1 + v^2\partial_2 \in \mathcal{V}(M)$

- $\mathrm{Grad} f = \nabla_\Gamma f = \nabla^\sharp f = (\mathbf{d}f)^\sharp$
  $[\mathrm{Grad} f]^i = \nabla^i f = g^{ij}\nabla_j f = g^{ij}\partial_j f$
- $\mathrm{Div}\vec{v} = -\delta\vec{v}^\flat = *\mathbf{d}*\vec{v}^\flat = \nabla_i v^i = \partial_i v^i + v^k\Gamma^i_{ik} = \partial_i v^i + v^k\partial_k \log\sqrt{|g|}$
  $$= \sum_{i=1,2}\frac{1}{\sqrt{|g|}}\partial_i\sqrt{|g|}v^i = \sum_{i=1,2}\frac{v^i}{\sqrt{|g|}}\partial_i\sqrt{|g|} + \partial_i v^i$$
- $\delta\left(f\alpha\right) = f\delta\alpha - \langle\mathbf{d}f, \alpha\rangle$
- $\mathrm{Div}(f\vec{v}) = f\mathrm{Div}\vec{v} + \langle v, \nabla f\rangle = f\mathrm{Div}\vec{v} + \nabla_{\vec{v}} f = f\nabla_i v^i + v^i\nabla_i f$
- $\mathrm{Rot}(f\vec{v}) = f\mathrm{Rot}\vec{v} + \langle v, \mathrm{Rot} f\rangle$
- **Laplace-Beltrami operator**:
  $\Delta_B f = -\delta\mathbf{d}f = *\mathbf{d}*\mathbf{d}f = \mathrm{DivGrad}f = \nabla_i\nabla^i f = \frac{1}{\sqrt{|g|}}\partial_j\left(g^{ij}\sqrt{|g|}\partial_i f\right)$
- **Laplace-de Rham operator**:
  $\Delta_{dR}\alpha = \left(\delta\mathbf{d} + \mathbf{d}\delta\right)\alpha =: -\left(\Delta_B + \Delta_{CB}\right)\alpha$
  $\Delta_{dR}\vec{v} = \left(\Delta_{dR}\vec{v}^\flat\right)^\sharp$

- **Giaquinta-Hildebrandt operator**:
  $\Delta_{GH} f = \Box f = \mathrm{Div}\left(KII^{-1}\mathbf{d}f\right) = -\delta\left(KS^{-T}\mathbf{d}f\right) = \frac{1}{\sqrt{|g|}}\partial_j\left(\sqrt{|g|}Kb^{ij}\partial_i f\right)$

- $-\delta\left(S^T\alpha\right) = -H\mathrm{Div}\alpha^\sharp - \mathrm{Div}\left(KII^{-1}\alpha\right) - \nabla_{\alpha^\sharp}H = H\delta\alpha + \delta\left(KS^{-T}\alpha\right) - \langle\alpha, \mathbf{d}H\rangle$
  $-\delta\left(S^T\mathbf{d}f\right) = -H\Delta_B f - \Delta_{GH} f - \langle\mathbf{d}H, \mathbf{d}f\rangle$

1.12 Moving Surfaces $M(t)$

$\vec{V} := \vec{v} + v_n\vec{N} = \partial_t\vec{X}$ (surface velocity), $\vec{X} : M \to E^3$ (parametrization)

- $\partial_i\vec{V}\cdot\partial_j\vec{X} = g_{jk}\nabla_i v^k - v_n h_{ij} = \left[(\nabla\vec{v} - v_n S)^\flat\right]_{ij}$

- (rate-of-deformation tensor $\boldsymbol{d}$)
  $\frac{d}{dt}g = \left(\nabla\vec{v}^\flat\right) + \left(\nabla\vec{v}^\flat\right)^T - 2v_n II = \mathcal{L}_{\vec{V}}g = 2\boldsymbol{d}$
  $\frac{d}{dt}g_{ij} = g_{ik}\nabla_j v^k + g_{jk}\nabla_i v^k - 2v_n h_{ij} = 2d_{ij}$

