Discrete Exterior Calculus (DEC) for the Surface Navier-Stokes Equation

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Navier-Stokes Equation[1]

- Smooth Riemannian surface $\mathcal{S}$ without boundary
- Inextensible homogeneous medium
- No external forces
- Tangential surface velocity field: $\mathbf{v}(t) \in \mathrm{T}\mathcal{S}$
- Conservation of mass: $\operatorname{div} \mathbf{v} = 0$
- Conservation of linear momentum: $\rho \left(\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v}\right) = \operatorname{div} \sigma$
- Surface Cauchy stress tensor: ${}^\flat\sigma^\flat = -p\mathbf{g} + \mu \mathcal{L}_{\mathbf{v}} \mathbf{g}$
- $\Rightarrow \quad \partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\operatorname{grad} p + \frac{1}{\mathrm{Re}} \left(-\boldsymbol{\Delta}^{\mathrm{dR}} \mathbf{v} + 2\kappa \mathbf{v}\right)$
- Laplace-DeRham: $-\boldsymbol{\Delta}^{\mathrm{dR}} \mathbf{v} = \operatorname{div} \operatorname{grad} \mathbf{v} - \kappa \mathbf{v}$ (Weitzenböck)

[1] Marino Arroyo and Antonio DeSimone. „Relaxation dynamics of fluid membranes.“ In: *Physical Review E* 79 (2009), p. 031915

Vorticity Equation

$$\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\operatorname{grad} p + \frac{1}{\mathrm{Re}}\left(-\boldsymbol{\Delta}^{\mathrm{dR}}\mathbf{v} + 2\kappa\mathbf{v}\right) \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \qquad \text{(NSE)}$$

- Streamfunction: ψ with $\mathbf{v} = \operatorname{rot}\psi$
- Vorticity: $\operatorname{rot}\mathbf{v} = \Delta\psi$
- Applying rot on (NSE):

$$\partial_t \Delta\psi + \langle \operatorname{rot}\psi, \operatorname{grad}\Delta\psi\rangle = \frac{1}{\mathrm{Re}}\left(\Delta^2\psi + 2\operatorname{div}\left(\kappa\operatorname{grad}\psi\right)\right) \qquad \text{(VE)}$$

- Approaches: e. g.
  - Surface Finite Element Method[1 2] (SFEM)
  - Diffuse Interface[3] (DI)
  - Discrete Exterior Calculus (DEC)

[1] I. Nitschke, A. Voigt, and J. Wensch. „A finite element approach to incompressible two-phase flow on manifolds.“ In: *Journal of Fluid Mechanics* 708 (2012), pp. 418–438

[2] S. Reuther and A. Voigt. „The Interplay of Curvature and Vortices in Flow on Curved Surfaces.“ In: *Multiscale Modeling & Simulation* 13 (2015), pp. 632–643

[3] S. Reuther and A. Voigt. „Incompressible two-phase flows with an inextensible Newtonian fluid interface.“ In: *Journal of Computational Physics* 322 (2016), pp. 850–858

Vorticity Equation

$$\partial_t \Delta\psi + \langle \operatorname{rot}\psi, \operatorname{grad}\Delta\psi \rangle = \frac{1}{\mathrm{Re}}\left(\Delta^2\psi + 2\operatorname{div}(\kappa \operatorname{grad}\psi)\right) \tag{VE}$$

- **Drawback**: reduced solution space for genus $g(\mathcal{S}) \neq 0$ (e. g. Torus)
- Hodge decomposition: $\mathbf{v} = \operatorname{rot}\psi + \operatorname{grad}\varphi + \mathbf{v}_{\mathrm{Harm}}$
- Harmonic vector field $\mathbf{v}_{\mathrm{Harm}} \in \mathrm{T}_{\mathrm{Harm}}\mathcal{S}$: $\operatorname{div}\mathbf{v}_{\mathrm{Harm}} = \operatorname{rot}\mathbf{v}_{\mathrm{Harm}} = 0$
- For $g(\mathcal{S}) \neq 0$: $\dim_{\mathbb{R}} \mathrm{T}_{\mathrm{Harm}}\mathcal{S} \neq 0$
- $\Rightarrow$ On the Torus, $\mathbf{v} = 0$ is the only stationary solution for arbitrary Re. **Contradicting** the existence of stationary Killing vector Fields $\mathbf{v}_{\mathrm{Kill}} \neq 0$, where $\mathcal{L}_{\mathbf{v}_{\mathrm{Kill}}}\mathbf{g} = 0$.

