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Topics in Partial Differential Equations

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February 1, 2013

Contents

These lectures notes have been prepared for a graduate course given within the Warwick MASDOC Centre for Doctoral Training.

Because of the ubiquitous nature of PDE based mathematical models in biology, advanced materials, finance, physics and engineering much of mathematical analysis is devoted to their study. The complexity of the models means that finding formulae for solutions is impossible in most practical situations. Issues for mathematical analysis include: the formulation of well posed problems in appropriate function spaces, regularity and qualitative information about the solution.

The purpose of this module is to provide a wide ranging introduction to topics in the modern analysis of PDEs selected for relevance to applications (geometry, material science, theoretical biology, finance , continuum mechanics).Although the main emphasis is on applied analysis the notes make connections to numerical analysis via the use of discretisation and iteration to prove existence results and to applications by the use of examples. The tone is uneven with respect to the provision of detail and background.

The module follows on from the A1 MASDOC module

Analysis for Linear PDEs, see http://www.warwick.ac.uk/ masdh/

Useful texts are [23, 43, 34, 12, 16, 48, 29, 49, 54, 3, 1, 40, 45].

Chapter 1

Abstract variational problems

1.1 Variational Problems

Let V be a normed real vector space with norm $\|\cdot\|_V$. Let $\{u_m\}_{m=1}^{\infty} \subset V$ be a sequence.

If the sequence satisfies

$$\lim_{n,m\to\infty} \|u_m - u_n\|_V = 0,$$

then the sequence is said to be *Cauchy*.

If V is such that for any Cauchy sequence $\{u_m\}_{m=1}^{\infty}$ there exists $u \in V$ satisfying

$$\lim_{n\to\infty} \|u_n - u\|_V = 0 \text{ (i.e, } \lim_{n\to\infty} u_n = u),$$

then V is said to be a *complete normed vector space* or *Banach space*.

Let $l(\cdot) : V \to \mathbb{R}$ be a linear functional or linear form[1], i.e,

$$l(\alpha v + \beta w) = \alpha l(v) + \beta l(w), \ \forall \alpha, \beta \in \mathbb{R}, \ v, w \in V,$$

then $l(\cdot)$ is said to be *bounded* if

$$\exists c_l \in \mathbb{R} \text{ s.t } |l(v)| \leq c_l \|v\|_V, \ \forall v \in V.$$

Remark. We also say that a linear functional $l(\cdot)$ is *continuous* when it is bounded.

Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$. Then $a(\cdot,\cdot)$ is *bilinear* if for $\forall \alpha, \beta \in \mathbb{R}, \ u, v, w \in V$

$$a(\alpha v + \beta w, u) = \alpha a(v, u) + \beta a(w, u),$$
$$a(u, \alpha v + \beta w) = \alpha a(u, v) + \beta a(u, w).$$

[1] Note that if $l(\cdot) : V \to \mathbb{R}$ is linear, taking $\alpha = \beta = 0$ yields $l(0) = 0$.

We say the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is *symmetric* if

$$a(u,w) = a(w,u) \ \forall u, w \in V.$$

Let $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{R}$ be a symmetric bilinear form on V satisfying

$$\begin{aligned} &\langle v, v\rangle \geq 0 \ \forall v \in V, \\ &\langle v, v\rangle = 0 \Leftrightarrow v = 0, \end{aligned}$$

then $\langle \cdot,\cdot \rangle$ is said to be an *inner product* on V.

Lemma 1.1.1 (Cauchy-Schwarz inequality)**.** *Let $\langle \cdot,\cdot \rangle$ be an inner product on V. Then*

$$|\langle u, v\rangle| \leq \sqrt{\langle u,u\rangle}\sqrt{\langle v,v\rangle} \ \forall u, v \in V.$$

Proof.

$$\begin{aligned} 0 &\leq \langle u + \lambda v, u + \lambda v\rangle \\ &= \langle u, u + \lambda v\rangle + \lambda\langle v, u + \lambda v\rangle \text{ (by linearity w.r.t 1st variable)} \\ &= \langle u, u\rangle + \lambda\langle u, v\rangle + \lambda\langle v, u\rangle + \lambda^2\langle v, v\rangle \text{ (by linearity w.r.t 2nd variable)} \\ &= \langle u, u\rangle + 2\lambda\langle u, v\rangle + \lambda^2\langle v, v\rangle \text{ (by symmetry)} \\ &:= Q(\lambda). \end{aligned}$$

If $v = 0$, the required inequality is obvious since the both sides are zero. Suppose $v \neq 0$. Set

$$a = \langle v, v\rangle, b = \langle u, v\rangle, c = \langle u, u\rangle.$$

Noting

$$Q(\lambda) = a\lambda^2 + 2b\lambda + c \geq 0, \ \ a > 0,$$

we require a non-positive discriminant $b^2 \leq ac$, which is

$$\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle.$$

□

Given a vector space V with an inner product $\langle \cdot,\cdot \rangle$ we can set

$$\|v\|_V := \sqrt{\langle v, v\rangle} \ \forall v \in V$$

to be a norm.

Lemma 1.1.2. *$\|\cdot\|_V$ defines a norm on V.*

Proof. i) $\|v\|_V \geq 0 \ \forall v \in V$.

ii) $\|v\|_V = 0 \Leftrightarrow v = 0$.

iii) $\|\lambda v\|_V = \sqrt{\langle \lambda v, \lambda v\rangle} = \sqrt{\lambda^2 \langle v, v\rangle} = |\lambda| \|v\|_V$.

iv) Triangle inequality;

$$\begin{aligned}
\|u+v\|_V &= \sqrt{\langle u+v, u+v\rangle} \\
&= \sqrt{\langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle} \\
&\leq \sqrt{\|u\|_V^2 + 2\|u\|_V\|v\|_V + \|v\|_V^2} \\
&\qquad \text{(by Cauchy-Schwarz inequality)} \\
&= \sqrt{(\|u\|_V + \|v\|_V)^2} \\
&= \|u\|_V + \|v\|_V.
\end{aligned}$$

□

Suppose V is an inner product space with $\langle v, v\rangle$ $\forall v \in V$ and suppose V is a Banach space with the norm $\|v\|_V = \sqrt{\langle v, v\rangle}$ $\forall v \in V$. Then V is a *Hilbert space.*

Remark. We say that the normed vector space V is *complete* if all Cauchy sequences converge in V, i.e, a Banach space is a complete normed vector space, a Hilbert space is a complete inner product space.

Let V^* be the space of all bounded linear functionals on V. Then V^* is a linear space. Indeed, for $l_1, l_2 \in V^*$ define

$$l(v) := \alpha l_1(v) + \beta l_2(v) \ \forall v \in V, \alpha, \beta \in \mathbb{R}.$$

Then l is a bounded linear functional on V, thus $l \in V^*$. This implies that V^* is a linear space.

For V^* we use the following norm;

$$\|l\|_{V^*} := \sup_{v \in V, v \neq 0} \frac{l(v)}{\|v\|_V} \ \forall v \in V.$$

Clearly

$$\begin{aligned}
\|l\|_{V^*} &\geq \frac{l(v)}{\|v\|_V}, \\
&\Rightarrow |l(v)| \leq \|l\|_{V^*}\|v\|_V.
\end{aligned}$$

Hence $\|l\|_{V^*}$ is the least upper bound of all c_l such that $|l(v)| \leq c_l\|v\|_V$ $\forall v \in V$.

Consider $\{v_m\}_{m=0}^{\infty} \subset V$ satisfying $\lim_{m\to\infty} v_m = v \in V$ (i.e. $\lim_{m\to\infty} \|v_m - v\|_V = 0$). Then,

$$|l(v) - l(v_m)| = |l(v - v_m)| \leq \|l\|_{V^*}\|v - v_m\|_V \to 0 \text{ as } m \to \infty.$$

Thus, a bounded linear functional is a continuous linear functional.

The Abstract Variational Problem

Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\|v\|_V := \sqrt{\langle v, v \rangle}\ \forall v \in V$. Let $l(\cdot) : V \to \mathbb{R}$ be a bounded linear functional. Let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bilinear form.

We consider the following variational problem.

(**P**) Find $u \in V$ such that $a(u, v) = l(v)\ \forall v \in V$.

Theorem 1.1.3 (Lax-Milgram)**.** *Let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bilinear form such that*

1) $a(\cdot, \cdot)$ is bounded, i.e.,

$$\exists \gamma > 0 \ s.t.\ |a(v, w)| \leq \gamma \|v\|_V \|w\|_V \ \forall v, w \in V.$$

2) $a(\cdot, \cdot)$ is coercive (= V-elliptic), i.e.,

$$\exists \alpha > 0 \ s.t.\ a(v, v) \geq \alpha \|v\|_V^2 \ \forall v \in V.$$

Let $l(\cdot) : V \to \mathbb{R}$ be a bounded linear functional, i.e.,

$$\exists M > 0 \ s.t.\ |l(v)| \leq M \|v\|_V \ \forall v \in V.$$

*Then the variational problem (**P**) has a unique solution.*

Proof. (Uniqueness)

Suppose u_1, u_2 are two solutions of (**P**).

$$\begin{aligned} \Rightarrow a(u_1, v) &= l(v), \\ a(u_2, v) &= l(v)\ \forall v \in V. \end{aligned}$$

Subtraction and linearity give

$$a(u_2, v) - a(u_1, v) = a(u_2 - u_1, v) = 0\ \forall v \in V.$$

In particular, choose $v = u_2 - u_1$, then we see

$$a(u_2 - u_1, u_2 - u_1) = 0.$$

By coercivety

$$\begin{aligned} & 0 \geq \alpha \|u_2 - u_1\|_V^2, \\ \Rightarrow\ & 0 \geq \|u_2 - u_1\|_V, \\ \Rightarrow\ & u_1 = u_2. \end{aligned}$$

(Existence)

This is proved by a direct consequence of the Riesz representation theorem (see Functional Analysis). Since we will now fix $v \in V$, then $a(v, \cdot) : V \to \mathbb{R}$ is a bounded linear functional. The Riesz representation theorem says that there uniquely exists a point (which we will call Av) in V such that

$$a(v, w) = \langle Av, w \rangle \ \forall w \in V.$$

Clearly this defines a map $A : V \to V$, which is linear. Furthermore it holds that

$$|\langle Av, w \rangle| = |a(v, w)| \le \gamma \|v\|_V \|w\|_V \ \forall v, w \in V.$$

Now, take $w = Av$, then

$$\begin{aligned} \text{LHS} = \|Av\|_V^2 &\le \gamma \|v\|_V \|Av\|_V = \text{RHS} \\ \Rightarrow \|Av\|_V &\le \gamma \|v\|_V. \end{aligned}$$

This implies that A is a bounded linear operator

$$A : V \to V \text{ with } |A| \le \gamma,$$

where

$$|A| := \sup_{v \in V, v \neq 0} \frac{\|Av\|_V}{\|v\|} \le \sup_{v \in V, v \neq 0} \frac{\gamma \|v\|_V}{\|v\|_V} = \gamma.$$

It follows that

$$\begin{aligned} & a(u, v) = l(v) \ \forall v \in V \\ \Leftrightarrow \ & \langle Au, v \rangle = \langle L, v \rangle \ \forall v \in V \\ \Leftrightarrow \ & \langle Au - L, v \rangle = 0 \ \forall v \in V \\ \Leftrightarrow \ & Au = L : \text{ a linear operator equation }. \end{aligned}$$

Now let u be a solution of $(\mathbf{P})$.
$\Leftrightarrow$ u solves $Au = L$.
$\Leftrightarrow$ u solves $u = u - \rho(Au - L), \ \rho > 0$.

Consider

$$\begin{cases} u_{k+1} = u_k - \rho(Au_k - L), \\ u_0\text{: given.} \end{cases}$$

Then we have an infinite sequence $\{u_k\}_{k=0}^{\infty}$.

Let w_ρ be a map defined by $w_\rho(v) = v - \rho(Av - L)$. Then we see $u_{k+1} = w_\rho(u_k)$ and

$$\begin{aligned} \|w_\rho(v_2) - & w_\rho(v_1)\|_V^2 \\ &= \|v_2 - v_1\|_V^2 - 2\rho a(v_1 - v_2, v_1 - v_2) + \rho^2 \|A(v_2 - v_1)\|_V^2. \\ \Rightarrow \|w_\rho(v_2) - w_\rho(v_1)\|_V^2 &\le (1 - 2\rho\alpha + \rho^2 |A|^2) \|v_2 - v_1\|^2. \end{aligned}$$

Choose $\rho \in (0, 2\alpha/|A|^2)$, then w_ρ is a strict contraction. Now the contraction mapping theorem for Hilbert spaces assures that w_ρ has a unique fixed point. $\square$

1.2 Remarks on the Lax-Milgram Result

1. Uniqueness: by our usual methods this follows from the linearity of $l(\cdot)$, the bilinearity of $a(\cdot,\cdot)$ and the coercivity of $a(\cdot,\cdot)$.

2. Stability estimate: we know that

$$\alpha\|u\|_V^2 \leq a(u,u) = l(u) \leq ||l||_{V^*}\|u\|_V$$

so we can deduce that the solution to our BVP satisfies

$$\|u\|_V \leq \frac{||l||_{V*}}{\alpha}\|f\|_{L^2(\Omega)}. \tag{1.1}$$

3. Continuity with respect to $l(\cdot)$. Consider the two problems

$$\begin{aligned} u_1 \in V \ s.t. \quad & a(u_1,v) = l_1(v) \quad && \forall\, v \in V \\ u_2 \in V \ s.t. \quad & a(u_2,v) = l_2(v) \quad && \forall\, v \in V. \end{aligned}$$

Then

$$a(u_1 - u_2, v) = l_1(v) - l_2(v) = \hat{l}(v). \tag{1.2}$$

Choosing $v = u_1 - u_2$:

$$\begin{aligned} c_0\|u_1 - u_2\|_V^2 \leq \hat{l}(u_1 - u_2) \leq \|l_1 - l_2\|_{V^*}\|u_1 - u_2\| \\ \Rightarrow \|u_1 - u_2\|_V \leq \frac{\|l_1 - l_2\|_{V^*}}{\alpha} \end{aligned}$$

4. If l is the zero element of V^* (i.e. $l(v) = 0\ \forall\, v \in V$) then $0 = a(u,u) \Rightarrow \|u\|_V = 0$ by coercivity and $u = 0$.

1.3 Calculus of Variations

Suppose $a(\cdot,\cdot)$ is also symmetric, i.e,

$$a(u,v) = a(v,u)\ \forall u,v \in V.$$

Define $J(\cdot) : V \to \mathbb{R}$ by

$$J(v) = \frac{1}{2}a(v,v) - l(v)\ \forall v \in V.$$

We say that $J(\cdot)$ is a *quadratic functional*.

Now consider the minimization problem;

(M) Find $u \in V$ such that

$$J(u) \leq J(v)\ \forall v \in V.$$

Theorem 1.3.1. *The problem (**P**) is equivalent to the problem (**M**).*

Proof. ((**P**) $\Rightarrow$(**M**)): Let u be a solution of (**P**).

$$\begin{aligned}J(v) &= J(u + (v-u))\\ &= \frac{1}{2}a(u,u) - l(u) + (a(u,v-u) - l(v-u)) + \frac{1}{2}a(v-u,v-u)\\ &= J(u) + (a(u,v-u) - l(v-u)) + \frac{1}{2}a(v-u,v-u).\end{aligned}$$

Since u solves (**P**),

$$a(u,v-u) - l(v-u) = 0 \ \forall v \in V.$$

Therefore, noting that $a(\cdot,\cdot)$ is coercive,

$$\begin{aligned}J(v) &= J(u) + (a(u,v-u) - l(v-u)) + \frac{1}{2}a(v-u,v-u)\\ &= J(u) + \frac{1}{2}a(v-u,v-u)\\ &\geq J(u) + \frac{\alpha}{2}\|v-u\|_V^2\\ &\geq J(u),\end{aligned}$$

for all $v \in V$. This means that u solves (**M**).

((**M**) $\Rightarrow$(**P**)): Let u denote a solution of (**M**). Since we have $J(u) \leq J(v)$ for all $v \in V$, for all $t \in \mathbb{R}$ we see

$$J(u) \leq J(u+tv). \tag{1.3}$$

Let us fix v and define

$$G(t) := J(u+tv).$$

Calculations yield

$$\begin{aligned}G(t) &= \frac{1}{2}a(u+tv,u+tv) - l(u+tv)\\ &= \frac{1}{2}a(u,u) - l(u) + t(a(u,v) - l(v)) + \frac{1}{2}t^2(a(v,v)),\end{aligned}$$

which means that G is quadratic in t.