- $\left\langle\frac{d}{dt}\vec{p}^\flat, \alpha\right\rangle = \dot{\vec{p}}\alpha + 2\vec{p}\boldsymbol{d}\alpha^\sharp = \dot{\vec{p}}\alpha + \vec{p}\left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2 * v_n S^T\right)\alpha$
  $= \dot{p}^i\alpha_i + p^j\left(\nabla_j v^i + \nabla^i v_j - 2v_n h^i_j\right)\alpha_i$

- $\frac{d}{dt}\alpha^\sharp = \left[\dot{\alpha} + \left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\alpha\right]^\sharp$

- $\frac{d}{dt} * \omega = *\left[\dot{\omega} - (\mathrm{Div}\vec{v} + v_n H)\,\omega\right]$

- $\frac{d}{dt} * \vec{p}^\flat = *\left[\dot{\vec{p}} + (\mathrm{Div}\vec{v} + v_n H)\,\vec{p}\right]^\flat$

- $\frac{d}{dt} * \alpha = *\left[\dot{\alpha} + \left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\alpha + (\mathrm{Div}\vec{v} + v_n H)\,\alpha\right]$

- $\frac{1}{2}\frac{d}{dt}\|\alpha\|^2 = \left\langle\dot{\alpha} + v_n S^T\alpha - (\nabla\vec{v})^T\alpha, \alpha\right\rangle = \dot{\alpha}\alpha^\sharp + \alpha\left(v_n S - \nabla\vec{v}\right)\alpha^\sharp$
  $= \dot{\alpha}_i\alpha^i + v_n\alpha_i h^i_j\alpha^j - \alpha_i\left(\nabla_j v^i\right)\alpha^j$

- $\frac{1}{2}\frac{d}{dt}\|\omega\|^2 = \frac{1}{2}\frac{d}{dt}(*\omega)^2 = \langle\dot{\omega}, \omega\rangle - (\mathrm{Div}\vec{v} + v_n H)\|\omega\|^2$

- $\frac{1}{2}\frac{d}{dt}\|\delta\alpha\|^2 = \left\langle\delta\left[\dot{\alpha} + \left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\alpha + (\mathrm{Div}\vec{v} + v_n H)\,\alpha\right], \delta\alpha\right\rangle - (\mathrm{Div}\vec{v} + v_n H)\|\delta\alpha\|^2$

- $\frac{1}{2}\frac{d}{dt}\left\|\delta\vec{p}^\flat\right\|^2 = \left\langle\delta\left[\dot{\vec{p}} + (\mathrm{Div}\vec{v} + v_n H)\,\vec{p}\right]^\flat, \delta\vec{p}^\flat\right\rangle - (\mathrm{Div}\vec{v} + v_n H)\left\|\delta\vec{p}^\flat\right\|^2$

- $\frac{d}{dt}\int_{M(t)} f\mu = \int_{M(t)}\dot{f} + f\left(\mathrm{Div}\vec{v} + v_n H\right)\mu$

- $\frac{d}{dt}\int_{M(t)} \frac{1}{2}\|\alpha\|^2 \mu = \int_{M(t)} \langle\dot{\alpha},\alpha\rangle + v_n \left\langle S^T\alpha + \frac{1}{2}H\alpha, \alpha\right\rangle + \left\langle \frac{1}{2}\left(\mathrm{Div}\vec{v}\right)\alpha - (\nabla\vec{v})^T\alpha, \alpha\right\rangle \mu$

- $\frac{d}{dt}\int_{M(t)} \frac{1}{2}\|\omega\|^2 \mu = \int_{M(t)} \langle\dot{\omega},\omega\rangle - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\|\omega\|^2 \mu$