Navier-Stokes Equation

$$\partial_t \mathbf{v} + \nabla_{\mathbf{v}} \mathbf{v} = -\operatorname{grad} p + \frac{1}{\mathrm{Re}} \left(-\mathbf{\Delta}^{\mathrm{dR}} \mathbf{v} + 2\kappa \mathbf{v} \right) \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \qquad \text{(NSE)}$$

- Approaches:
  - Vector Spherical Harmonics[1] (VSH)
    - Needs eigen function of $\Delta \rightsquigarrow$ difficult for arbitrary surfaces
  - SFEM[1] of coordinate function on the embedding space $\mathbb{R}^3$:
    - Huge amount of assembling effort
    - e. g. for $I, J, K \in \{x, y, z\}$: $\int_S \left\langle \operatorname{grad} \tilde{\mathbf{v}}, \operatorname{grad} \tilde{\Psi} \right\rangle \mu =$ $\int_S \Pi^I{}_J \tilde{v}_{I:K} \tilde{\Psi}^{J:K} + \nu_J B^K{}_I \tilde{v}^I{}_{:K} \tilde{\Psi}^J + \nu_I B^K{}_J \tilde{v}^I \tilde{\Psi}^J{}_{:K} + (\mathcal{H}^2 - 2\kappa)\, \nu_I \nu_J \tilde{v}^I \tilde{\Psi}^J \mu$
  - Special FE-Spaces
    - e. g. Brezzi-Douglas-Marini or Raviart-Thomas elements[2]
  - Discrete Exterior Calculus[1,3] (DEC)

[1] M. Nestler et al. „Orientational order on surfaces - the coupling of topology, geometry and dynamics.“ In: *arXiv:1608.01343* (2016)

[2] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. „Finite element exterior calculus, homological techniques, and applications.“ In: *Acta Numerica* 15 (2006), pp. 1–155

[3] A. N. Hirani. „Discrete Exterior Calculus.“ PhD thesis. Pasadena, CA, USA: California Institute of Technology, 2003

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Navier-Stokes Equation - Exterior Calculus Description

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Navier-Stokes Equation - Exterior Calculus Description

$$\partial_t \mathbf{u} + \frac{1}{2}\mathbf{d}\left\|\mathbf{u}\right\|^2 + (*\mathbf{du})(*\mathbf{u}) = -\mathbf{d}p + \frac{1}{\mathrm{Re}}\left(*\mathbf{d}*\mathbf{du} + 2\kappa\mathbf{u}\right) \quad \text{and} \quad *\mathbf{d}*\mathbf{u} = 0 \tag{NSE}$$

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- $-\mathbf{\Delta}^{\mathrm{dR}}\mathbf{u} = *\mathbf{d}*\mathbf{du} + \mathbf{d}*\mathbf{d}*\mathbf{u} = *\mathbf{d}*\mathbf{du}$
- $\nabla_{\mathbf{v}}\mathbf{u} = \frac{1}{2}\mathbf{d}\left\|\mathbf{u}\right\|^2 + (*\mathbf{du})(*\mathbf{u})$

Time-discrete equations

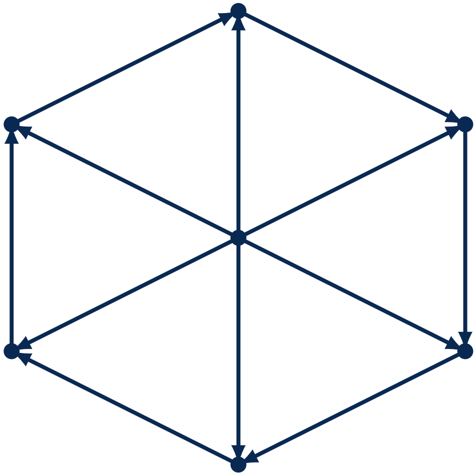
- Solution at time t_k: $\mathbf{u}_k \in \Lambda^1\mathcal{S}$
- Initial condition for $k = 0$: $\mathbf{u}_0 := \mathbf{u}(t = 0)$
- $\nabla_{\mathbf{u}^\sharp_{k+1}} \mathbf{u}_{k+1} = \nabla_{\mathbf{u}^\sharp_{k}} \mathbf{u}_{k+1} + \nabla_{\mathbf{u}^\sharp_{k+1}} \mathbf{u}_{k} - \nabla_{\mathbf{u}^\sharp_{k}} \mathbf{u}_{k}$
  $= \mathbf{d}\left(\langle \mathbf{u}_{k+1}, \mathbf{u}_k\rangle - \frac{1}{2}\|\mathbf{u}_k\|^2\right) + (*\mathbf{d}\mathbf{u}_{k+1} - *\mathbf{d}\mathbf{u}_k)(*\mathbf{u}_k) - (*\mathbf{d}**\mathbf{u}_k)(*\mathbf{u}_{k+1}) \in \Lambda^1\mathcal{S}$
- Generalized pressure: $q_{k+1} := p_{k+1} + \langle \mathbf{u}_{k+1}, \mathbf{u}_k\rangle - \frac{1}{2}\|\mathbf{u}_k\|^2 \in \Lambda^0\mathcal{S}$