Then, by (1.3)

$$G(0) = J(u) \leq J(u+tv) = G(t)$$

for all $t \in \mathbb{R}$. Note that $G(t)$ has a critical point (a minimum) at $t = 0$. Thus, $G'(0) = 0$. Since

$$G'(t) = a(u,v) - l(v) + ta(v,v),$$

we obtain

$$0 = a(u,v) - l(v),$$

which implies

$$a(u, v) = l(v) \ \forall v \in V.$$

Hence, u solves ($\mathbf{P}$). $\square$

Note that we say ($\mathbf{P}$) is the variational *Euler-Lagrange* equation for ($\mathbf{M}$).

1.4 Abstract Galerkin or Finite Element Method

1.4.1 Abstract FEM

Let h be a parameter that we shall send to zero. We have a family V_h of finite dimensional subspaces of V, i.e,

- V_h is a linear space.
- $V_h \subset V$.
- $\dim V_h = N_h$ (integer depending on h), $N_h \to +\infty$ as $h \to 0$.

($\mathbf{P}_h$) Find $u_h \in V_h$ such that $a(u_h, v_h) = l(v_h) \ \forall v_h \in V_h$.

Theorem 1.4.1. *If $a(\cdot,\cdot)$ is a coercive bilinear form and $l(\cdot)$ is linear, there uniquely exists $u_h \in V_h$ solving ($\mathbf{P}_h$).*

Proof. (Uniqueness): Suppose u_h^1, u_h^2 solve ($\mathbf{P}_h$). Then we see

$$\begin{aligned}
&a(u_h^i, v_h) = l(v_h) \ \forall v_h \in V_h, \ i = 1, 2. \\
&\Longrightarrow a(u_h^1 - u_h^2, v_h) = 0 \ \forall v_h \in V_h. \\
&\Longrightarrow \alpha \|u_h^1 - u_h^2\|_V^2 \le a(u_h^1 - u_h^2, u_h^1 - u_h^2) = 0. \\
&\Longrightarrow u_h^1 - u_h^2 = 0.
\end{aligned}$$

(Existence): Now, V_h is finite dimensional, so it has a basis $\{\phi_j^h\}_{j=1}^{N_h}$. For all u_h there uniquely exists a vector $(\alpha_1, \cdots, \alpha_{N_h}) \in \mathbb{R}^{N_h}$ such that

$$u_h = \sum_{j=1}^{N_h} \alpha_j \phi_j^h.$$

If u_h is a solution of ($\mathbf{P}_h$),

$$\begin{aligned}
&a(u_h, \phi_k^h) = l(\phi_k^h) \ k = 1, 2, \cdots, N_h. \\
&\Longleftrightarrow \sum_{j=1}^{N_h} \alpha_j a(\phi_j^h, \phi_k^h) = l(\phi_k^h) \ k = 1, 2, \cdots, N_h. \\
&\Longleftrightarrow A\alpha = \mathbf{b},
\end{aligned}$$

where $A = (a(\phi_j^h, \phi_k^h))_{j,k=1,\cdots,N_h}$, $\alpha = (\alpha_{\mathbf{1}}, \cdots, \alpha_{\mathbf{N_h}})^{\mathbf{T}}$,
$\mathbf{b} = (l(\phi_1^h), \cdots, l(\phi_{N_h}^h))^T$. Note that the matrix A is non-singular since $a(\cdot,\cdot)$ is coercive. Therefore, the vector $\mathbf{alpha} = \mathbf{A^{-1}b}$ gives a solution u_h of $(\mathbf{P}_h)$. □

Remark.

1) Similarly we can propose a problem
   $(\mathbf{P}^*)$ Find $u^* \in V$ such that $a(v, u^*) = l(v)$ for all $v \in V$.
   By setting $a^*(w, v) := a(v, w)$ and using Lax-Milgram's theorem we can show the unique existence of a solution of $(\mathbf{P}^*)$.

2) If $a(\cdot,\cdot)$ is symmetric, then $(\mathbf{P}^*)$ is equivalent to $(\mathbf{P})$.

3) If $a(\cdot,\cdot)$ is symmetric, then A is symmetric. If $a(\cdot,\cdot)$ is coercive, A is positive definite and satisfies $\beta \mathbf{A} \beta^{\mathbf{T}} > \mathbf{0}\ \forall \beta \in \mathbb{R}^{\mathbf{N_h}} \backslash \{\mathbf{0}\}$.

4) For symmetric $a(\cdot,\cdot)$ we propose a minimization problem:-
   $(\mathbf{M}_h)$ Find $u_h \in V_h$ such that $J(u_h) \le J(v_h)$ for all $v_h \in V_h$.
   The equivalence between $(\mathbf{M}_h)$ and $(\mathbf{P}_h)$ can be proved in the same way as the equivalence between $(\mathbf{M})$ and $(\mathbf{P})$.

Exercise Show that $(\mathbf{M}_h)$ can be formulated as:-
Find $\alpha\mathbf{a}$ such that

$$\frac{1}{2}\alpha^{\mathbf{T}}\mathbf{A}\alpha - \mathbf{b}^{\mathbf{T}}\alpha \le \frac{\mathbf{1}}{\mathbf{2}}\beta^{\mathbf{T}}\mathbf{A}\beta - \mathbf{b}^{\mathbf{T}}\beta$$

for all $\beta \in \mathbb{R}^{\mathbf{N_h}}$.

1.4.2 Galerkin Orthogonality

The idea of Galerkin orthogonality is the main ingredient for proving an error bound in the 'energy' norm, i.e. an error bound in the V norm.

Clearly $e_h = u - u_h$ is the error, which in general is not 0. In order to evaluate the quality of our approximation we try to estimate the error. In order to do this we find an equation that the error satisfies. Observe that since $V_h \subset V$ we have from (**??**) that by choosing $v = v_h \in V_h \subset V$ we have for $u \in V$

$$a(u, v_h) = l(v_h) \qquad \forall\, v_h \in V_h. \tag{1.4}$$

Now since u and u_h essentially solve the same variational equation subtracting (**??**) from (1.4) we have

$$a(u, v_h) - a(u_h, v_h) = l(v_h) - l(v_h) \tag{1.5}$$

$$\Rightarrow a(u - u_h, v_h) = 0 \qquad \forall\, v_h \in V_h. \tag{1.6}$$

This is called Galerkin orthogonality.

We wish to estimate a norm of e_h. So consider

$$a(e_h, e_h) = a(u - u_h, u - u_h) = a(u - u_h, u - v_h + v_h - u_h)$$
$$= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h).$$

We use Galerkin orthogonality to deduce that the last term here is zero beacuse $v_h - u_h \in V_h$! Thus

$$a(e_h, e_h) = a(e_h, u - v_h) \qquad \forall\, v_h \in V_h.$$

Lemma 1.4.2 (Cea's Lemma)**.** *The Galerkin approximation u_h satisifies the error bound*

$$\|u - u_h\|_V \le \frac{c_1}{c_0} \min_{v_h \in V_h} \|u - v_h\|_V. \tag{1.7}$$

Proof: For $e_h = u - u_h$ we have seen that by Galerkin orthogonality $a(e_h, e_h) = a(e_h, u - v_h)\ \forall\, v_h \in V_h$. By the coercivity and boundedness of $a(\cdot, \cdot)$

$$c_0\|e_h\|_V^2 \le a(e_h, e_h) = a(e_h, u - v_h) \le c_1\|e_h\|_V\|u - v_h\|_V \qquad \forall\, v_h \in V_h$$
$$\Rightarrow \|e_h\|_V \le \frac{c_1}{c_0} \min_{v_h \in V_h} \|u - v_h\|_V.$$

■

Remark 1.4.3.

1. This shows us that the discretisation error e_h measured in the V norm is of the 'same size' as the best approximation to u in V_h. The best approximation to u in V_h is defined as u_h^B where $\|u - u_h^B\|_V = \min_{v_h \in V_h} \|u - v_h\|_V$.
2. We use Cea's lemma by using an element of V_h which is a natural approximation to u and for which we can estimate the error.
3. In general finite element spaces V_h have approximation properties like $\min_{v_h \in V} \|u - v_h\|_V \le c(u)h^s$ where h is the element size and s in an integer depending on the polynomials used to define V_h. The higher s is the more derivatives in u that are required in order to define $c(u)$.

1.4.3 Abstract Aubin-Nitsche Lemma

Suppose that the assumptions of the Lax-Milgram theorem hold with respect to the bilinear form $a(\cdot, \cdot)$ and the Hilbert space V. Suppose that there exists a Hilbert space H with inner product $< \cdot, \cdot >_H$ into which V can be continuously embedded. It follows that for any $g \in H$ we may define a bounded linear functional $l_g(\cdot)$ by

$$l_g(v) :=< g, v >_H \quad \text{and} \quad |l_g(v)| = | < g, v >_H | \le ||g||_H ||v||_H \le c_H ||g||_H ||v||_V.$$

- **Dual regularity**

  There exists a positive constant c_s such that for any $g \in H$, the *adjoint problem* find $w(g) \in V$ such that
  $$a(v, w(g)) = l_g(v) \forall v \in V$$
  has a unique solution satisfying the regularity result
  $$||w(g)||_Z \leq c_Z ||g||_H.$$

- **Approximation**

  Let V_h be a subspace of V which satisfies the following approximation property for a subspace $Z \subset V$: There exists a linear operator $\mathcal{I}_h : Z \to V_h$ for which there is a constant $\mathcal{K}$ independent of v and h such that for all $v \in Z$,
  $$||v - \mathcal{I}_h v||_V \leq \mathcal{K} h ||v||_Z.$$

Lemma 1.4.4. *(Aubin-Nitsche lemma) Under the above assumptions,*
$$||u - u_h||_H \leq \mathcal{K} c_Z \gamma h ||u - u_h||_V.$$

Proof

Setting $e := u - u_h \in V$ we have that
$$||e||_H^2 = < e, e >_H = a(e, w(e))$$
and using Galerkin orthogonality,
$$\begin{aligned} < e, e >_H = a(e, w(e)) = a(u - u_h, w(e)) &= a(u - u_h, w(e) - w(e)_h^*) \\ &\leq \gamma ||u - u_h||_V ||w(e) - w(e)_h^*||_V \end{aligned}$$
so that, by approximation,
$$||e||_H^2 \leq \gamma ||u - u_h||_V \mathcal{K} h ||w(e)||_Z$$
and applying the dual regularity result,
$$||e||_H^2 \leq \gamma c_Z \mathcal{K} h ||e||_H$$
from which we have the desired result. □

1.4.4 Abstract Error Bound

- **Regularity**

  There exists a positive constant c_R such that for any $f \in H$, the *problem* find $w(f) \in V$ such that
  $$a(w(f), v) = l_f(v) \forall v \in V$$
  has a unique solution satisfying the regularity result
  $$||w(f)||_Z \leq c_R ||f||_H.$$

From Cea's lemma we have

$$||u - u_h||_V \le \frac{\gamma}{\alpha}||u - v||_V \quad \forall v \in V_h$$

and using the approximation assumption together with the regularity assumption $u \in Z$ we have

$$||u - u_h||_V \le \frac{\gamma}{\alpha}\mathcal{K}h||u||_Z$$

and

$$||u - u_h||_H \le Ch^2||u||_Z.$$

Higher order approximation of u

Suppose that

- u is sufficently smooth in a space $Z(k)$, say, and the space V_h has a higher power of approximation so that

$$||v - \mathcal{I}_h v||_V \le \mathcal{K}h^k||v||_{Z(k)}.$$

then we have the a priori error bound

$$||u - u_h||_H + h||u - u_h||_V \le Ch^{k+1}||u||_{Z(k)}$$

Chapter 2

Variational Formulation of Boundary Value Problems

2.1 Elements of Function Spaces

2.1.1 Space of Continuous Functions

- $\mathbb{N}$ is a set of non-negative integers.

- 1) An n-tuple $\alpha = (\alpha_{\mathbf{1}}, \cdots, \alpha_{\mathbf{n}}) \in \mathbb{N}^{\mathbf{n}}$ is called a *multi-index.*

  2) The length of α is

$$|\alpha| := \sum_{\mathbf{j=1}}^{\mathbf{n}} \alpha_{\mathbf{j}}.$$

  3) $\underline{0} = (0, \cdots, 0)$.

- Set $D^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

Example 2.1.1. *Assume $n = 3$, $\alpha = (\alpha_{\mathbf{1}}, \alpha_{\mathbf{2}}, \alpha_{\mathbf{3}}) \in \mathbb{N}^{\mathbf{3}}$, $u(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$. What is*

$\sum_{|\alpha|=\mathbf{3}} D^\alpha u$?

$$\begin{aligned}
&|\alpha| = \mathbf{3} \implies \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{3}} \alpha_{\mathbf{j}} = \mathbf{3}.\\
&\implies \alpha = (\mathbf{3},\mathbf{0},\mathbf{0}), (\mathbf{0},\mathbf{3},\mathbf{0}), (\mathbf{0},\mathbf{0},\mathbf{3}), (\mathbf{2},\mathbf{1},\mathbf{0}), (\mathbf{2},\mathbf{0},\mathbf{1}), (\mathbf{0},\mathbf{2},\mathbf{1}),\\
&\qquad\qquad (1,2,0), (1,0,2), (0,1,2), (1,1,1).\\
&\implies \sum_{|\alpha|=\mathbf{3}} D^\alpha u = \frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3} + \frac{\partial^3 u}{\partial x_3^3} + \frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1^2 \partial x_3} + \frac{\partial^3 u}{\partial x_2^2 \partial x_3}\\
&\qquad\qquad + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u}{\partial x_1 \partial x_3^2} + \frac{\partial^3 u}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3}.
\end{aligned}$$

This sort of list can get very long. Hence D^α is useful notation.

Definition 2.1.2. Let Ω be an open set in $\mathbb{R}^n$. Let $k \in \mathbb{N}$. Define spaces $C^k(\Omega)$, $C^k(\overline{\Omega})$ and $C^\infty(\Omega)$ by

$$\begin{aligned}
C^k(\Omega) &:= \{u : \Omega \to \mathbb{R} \mid D^\alpha u \text{ is continuous in } \Omega \text{ for all } |\alpha| \le \mathbf{k}\},\\
C^k(\overline{\Omega}) &:= \{u : \overline{\Omega} \to \mathbb{R} \mid D^\alpha u \text{ is continuous in } \overline{\Omega} \text{ for all } |\alpha| \le \mathbf{k}\},\\
C^\infty(\Omega) &:= \{u : \Omega \to \mathbb{R} \mid D^\alpha u \text{ is continuous in } \Omega \text{ for all } \alpha \in \mathbb{N}^{\mathbf{n}}\},
\end{aligned}$$

where $\overline{\Omega}$ is the closure of Ω. If Ω is bounded, $\overline{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega$ is the boundary of Ω. We denote $C(\Omega) = C^0(\Omega)$ and $C(\overline{\Omega}) = C^0(\overline{\Omega})$.

Example 2.1.3. *Set $I := (0,1)$ and $u(x) := 1/x^2$ $\forall x \in I$. Then clearly for all $k \ge 0$ $u \in C^k(I)$. However, in $\overline{I} = [0,1]$ u is not continuous at 0. Thus, $u \notin C(\overline{I})$.*

Definition 2.1.4. For a bounded open set $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $u \in C^k(\overline{\Omega})$, the norm $\|u\|_{C^k(\overline{\Omega})}$ is defined by

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|\alpha| \le \mathbf{k}} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)|.$$

Example 2.1.5. *Let $I = (0,1)$, $u(x) := x$, $u \in C(\overline{I})$. Then, $\sup_{x \in \overline{I}} |u(x)| = 1$.*

Definition 2.1.6. For an open set $\Omega \subset \mathbb{R}^n$ and $\eta \in C(\Omega)$ the support of η denoted by *support* η $(\subset \mathbb{R}^n)$ is defined by

$$\textit{support}\ \eta := \text{ the closure of } \{x \in \Omega \mid \eta(x) \neq 0\}.$$

Remark.

- The support of η is the smallest closed subset of $\overline{\Omega}$ such that $\eta = 0$ in $\Omega \backslash$*support* u.
- If *support* η is bounded then we say that η has compact support.