- $$\begin{aligned}\frac{d}{dt}\int_{M(t)} \frac{1}{2}\left\|\mathbf{d}\vec{p}^\flat\right\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt}\vec{p}^\flat, \delta\mathbf{d}\vec{p}^\flat\right\rangle - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\mathbf{d}\vec{p}^\flat\right\|^2 \mu \\ &= \int_{M(t)} \dot{\vec{p}}\delta\mathbf{d}\vec{p}^\flat + \vec{p}\left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2v_n S^T\right)\delta\mathbf{d}\vec{p}^\flat \\ &\quad - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\mathbf{d}\vec{p}^\flat\right\|^2 \mu\end{aligned}$$

- $$\begin{aligned}\frac{d}{dt}\int_{M(t)} \frac{1}{2}\left\|\delta\vec{p}^\flat\right\|^2 \mu &= \int_{M(t)} \left\langle \frac{d}{dt}\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat\right\rangle + \left\langle\left(2v_n S^T - (\nabla\vec{v})^T - \left(\nabla^\sharp\vec{v}^\flat\right)\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat\right\rangle \\ &\quad + \left\langle\left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat\right\rangle - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\delta\vec{p}^\flat\right\|^2 \mu \\ &= \int_{M(t)} \dot{\vec{p}}\mathbf{d}\delta\vec{p}^\flat + \left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}\mathbf{d}\delta\vec{p}^\flat - \frac{1}{2}\left(\mathrm{Div}\vec{v} + v_n H\right)\left\|\delta\vec{p}^\flat\right\|^2 \mu\end{aligned}$$

- $$\begin{aligned}\frac{d}{dt}\int_{M(t)} \frac{1}{2}\left(\left\|\mathbf{d}\vec{p}^\flat\right\|^2 + \left\|\delta\vec{p}^\flat\right\|^2\right)\mu &= \frac{d}{dt}\int_{M(t)} \frac{1}{2}\left(\|\mathrm{Rot}\vec{p}\|^2 + \|\mathrm{Div}\vec{p}\|^2\right)\mu \\ &= \int_{M(t)} \left\langle \frac{d}{dt}\vec{p}^\flat, \Delta_{dR}\vec{p}^\flat\right\rangle - \frac{\mathrm{Div}\vec{v} + v_n H}{2}\left(\left\|\mathbf{d}\vec{p}^\flat\right\|^2 + \left\|\delta\vec{p}^\flat\right\|^2\right) \\ &\quad + \left\langle\left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat\right\rangle \\ &\quad - \left\langle\left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2v_n S^T\right)\vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat\right\rangle \mu \\ &= \int_{M(t)} \left\langle\dot{\vec{p}}, \Delta_{dR}\vec{p}\right\rangle - \frac{\mathrm{Div}\vec{v} + v_n H}{2}\left(\|\mathrm{Rot}\vec{p}\|^2 + \|\mathrm{Div}\vec{p}\|^2\right) \\ &\quad + \left(\mathrm{Div}\vec{v} + v_n H\right)\vec{p}\mathbf{d}\delta\vec{p}^\flat \\ &\quad + \vec{p}\left((\nabla\vec{v})^T + \left(\nabla^\sharp\vec{v}^\flat\right) - 2v_n S^T\right)\delta\mathbf{d}\vec{p}^\flat\mu\end{aligned}$$

- $$\begin{aligned}\frac{d}{dt}\int_{M(t)} \frac{1}{4}\left(\|\vec{p}\|^2 - 1\right)^2 \mu &= \int_{M(t)} \left(\left\|\vec{p}^\flat\right\|^2 - 1\right)\left[\left\langle \frac{d}{dt}\vec{p}^\flat, \vec{p}^\flat\right\rangle + \left\langle\left(v_n S^T - (\nabla\vec{v})^T\right)\vec{p}^\flat, \vec{p}^\flat\right\rangle\right. \\ &\quad \left. + \frac{1}{4}\left(\left\|\vec{p}^\flat\right\|^2 - 1\right)\left(\mathrm{Div}\vec{v} + v_n H\right)\right]\mu \\ &= \int_{M(t)} \left(\left\|\vec{p}^\flat\right\|^2 - 1\right)\left[\left\langle\dot{\vec{p}}, \vec{p}\right\rangle + \left\langle\left(\left(\nabla^\sharp\vec{v}^\flat\right)^T - v_n S\right)\vec{p}, \vec{p}\right\rangle\right. \\ &\quad \left. + \frac{1}{4}\left(\|\vec{p}\|^2 - 1\right)\left(\mathrm{Div}\vec{v} + v_n H\right)\right]\mu\end{aligned}$$