$$
\begin{aligned}
\frac{1}{\tau_k}\mathbf{u}_{k+1} + \mathbf{d}q_{k+1} + (*\mathbf{d}\mathbf{u}_{k+1})(*\mathbf{u}_k) - (*\mathbf{d}*)(*\mathbf{u}_k)(*\mathbf{u}_{k+1}) & \\
-\frac{1}{\mathrm{Re}}\left((*\mathbf{d}*\mathbf{d})\mathbf{u}_{k+1} + 2\kappa\mathbf{u}_{k+1}\right) &= \frac{1}{\tau_k}\mathbf{u}_k + (*\mathbf{d}\mathbf{u}_k)(*\mathbf{u}_k) \\
\langle \mathbf{u}_{k+1}, \mathbf{u}_k\rangle + p_{k+1} - q_{k+1} &= \frac{1}{2}\|\mathbf{u}_k\|^2 \\
\mathbf{d}\mathbf{u}_{k+1} &= 0
\end{aligned}
\tag{TDNSE}
$$

Content

Surface Discretization

- Simplicial Complex:
  $\mathcal{K} = \mathcal{V} \sqcup \mathcal{E} \sqcup \mathcal{T}$



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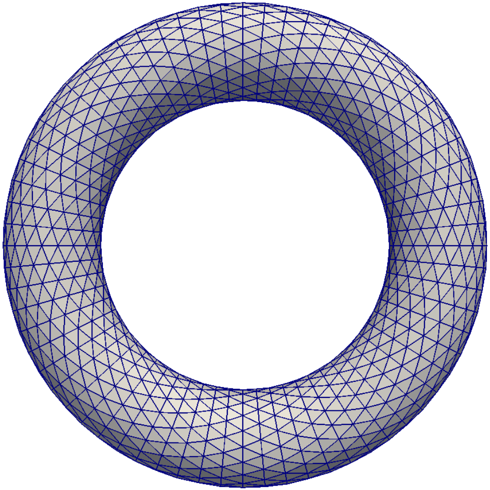

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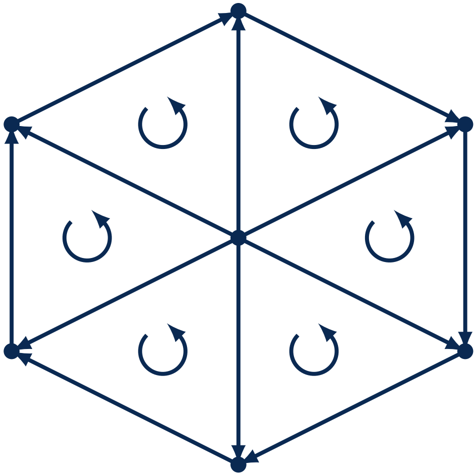
- well-centered, orientable



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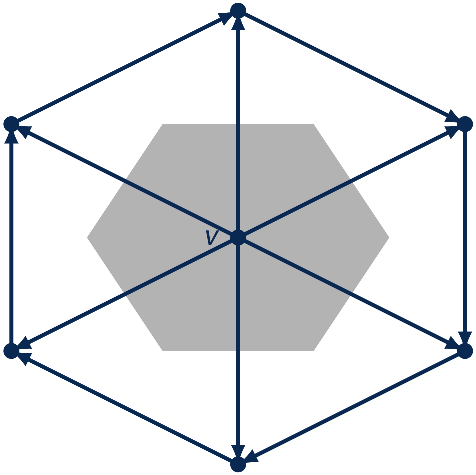