Definition 2.1.7. Define $C_0^k(\Omega)$ $(\subset C^k(\Omega))$ by

$$C_0^k(\Omega) := \{u \in C^k(\Omega) \mid \textit{support}\ u \text{ is a bounded subset of } \Omega\}.$$

Lemma 2.1.8. *$C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \le p < \infty$.*

This means that every function in $L^p(\Omega)$ can be arbitrarily closely approximated by a function from $C_0^\infty(\Omega)$ (with the error measured in the $L^p(\Omega)$ norm).

Example 2.1.9. *1) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a patition of $[0, 1]$. Define $\phi_j(x) : [0, 1] \to \mathbb{R}$ by*

$$\phi_j(x) := \begin{cases} \dfrac{x - x_{j-1}}{h} & x \in (x_{j-1}, x_j), \\ \dfrac{x_{j+1} - x}{h} & x \in (x_j, x_{j+1}), \\ 0 & \textit{elsewhere.} \end{cases}$$

Then we see $\phi_j \in C(\bar{I})$ and support $\phi_j = [x_{j-1}, x_{j+1}]$.

2) Define $w(x) : \mathbb{R}^n \to \mathbb{R}$ by

$$w(x) := \begin{cases} e^{-\dfrac{1}{1-|x|^2}} & |x| < 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Then we see support $w = \{x \in \mathbb{R}^n \mid |x| \le 1\}$ and $w \in C_0^\infty(\Omega)$ for any Ω containing $\bar{B}(0, 1) := \{x \in \mathbb{R}^n : |x| \le 1\}$.

Proof For $|x| < 1$, write $w(x) = e^{-t}$ with $t = (1-|x|^2)^{-1}$ and show that $w_{x_j} = -2e^{-t}t^2 x_j$. Prove by induction that for all multi-indices α, that there exists a polynomial P_α such that $D^\alpha w(x) = P_\alpha(x)e^{-t}t^{2|\alpha|}, |x| < 1$.

2.1.2 Spaces of Integrable Functions

Definition 2.1.10. Let Ω denote an open subset of $\mathbb{R}^n$ and assume $1 \le p < \infty$. We define a space of integrable functions $L^p(\Omega)$ by

$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} \mid \int_\Omega |v(x)|^p dx < +\infty \right\}.$$

The space $L^p(\Omega)$ is a Banach space with norm $\|\cdot\|_{L^p(\Omega)}$ defined by

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p dx \right)^{1/p}.$$

Especially the space $L^2(\Omega)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ defined by

$$\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega u(x)v(x)dx$$

and norm $\|\cdot\|_{L^2(\Omega)}$ defined by $\|u\|_{L^2(\Omega)} := \sqrt{\langle u, v\rangle_{L^2(\Omega)}}$.

We have *Minkowski's inequality* as follows. For $u, v \in L^p(\Omega)$, $1 \le p < \infty$

$$\|u+v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

We also have *Hölder's inequality.* For $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, $1 \le p, q < \infty$ with $1/p + 1/q = 1$

$$\Big|\int_\Omega u(x)v(x)dx\Big| \le \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

Now, any two integrable functions are equivalent if they are equal *almost everywhere*, that is, they are equal except on a set of zero measure. Strictly speaking, $L^p(\Omega)$ consists of equivalent classes of functions.

Example 2.1.11. *Let $u,\ v : (-1,1) \to \mathbb{R}$ be*

$$u(x) = \begin{cases} 1 & x \in (0,1), \\ 0 & x \in (-1,0], \end{cases} \quad v(x) = \begin{cases} 1 & x \in [0,1), \\ 0 & x \in (-1,0), \end{cases}$$

The functions u and v are equal almost everywhere, since the set $\{0\}$ where $u(0) \ne v(0)$ has zero measure in the interval $(-1,1)$. So u and v are equal as integrable functions in $(-1,1)$.

Suppose that $u \in C^k(\Omega)$, where Ω is an open set of $\mathbb{R}^n$. Let $v \in C_0^\infty(\Omega)$. Then we see by integration by parts

$$\int_\Omega D^\alpha u(x)v(x)dx = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha v(x)dx,$$

where $|\alpha| \le \mathbf{k}$.

Definition 2.1.12. A function $\eta : \Omega \to \mathbb{R}$ is locally integrable if $\eta \in L^1(K)$ for every bounded open set K such that $\overline{K} \subset \Omega$. The space $L^1_{loc}(\Omega)$ consists of locally integrable functions.

Definition 2.1.13. Weak derivative Suppose $\eta : \Omega \to \mathbb{R} \in L^1_{loc}(\Omega)$ and there is a locally integrable function $w_\alpha : \Omega \to \mathbb{R}$ such that

$$\int_\Omega w_\alpha(x)\phi(x)dx = (-1)^{|\alpha|}\int_\Omega \eta(x)D^\alpha\phi(x)dx$$

for all $\phi \in C_0^\infty(\Omega)$.

Then the weak derivative of η of order α denoted by $D^\alpha u$ is defined by $D^\alpha u = w_\alpha$.

Note that at most only one w_α satisfies (2.1.13) so the weak derivative of u is well-defined. Indeed, the following *DuBois-Raymond* lemma shows such w_α is unique.

Lemma 2.1.14. *(DuBois-Raymond) Suppose Ω is an open set in $\mathbb{R}^n$ and $w : \Omega \to \mathbb{R}$ is locally integrable. If*

$$\int_\Omega w(x)\phi(x)dx = 0$$

for all $\phi \in C_0^\infty(\Omega)$, then $w(x) = 0$ for a.e $x \in \Omega$.

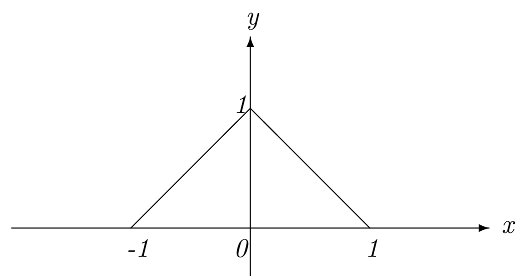
We will use D for both classical and weak derivatives.

Example 2.1.15. *Let $\Omega = \mathbb{R}$. Set $u(x) = (1 - |x|)_+$, $x \in \Omega$, where*

$$(x)_+ := \begin{cases} x & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Thus,

$$u(x) := \begin{cases} 0 & x \leq -1, \\ 1 + x & -1 \leq x \leq 0, \\ 1 - x & 0 \leq x \leq 1, \\ 0 & 1 \leq x. \end{cases}$$



Clearly we see that u is locally integrable, $u \in C(\Omega)$ and $u \notin C^1(\Omega)$. However, it may have a weak derivative. Take any $\phi \in C_0^\infty(\Omega)$ and $\alpha = 1$. Then,

$$\begin{aligned} (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \phi(x) dx &= -\int_{-\infty}^{\infty} u(x)\phi'(x)dx \\ &= -\int_{-1}^{1} (1 - |x|)\phi'(x)dx \\ &= -\int_{-1}^{0} (1 + x)\phi'(x)dx - \int_0^1 (1 - x)\phi'(x)dx \\ &= \int_{-1}^{0} 1 \cdot \phi(x)dx + \int_0^1 (-1)\phi(x)dx \\ &= \int_\Omega w(x)\phi(x)dx, \end{aligned}$$

where

$$w(x) := \begin{cases} 0 & x < -1, \\ 1 & -1 < x < 0, \\ -1 & 0 < x < 1, \\ 0 & 1 < x. \end{cases}$$

Here we do not worry about the points $x = -1, 0, 1$, since they have zero measure. Thus, u has its weak derivative $Du = w$.

Definition 2.1.16. Let k be a non-negative integer and $p \in [0, \infty)$. The space $W^{k,p}(\Omega)$ defined by

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \ \forall |\alpha| \leq \mathbf{k}\}$$

is called a *Sobolev space*. It is a Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq \mathbf{k}} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

Especially, when $p = 2$, we denote $H^k(\Omega)$ as $W^{k,2}(\Omega)$. It is a Hilbert space with the inner product

$$\langle u, v \rangle_{H^k(\Omega)} := \sum_{|\alpha| \leq \mathbf{k}} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)}.$$

Of special interest are $H^1(\Omega)$ and $H^2(\Omega)$. If $\Omega = (a, b), \subset \mathbb{R}$, we see that

$$\begin{aligned}
\langle u, v \rangle_{H^1(\Omega)} &= \langle u, v \rangle_{L^2(\Omega)} + \langle Du, Dv \rangle_{L^2(\Omega)} \\
&= \int_a^b u(x)v(x)dx + \int_a^b Du(x)Dv(x)dx. \\
\langle u, v \rangle_{H^2(\Omega)} &= \langle u, v \rangle_{L^2(\Omega)} + \langle Du, Dv \rangle_{L^2(\Omega)} + \langle D^2u, D^2v \rangle_{L^2(\Omega)} \\
&= \int_a^b u(x)v(x)dx + \int_a^b Du(x)Dv(x)dx + \int_a^b D^2u(x)D^2v(x)dx.
\end{aligned}$$

Remark. 1) By using Hölder's inequality, we can prove *Cauchy-Schwarz* inequality for the inner product of $H^k(\Omega)$ as follows.

$$\begin{aligned}
|\langle u, v \rangle_{H^k(\Omega)}| &\leq \sum_{|\alpha| \leq \mathbf{k}} |\langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)}| \\
&\leq \sum_{|\alpha| \leq \mathbf{k}} \|D^\alpha u\|_{L^2(\Omega)} \|D^\alpha v\|_{L^2(\Omega)} \\
&\leq \sqrt{\sum_{|\alpha| \leq \mathbf{k}} \|D^\alpha u\|^2_{L^2(\Omega)}} \sqrt{\sum_{|\alpha| \leq \mathbf{k}} \|D^\alpha v\|^2_{L^2(\Omega)}} \\
&= \|u\|_{H^k(\Omega)} \|v\|_{H^k(\Omega)}.
\end{aligned}$$

2) Let $\Omega = (a, b) \subset \mathbb{R}$ and $u \in H^1(\Omega)$. Then $u \in C(\overline{\Omega})$. In higher space dimensions this statement is no longer true.

Definition 2.1.17. $H^k_0(\Omega)$ is defined to be the closure of $C^\infty_0(\Omega)$ with respect to $|| \cdot ||_{H^k(\Omega)}$.

Loosely speaking $H^k_0(\Omega)$ consists of those functions in $H^k(\Omega)$ whose derivatives up to order $|\alpha| \leq k - 1$ vanish on the boundary

2.2 Poincare inequality and trace theorem

Theorem 2.2.1. Poincare Inequality

If Ω is a bounded domain then then there exists a constant $C = C(\Omega)$ (depending on Ω) such that

$$||v||_{L^2(\Omega)} \leq C||\nabla v||_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

Theorem 2.2.2. Trace theorem

Let Ω be bounded with a Lipschitz boundary $\partial\Omega$. Then there exists a bounded linear operator $tr : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ with the property that $tr(v)$ and $v|_{\partial\Omega}$ coincide on $\partial\Omega$ for all $v \in C(\bar{\Omega}) \cap H^1(\Omega)$. It follows that there exists a constant $c = C(\Omega)$ such that

$$||tr(v)||_{L^2(\partial\Omega)} \leq C||v||_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

The following inequality is also true:

$$||v||_{L^2(\partial\Omega)} \leq C||v||^{1/2}_{L^2(\Omega)}||v||^{1/2}_{H^1(\Omega)}.$$

2.2.1 The space $H_0^1(\Omega)$ in variational problems

Because of the Poincare inequality there exists a constant c such that

$$c||v||_{H^1(\Omega)} \leq ||\nabla v||_{L^2(\Omega)} \leq ||v||_{H^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

It follows that we may take in a variational problem $V = H_0^1(\Omega)$ with norm either $||v||_V = ||v||_{H^1(\Omega)}$ or $||v||_V = ||\nabla v||_{L^2(\Omega)}$.

Note that for

$$a(w, v) = \int_\Omega (p\nabla w \cdot \nabla v + qwv)dx$$

with $p(x) \geq p_m > 0$ and $q(x) \geq q_m \geq 0$ we can use the inequalities

$$a(v, v) \geq p_m||\nabla v||^2_{L^2(\Omega)} \text{ if } q_m = 0$$

or

$$a(v, v) \geq \min(p_m, q_m)||v||^2_{H^1(\Omega)} \text{ if } q_m > 0$$

to deduce that $a(\cdot, \cdot)$ is coercive on $H_0^1(\Omega)$.

2.3 Divergence theorem and integration by parts

Theorem 2.3.1. Divergence theorem *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $Q : \overline{\Omega} \to \mathbb{R}^n$ be a vector field whose components are in $H^1(\Omega)$. The following equality holds.*

$$\int_\Omega \nabla \cdot Q dx = \int_{\partial\Omega} Q \cdot \nu ds$$

where ν is the unit outward pointing normal to $\partial\Omega$.

Remark.

- Suppose $Q = f\mathbf{e_i}$ with the coordinate vector $\mathbf{e}_i = (0, \cdots, 1, \cdots, 0)^T$, i.e, the jth component is $\{\mathbf{e}_i\}_j = \delta_{i,j}$. Then we see

$$\nabla Q = \frac{\partial}{\partial x_i} f.$$

  So by the Divergence theorem

$$\int_\Omega \frac{\partial f}{\partial x_i} dx = \int_{\partial\Omega} f \nu_i ds.$$

  In one dimensional case where $\Omega = (a, b)$, $\partial\Omega = \{a, b\}$, the Divergence theorem becomes

$$\int_a^b \frac{\partial f}{\partial x} dx = f(b) - f(a).$$

- Similarly $Q = wv\mathbf{e_i}$ yields

$$\int_\Omega \frac{\partial w}{\partial x_i} v dx = -\int_\Omega w \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} wv\nu_i dS.$$

Let us derive the integration by parts formula.

Proposition 2.3.1. Integration by parts
For $Q \in H^1(\overline{\Omega}; \mathbb{R}^n)$, $g \in H^1(\Omega)$,

$$\int_\Omega Q \cdot \nabla g dx = \int_{\partial\Omega} gQ \cdot \nu ds - \int_\Omega g\nabla \cdot Q dx.$$

Proof. By the divergence theorem we see that

$$\int_\Omega \nabla(Qg) dx = \int_{\partial\Omega} Q \cdot \nu g ds.$$

Alternatively,

$$\nabla(Qg) = g\nabla Q + Q \cdot \nabla g.$$

By combining these equality we get the desired formula. $\square$

For example, if $Q = \nabla u$ and $g = v$, we have by integration by parts formula that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v \nabla u \cdot \nu ds - \int_\Omega v \Delta u dx.$$

where Δ is the Laplacian and noting that

$$\nabla u \cdot \nu = \frac{\partial \mathbf{u}}{\partial \nu}$$

we obtain

$$\int_\Omega v \Delta u dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} ds - \int_\Omega \nabla u \cdot \nabla v dx.$$

Similarly we have that

$$-\int_\Omega \nabla(p \nabla u) v dx = \int_\Omega p \nabla u \cdot \nabla v dx - \int_{\partial\Omega} p \frac{\partial u}{\partial \nu} v ds.$$

2.4 Sobolev embedding, Sobolev inequalities, compactness

Definition 2.4.1. Let $1 \leq p < n$ then the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p.$$

Theorem 2.4.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with a locally Lipschitz boundary.*

- *If $p > n$ and $\gamma = 1 - \frac{n}{p}$ then the space $W^{1,p}(\Omega)$ is continuously embedded in $C^{0,\gamma}(\bar{\Omega})$ and*

  $$||v||_{C^{0,\gamma}(\bar{\Omega})} \leq C||v||_{W^{1,p}(\Omega)}, \quad C = C(\Omega, p, n).$$

  In the case $n = 1$ the space $W^{1,1}(\Omega)$ is continuously embedded in $C^0(\bar{\Omega})$ and

  $$||v||_{C^0(\bar{\Omega})} \leq C||v||_{W^{1,1}(\Omega)}, \quad C = C(\Omega).$$

- *For*

  $$p \geq n \quad \text{and} \quad p \leq q$$

  the space $W^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ and

  $$||v||_{L^q(\Omega)} \leq C||v||_{W^{1,p}(\Omega)}, \quad C = C(\Omega, p).$$

- *For*

  $$p < n \quad \text{and} \quad p < q \leq p^* = \frac{np}{n-p}$$

  the space $W^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ and

  $$||v||_{L^q(\Omega)} \leq C||v||_{W^{1,p}(\Omega)}, \quad C = C(\Omega, p).$$

Theorem 2.4.3.