- $\int_{M(t)} \left\langle (\mathrm{Div}\vec{v} + v_n H)\, \vec{p}^\flat, \mathbf{d}\delta\vec{p}^\flat \right\rangle - \frac{1}{2}(\mathrm{Div}\vec{v} + v_n H) \left\| \delta\vec{p}^\flat \right\|^2 \mu$
$$= \int_{M(t)} \mathcal{L}_{\vec{p}} \left\langle \vec{p}^\flat, \mathbf{d}\delta\vec{v}^\flat + \mathbf{d}(v_v H) \right\rangle + \frac{1}{2}(\mathrm{Div}\vec{v} + v_n H) \left\| \delta\vec{p}^\flat \right\|^2 \mu$$

2 Tensors

2.1 Flat / Sharp

- $t := t^i{}_j \partial_i \otimes dx^j$
- ${}^\flat t = gt = g_{ik} t^k{}_j dx^i \otimes dx^j = t_{ij} dx^i \otimes dx^j$
- $t^\sharp = tg^{-1} = t^i{}_k g^{kj} \partial_i \otimes \partial_j = t^{ij} \partial_i \otimes \partial_j$
- ${}^\flat t^\sharp = gtg^{-1} = g_{ik} t^k{}_l g^{lj} dx^i \otimes \partial_j = t_i{}^j dx^i \otimes \partial_j$

2.2 Product / Contraction

- $(s \cdot t)_i{}^j := s_{ik} t^{kj}$
- $s : t := s_{ij} t^{ij} = \mathrm{Tr}(s \cdot t^T) = \mathrm{Tr}(s^T \cdot t) = \dots$

2.3 Conclusions

$\alpha = \vec{v}^\flat = {}^\flat\vec{v}$, $\vec{w} = \beta^\sharp = {}^\sharp\beta$, $s = \vec{v} \otimes \beta$:

- t symmetric ($t_{12} = t_{21}$ resp. $t^{12} = t^{21}$): ${}^\flat t^\sharp = t^T$
- $\alpha t \vec{w} = \vec{v}^\flat t \vec{w} = \alpha t^\sharp \beta = \vec{v}^\flat t^\sharp \beta$ (Associativity referring to arguments)

$\Rightarrow$ $t(\alpha, \vec{w}) = {}^\flat t(\vec{v}, \vec{w}) = t^\sharp(\alpha, \beta) = {}^\flat t^\sharp(\vec{v}, \beta)$

- $s = \vec{v} \otimes \beta$: ${}^\flat s = \alpha \otimes \beta$, $s^\sharp = \vec{v} \otimes \vec{w}$, ${}^\flat s^\sharp = \alpha \otimes \vec{w}$ (Associativity referring to factors, tensor product is metric compatible)
- $t^T := {}^\sharp({}^\flat t)^T = ((t^\sharp)^T)^\flat = g^{-1}(gt)^T = (tg^{-1})^T g = \{t_j{}^i\}^i{}_j \partial_i \otimes dx^j$
- $(t^T)^T = t^T$
- $tt^T = (tt^T)^T = t^T t$
- $|t| = |t^T| = |{}^\flat t^\sharp| = |g||t^\sharp| = \frac{|{}^\flat t|}{|g|}$
- $(\alpha \otimes \beta)^T = \beta \otimes \alpha$
- $\mathrm{Tr}[\alpha \otimes \beta] = \langle \alpha, \beta \rangle$
- $(\alpha \otimes \beta)\gamma^\sharp = \langle \beta, \gamma \rangle \alpha$