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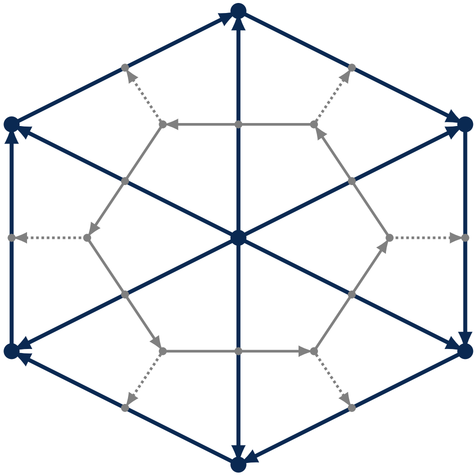

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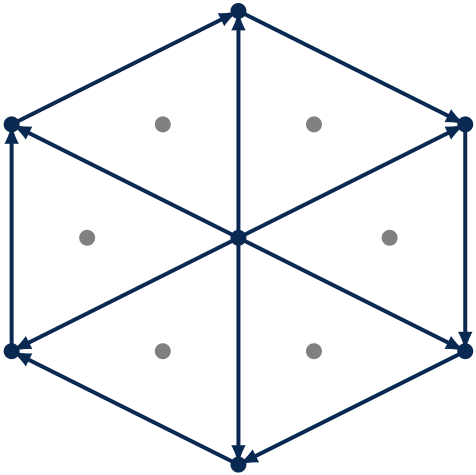

Surface Discretization

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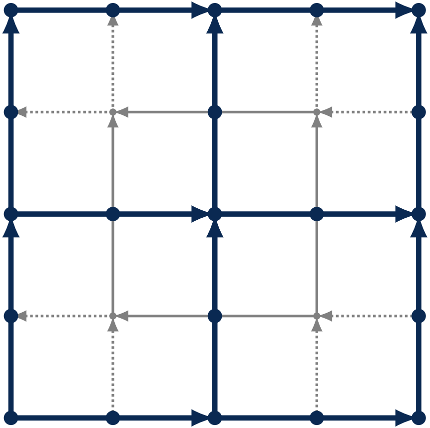
- Not restricted to triangle faces



Degrees Of Freedom (DOFs)

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- 1-form $u_h \in \Lambda_h^1\mathcal{K}$: $\quad u_h(e) = \int_{\pi(e)} \mathbf{u}$
  - $u_h(e) \approx \mathbf{u}(\mathbf{e}) = \langle \mathbf{v}, \mathbf{e} \rangle$ on an intermediate point $\xi \in \pi(e) \subset \mathcal{S}$.

Discrete Hodge Operator

- $p_h \in \Lambda_h^0\mathcal{K}, \quad u_h \in \Lambda_h^1\mathcal{K}, \quad \omega_h \in \Lambda_h^2\mathcal{K}$

[1] M. S. Mohamed, A. N. Hirani, and R. Samtaney. „Comparison of discrete Hodge star operators for surfaces.“ In: *Computer-Aided Design* (2016). DOI: 10.1016/j.cad.2016.05.002

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- e. g. $(*\mathbf{u})_h(e) \approx \circledast u_h(e)$

$$:= \frac{1}{4}\sum_{T>e}\sum_{\substack{\tilde{e}<T\\ \tilde{e}\neq e}} \frac{s_{e\tilde{e}}}{\sqrt{|e|^2\,|\tilde{e}|^2 - (\mathbf{e}\cdot\tilde{\mathbf{e}})^2}}\Big((\mathbf{e}\cdot\tilde{\mathbf{e}})\,u_h(e) - |e|^2\,u_h(\tilde{e})\Big)$$

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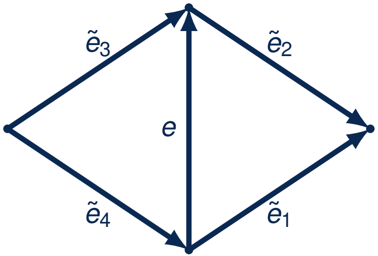
Discrete Hodge Operator