- *If $p > n$ and $\gamma < 1 - \frac{n}{p}$ then the space $W^{1,p}(\Omega)$ is compactly embedded in $C^{0,\gamma}(\bar{\Omega})$.*
- *For*
$$p < n \quad \text{and} \quad 1 < q < p^* = \frac{np}{n-p}$$
the space $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.
- *For*
$$p \geq n \quad \text{and} \quad 1 \leq q < \infty$$
the space $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.

This theorems are versions of the *Sobolev inequalities* and the embedding theorems (including the *Rellich-Kondrachov* theorem). Many more such results may be found in the book by Adams and Fournier [1].

Theorem 2.4.4. *Let X, Y, Z be three Banach spaces where X and Y are reflexive. Suppose $X \subset Z \subset Y$ with continuous embeddings. Suppose the embedding of X into Z is compact. For any $1 < p, q < \infty$ set*

$$W : \{v : v \in L^p(0, T; X), v' \in L^q(0, T; Y)\}.$$

Then the embedding fom W into $L^p(0, T : Z)$ is compact.

Remark 2.4.5. This compactness result is due to Aubin. A setting might be $p = q = 2$, $X = H^1(\Omega), Z = L^2(\Omega), Y = (H^1(\Omega))'$.

Lemma 2.4.6. *Let X, Y be Banach spaces, $X = Y'$ and $1 < p \leq \infty$. Suppose*

$$\begin{cases} u_n \to u \quad \text{weak star in} \quad L^p(0, T; X) \\ u_n' \to u' \quad \text{weak star in} \quad L^p(0, T; X) \end{cases}$$

then

$$u_n(0) \to u(0) \quad \text{weak star in} \quad X.$$

If X is reflexive then the weak star convergence is equivalent to weak convergence.

2.5 Elliptic regularity

Theorem 2.5.1. *Let Ω be bounded and convex. Let $\mathcal{A}$ be strictly positive definite and $\mathcal{A}_{ij} \in C^{0,1}(\bar{\Omega})$.*

- *If $u \in H_0^1(\Omega)$ then*
$$\Big(\sum_{|\alpha|=2} |D^\alpha u|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \leq ||\Delta u||_{L^2(\Omega)}$$

- *If* $u \in H_0^1(\Omega)$ *and*

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v dx = (f, v) = 0 \quad \forall v \in H_0^1(\Omega)$$

then

$$(\sum_{|\alpha|=2} |D^\alpha u|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq C(||f||_{L^2(\Omega)} + ||\nabla u||_{L^2(\Omega)})$$

where $C = C(\Omega)$ *depends only on the diameter of* Ω*.*

- *If* $u \in H^1(\Omega)$ *and*

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v dx = (f, v) = 0 \quad \forall v \in H^1 0(\Omega)$$

then

$$||u||_{H^2(\Omega)} \leq C||f||_{L^2(\Omega)}$$

where $C = C(\lambda, \mathcal{A}, \Omega)$*.*

Theorem 2.5.2. *Let* Ω *be bounded and* $\partial\Omega \in C^2$*. Let* $\mathcal{A}$ *be strictly positive definite and* $\mathcal{A}_{ij} \in C^1(\bar{\Omega})$*. Let* $u \in H_0^1(\Omega)$ *solve*

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v dx + \lambda \int_\Omega uv = (f, v) = 0 \quad \forall v \in H_0^1(\Omega)$$

then

$$||u||_{H^2(\Omega)} \leq C(||f||_{L^2(\Omega)} + ||u||_{L^2(\Omega)})$$

where $C = C(\Omega, \mathcal{A}, \lambda)$ *depends only on* Ω*,* $\mathcal{A}$ *and* λ*.*

2.6 One Dimensional Problem

2.6.1 One Dimensional H^1 inequalities

Here we derive some inequalities in the one dimensional case $\Omega = (a, b)$. We define a function space $H^1_{\epsilon_0}(\Omega)$ by

$$H^1_{\epsilon_0}(\Omega) := \{\phi \in H^1(\Omega) \mid \phi(a) = 0\}.$$

The following inequality is one example of *Poincaré-Friedrichs inequality.*

Proposition 2.6.1. *For all* $\phi \in H^1_{\epsilon_0}(\Omega)$*,*

$$\|\phi\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{2}}(b - a)\|D\phi\|_{L^2(\Omega)}.$$

Proof. We can write that for $a \le \forall x \le b$

$$\phi(x) = \int_a^x D\phi(\eta)d\eta.$$

Then we see that

$$\begin{aligned}
\|\phi\|_{L^2(\Omega)}^2 &= \int_a^b \phi(x)^2 dx \\
&= \int_a^b \left(\int_a^x D\phi(\eta)d\eta\right)^2 dx \\
&\le \int_a^b \left(\int_a^x 1^2 dx\right)\left(\int_a^x D\phi(\eta)^2 d\eta\right) dx \\
&= \int_a^b (x-a) \int_a^x D\phi(\eta)^2 d\eta dx \\
&\le \int_a^b (x-a) \int_a^b D\phi(\eta)^2 d\eta dx \\
&= \frac{1}{2}(b-a)^2 \|D\phi\|_{L^2(a,b)}^2.
\end{aligned}$$

□

Example 2.6.2. *We can apply this inequality to prove the unique solvability of the Dirichlet problem with a functional space V, a bilinear form $a(\cdot,\cdot)$ and a linear functional $l(\cdot)$ defined by*

$$\begin{aligned}
V &:= \{\phi \in H^1(0,1) \mid \phi(0) = \phi(1) = 0\}, \\
a(u,v) &:= \int_0^1 pDuDvdx \text{ for } \forall u, v \in V, \\
l(u) &:= \int_0^1 fudx \text{ for } \forall u \in V.
\end{aligned}$$

We only check that the bilinear form a is bounded and coercive. We see that

$$|a(u,v)| \le \sup_{x\in(0,1)} |p(x)| \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} \le \sup_{x\in(0,1)} |p(x)| \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

and

$$\begin{aligned}
a(v,v) &\ge p_0 \|Dv\|_{L^2(\Omega)}^2 \\
&= \frac{p_0}{2}\|Dv\|_{L^2(\Omega)}^2 + \frac{p_0}{2}\|Dv\|_{L^2(\Omega)}^2 \\
&\ge \frac{p_0}{2}\left(\|Dv\|_{L^2(\Omega)}^2 + \frac{2\|v\|_{L^2(\Omega)}^2}{(b-a)^2}\right) \\
&\ge \alpha \|v\|_V^2,
\end{aligned}$$

where $\alpha := p_0 \min(1, 2/(b-a)^2)/2$.

Proposition 2.6.3. *The following Agmon's inequality holds. For all* $\phi \in H^1_{\epsilon_0}(\Omega)$

$$\max_{x\in\overline{\Omega}} |\phi(x)|^2 \le 2\|\phi\|_{L^2(\Omega)}\|D\phi\|_{L^2(\Omega)}.$$

Proof.

$$\begin{aligned}
\phi(x)^2 &= \int_a^x \frac{d\phi(\eta)^2}{d\eta}d\eta \\
&= 2\int_a^x \phi(\eta)D\phi(\eta)d\eta \\
&\le 2\left(\int_a^x \phi(\eta)^2 d\eta\right)^{1/2}\left(\int_a^x D\phi(\eta)^2 d\eta\right)^{1/2} \\
&\le 2\left(\int_a^b \phi(\eta)^2 d\eta\right)^{1/2}\left(\int_a^b D\phi(\eta)^2 d\eta\right)^{1/2} \\
&\le 2\|\phi\|_{L^2(\Omega)}\|D\phi\|_{L^2(\Omega)},
\end{aligned}$$

which gives the inequality. □

Noting that

$$\|\phi\|_{L^2(\Omega)} \le \sqrt{b-a}\max_{x\in[a,b]}|\phi(x)|,$$

this Agmon's inequality yields

$$\max_{x\in\overline{\Omega}}|\phi(x)|^2 \le 2\sqrt{b-a}\max_{x\in[a,b]}|\phi(x)|\|D\phi\|_{L^2(\Omega)},$$

or

$$\max_{x\in\overline{\Omega}}|\phi(x)| \le 2\sqrt{b-a}\|D\phi\|_{L^2(\Omega)}$$

for any $\phi \in H^1_{\epsilon_0}(\Omega)$.

2.6.2 Dirichlet condition

Let $\Omega = (0,1)$, $p(\cdot), q(\cdot) \in C(\overline{\Omega})$ and $f(\cdot) \in L^2(\Omega)$. Note that $\partial\Omega = \{x=0\}\cup\{x=1\}$. We consider the following problem.

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\dfrac{d}{dx}(p\dfrac{du}{dx}) + qu = f, & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega. \end{cases} \tag{BVP}$$

Specifying the value of u at boundary points is said to be a *Dirichlet boundary condition.* Now the methodology is

1) multiply the equation by a test function, integrate by parts and use boundary conditions appropriately,

2) identify V, $a(\cdot,\cdot)$ and $l(\cdot)$,

3) verify, if possible, the assumptions of Lax-Milgram.

$\Longrightarrow$ Unique existence to the variational formulation of the BVP.

Let $\phi : \overline{\Omega} \to \mathbb{R}$ be sufficiently smooth. We will call ϕ our test function. Let us follow the methodology.

1)

$$\int_\Omega f(x)\phi(x)dx = \int_\Omega \left(-\frac{d}{dx}(p(x)\frac{du(x)}{dx})\phi(x) + q(x)u(x)\phi(x)\right) dx$$
$$= \left[-p(x)\frac{du(x)}{dx}\phi(x)\right]_{x=0}^{x=1} + \int_\Omega \left(p(x)\frac{du(x)}{dx}\frac{d\phi(x)}{dx} + q(x)u(x)\phi(x)\right) dx.$$

We want to eliminate the term $[p(x)du(x)/dx\phi(x)]_{x=0}^{x=1}$, so we suppose that the test function ϕ satisfies the same Dirichlet conditions as u, i.e, $\phi(0) = \phi(1) = 0$. Then we have that

$$\int_\Omega \left(p(x)\frac{du(x)}{dx}\frac{d\phi(x)}{dx} + q(x)u(x)\phi(x)\right) dx = \int_\Omega f(x)\phi(x)dx$$

for any test function ϕ. We want u, ϕ to be from the same space. For the term $\int_\Omega u\phi dx$ to make sense, we need $u, \phi \in L^2(\Omega)$. For the derivatives $du/dx, d\phi/dx$ to make sense, we take this further, so $u, \phi \in H^1(\Omega)$.

2) Let us choose

$$V := \left\{\phi \in H^1(\Omega) \mid \phi(0) = \phi(1) = 0\right\},$$

where

$$H^1(\Omega) = \left\{\phi \in L^2(\Omega) \mid D\phi \in L^2(\Omega)\right\}.$$

We equip V with the inner product $\langle\cdot,\cdot\rangle_V := \langle\cdot,\cdot\rangle_{H^1(\Omega)}$. Let us define

$$a(u,v) := \int_\Omega (pDuDv + quv)dx,$$
$$l(v) := \int_\Omega fv\,dx.$$

Moreover, assume that $p(x) \geq p_0 > 0$, $q(x) \geq q_0 > 0$ for all $x \in \overline{\Omega}$.

3) We will verify the assumptions of Lax-Milgram's theorem.

i) For $\phi \in V$ and $f \in L^2(\Omega)$, we see by Cauchy-Schwarz inequality that

$$
\begin{aligned}
|l(\phi)| &= |\int_\Omega f\phi dx| \\
&\leq \|f\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \\
&\leq \|f\|_{L^2(\Omega)} \quad \left(\|\phi\|^2_{L^2(\Omega)} + \|D\phi\|^2_{L^2(\Omega)}\right)^{1/2} \\
&= c_l\|\phi\|_V,
\end{aligned}
$$

where we have set $c_l := \|f\|_{L^2(\Omega)}$. Thus, $l : V \to \mathbb{R}$ is bounded. Clearly l is linear, i.e, $l(\alpha\phi + \beta\psi) = \alpha l(\phi) + \beta l(\psi)$ for any $\phi, \psi \in V$ and $\alpha, \beta \in \mathbb{R}$.

ii) Obviously $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ is bilinear. Moreover $a(\cdot, \cdot)$ is bounded. Indeed,

$$
\begin{aligned}
&|a(\phi, \psi)| \leq |\int_\Omega pD\phi D\psi dx| + |\int_\Omega q\phi\psi dx| \\
&\leq \max_{x\in\overline{\Omega}} |p(x)| \int_\Omega |D\phi D\psi| dx + \max_{x\in\overline{\Omega}} |q(x)| \int_\Omega |\phi\psi| dx \\
&\leq \max_{x\in\overline{\Omega}} |p(x)| \|D\phi\|_{L^2(\Omega)}\|D\psi\|_{L^2(\Omega)} + \max_{x\in\overline{\Omega}} |q(x)| \|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} \\
&\leq C(\|D\phi\|_{L^2(\Omega)}\|D\psi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}) \\
&\leq C\sqrt{\|D\phi\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)}}\sqrt{\|D\psi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)}} \\
&= C\|\phi\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)},
\end{aligned}
$$

where we have set

$$C := \max\{\max_{x\in\overline{\Omega}} |p(x)|, \max_{x\in\overline{\Omega}} |q(x)|\}.$$

The bilinear form $a(\cdot, \cdot)$ is coercive, since for all $\phi \in V$

$$
\begin{aligned}
a(\phi, \phi) &= \int_\Omega p|D\phi|^2 dx + \int_\Omega q|\phi|^2 dx \\
&\geq p_0 \int_\Omega |D\phi|^2 dx + q_0 \int_\Omega |\phi|^2 dx \\
&= \hat{C}\|\phi\|^2_V,
\end{aligned}
$$

where we have set $\hat{C} := \min\{p_0, q_0\}$.

We can now apply Lax-Milgram's theorem to see that there uniquely exists a solution to the following problem

(P) Find $u \in V$ such that

$$\int_\Omega (pDuD\phi + qu\phi) dx = \int_\Omega f\phi dx,$$

for any $\phi \in V$.

Remark. For $V = H_0^1(\Omega)$ we can use the norm $\|\cdot\|_V$ defined by

$$\|\phi\|_V^2 = \int_\Omega |D\phi|^2 dx,$$

since the following Poincare inequality holds:- there exists $C > 0$ such that

$$\int_\Omega |\phi|^2 dx \leq C \int_\Omega |D\phi|^2 dx \text{ for } \forall \phi \in V.$$

By using this inequality we can prove the unique existence of the solution solving (**P**) with $q \equiv 0$ in the same way as above.

2.6.3 One Dimensional Problem: Neumann condition

Let $\Omega = (0,1)$, $p(x), q(x) \in C(\overline{\Omega})$ and $f(x) \in L^2(\Omega)$. We consider the following problem.

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\dfrac{d}{dx}\left(p\dfrac{du}{dx}\right) + qu = f, & x \in \Omega, \\ \dfrac{d}{dx}u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{NBVP}$$

Specifying the value of du/dx at boundary points is said to be a *Neumann boundary condition.* We assume the same conditions for p, q as before, i.e, $p(x) \geq p_0 > 0$, $q(x) \geq q_0 > 0$ in Ω. Let us derive the variational form. Take a sufficiently smooth test function ϕ, multiply (NBVP) by ϕ and integrate.

$$\begin{aligned} &\int_\Omega f(x)\phi(x)dx = \int_\Omega \left(-\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right)\phi(x) + q(x)u(x)\phi(x)\right)dx \\ &= \left[-p(x)\frac{du(x)}{dx}\phi(x)\right]_{x=0}^{x=1} + \int_\Omega \left(p(x)\frac{du(x)}{dx}\frac{d\phi(x)}{dx} + q(x)u(x)\phi(x)\right)dx \\ &= \int_\Omega \left(p(x)\frac{du(x)}{dx}\frac{d\phi(x)}{dx} + q(x)u(x)\phi(x)\right)dx. \end{aligned}$$

We have eliminated the term $[p(x)du(x)/dx\phi(x)]_{x=0}^{x=1}$ by taking into account the Neumann boundary conditions $du(x)/dx = 0$ for $x = 0, 1$. Let us choose the functional space $V := H^1(\Omega)$ in this case and define

$$\begin{aligned} a(u,v) &:= \int_\Omega (pDuDv + quv)dx, \\ l(v) &:= \int_\Omega fv\,dx. \end{aligned}$$

The corresponding variational problem is that:-

($\mathbf{P}$) Find $u \in V$ such that

$$\int_\Omega (pDuD\phi + qu\phi)dx = \int_\Omega f\phi dx,$$

for any $\phi \in V$.