- $(*\alpha) \otimes (*\beta) + \beta \otimes \alpha = \langle \alpha, \beta \rangle \, g$
- $(*\alpha) \otimes (*\alpha) + \alpha \otimes \alpha = \|\alpha\|^2 \, g$
- $\alpha \otimes (*\beta) - (*\beta) \otimes \alpha = \langle \alpha, \beta \rangle \, E$ MProved?
- $\alpha \otimes (*\alpha) - (*\alpha) \otimes \alpha = \|\alpha\|^2 \, E$

2.4 Levi-Civita-Tensor

- Levi-Civita-Symbols: $\epsilon_{ij} = \epsilon^{ij} = \begin{cases} 1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{else} \end{cases}$
- Levi-Civita-Tensor: $E_{ij} = \sqrt{|g|}\epsilon_{ij}$
- $E_{ij} = -E_{ji}$
- $\nabla E = \nabla g = \mathcal{O}$
- $E \otimes E = |g| \epsilon \otimes \epsilon$
- $E_{ij} E_{kl} = g_{ik} g_{jl} - g_{il} g_{jk}$
- $E_{ij} E_k{}^j = g_{ik}$
- $(*\alpha)_i = -E_{ij}\alpha^j = \alpha^j E_{ji}$
- $|t| E_{ij} = E^{kl} t_{ik} t_{jl}$
- $|t| = \frac{1}{2} E_{ij} E_{kl} t^{ik} t^{jl} = \frac{1}{2}\left((\mathrm{Tr} t)^2 - \mathrm{Tr} t^2\right)$
- $0 = t^2 - (\mathrm{Tr} t) t + |t| g$ ...... teilweise um factor $|g|$ falsch für vollcovariantes t

2.5 Covariant Derivative $\nabla_\bullet = g_{\bullet i} \nabla^\bullet$

- $\nabla_k f = \partial_k f$
- $\nabla_k \sqrt{|g|} = \sqrt{|g|} \Gamma_{kl}{}^l$
- $\nabla_k v^i = \partial_k v^i + \Gamma_{kl}{}^i v^l$
- $\nabla_k v_i = \partial_k v_i - \Gamma_{ki}{}^l v_l$
- $\nabla_k t^i{}_j = \partial_k t^i{}_j + \Gamma_{kl}{}^i t^l{}_j - \Gamma_{kj}{}^l t^i{}_l$
- $\nabla_k t_i{}^j = \partial_k t_i{}^j - \Gamma_{ki}{}^l t_l{}^j + \Gamma_{kl}{}^j t_i{}^l$
- $\nabla_k t^{ij} = \partial_k t^{ij} + \Gamma_{kl}{}^i t^{lj} + \Gamma_{kl}{}^j t^{il}$
- $\nabla_k t_{ij} = \partial_k t_{ij} - \Gamma_{ki}{}^l t_{lj} - \Gamma_{kj}{}^l t_{il}$

- $\nabla * \alpha = -E\nabla\alpha^\sharp \rightsquigarrow (*\alpha)_{i|k} = -E_{ij}\alpha^j{}_{|k}$
- $*\nabla_\beta \alpha = \nabla_\beta * \alpha$
- $\nabla(\alpha \otimes \beta) = \alpha \otimes \nabla\beta + (\beta \otimes \nabla\alpha)^T$, $(t^T_{ijk} = t_{jik})$,$\rightsquigarrow (\alpha_i\beta_j)_{|k} = \alpha_i\beta_{j|k} + \beta_j\alpha_{i|k}$
- $\mathrm{Rot}(f) = -\frac{1}{\sqrt{|g|}} g_{ik}\epsilon^{kl}\partial_l f dx^i = -\frac{1}{\sqrt{|g|}} g\epsilon \mathbf{d} f = -\sqrt{|g|}\epsilon\nabla f = -E \cdot \nabla f$
- $\mathrm{Div}(v) = \nabla_i v^i = \nabla^i v_i = \mathrm{Tr}(\nabla\vec{v}) = \mathrm{Tr}({}^\flat\nabla^\sharp\alpha) = \sqrt{\nabla\vec{v} : g \otimes g : \nabla\vec{v}}$
- $\mathrm{Rot}(v) = (\sqrt{|g|})^{-1}\epsilon^{ki}\nabla_k v_i = (\sqrt{|g|})^{-1}\mathrm{Tr}((\nabla\alpha)\epsilon) = \sqrt{\nabla\vec{v} : E \otimes E : \nabla\vec{v}}$
- $\mathrm{Rot}(v) = -\mathrm{Tr}(\nabla * \alpha) = \mathrm{Tr}(E\nabla v) = E_{ki}\nabla^k v^i$
- $\mathrm{Rot}(v)^2 = \nabla_j v^i(\nabla^j v_i - \nabla_j v^i) = |\nabla v| + \mathrm{Tr}(\nabla v(\nabla v)^T) - \mathrm{Div}(v)^2$
- $\mathrm{Div}(t) = \nabla^j t^i{}_j \partial_i = \nabla_j t_i{}^j dx^i = \nabla_j t^{ij}\partial_i = \nabla^j t_{ij} dx^i$