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:= \frac{1}{4}\sum_{T>e}\sum_{\substack{\tilde{e}<T\\ \tilde{e}\neq e}} \frac{s_{e\tilde{e}}}{\sqrt{|e|^2\,|\tilde{e}|^2 - (\mathbf{e}\cdot\tilde{\mathbf{e}})^2}}\Big((\mathbf{e}\cdot\tilde{\mathbf{e}})\,u_h(e) - |e|^2\,u_h(\tilde{e})\Big)
$$

[1] M. S. Mohamed, A. N. Hirani, and R. Samtaney. „Comparison of discrete Hodge star operators for surfaces.“ In: *Computer-Aided Design* (2016). DOI: 10.1016/j.cad.2016.05.002

Operator Discretizations[1]

$$\begin{aligned}\frac{1}{\tau_k}\mathbf{u}_{k+1} + \mathbf{d}q_{k+1} + (*\mathbf{d}\mathbf{u}_{k+1})(*\mathbf{u}_k) - (*\mathbf{d}*)(*\mathbf{u}_k)(*\mathbf{u}_{k+1})& \\ -\frac{1}{\mathrm{Re}}\left(({*\mathbf{d}*\mathbf{d}})\mathbf{u}_{k+1} + 2\kappa\mathbf{u}_{k+1}\right) &= \frac{1}{\tau_k}\mathbf{u}_k + (*\mathbf{d}\mathbf{u}_k)(*\mathbf{u}_k) \\ \langle \mathbf{u}_{k+1}, \mathbf{u}_k\rangle + p_{k+1} - q_{k+1} &= \frac{1}{2}\|\mathbf{u}_k\|^2 \\ *\mathbf{d}*\mathbf{u}_{k+1} &= 0\end{aligned} \tag{TDNSE}$$

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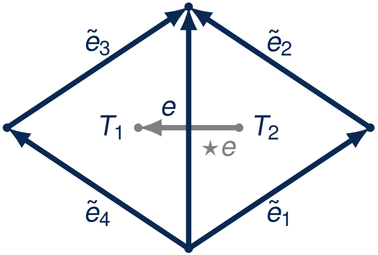
[1] I. Nitschke, S. Reuther, and A. Voigt. „Discrete exterior calculus (DEC) for the surface Navier-Stokes equation.“ In: *ArXiv e-prints* (Nov. 2016). arXiv: `1611.04392 [math.NA]`

Operator Discretizations[1]



$$(*\mathbf{d} * \mathbf{d}\mathbf{u}_{k+1})_h(e) \approx -\frac{|e|}{|\star e|} \sum_{T \succ e} \frac{s_{T,e}}{|T|} \sum_{\tilde{e} \prec T} s_{T,\tilde{e}} \, (\mathbf{u}_{k+1})_h(\tilde{e}) =: (*\mathbf{d} * \mathbf{d})_h(\mathbf{u}_{k+1})_h(e)$$

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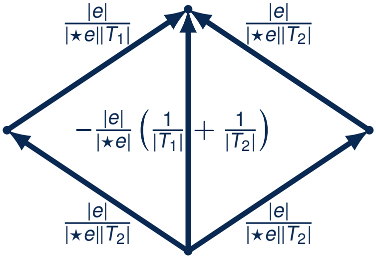

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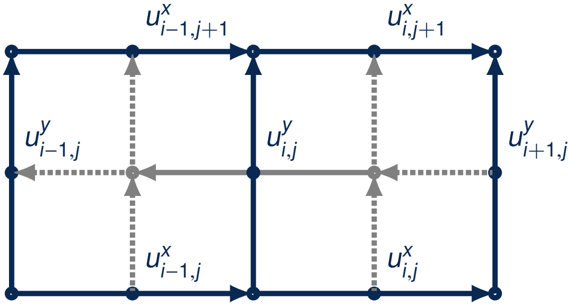

$$(*\mathbf{d} * \mathbf{d}u_{k+1})_h(e) \approx -\frac{|e|}{|\star e|} \sum_{T>e} \frac{s_{T,e}}{|T|} \sum_{\tilde{e}<T} s_{T,\tilde{e}} \left(\mathbf{u}_{k+1}\right)_h(\tilde{e}) =: (*\mathbf{d} * \mathbf{d})_h (\mathbf{u}_{k+1})_h(e)$$