Again the linear form $l(\cdot) : V \to \mathbb{R}$ and the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ satisfy the assumptions of Lax-Milgram's theorem, hence the problem ($\mathbf{P}$) has the unique solution.

Remark. Consider (NBVP) with $q \equiv 0$. Then we see

$$\begin{aligned}\int_\Omega f(x)dx &= -\int_\Omega \frac{d}{dx}(p(x)\frac{du(x)}{dx})dx \\ &= \left[p(x)\frac{du(x)}{dx}\right]_{x=0}^{x=1} = 0.\end{aligned}$$

Thus, we need to assume $\int_\Omega f(x)dx = 0$ as a compatibility condition in this case. In order to prove the unique existence of the solution, we need to modify the functional space. Let us define $H_m^1(\Omega)$ by

$$H_m^1(\Omega) := \{\phi \in H^1(\Omega) \mid \int_\Omega \phi dx = 0\},$$

and equip the same inner product as $H^1(\Omega)$. Again *Poincare's inequality* is available for this space $H_m^1(\Omega)$, i.e, for all $\phi \in H_m^1(\Omega)$

$$\int_\Omega |\phi|^2 dx \le C \int_\Omega |D\phi|^2 dx,$$

where $C > 0$ is a constant. By using this inequality we can prove the unique existence of the solution solving ($\mathbf{P}$) with $q \equiv 0$ and $V = H_m^1(\Omega)$ in the same way as above on the assumption $\int_\Omega f(x)dx = 0$.

2.6.4 One Dimensional Problem: Robin/Newton Condition

Let $\Omega = (0,1)$, $p(x), q(x) \in C(\overline{\Omega})$, $f(x) \in L^2(\Omega)$, $\delta, g_0, g_1 \in \mathbb{R}$ be constants. We consider the following problem. Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\dfrac{d}{dx}\left(p\dfrac{du}{dx}\right) + qu = f, \qquad x \in \Omega, \\ -p\dfrac{d}{dx}u(0) + \delta u(0) = g_0, \\ p\dfrac{d}{dx}u(1) + \delta u(1) = g_1. \end{cases} \tag{RNBVP}$$

Let us derive the variational form. Take sufficiently smooth ϕ. This kind of boundary condition is said to be a *Robin/Newton boundary condition.* We assume the same conditions for p, q as before, i.e, $p(x) \geq p_0 > 0$, $q(x) \geq q_0 > 0$ in Ω and that $\delta \geq 0$.

$$\begin{aligned}
&\int_\Omega f(x)\phi(x)dx = \int_\Omega \left(-\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right)\phi(x) + q(x)u(x)\phi(x)\right)dx \\
&= \left[-p(x)\frac{du(x)}{dx}\phi(x)\right]_{x=0}^{x=1} + \int_\Omega \left(p(x)\frac{du(x)}{dx}\frac{d\phi(x)}{dx} + q(x)u(x)\phi(x)\right)dx \\
&= (-g_1 + \delta u(1))\phi(1) - (g_0 - \delta u(0))\phi(0) \\
&\quad + \int_\Omega \left(p(x)\frac{du(x)}{dx}\frac{d\phi(x)}{dx} + q(x)u(x)\phi(x)\right)dx,
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\int_\Omega \left(p(x)\frac{du(x)}{dx}\frac{d\phi(x)}{dx} + q(x)u(x)\phi(x)\right)dx + \delta u(1)\phi(1) + \delta u(0)\phi(0) \\
&\quad = \int_\Omega f(x)\phi(x)dx - g_1\phi(1) + g_0\phi(0).
\end{aligned}$$

This suggests that we should define $a(\cdot,\cdot)$, $l(\cdot)$ and the functional space V as following.

$$\begin{aligned}
a(u, v) &:= \int_\Omega (pDuDv + quv)dx + \delta u(1)v(1) + \delta u(0)v(0), \\
l(v) &:= \int_\Omega fvdx + g_1v(1) + g_0v(0), \\
V &:= H^1(\Omega).
\end{aligned}$$

As usual we need to show that the (bi)linear forms a, l are bounded and a is coercive to establish the unique solvability of $(RNBVP)$. Let us assume the following inequality holds true for a while.

$$|\phi(x)| \leq C\|\phi\|_{H^1(\Omega)} \tag{2.1}$$

for all $\phi \in H^1(\Omega)$. Then it is easy to see that $a(\cdot,\cdot), l(\cdot)$ are bounded. Now

$$\begin{aligned}
a(\phi, \phi) &\geq \min(p_0, q_0)\|\phi\|_V^2 + \delta(\phi(1)^2 + \phi(0)^2) \\
&\geq \min(p_0, q_0)\|\phi\|_V^2 \\
&= \alpha\|\phi\|_V^2,
\end{aligned}$$

where $\alpha = \min(p_0, q_0)$. Hence the bilinear form a becomes coercive and Lax-Milgram's theorem assures the unique existence of the solution.

2.7 Generalised coercivity lemma

Lemma 2.7.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ let $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form*

$$a(v, w) := a_1(v, w) + a_0(v, w) \tag{2.2}$$

where $a_i(\cdot,\cdot), i = 1,2$ *are continuous and bilinear such that:* $\exists \beta > 0$

$$a_1(v,v) \geq \beta ||\nabla v||^2_{L^2(\Omega)} \tag{2.3}$$

and

$$a_0(v,v) \geq 0 \quad \forall v \in H^1(\Omega) \tag{2.4}$$

$$a_0(v,v) = 0 \quad \textit{and} \quad v = c \in \mathbb{R} \implies v = 0. \tag{2.5}$$

Then there exists $\alpha > 0$ *such that*

$$a(v,v) \geq \alpha ||v||^2_{H^1(\Omega)}, \ \forall v \in H^1(\Omega). \tag{2.6}$$

Proof. We use an abstract compactness argument based on the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$. If such an α does not exist then there exists a sequence $v_n \in H^1(\Omega)$ such that

$$||v_n||_{H^1(\Omega)} = 1 \tag{2.7}$$

$$\lim_{n\to\infty} a(v_n, v_n) = 0. \tag{2.8}$$

Since the sequence is bounded it follows that there is a subsequence v_n (still denoted byv_n) which converges weakly in $H^1(\Omega)$ to a limit v^*. Using the non-negativity of the bilinear form

$$a(v^*, v_n) + a(v_n, v^*) - a(v^*, v^*) \leq a(v_n, v_n)$$

weak convergence and continuity of the bilinear form implies

$$0 \leq \beta ||\nabla v^*||_{L^2(\Omega)} \leq a(v^*, v^*) \leq \liminf_{n\to\infty} a(v_n, v_n) = \lim_{n\to} a(v_n, v_n) = 0.$$

It follows that $a_i(v^*, v^*) = 0, \ i = 0,1$ from which we deduce that

$$\nabla v^* = 0 \ a.e. \ \Omega \implies v^* = c \in \mathbb{R}$$

and

$$a_0(v^*, v^*) = 0$$

from which

$$v^* = 0.$$

Note that since we have weak convergence in L^2 of the gradient and convergence of the L^2 norm of the gradient we infer

$$\nabla v_n \to 0 \ \text{ strongly in } \ L^2(\Omega).$$

Furthermore the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ implies that

$$v_n \to 0 \quad \text{strongly in} \ \ L^2(\Omega)$$

from which we find

$$v_n \to 0 \ \text{ strongly in } \ H^1(\Omega)$$

which contradicts $||v_n||_{H^1(\Omega)} = 1, \ \forall n.$

□

2.8 Variational formulation of elliptic equations

2.8.1 Weak Solutions to Elliptic Problems

The simplest elliptic equation is Laplaces' equation:

$$\Delta u = 0, \tag{2.9}$$

where $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator. A general second order elliptic equation is: given a bounded open set $\Omega \subset \mathbb{R}^n$ find u such that:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) \qquad x \in \Omega, \tag{2.10}$$

where classically $a_ij \in C^1(\Omega), i,j = 1,\dots,n; b_i \in C(\Omega), i = 1,\dots,n; c \in C(\Omega); f \in C(\Omega)$. For the equation to be elliptic we require

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \tilde{C}\sum_{i=1}^{n}\xi_i^2 \qquad \forall\, \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n, \tag{2.11}$$

where $\tilde{C} > 0$ is independent of x, ξ. Condition (2.11) is called uniform ellipticity.

The equation is usually supplemented with boundary conditions - Dirichlet, Neumann, Robin, or a mixed Dirichlet/Neumann boundary.

In the case of the homogeneous Dirichlet problem ($u = 0$ on $\partial\Omega$) u is said to be a classical solution provided $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Elliptic theory tells us that there exists a unique classical solution provided a_{ij}, b_i, c, f and $\partial\Omega$ are sufficiently smooth. However we are only intersted in problems where the data is not smooth, for example $f = sign(1/2 - |x|), \Omega = (-1,1)$. This problem can't have $u \in C^2(\Omega)$ because Δu has a jump discontinuity are $|x| = 1/2$. With the help of functional analysis the existence/uniqueness theory for 'weak', 'variational' solutions turn out to be easy and is good for FEM.

2.8.2 Variational Formulation of Elliptic Equation: Neumann Condition

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $p, q \in C(\overline{\Omega})$ such that

$$p(x) \geq p_0 > 0,\ q(x) \geq q_0 > 0\ \forall x \in \overline{\Omega},$$

and $f \in L^2(\Omega)$.

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\nabla\cdot(p\nabla u) + qu = f, & x \in \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{NBVP}$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$. Note that

$$
\begin{aligned}
\nabla \cdot (p\nabla u) &= \sum_{i=1}^{n} \frac{\partial}{\partial_{x_i}} \left(p\frac{\partial u}{\partial x_i} \right) \\
&= \sum_{i=1}^{n} \left(p\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial p}{\partial x_i}\frac{\partial u}{\partial x_i} \right) \\
&= p\Delta u + \nabla p \cdot \nabla u, \\
\frac{\partial u}{\partial \mathbf{n}} &= \nabla u \cdot \mathbf{n}.
\end{aligned}
$$

So we have a second order PDE. In one dimensional problem, in order to derive the variational formulation we used integration by parts. Let us revise some formulae related to the integration by parts.

Notation:

$$
\begin{aligned}
&\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_1} \right)^T \\
&\nabla \cdot \nabla v = \nabla^2 v = \Delta v \\
&\nabla \cdot \mathbf{A} = \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i} \\
&(D^2 v)_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j} \\
&\mathrm{Tr}(D^2 v) = \Delta v.
\end{aligned}
$$

Let v be a sufficiently smooth test function. Multiply (NBVP) by v and integrate using Divergence theorem.

$$
\begin{aligned}
\int_\Omega f v dx &= \int_\Omega (-\nabla \cdot (p\nabla u) + qu) v dx \\
&= \int_\Omega p\nabla u \nabla v dx - \int_{\partial\Omega} p\frac{\partial u}{\partial \mathbf{n}} v ds + \int_\Omega quv dx.
\end{aligned}
$$

Since $\partial u/\partial \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, we do not need to place a restriction on the test function v. So if u solve (BVP), then

$$
\int_\Omega (p\nabla u \nabla v + quv) dx = \int_\Omega f v dx,
$$

for any sufficiently smooth function v.

Now to use Lax-Milgram, we have to set up V, $a(\cdot,\cdot)$ and $l(\cdot)$. In order for the two inner products on the left hand side to make sense, we take

$$
\begin{aligned}
V &= H^1(\Omega), \\
a(u,v) &= \int_\Omega (p\nabla u \nabla v + quv)dx, \\
l(v) &= \int_\Omega fv dx,
\end{aligned}
$$

for all $u, v \in V$. Note that V is a real Hilbert space with the norm

$$
\|v\|_V = \|v\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla v|^2 dx + \int_\Omega v^2 dx \right)^{1/2}
$$

and obviously $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is bilinear and $l(\cdot) : V \to \mathbb{R}$ is linear. Moreover we observe

$$
\begin{aligned}
a(v,v) &= \int_\Omega (p|\nabla v|^2 + qv^2)dx \\
&\geq p_0 \int_\Omega |\nabla v|^2 dx + q_0 \int_\Omega v^2 dx \\
&\geq \min\{p_0, q_0\} \|v\|^2_{H^1(\Omega)} \\
&= \alpha \|v\|^2_{H^1(\Omega)},
\end{aligned}
$$

where we have put $\alpha := \min\{p_0, q_0\}$. Thus $a(\cdot,\cdot)$ is coercive.

$$
\begin{aligned}
|a(v,w)| &= \left| \int_\Omega (p\nabla v \cdot \nabla w + qvw)\, dx \right| \\
&\leq \int_\Omega (|p\nabla u \cdot \nabla w| + |qvw|)dx \\
&\leq C \int_\Omega (|\nabla v|^2 + v^2)^{1/2} (|\nabla w|^2 + w^2)^{1/2} dx \\
&\leq C \left(\int_\Omega (|\nabla v|^2 + v^2)dx \right)^{1/2} \left(\int_\Omega (|\nabla w|^2 + w^2)dx \right)^{1/2} \\
&= C \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}.
\end{aligned}
$$

Therefore, $a(\cdot,\cdot)$ is bounded. Finally let us check the boundedness of $l(\cdot)$.

$$
\begin{aligned}
|l(v)| &= \left| \int_\Omega fv dx \right| \\
&\leq \int_\Omega |f||v| dx \\
&\leq \left(\int_\Omega f^2 dx \right)^{1/2} \left(\int_\Omega v^2 dx \right)^{1/2} \\
&= \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \|f\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\
&= C_l \|v\|_{H^1(\Omega)}.
\end{aligned}
$$

Thus, $l(\cdot)$ is bounded. Now we can apply Lax-Milgram to prove that there exists a unique solution $u \in V$ to the following problem (**P**).

(P) Find $u \in V$ such that

$$a(u,v) = l(v)$$

for all $v \in V$.

2.8.3 Variational Formulation of Elliptic Equation: Dirichlet Problem

On the same assumptions on Ω, p, q, f, we consider the following problem.

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\nabla \cdot (p\nabla u) + qu = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{DBVP}$$

Let us derive the variational form of (DBVP) as in the section 2.5. Multiply (DBVP) by a sufficiently smooth test function v and integrate. Then we see

$$\int_\Omega f v dx = \int_\Omega (p\nabla u \nabla v + quv) dx - \int_{\partial\Omega} p \frac{\partial u}{\partial n} v dx.$$

Since we have $u \equiv 0$ on $\partial\Omega$, we have to force our test function v to satisfy the same condition; $v \equiv 0$ on $\partial\Omega$. Then we obtain

$$\int_\Omega (p\nabla u \nabla v + quv) dx = \int_\Omega f v dx$$

for any sufficient smooth function v with $v \equiv 0$ on $\partial\Omega$. Set

$$\begin{aligned} V &:= \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\} \\ &= H_0^1(\Omega). \end{aligned}$$

Note that V is a real Hilbert space with the inner product

$$\langle v, w \rangle_V := \langle \nabla v, \nabla w \rangle_{L^2(\Omega)} + \langle v, w \rangle_{L^2(\Omega)}$$

and $\|v\|_V = \|v\|_{H^1(\Omega)}$. As before we define

$$\begin{aligned} a(v,w) &:= \int_\Omega (p\nabla v \nabla w + qvw)\ dx, \\ l(v) &= \int_\Omega f v\ dx. \end{aligned}$$

The same argument as the previous section shows that $a(\cdot,\cdot)$ is a coercive and bounded bilinear form and $l(\cdot)$ is a bounded linear functional on V. Therefore Lax-Milgram's theorem tells us that there uniquely exists a solution to the variational problem of (DBVP).

2.8.4 A general second order elliptic problem

Consider the problem

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) \qquad \forall\, x \in \Omega \tag{2.12}$$

with $u = 0$ on $\partial\Omega$. Multiply by a test funciton and integrate by parts in the second order term using the divergence theorem. The result is the weak (variational) form of the BVP: find $u \in V$ such that $a(u, v) = l(v)\ \forall\, v \in V$ where $V = H_0^1(\Omega)$ and

$$\begin{aligned} a(w, v) &:= \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{n} \int_\Omega b_i(x)\frac{\partial w}{\partial x_i} v(x) + \int_\Omega c(x)w(x)v(x), \\ l(v) &:= \int_\Omega f(x)v(x) = (f, v). \end{aligned}$$

We seek to apply the Lax-Milgram theorem. Recall $(v, w)_{H_0^1(\Omega)} = \int_\Omega vw + \nabla v \nabla w = (v, w) + (\nabla v, \nabla w)$. We have three conditions to check to satisfy the theorem.