2.6 Conclusion Stuff

- $\nabla \|\alpha\|^2 = \mathbf{d}\|\alpha\|^2 = 2\alpha \cdot \nabla\alpha \rightsquigarrow \partial_k(\alpha_i\alpha^i) = 2\alpha^i\alpha_{i|k}$
- $\nabla\langle\alpha, \beta\rangle = \mathbf{d}\langle\alpha, \beta\rangle = \alpha \cdot \nabla\beta + \beta \cdot \nabla\alpha \rightsquigarrow \partial_k(\alpha_i\beta^i) = \alpha^i\beta_{i|k} + \beta^i\alpha_{i|k}$
- $*(\beta \cdot \nabla\alpha) = (\mathrm{Div}\alpha)(*\beta) - \nabla_{*\beta}\alpha = \beta \cdot \nabla * \alpha + (\mathrm{Rot}\alpha)\beta + (\mathrm{Div}\alpha)(*\beta)$
  $\rightsquigarrow -E_i{}^k\beta^j\alpha_{j|k} = (*\beta)_i\alpha_k{}^{|k} - (*\beta)^k\alpha_{i|k} = \beta^l(*\alpha)_{l|i} + E^{kj}\alpha_{j|k}\beta_i + \alpha_k{}^{|k}(*\beta)_i$
- $\beta \cdot \nabla\alpha = \nabla_{*\beta} * \alpha + (\mathrm{Div}\alpha)\beta = \nabla_\beta\alpha - (\mathrm{Rot}\alpha)(*\beta)$
  $\rightsquigarrow \beta^i\alpha_{i|k} = (*\beta)^k(*\alpha)_{i|k} + \beta_i\alpha_k{}^{|k} = \beta^i\alpha_{k|i} - E_{ik}E_{lj}\beta^i\alpha^{j|l}$

2.7 Rotation $R : T_pM \to (T_pM)'$

- $Rv := v' := \cos(\varphi)v + \sin(\varphi)(*v) = \cos(\varphi)v + \sin(\varphi)(*v^\flat)^\sharp$
- $R = \cos(\varphi)I + \frac{\sin(\varphi)}{\sqrt{|g|}}\begin{bmatrix} -g^{12} & -g^{22} \\ g^{11} & g^{12} \end{bmatrix} = R^i{}_j\partial_i \otimes dx^j$indices unten? nutze levi-cevita tensor!
- $R = \cos(\varphi)g - \sin(\varphi)E \in \mathcal{T}^0{}_2$
- $R^{-1} = R^T$
- $\langle v', w'\rangle = \langle v, w\rangle \quad \Rightarrow R \in O(T_pM)$
- $\nabla v' = R\nabla v$

2.7.1 Push-forward $R_* = R^{*T}$

- $R_* v = Rv = v'$
- $R_* \alpha = \alpha R^T = \alpha'$
- $R_* {}^\bullet t^\bullet = R {}^\bullet t^\bullet R^T = {}^\bullet t'^\bullet$
- $R_*(\bullet \otimes \bullet) = (R_* \bullet) \otimes (R_* \bullet)$