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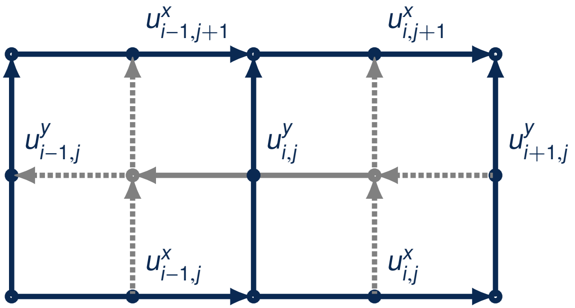

$$(\operatorname{rot}\operatorname{rot} u)^y_{i,j} = \frac{1}{h^2}\left(-2u^y_{i,j} + u^y_{i+1,j} + u^y_{i-1,j} - u^x_{i,j} + u^x_{i,j+1} - u^x_{i-1,j+1} + u^x_{i-1,j}\right) + \mathcal{O}(h^2)$$

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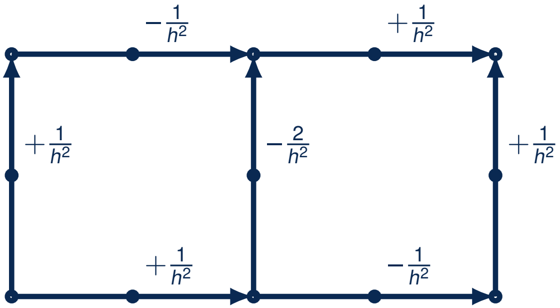

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$$\begin{aligned}\frac{1}{\tau_k}\mathbf{u}_{k+1} + \mathbf{d}q_{k+1} + (*\mathbf{d}\mathbf{u}_{k+1})(*\mathbf{u}_k) - (*\mathbf{d}*)(*\mathbf{u}_k)(*\mathbf{u}_{k+1}) & \\ -\frac{1}{\mathrm{Re}}\left((*\mathbf{d}*\mathbf{d})\mathbf{u}_{k+1} + 2\kappa\mathbf{u}_{k+1}\right) &= \frac{1}{\tau_k}\mathbf{u}_k + (*\mathbf{d}\mathbf{u}_k)(*\mathbf{u}_k) \\ \langle\mathbf{u}_{k+1},\mathbf{u}_k\rangle + p_{k+1} - q_{k+1} &= \frac{1}{2}\|\mathbf{u}_k\|^2 \\ *\mathbf{d}*\mathbf{u}_{k+1} &= 0\end{aligned} \tag{TDNSE}$$

$$\left((*\mathbf{d}\mathbf{u}_{k+1})(*\mathbf{u}_k)\right)_h(e) \approx \frac{(*\mathbf{u}_k)_h(e)}{\sum_{T\succ e}|T|}\sum_{T\succ e}\sum_{\tilde{e}\prec T} s_{T,\tilde{e}}(\mathbf{u}_{k+1})_h(\tilde{e}) =: \left((*\mathbf{u}_k)(*\mathbf{d})\right)_h(\mathbf{u}_{k+1})_h$$

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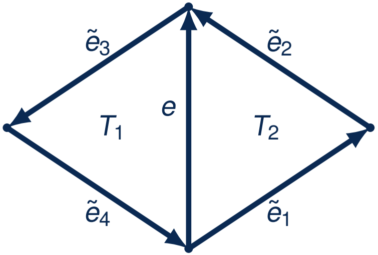

$$\left((*\mathbf{d}\mathbf{u}_{k+1})(*\mathbf{u}_k)\right)_h(e) \approx \frac{(*\mathbf{u}_k)_h(e)}{\sum_{T\succ e}|T|}\sum_{T\succ e}\sum_{\tilde{e}\prec T} s_{T,\tilde{e}}(\mathbf{u}_{k+1})_h(\tilde{e}) =: \left((*\mathbf{u}_k)(*\mathbf{d})\right)_h(\mathbf{u}_{k+1})_h$$

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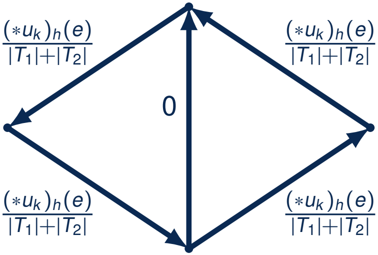

$$\left((*\mathbf{du}_{k+1})(*\mathbf{u}_k)\right)_h(e) \approx \frac{(*\mathbf{u}_k)_h(e)}{\sum_{T>e}|T|}\sum_{T>e}\sum_{\tilde{e}<T} s_{T,\tilde{e}}(\mathbf{u}_{k+1})_h(\tilde{e}) =: \left((*\mathbf{u}_k)(*\mathbf{d})\right)_h(\mathbf{u}_{k+1})_h$$