(1) Is $l(\cdot)$ a bounded linear functional? Clearly

$$l(\alpha v + \beta w) = (f, \alpha v + \beta w) = \alpha(f, v) + \beta(f, w) = \alpha l(v) + \beta l(w)$$

so $l(\cdot)$ is a linear functional on V and

$$|l(v)| = \left|\int_\Omega f(x)v(x)\ dx\right| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{H_0^1(\Omega)}$$

where we have used the Cauchy-Schwartz inequality and thus $l(\cdot)$ is bounded.

(2) Is $a(\cdot, \cdot)$ bounded? Assume that $\|a_{ij}\|_{L^\infty(\Omega)}, \|b_i\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}$ are all bounded for all

i, j and that $f \in L^2(\Omega)$. Then

$$
\begin{aligned}
|a(w,v)| &\leq \left| \sum_{i,j=1}^{n} \int_\Omega a_{ij} w_{x_i} v_{x_j} \ dx \right| + \left| \sum_{i=1}^{n} \int_\Omega b_i w_{x_i} v \ dx \right| + \left| \int_\Omega cwv \ dx \right| \\
&\leq \sum_{i,j=1}^{n} \max_{x\in\Omega} |a_{ij}(x)| \int_\Omega |w_{x_i}||v_{x_j}| \ dx \\
&+ \sum_{i=1}^{n} \max_{x\in\Omega} |b_i(x)| \int_\Omega |w_{x_i}||v| \ dx + \max_{x\in\Omega} |c(x)| \int_\Omega |w||v| \ dx \\
&\leq \tilde{c} \left(\sum_{i,j=1}^{n} \int_\Omega |w_{x_i}||v_{x_j}| \ dx + \sum_{i=1}^{n} \int_\Omega |w_{x_i}||v| \ dx + \int_\Omega |w||v| \ dx \right) \\
&\leq \tilde{c} \left(\sum_{i,j=1}^{n} \|w_{x_i}\| \|v_{x_j}\| + \sum_{i=1}^{n} \|w_{x_i}\| \|v\| + \|w\| \|v\| \right) \\
&\leq \tilde{c} \left(\sum_{i,j=1}^{n} \|w\|_V \|v\|_V + \sum_{i=1}^{n} \|w\|_V \|v\|_V + \|w\|_V \|v\|_V \right) \\
&= c_1 \|w\|_V \|v\|_V
\end{aligned}
$$

where $\tilde{c} = \max\{\max_{ij} \max_\Omega |a_{ij}(x)|, \max_i \max_\Omega |b_i(x)|, \max_\Omega |c(x)|\}$ and $c_1 = \tilde{c}(n^2 + n + 1)$.

(3) Is $a(\cdot,\cdot)$ coercive? The crucial assumption is that the a_{ij} satisfies the ellipticity assumption

$$
\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \tilde{c} \sum_{i=1}^{n} \xi_i^2 \qquad \forall\, (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,\ \forall\, x \in \overline{\Omega}, \tag{2.13}
$$

i.e. for all $x \in \overline{\Omega}$ we must have

$$
\xi^T A(x) \xi \geq \hat{c} \|\xi\|^2 = \hat{c} \xi^T \xi. \tag{2.14}
$$

We also assume that

$$
c(x) - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial b_i(x)}{\partial x_i} \geq 0 \forall\, x \in \Omega. \tag{2.15}
$$

Then

$$
\begin{aligned}
a(v,v) &= \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) v_{x_i} v_{x_j} + \sum_{i=1}^{n} \int_\Omega b_i(x) v_{x_i} v + \int_\Omega c(x) v(x)^2 \\
&\geq \tilde{c} \int_\Omega \sum_{i=1}^{n} v_{x_i}^2 + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial v^2/2}{\partial x_i} + \int cv^2.
\end{aligned}
$$

The middle integral here is $\frac{1}{2}\int_\Omega b\cdot\nabla(v^2)$, which after integration by parts equals $-\frac{1}{2}\int_\Omega v^2\nabla\cdot b$ so that

$$
\begin{aligned}
a(v,v) &\geq \tilde{c}\sum_{i=1}^n \int_\Omega v_{x_i}^2 + \int_\Omega v^2(c(x) - \frac{1}{2}\nabla\cdot b(x)) \\
&\geq \tilde{c}\sum_{i=1}^n \int_\Omega v_{x_i}^2 \\
&= \tilde{c}\|\nabla v\|^2
\end{aligned}
$$

Note that we need $\nabla\cdot b \in L^\infty(\Omega)$ for this to work. We wish to show that

$$
a(v,v) \geq c_0\|v\|_V^2 = c_0(\|v\| + \|\nabla v\|). \tag{2.16}
$$

Recall the Poincare-Friedrichs inequalities

$$
\|v\|^2 \leq c_*\|\nabla v\|^2 \qquad \forall\, v \in H_0^1(\Omega).
$$

Hence

$$
\begin{aligned}
a(v,v) \geq \tilde{c}\|\nabla v\|^2 &\geq \frac{\tilde{c}}{c_*}\|v\|^2 \\
\frac{1}{2}a(v,v) + \frac{1}{2}a(v,v) &\geq \frac{\tilde{c}}{2}\|\nabla v\|^2 + \frac{\tilde{c}}{2c_*}\|v\|^2 \\
&\geq c_0\left(\|\nabla v\|^2 + \|v\|^2\right)
\end{aligned}
$$

2.9 Inhomogeneous Boundary Conditions

Consider the elliptic problem

$$
-\nabla\cdot(p\nabla u) + qu = f \qquad x \in \Omega \tag{2.17}
$$

$$
u = g \qquad x \in \partial\Omega \tag{2.18}
$$

where Ω is a bounded open subset of $\mathbb{R}^2$. We assume that the data p, q, f, g are sufficiently smooth and that

$$
\begin{aligned}
p_M \geq p(x) \geq p_0 > 0 \qquad \forall\, x \in \Omega \\
q_M \geq q(x) \geq q_0 > 0 \qquad \forall\, x \in \Omega
\end{aligned}
$$

Let v be a test function. Multiply by v and integrate:

$$
\begin{aligned}
0 &= -\int_\Omega v\nabla\cdot(p\nabla u) + \int_\Omega quv - \int_\Omega fv \\
&= I_1 + I_2 + I_3
\end{aligned}
$$

Now choosing $\varphi = v, \mathbf{f} = p\nabla u$ in:

$$\begin{aligned}\nabla \cdot (\varphi \mathbf{f}) &= \varphi \nabla \cdot \mathbf{f} + \nabla \varphi \cdot \mathbf{f} \\ \nabla \cdot (vp\nabla u) &= v\nabla \cdot (p\nabla u) + \nabla v \cdot p\nabla u \\ I_1 &= -\int_\Omega \nabla \cdot (vp\nabla u) + \nabla v \cdot p\nabla u \\ &= \int_\Omega p\nabla v \cdot \nabla u - \int_{\partial\Omega} vp\nabla u \cdot \nu\end{aligned}$$

Choosing $v = 0$ on $\partial\Omega$ we have $I_1 = \int_\Omega p\nabla v \cdot \nabla u$. Thus

$$0 = \int_\Omega p\nabla v \cdot \nabla u + quv - fv \qquad \forall\, v \in H_0^1(\Omega). \tag{2.19}$$

Set $V_0 = H_0^1(\Omega), a(u,v) = \int_\Omega p\nabla v \cdot \nabla u + quv, l(v) = \int_\Omega fv$. Note that $u \notin V_0$. However $g \in H^1(\Omega)$ so $u - g \in H^1(\Omega)$ and $u - g \in H_0^1(\Omega) = V_0$, i.e $u \in V_g := \{w \in V = H^1(\Omega) : w = g + v, v \in V_0\}$.

Thus our variational problem (P) is to find $u \in V_g$ such that $a(u,v) = l(v)\; \forall\, v \in V_0$. Observe that V_0 is a linear space but $V_g = g + V_0$ is an affine space. We can't apply Lax-Milgram directly. Consider $u^* = u - g \in V_0$:

$$a(u^* + g, v) = a(u,v) = l(v) \qquad \forall\, v \in V_0$$

so $u^* \in V_0$ solves

$$a(u^*, v) = l(v) - a(g,v) =: l^*(v) \qquad \forall\, v \in V_0$$

Now we just need to check Lax-Milgram for this problem. Clearly $a(\cdot,\cdot)$ is bilinear (and symmetric). Coercivity:

$$a(v,v) = \int_\Omega (p|\nabla v|^2 + qv^2)\, dx \geq p_0 \int_\Omega |\nabla v|^2\, dx + q_0 \int_\Omega v^2\, dx \geq \min(p_0, q_0) \int_\Omega (|\nabla v|^2 + v^2)\, dx \geq c_0 \|v\|_{H^1(}$$

Boundedness: using the Cauchy-Schwartz inequality we have

$$\begin{aligned}|a(w,v)| &= \left| \int_\Omega p\nabla w \nabla v + qwv \right| \leq p_M \int_\Omega |\nabla w||\nabla v| + q_M \int_\Omega |w||v| \\ &\leq \max(p_M, q_M)(\|\nabla w\|\|\nabla v\| + \|w\|\|v\|) \\ &\leq \tilde{c}\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.\end{aligned}$$

Clearly l^* is linear and

$$|l^*(v)| = |l(v) - a(g,v)| \leq |l(v)| + |a(g,v)| \leq \|f\|\|v\| + \tilde{c}\|g\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \leq L^*\|v\|_{H^1(\Omega)}.$$

Thus there exists a unique u^* and we conclude therefore that there exists a unique $u = u^* + g$.

The bilinear form is symmetric so there is an energy and associated minimisation problem:

$$\begin{aligned}&J(v) = \frac{1}{2}a(v,v) - l(v) \\ &\text{Find } u \in V_g \; s.t. \qquad J(u) \leq J(v) \qquad \forall\, v \in V_g.\end{aligned}$$

Exercise: Prove that these two problems are equivalent.

Chapter 3

An introduction to nonlinear elliptic equations

Ω is a bounded domain in $\mathbb{R}^n$.

Theorem 3.0.1 (*Dominated convergence theorem*)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Let g_j be a sequence of functions in $L^p(\Omega)$ with*

$$||g_j||_{L^p(\Omega)} \leq C \quad \forall \quad j.$$

If $g \in L^p(\Omega)$ and $g_j \to g \quad a.e.$ then

$$g_j \to g \quad in \quad L^p(\Omega).$$

3.1 Elementary functions on function spaces

Since we are interested in nonlinear partial differential equations it is necessary to introduce $f(u)$ for $u \in L^p(\Omega)$. Given $f \in C(\mathbb{R})$ we set $f(u)$ to be $f(u)(x) = f(u(x))$ $x \in \Omega$.

Lemma 3.1.1. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be C^1 and $\theta' \in L^\infty(\mathbb{R})$. Then $\Theta : H^1(\Omega) \to H^1(\Omega)$ is continuous and $\Theta : H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous in the case $\theta(0) = 0$ where $\Theta(u)(x) := \theta(u(x))$ It holds that*

$$\frac{\partial}{\partial x_i}\theta(u) = \theta'(u)\frac{\partial u}{\partial x_i} \tag{3.1}$$

where we use the notation $\theta(u)$ for convenience.

Proof. See [16, 13]. We have $|\theta'(\cdot)| \leq M$.

Since $u \in H^1(\Omega)$ there exists a sequence u_m of $C^1(\Omega)$ functions such that u_m converges to u in $H^1(\Omega)$ and also u_m converges to u *a.e.* in Ω. Obviously $\theta(u_m) \in C^1(\bar{\Omega})$ and since

$$|\theta(u_m) - \theta(u)| \leq M|u_n - u|$$

$\theta(u_m)$ converges to $\theta(u)$ in $L^2(\Omega)$.

On the other hand

$$\theta'(u_m)\frac{\partial u_m}{\partial x_i} = \frac{\partial}{\partial x_i}\theta(u_m) \to \theta'(u)\frac{\partial u}{\partial x_i} \quad \text{in} \quad L^2(\Omega).$$

This follows by:- (i)

$$\theta'(u_m)\frac{\partial u_m}{\partial x_i} - \theta'(u)\frac{\partial u}{\partial x_i} = \theta'(u_m)[\frac{\partial u_m}{\partial x_i} - \frac{\partial u}{\partial x_i}] + [\theta'(u_m) - \theta'(u)]\frac{\partial u}{\partial x_i} = A_m + B_m$$

and

(ii)

observing that A_m converges to zero in $L^2(\Omega)$ and B_m converges to zero a.e. in Ω and

$$|B_m|^2 \leq (2M)^2|]\frac{\partial u}{\partial x_i}|^2$$

so by the dominated convergence theorem, B_m converges to zero in $L^2(\Omega)$.

Since the derivatives in the sense of distributions of $\theta(u)$ are the limit in $L^2(\Omega)$ of $\frac{\partial}{\partial x_i}\theta(u_m)$ we have proved (3.1). $\square$

Definition 3.1.2. Max, Min, and Mod in $H^1(\Omega)$ Set

$$u^+ := max(u,0) \quad = \quad \begin{cases} u(x) \text{ if } \ u(x) \geq 0 \\ 0 \text{ if } \ u(x) \leq 0 \end{cases} \tag{3.2}$$

$$u^- := max(-u,0) = -min(u,0) \quad = \quad \begin{cases} -u(x) \text{ if } \ -u(x) \geq 0 \\ 0 \text{ if } \ u(x) \geq 0 \end{cases} \tag{3.3}$$

Then

$$u = u^+ - u^-, \qquad |u| = u^+ + u^-.$$

$$\chi_{\{u>0\}}(x) \begin{cases} = 1 \ \text{ if } u(x) > 0 \\ = 0 \ \text{ if } u(x) \leq 0. \end{cases}$$

$$sign(u)(x) \begin{cases} = 1 \ \text{ if } u(x) > 0 \\ = -1 \ \text{ if } u(x) < 0 \\ = 0 \ \text{ if } u(x) = 0. \end{cases}$$

Theorem 3.1.3. *If Ω is a bounded domain and $u \in H^1(\Omega)$ then u^-, u^- and $|u| \in H^1(\Omega)$.*

Proof. Consider the global Lipschitz C^1 functions $\theta_\epsilon(\cdot) : \mathbb{R} \to \mathbb{R}$

$$\theta_\epsilon(\cdot) = \begin{cases} |\cdot|_\epsilon \\ (\cdot)_\epsilon^+ \end{cases}$$

defined by

$$|r|_\epsilon := (r^2+\epsilon^2)^{\frac{1}{2}}, \quad (r)_\epsilon^+ = \begin{cases} (r^2+\epsilon^2)^{\frac{1}{2}} - \epsilon & \text{if} \ \ r > 0 \\ 0 \ \ \text{if } r \le 0 \end{cases}$$

which have Lipschitz constant 1. Then

$$\frac{\partial}{\partial x_i}\theta_\epsilon(u) = \theta_\epsilon'(u)\frac{\partial u}{\partial x_i}$$

and

$$\theta_\epsilon(u) \in H^1(\Omega)$$

$$\theta_\epsilon(u) \to \theta(u) \ a.e. \ \text{ in } \Omega, \quad |\theta_\epsilon(u)| \le |u|$$

so

$$\theta_\epsilon(u) \to \theta(u) \ \text{ in } \ L^2(\Omega).$$

Here $\theta(r) = |r|$ or $\theta(r) = (r)^+$.