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$$\begin{aligned}\frac{1}{\tau_k}\mathbf{u}_{k+1} + \mathbf{d}q_{k+1} + (*\mathbf{d}\mathbf{u}_{k+1})(*\mathbf{u}_k) - (*\mathbf{d}*)(*\mathbf{u}_k)(*\mathbf{u}_{k+1}) \\ -\frac{1}{\mathrm{Re}}\left((*\mathbf{d}*\mathbf{d})\mathbf{u}_{k+1} + 2\kappa\mathbf{u}_{k+1}\right) &= \frac{1}{\tau_k}\mathbf{u}_k + (*\mathbf{d}\mathbf{u}_k)(*\mathbf{u}_k) \\ \langle \mathbf{u}_{k+1}, \mathbf{u}_k\rangle + p_{k+1} - q_{k+1} &= \frac{1}{2}\|\mathbf{u}_k\|^2 \\ *\mathbf{d}*\mathbf{u}_{k+1} &= 0\end{aligned} \tag{TDNSE}$$

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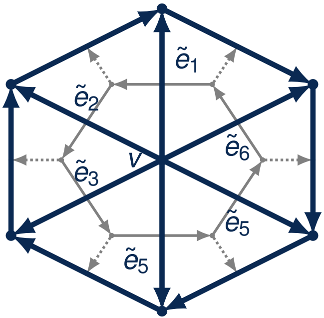

$$(*\mathbf{d} * \mathbf{u}_{k+1})_h(v) \approx -\frac{1}{|\star v|} \sum_{\tilde{e} \succ v} s_{v,\tilde{e}} \frac{|\star \tilde{e}|}{|\tilde{e}|} (\mathbf{u}_{k+1})_h(\tilde{e}) =: (*\mathbf{d}*)_h (\mathbf{u}_{k+1})_h(v)$$

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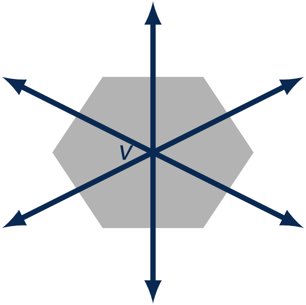

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\begin{aligned}
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\tag{TDNSE}
$$

$$
((*\mathbf{d}*)(*\mathbf{u}_k)(*\mathbf{u}_{k+1}))_h(e) \approx -\frac{1}{2}\left(\sum_{v<e}\frac{1}{|\star v|}\sum_{\tilde{e}>v} s_{v,\tilde{e}}\frac{|\star\tilde{e}|}{|\tilde{e}|}(*\mathbf{u}_k)_h(\tilde{e})\right)(*\mathbf{u}_{k+1})_h(e)
$$

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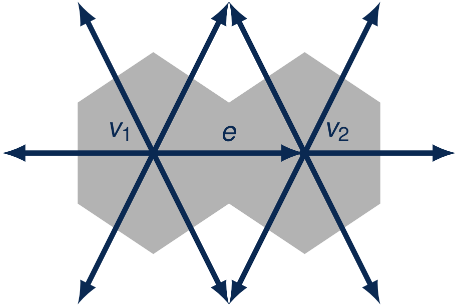

$$\left((*\mathbf{d}*)(*\mathbf{u}_k)(*\mathbf{u}_{k+1})\right)_h(e) \approx \frac{1}{2}\left(\sum_{v<e}(*\mathbf{d}*)_h(*\mathbf{u}_k)_h(v)\right)(*\mathbf{u}_{k+1})_h(e) =: \left((*\mathbf{d}*)(*\mathbf{u}_k)\right)_h(*\mathbf{u}_{k+1})_h$$

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\mathbf{d}\mathbf{u}_{k+1} &= 0
\end{aligned}
\tag{TDNSE}
$$

$$
\langle \mathbf{u}_{k+1}, \mathbf{u}_k \rangle_h (v) \approx \frac{1}{4|\star v|}\sum_{\tilde{e}\succ v}\frac{|\star\tilde{e}|}{|\tilde{e}|}\left((\mathbf{u}_k)_h(\tilde{e})(\mathbf{u}_{k+1})_h(\tilde{e}) + (*\mathbf{u}_k)_h(\tilde{e})(*\mathbf{u}_{k+1})_h(\tilde{e})\right)
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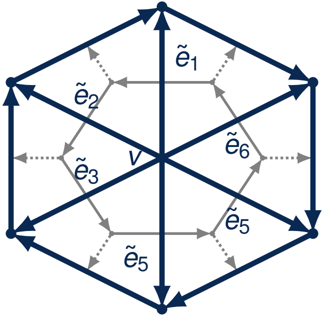


$$\langle \mathbf{u}_{k+1}, \mathbf{u}_k \rangle_h (v) \approx \frac{1}{4\,|\star v|} \sum_{\tilde{e} \succ v} \frac{|\star \tilde{e}|}{|\tilde{e}|} \left((\mathbf{u}_k)_h(\tilde{e})(\mathbf{u}_{k+1})_h(\tilde{e}) + (*\mathbf{u}_k)_h(\tilde{e})\,(*\mathbf{u}_{k+1})_h(\tilde{e}) \right)$$

[1] I. Nitschke, S. Reuther, and A. Voigt. „Discrete exterior calculus (DEC) for the surface Navier-Stokes equation.“ In: *ArXiv e-prints* (Nov. 2016). arXiv: 1611.04392 [math.NA]

Fully-discrete equations

- For $k = 0, 1, \ldots$ and given initial values $(\mathbf{u}_0)_h$ and $(*\mathbf{u}_0)_h$, find $(\mathbf{u}_{k+1})_h, (*\mathbf{u}_{k+1})_h \in \Lambda_h^1\mathcal{K}$ and $(p_{k+1})_h, (q_{k+1})_h \in \Lambda_h^0\mathcal{K}$ s.t.

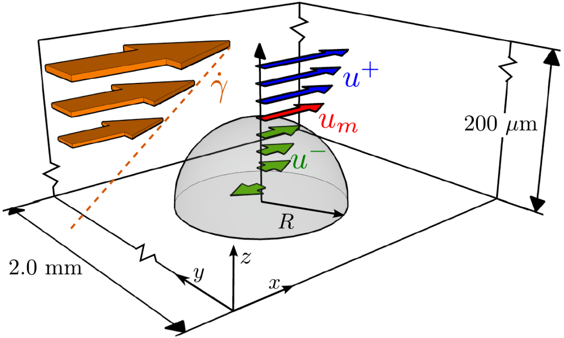
$$
\begin{aligned}
\circledast(\mathbf{u}_{k+1})_h - (*\mathbf{u}_{k+1})_h &= 0 && \text{in } \mathcal{E}\\
\frac{1}{\tau_k}(\mathbf{u}_{k+1})_h + (\mathbf{d}q_{k+1})_h + ((*\mathbf{u}_k)(*\mathbf{d}))_h(\mathbf{u}_{k+1})_h - ((*\mathbf{d}*)(*\mathbf{u}_k))_h(*\mathbf{u}_{k+1})_h & && \\
-\frac{1}{\mathrm{Re}}((*\mathbf{d}*\mathbf{d})_h(\mathbf{u}_{k+1})_h + 2\kappa(\mathbf{u}_{k+1})_h) &= \frac{1}{\tau_k}(\mathbf{u}_k)_h + ((*\mathbf{u}_k)(*\mathbf{d}))_h(\mathbf{u}_k)_h && \text{in } \mathcal{E}\\
\langle\cdot,\mathbf{u}_k\rangle_h\left[(\mathbf{u}_{k+1})_h, (*\mathbf{u}_{k+1})_h\right] + (p_{k+1})_h - (q_{k+1})_h &= \langle\cdot,\mathbf{u}_k\rangle_h\left[(\mathbf{u}_k)_h, (*\mathbf{u}_k)_h\right] && \text{in } \mathcal{V}\\
(*\mathbf{d}*)_h(\mathbf{u}_{k+1})_h &= 0 && \text{in } \mathcal{V}
\end{aligned}
$$

(DECNSE)

Content

Experimental Setup

- Experiment and Model[1]
- Shear flow through chamber
- Qualitative and quantitative results



[1] Honerkamp-Smith et al. „Membrane Viscosity Determined from Shear-Driven Flow in Giant Vesicles.“ In: *Phys. Rev. Lett.* 111 (3 2013)

Content

Content

Thank you for your attention!