Also we have

$$\frac{\partial}{\partial x_i}\theta_\epsilon(u) = \theta_\epsilon'(u)\frac{\partial u}{\partial x_i}$$

and

$$\theta_\epsilon'(u) \to \begin{cases} \chi_{\{u>0\}} \\ sign(u) \end{cases} \qquad a.e. \ \text{ in } \ \Omega$$

so we deduce that

$$\frac{\partial}{\partial x_i}\theta_\epsilon(u) = \theta_\epsilon'(u)\frac{\partial u}{\partial x_i} \to \begin{cases} \chi_{\{u>0\}}\frac{\partial u}{\partial x_i} \\ sign(u)\frac{\partial u}{\partial x_i} \end{cases} \qquad \text{in } \ L^2(\Omega).$$

□

Lemma 3.1.4. *Let $u \in L^1_{loc}(\Omega)$, $\frac{\partial u}{\partial x_i} \in L^1_{loc}(\Omega)$. Then u^-, u^- and $|u| \in L^1_{loc}(\Omega)$ and*

$$\frac{\partial u^+}{\partial x_i} = \chi_{\{u>0\}}\frac{\partial u}{\partial x_i}$$

where $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$, i.e.

$$\chi_{\{u>0\}}(x) \begin{cases} = 1 & \text{if } u(x) > 0 \\ = 0 & \text{if } u(x) \le 0. \end{cases}$$

Let c be a constant then for $u \in H^1(\Omega)$

$$\nabla u = 0 \quad a.e. \quad in \quad \{u(x) = c\}.$$

Definition 3.1.5. Let $v \in H^1(\Omega)$. Then we say that

$$v \leq 0 \quad \text{on} \quad \partial\Omega$$

if and only if $v^+ \in H_0^1(\Omega)$.

3.2 Weak maximum/comparison principle

Let $\boldsymbol{A}$ be an $n \times n$ matrix with coefficients

$$a_{ij} \in L^\infty(\Omega)$$

and for all $\boldsymbol{\xi} \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} \xi_i a_{ij}(x) \xi_j \geq a_0 |\boldsymbol{\xi}|^2 \quad a.e. \quad \text{in} \quad \Omega.$$

Set

$$a(u.v) := \int_\Omega \boldsymbol{A} \nabla u \cdot \nabla v dx.$$

Theorem 3.2.1. *Let $f_1, f_2 \in L^2(\Omega)$ and $\phi_1, \phi_2 \in H^1(\Omega)$. Suppose*

$$f_1 \leq f_2 \ a.e. \quad in \quad \Omega, \quad \phi_1 \leq \phi_2 \quad on \quad \partial\Omega.$$

Then the unique solutions of the boundary value problem

$$u_i = \phi_i \quad on \quad \partial\Omega, \quad -div(\boldsymbol{A}\nabla u_i) = f_i \quad in \quad \Omega$$

written in variational form as

$$u_i - \phi_i \in H^1(\Omega): \ a(u_i, v) = (f, v) \quad \forall v \in H^1(\Omega)$$

satisfy

$$u_1 \leq u_2 \quad a.e. \quad in \quad \Omega.$$

Proof. By subtraction

$$a(u_1 - u_2, v) = (f_1 - f_2, v) \quad \forall v \in H^1(\Omega)$$

and since $(u_1 - u_2)^+ \in H_0^1(\Omega)$ it follows that

$$a(u_1 - u_2, (u_1 - u_2)^+) = (f_1 - f_2, (u_1 - u_2)^+) \leq 0.$$

The result follows by noting that

$$\int_\Omega \boldsymbol{A}\nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)^+ dx = \int_\Omega \boldsymbol{A}\nabla(u_1 - u_2)^+ \cdot \nabla(u_1 - u_2)^+ dx.$$

$\square$

3.3 A compactness and finite dimensional approximation method

Definition 3.3.1. Caretheodory function The function $a : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is said to be a *Caretheodory function* provided it satisfies

1. for $a.e.\ x \in \Omega, \quad u \to a(x,u)$ is continuous from $\mathbb{R}$ into $\mathbb{R}$
2. $\forall u \in \mathbb{R}, \quad x \to a(x,u)$ is measurable

Let $a : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a *Caretheodory function* and satisfy

$$0 \leq A_0 \leq a(x,u) \leq A_M \quad a.e.\ x \in \Omega \quad \forall u \in \mathbb{R} \tag{3.4}$$

for positive constants A_0 and A_M.This is a nonlinear system of equations for the coefficients α_j

Consider the boundary value problem

$$\begin{cases} -\nabla(a(x,u)\nabla u) = f & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.5}$$

where $f \in L^2(\Omega)$ is given.

Theorem 3.3.2. *There exists a weak solution $u \in H_0^1(\Omega)$ of (3.5).*

Proof. We seek a solution in $V := H_0^1(\Omega)$ which, recalling Poincare's inequality,

$$||v||_{L^2(\Omega)} \leq C_P ||\nabla v||_{L^2(\Omega)} \quad \forall v \in H^1(\Omega)$$

we endow with the norm $||v||_V := ||\nabla v||_{L^2(\Omega)}$.

We use a *Galerkin method*, a fixed point theorem in finite dimensions and compactness. Let V_m be a finite dimensional subspace of $V := H_0^1(\Omega)$ with the approximation property that

$$\forall v \ \in V \ \exists\ v_m \in V_m \text{ such that } v_m \to v \ \text{ in } \ V.$$

V_m could be a finite element space or be spanned by eigenfunctions of a linear elliptic operator. Set

$$a(w : u, v) := \int_\Omega a(x,w)\nabla u \cdot \nabla v dx, \ \ u, v \ \in V$$

where w is given in V.

Variational problem

($\mathcal{P}$) Find $u \in V$ such that
$$a(u; u, v) = (f, v) \;\; \forall \;\; v \in V. \tag{3.6}$$

Finite dimensional approximation

($\mathcal{P}_{\mathbf{m}}$) Find $u_m \in V_m$ such that
$$a(u_m; u_m, v_m) = (f, v_m) \;\; \forall \;\; v_m \in V_m. \tag{3.7}$$

Fixed point problem in finite dimensions

Given $w_m = \sum_{j=1}^{m} \beta_j \phi_j^m \in V_m$ where the ϕ_j^m are the basis functions of V_m, set $U_m = \sum_{j=1}^{m} \alpha_j \phi_j^m \in V_m$ to be the unique solution of

$$U_m \in V_m : \quad a(w_m; U_m, v_m) = (f, v_m) \forall v_m \in V_m. \tag{3.8}$$

That U_m exists and is unique follows by the Lax-Milgram theorem and the following standard estimate holds

$$||\nabla U_m||_{L^2(\Omega)} \leq C_P \frac{||f||_{L^2(\Omega)}}{A_0} := c^*. \tag{3.9}$$

Thus we have constructed a map $G_m : V_m \to V_m$ by $G_m(w_m) := U_m$. and if we can show that G_m has a *fixed point* i.e.
$$G_m(u_m^*) = u_m^*$$
then we have constructed a solution of $\mathcal{P}_m$. In order to do this we will apply the following Brouwer fixed-point theorem. Of course it would be nice to use the contraction mapping theorem whenever it is applicable. However in general in this setting the mapping is not a contraction.

Theorem 3.3.3. Brouwer fixed-point theorem

Let $K \subset \mathbb{R}^n$ be a compact convex set and $F : K \to K$ be a continuous mapping. Then there exists a fixed point $u \in K$ of F i.e. $u = F(u)$.

Proof. A version is proved in [23].

□

To use this we formulate the fixed point theorem in terms of the coefficients α_j and β_j. Note that the discrete problem is

$$\sum_{j=1}^{m} \alpha_j \int_\Omega a(x, \sum_{j=1}^{m} \beta_j \phi_j^m) \nabla \phi_j^m \nabla \phi_i^m dx = (f, \phi_i^m) \;\; \forall i = 1, 2 ... m. \tag{3.10}$$

We may write this as

$$\mathcal{S}(\boldsymbol{\beta}) := \boldsymbol{\alpha}$$

and noting the a priori estimate (3.9) we define K_m and $\hat{K}_m$ to be the closed convex sets of H_0^1 and $\mathbb{R}^m$ by

$$K_m := \{v_m \in V_m \mid v_m := \sum_{j=1}^{m} \gamma_j \phi_j^m \text{ satisfies } ||\nabla v_m||_{L^2(\Omega)} \leq c^*\}$$

$$\hat{K}_m := \{\boldsymbol{\gamma} \mid v_m := \sum_{j=1}^{m} \gamma_j \phi_j^m \text{ satisfies } ||\gamma||_m := ||\nabla v_m||_{L^2(\Omega)} \leq c^*\}$$

and note that $\mathcal{S} : \hat{K}_m \to \hat{K}_m$.

We wish to apply the Brouwer fixed point theorem to obtain a fixed point of $\mathcal{S}$ and hence obtain a u_m solving $\mathcal{P}_{\mathbf{m}}$.

(1) It is straightforward to see that $\hat{K}_m$ is convex and compact. (Show that $||\cdot||_m$ is a norm on $\mathbb{R}^m$.)

(2) We now show that $\mathcal{S}(\cdot)$ is continuous.

Let $\boldsymbol{\beta}^n$ be a sequence converging to $\boldsymbol{\beta}$ then we may define $\boldsymbol{\alpha}^n$ given by

$$\sum_{l=1}^{m} \alpha_l^n \int_\Omega a(x, \sum_{j=1}^{m} \beta_j^n \phi_j^m) \nabla \phi_l^m \nabla \phi_i^m dx = (f, \phi_i^m) \quad \forall i = 1, 2 \ldots m. \tag{3.11}$$

Continuity of $\mathcal{S}(\cdot)$ follows from showing that

$$\mathcal{S}(\boldsymbol{\beta}^n) = \boldsymbol{\alpha}^n \to \boldsymbol{\alpha}$$

where $S(\beta) = \alpha$.

Since $U^n = \sum_{j=1}^m \alpha_j^n \phi_j^m \in K_m$ and $\hat{K}_m$ is compact, there is a subsequence $\boldsymbol{\alpha}^{n_k}$ which converges to an $\boldsymbol{\alpha}^*$ and $U^{n_k} \to U^* = \sum \alpha_l^* \phi_l^m$ in V_m. Also since for *a.e.* $x \in \Omega$ $a(x, r)$ is continuous in r we have that

$$\int_\Omega a(x, \sum_{j=1}^{m} \beta_j^{n_k} \phi_j^m) \sum_l \alpha_l^{n_k} \nabla \phi_l^m \nabla \phi_i^m \to \int_\Omega a(x, \sum_{j=1}^{m} \beta_j \phi_j^m) \sum_l \alpha_l^* \nabla \phi_l^m \nabla \phi_i^m.$$

Thus U^* satisfies

$$U^* \in V_m : \quad a(w_m; U^*, v_m) = (f, v_m) \forall v_m \in V_m$$

and by the uniqueness of U it holds that $U^* = U = \sum_l \alpha_l \phi_l^m$, i.e. $\alpha^* = \alpha$. Thus we have shown that $\mathcal{S}(\boldsymbol{\beta}^{n_k}) \to \alpha^* = \alpha = S(\boldsymbol{\beta})$. Since the limit function U is unique we have that the whole sequence converges and we have proved that $\mathcal{S}(\cdot)$ is continuous.

Let $u_m := \sum_{j=1}^m \boldsymbol{\gamma}_j \phi_j^m$ where $\boldsymbol{\gamma}$ is a fixed point of $\mathcal{S}(\cdot)$. It follows that u_m solves $\mathbf{P_m}$.

Passage to the limit

We now wish to consider the convergence of u_m as $m \to \infty$. First observe that

$$||u_m||_V \leq c^* \quad \forall m.$$

Using the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ there exists a subsequence u_{m_k} such that

$$u_{m_k} \to u \ \text{ weakly in } \ H_0^1(\Omega), \ u_{m_k} \to u \text{ in } L^2(\Omega), \ u_{m_k} \to u \ \ a.e. \ \text{ in } \ \Omega.$$

Let $v \in V$ and

$$v_m \to v \ \text{ in } \ V.$$

Applying the dominated convergence theorem we have

$$a(\cdot, u_{m_k})\nabla v_{m_k} \to a(\cdot, u)\nabla v \ \text{ in } \ L^2(\Omega)$$

which allows us to pass to the limit in

$$\int_\Omega a(x, u_{m_k})\nabla u_{m_k} \cdot \nabla v_{m_k} dx = \int_\Omega f v_{m_k} dx$$

and obtain

$$\int_\Omega a(x, u)\nabla u \cdot \nabla v dx = \int_\Omega f v dx.$$

Thus we have shown existence of a solution.

□

Remark 3.3.4. Let $a(x, u) = k(u)$, i.e. there is no x dependence in the coefficient a. We suppose that

$$K_M \geq k(r) \geq K_m > 0 \ \ \forall r.$$

By considering the *Kirchoff* transformation

$$w := \int_0^u k(r)dr \tag{3.12}$$

we may show the existence and uniqueness of a solution to

$$\begin{cases} -\nabla(k(u)\nabla u) = f & \text{in} \quad \Omega \\ u = 0 \ \text{ on } \ \partial\Omega \end{cases} \tag{3.13}$$

where $f \in L^2(\Omega)$ is given.

3.4 Monotonicity method

In this section we consider an example of a *monotone operator*. Our setting is that of a quasilinear second order elliptic equation.

Definition 3.4.1.

- A vector field $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone* provided

$$\sum_{k=1}^{n}(A_k(\boldsymbol{p}) - A_k(\boldsymbol{q}))(p_k - q_k) \geq 0 \tag{3.14}$$

  for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$.

- A vector field $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is called *strictly monotone* provided there exists $\delta > 0$ such that

$$\sum_{k=1}^{n}(A_k(\boldsymbol{p}) - A_k(\boldsymbol{q}))(p_k - q_k) \geq \delta|\boldsymbol{p} - \boldsymbol{q}|^2 \tag{3.15}$$

  for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$.

- We say that $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is *bounded* provided

$$|\boldsymbol{A}(\mathbf{p})| \leq C(|\mathbf{p}| + 1). \tag{3.16}$$

- We say that $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ is *coercive* provided there exist $\gamma > 0, \beta \geq 0$ such that

$$\boldsymbol{A}(\mathbf{p}) \cdot \mathbf{p} \geq \gamma|\mathbf{p}|^2 - \beta. \tag{3.17}$$

PDE and variational form

Let $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping , $f \in L^2(\Omega)$ and Ω be a bounded domain in $\mathbb{R}^n$. We seek a solution to the boundary value problem

$$\begin{aligned} -\nabla \cdot \boldsymbol{A}(\nabla u) &= f && (3.18)\\ u &= 0 \ \text{ on } \ \partial\Omega && (3.19) \end{aligned}$$

which has the variational formulation

$$u \in V : \quad \int_\Omega \boldsymbol{A}(\nabla u) \cdot \nabla v dx = \int_\Omega f v dx \ \ \forall v \in V. \tag{3.20}$$

Finite dimensional approximation

Let V_m be a finite dimensional subspace of $V := H_0^1(\Omega)$ with a basis $\{\phi_j^m\}$. We assume the approximation property that

$$\forall v \ \in V \ \exists \ v_m \in V \text{ such that } v_m \to v \ \text{ in } \ V.$$

Consider the Galerkin approximation:

$$u_m = \sum_{j=1}^{m} \alpha_j \phi_j^m \in V_m : \quad \int_\Omega \boldsymbol{A}(\nabla u_m) \cdot \nabla v_m dx = \int_\Omega f v_m dx \quad \forall v_m \in V_m. \tag{3.21}$$

Lemma 3.4.2. *If $\boldsymbol{A}$ is coercive then the discrete problem has a solution. If $\boldsymbol{A}$ is strictly monotone then the discrete problem has at most one solution. If $\boldsymbol{A}$ is coercive then the discrete solution satisfies*

$$||\nabla u_m||_{L^2(\Omega)} \leq C(1 + ||f||_{L^2(\Omega)}) \tag{3.22}$$

where C depends on $\boldsymbol{A}$ and Ω.

Proof. Existence

The Galerkin approximation is equivalent to

$$\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0}$$

where

$$\boldsymbol{F}(\boldsymbol{\alpha})_i := \int_\Omega \boldsymbol{A}(\nabla \sum_{j=1}^{m} \alpha_j \phi_j^m) \cdot \nabla \phi_i^m dx - \int_\Omega f \phi_i^m dx.$$

Observe that

$$\boldsymbol{F}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} = \int_\Omega \boldsymbol{A}(\nabla \sum_{j=1}^{m} \alpha_j \phi_j^m) \cdot \sum_{i=1}^{m} \alpha_i \nabla \phi_i^m dx - \int_\Omega f \sum_{i=1}^{m} \alpha_i \phi_i^m dx.$$

Using the coercivity of $\boldsymbol{A}$ we find that

$$\boldsymbol{F}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} \geq \gamma \boldsymbol{\alpha} \cdot \mathcal{S} \boldsymbol{\alpha} - \beta |\Omega| - \boldsymbol{\alpha} \cdot \boldsymbol{f}$$

where $\mathcal{S}_{ij} = \int_\Omega \nabla \phi_i^m \nabla \phi_j^m dx$ and $\boldsymbol{f}_j = \int_\Omega f \phi_i^m dx$. The (stiffness) matrix $\mathcal{S}$ is positive definite so for all $|\boldsymbol{\alpha}|$ sufficiently large $\boldsymbol{F}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} \geq 0$ and we can apply the lemma 10.12.2 to yield the existence of a solution to $\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0}$.

Lemma 3.4.3. *Let a continuous mapping $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ satisfy*

$$\mathbf{F}(\boldsymbol{x}) \cdot \boldsymbol{x} \geq 0 \quad \textit{if} \quad |\boldsymbol{x}| = r \tag{3.23}$$

for some $r > 0$. Then there exists $\boldsymbol{x} \in B(\boldsymbol{0}, r)$ such that

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}.$$

Proof. Suppose the assertion is false. Then $\boldsymbol{F}(\boldsymbol{x}) \neq 0 \ \forall \boldsymbol{x} \in B(\boldsymbol{0}, r)$. Define the continuous mapping $S : B(\boldsymbol{0}, r) \to \partial B(\boldsymbol{0}, r)$ by

$$S(\boldsymbol{x}) := -\frac{r}{|\boldsymbol{F}(\boldsymbol{x})|} \boldsymbol{F}(\boldsymbol{x}) \quad \boldsymbol{x} \in B(\boldsymbol{0}, r).$$

By the Brouwer fixed point theorem there exists a point $\boldsymbol{z} \in B(\mathbf{0}, r)$ such that

$$S(\boldsymbol{z}) = \boldsymbol{z}.$$

But it also holds that $\boldsymbol{z} \in \partial B(\mathbf{0}, r)$ so that

$$r^2 = |\boldsymbol{z}|^2 = S(\boldsymbol{z}) \cdot \boldsymbol{z} = -\frac{r}{|\boldsymbol{F}(\boldsymbol{x})|} \boldsymbol{F}(\boldsymbol{x}) \cdot \boldsymbol{z} \leq 0$$

which is a contradiction.

$\square$

Uniqueness

Strict monotonicity of $\boldsymbol{A}$ immediately implies uniqueness since for two solutions we have

$$\int_\Omega (\boldsymbol{A}(\nabla u_m^1) - \boldsymbol{A}(\nabla u_m^2)) \cdot \nabla v_m dx = 0, \ \forall v_m \in V_m$$

and we may take $v_m = u_m^1 - u_m^2$.

Energy estimate

Take $v_m = u_m$ in the variational form to give

$$\int_\Omega \boldsymbol{A}(\nabla u_m) \cdot \nabla u_m dx = \int_\Omega f u_m dx$$

and using coercivity on the left hand side and Young's inequality and Poincare on the left hand side gives the desired bound.

$\square$

Passage to the limit

The uniform $H_0^1(\Omega)$ a priori bound on the sequence $\{u_m\}$ implies that there is a subsequence $\{u_{m_k}\}$ converging weakly in $H_0^1(\Omega)$ to $u \in H_0^1(\Omega)$ so that

$$u_{m_k} \to u \text{ in } L^2(\Omega), \ \nabla u_{m_k} \to \nabla u \ \text{ weakly in } \ L^2(\Omega).$$

In particular we have that $\xi_m := \boldsymbol{A}(\nabla u_m)$ is uniformly bounded in $L^2(\Omega)$ from which we deduce that

$$\boldsymbol{\xi}_{m_k} \to \boldsymbol{\xi} \ \text{ weakly in } \ L^2(\Omega)$$

and we easily deduce that

$$\int_\Omega \boldsymbol{\xi} \cdot \nabla v dx = \int_\Omega f v dx \ \ \forall v \in V.$$

However a fundamental issue in the study of nonlinear PDEs is that of passage to the limit in nonlinear functions with respect to weak convergence. That is we unable to deduce simply from the stated convergence facts that

$$\boldsymbol{\xi} = \boldsymbol{A}(\nabla u).$$

In order establish this we use the method of *monotonicity*. First we note that for a monotonic vector field $\boldsymbol{A}$

$$\int_\Omega (\boldsymbol{A}(\nabla u_m) - \boldsymbol{A}(\nabla w)) \cdot (\nabla u_m - \nabla w) dx \geq 0 \quad \forall w \in V.$$

Observe that from the discrete equation

$$\int_\Omega \boldsymbol{A}(\nabla u_m) \cdot \nabla u_m dx = \int_\Omega f u_m dx$$

which implies that

$$\int_\Omega \boldsymbol{A}(\nabla u_{m_k}) \cdot \nabla u_{m_k} dx = \int_\Omega f u_{m_k} dx \to \int_\Omega f u dx = \int_\Omega \boldsymbol{\xi} \cdot \nabla u dx.$$

Furthermore using the weak convergences of the subsequences we have

$$\int_\Omega \boldsymbol{A}(\nabla u_{m_k}) \cdot \nabla w dx \to \int_\Omega \boldsymbol{\xi} \cdot \nabla w dx, \quad \int_\Omega \boldsymbol{A}(\nabla w) \cdot \nabla u_{m_k} dx \to \int_\Omega \boldsymbol{A}(\nabla w) \cdot \nabla u dx.$$

Thus from the discrete problem, the weak convergence and the monotonicity of $\boldsymbol{A}$ we deduce

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla w)) \cdot (\nabla u - \nabla w) dx \geq 0 \quad \forall w \in V. \tag{3.24}$$

The method of *Minty and Browder* is the observation that the above inequality yields the desired equation by consideration of

$$w = u - \lambda v, \quad v \in V, \lambda > 0.$$

Immediately we have

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla u - \lambda \nabla v)) \cdot \nabla v dx \geq 0 \ \forall v \in V.$$

Passing to the limit $\lambda \to 0$ and using the continuity of $\boldsymbol{A}$ we find

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla u)) \cdot \nabla v dx \geq 0 \ \forall v \in V$$

and since this inequality is also true with v replaced by $-v$ we find

$$\int_\Omega (\boldsymbol{\xi} - \boldsymbol{A}(\nabla u)) \cdot \nabla v dx = 0 \ \forall v \in V$$

and

$$\boldsymbol{\xi} = \boldsymbol{A}(\nabla u) \quad \text{in} \quad L^2(\Omega).$$

Theorem 3.4.4. *Let $\boldsymbol{A}$ be a continuous, coercive, bounded and monotonic vector field and $f \in L^2(\Omega)$.. Then there exists a solution to*

$$u \in V : \int_\Omega \boldsymbol{A}(\nabla u) \cdot \nabla v = \int_\Omega f v dx \quad \forall v \in V. \tag{3.25}$$

The solution is unique provided $\boldsymbol{A}$ is strictly monotone.

3.5 Applications

3.5.1 Conservation with a diffusive flux

Consider a scalar quantity (temperature, concentration of mass) $u : \Omega \to \mathbb{R}$ for which there is a flux $\boldsymbol{q}$ such that in any material domain $D \subset \Omega$ the following conservation law holds:-

flux out of D is balanced by the production of the quantity u in Ω.

This is written as

$$\int_{\partial D} \boldsymbol{q} \cdot \nu = \int_D f, \quad \forall D \subset \Omega$$

where f denotes the production rate. This equation may be rewritten using the divergence theorem to obtain

$$\int_D \nabla \cdot \boldsymbol{q} - f = 0 \quad \forall D \subset \Omega$$

and since this is true for all arbitrary D we have

$$\nabla \cdot \boldsymbol{q} - f = 0 \text{ in } \Omega. \tag{3.26}$$

This is the fundamental conservation law.

If we now assume a *constitutive relation* of the form

$$\boldsymbol{q} = -A \nabla u \tag{3.27}$$

then we obtain the following PDE

$$-\nabla \cdot A \nabla u = f \text{ in } \Omega. \tag{3.28}$$

Here A may be an $n \times n$ matrix. We say that the flux $\boldsymbol{q}$ is a *diffusive* flux and that A is a diffusivity tensor. In the case that

$$A = a(x, u)\mathcal{I}$$

we obtain a nonlinear elliptic equation. We may obtain more complicated equations by assuming that the diffusivity depends on ∇u. On the other hand the production rate f may also depend on u and ∇u.

3.5.2 Steady state problems

The *heat (diffusion)* equation

$$u_t = \nabla \cdot k\nabla u + f(x,u) \ x \in \Omega, \ t > 0$$

for $u(x,t)$ *(temperature in the heat equation and density/concentration in the diffusion equation)* is usually posed as an initial value problem for given u_0 with

$$u(x,0) = u_0(x) \ \ x \in \Omega$$

and with some suitable boundary condition. If the solution tends to a time independent function w as $t \to \infty$ then we obtain the steady state elliptic equation

$$-\nabla \cdot k\nabla w = f(x,w) \ x \in \Omega.$$

Here k (*conductivity or diffusivity*) is a given positive constant and f is a source term modelling the generation of heat or a reaction term. Note that k might also depend on the solution.

3.5.3 Advection -diffusion equation

We place ourselves now in the context of the previous subsection but assume a constitutive law of the form

$$\boldsymbol{q} = -A\nabla u + \boldsymbol{v} \cdot u \tag{3.29}$$

where we interpret $\boldsymbol{v}$ as a material velocity field which transports (advects) the scalar field u. Then we are led to the equation

$$-\nabla \cdot A\nabla u + \nabla \cdot u\boldsymbol{v} = f \text{ in } \Omega. \tag{3.30}$$

We call this an advection-diffusion equation.

3.5.4 Surfaces of prescribed curvature

Let Γ be an $n-$dimensional hypersurface in $\mathbb{R}^{n+1}$ and which is a graph $x_{n+1} = u(x), x \in \Omega$ over the $n-$dimensional bounded domain Ω where $u : \Omega \to \mathbb{R}$ so that

$$\Gamma := \{x' \in \mathbb{R}^{n+1} : x' = (x, u(x)), x \in \Omega\}.$$

The area of Γ may be written as

$$|\Gamma| = \mathcal{E}(u) :== \int_\Omega (1 + |\nabla u|^2)^{\frac{1}{2}})dx. \tag{3.31}$$

The *mean curvature* of Γ is given by (see the later chapter on surface partial differential equations)

$$-\nabla \cdot \frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}. \tag{3.32}$$

Given u, v we may write $G(t) = \mathcal{E}(u+tv)$ and see that

$$G'(0) = \int_\Omega \frac{\nabla u \cdot \nabla v}{(1+|\nabla u|^2)^{\frac{1}{2}}} dx.$$

Graph like minimal surfaces

It follows that if we seek to find a graph like surface over the domain Ω which has a prescribed height at the boundary of Ω and which has minimal area we are led to the boundary value problem:

(**M**): Find u

$$-\nabla \cdot \frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}} = 0 \quad \text{in} \quad \Omega \tag{3.33}$$

$$u = g \quad \text{on} \quad \partial\Omega \tag{3.34}$$

or equivalently in the weak form

$$\int_\Omega \frac{\nabla u \cdot \nabla v}{(1+|\nabla u|^2)^{\frac{1}{2}}} dx = 0, \tag{3.35}$$

for v in a suitable test space.

Surfaces of prescribed curvature

Given f find a graph like surface Γ spanning Ω by solving the boundary value problem: (**P.C.**): Find u

$$-\nabla \cdot \frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}} = f \quad \text{in} \quad \Omega \tag{3.36}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.37}$$

Remark 3.5.1. The vector field

$$\boldsymbol{A}(\boldsymbol{p}) = \frac{\boldsymbol{p}}{(1+|\boldsymbol{p}|^2)^{1/2}}$$

is not coercive in the sense of the definition (3.4.1).

3.5.5 Flow in porous media

Flow in porous media is modelled using D'Arcy's law which states that the velocity field $\boldsymbol{q}$ in a saturated porous medium is given by

$$\boldsymbol{q} = -\frac{K}{\mu}\nabla p \tag{3.38}$$

where p is the pressure, K is the permeability tensor and μ is the fluid viscosity. In the case of an incompressible fluid

$$\nabla \cdot \boldsymbol{q} = 0. \tag{3.39}$$

This leads to an elliptic equation for p.

Chapter 4

Variational Inequalities

4.1 Projection theorem

Theorem 4.1.1. *Let K be a closed convex subset of a Hilbert space H. It follows that*

- *For all $w \in H$ there exists a unique $u \in K$ such that*

  $$||u - w|| = \inf_{\eta \in K} ||\eta - w||_H.$$

  We set

  $$u := \mathbb{P}_K w$$

  and call $\mathbb{P}_K : H \to K$ the projection operator from H onto K.

- $$u = \mathbb{P}_K w \iff u \in K \quad \textit{and} \quad \langle u, \eta - u\rangle_H \geq \langle w, \eta - u\rangle_H \ \forall \eta \in K.$$

- *The operator $\mathbb{P}$ is non-expansive:-*

  $$||\mathbb{P}_K w_1 - \mathbb{P}_K w_2||_H \leq ||w_1 - w_2||_H.$$

Proof. Set

$$\bar{J}(v) := \frac{1}{2}||v - w||^2.$$

Clearly the problem can be formulated in terms of minimzing $\bar{J}(\cdot)$ over K. Also note that

$$\bar{J}(v) = \frac{1}{2}\langle v, v\rangle - \langle w, v\rangle + \frac{1}{2}\langle w, w\rangle$$

so that we may also pose the problem as minimizing

$$J(v) = \frac{1}{2}\langle v, v\rangle - \langle w, v\rangle$$

over K. A more general case is considered in the next section. $\square$

4.2 Elliptic variational inequality

Let K be a closed convex subset of a Hilbert space V. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bounded coercive bilinear form satisfying

1) $a(\cdot,\cdot)$ is bounded, i.e.,

$$\exists \gamma > 0 \text{ s.t. } |a(v, w)| \leq \gamma \|v\|_V \|w\|_V \ \forall v, w \in V.$$

2) $a(\cdot,\cdot)$ is coercive i.e.,

$$\exists \alpha > 0 \text{ s.t. } a(v, v) \geq \alpha \|v\|_V^2 \ \forall v \in V.$$

and $l(\cdot) : V \to \mathbb{R}$ be a bounded linear functional, i.e.,

$$\exists c_l > 0 \text{ s.t. } |l(v)| \leq c_l \|v\|_V \ \forall v \in V.$$

Theorem 4.2.1. *There exists a unique $u \in K$ such that*

$$\textit{(VI)} \quad a(u, v-u) \geq l(v-u) \quad \forall \quad v \in K. \tag{4.1}$$

Furthermore

$$||u_1 - u_2||_V \leq \frac{1}{\alpha}||l_1 - l_2||_{V^*} \tag{4.2}$$

where $u_i, i = 1, 2$ solve (VI) for the linear forms l_1 and l_2, respectively.

Proof. Suppose $a(\cdot,\cdot)$ is symmetric.

Existence

Set $J(\cdot) : V \to \mathbb{R}$ to be the continuous quadratic functional

$$J(v) := \frac{1}{2}a(v, v) - l(v). \tag{4.3}$$

Note that

$$J(v) \geq \frac{\alpha}{2}||v||_V^2 - c_l||v||_V \geq \frac{\alpha}{2}||v||_V^2 - \frac{1}{2\alpha}c_l^2 - \frac{\alpha}{2}||v||_V^2$$

which implies that, since K is non-empty,

$$d := \inf_K J(v) \geq -\frac{1}{2\alpha} c_l^2 > -\infty$$

so that there is a minimizing sequence $u_n \in K$ such that

$$J(u_n) \to d.$$

We wish to establish that there exists $u \in K$ such that

$$u_n \to u \text{ and} J(u) = d.$$

We may choose the minimizing sequence so that

$$d \leq J(u_n) \leq d + \frac{1}{n}.$$

The bilinear form is symmetric so we have (e.g. from the parallelogram law)

$$\alpha||u_n - u_m||^2 \leq a(u_n - u_m, u_n - u_m) = 4J(u_n) + 4J(u_m) - 8J(\frac{1}{2}(u_n + u_m)) \leq 4(\frac{1}{n} + \frac{1}{m}).$$

Hence the sequence $\{u_n\}$ is Cauchy and has a limit u which because K is closed lies in K. Furthermore from the contiuity of $J(\cdot)$ we have

$$J(u_n) \to J(u) = d.$$

We now show that u solves the variational inequality (4.1). For any $v \in K$, because K is convex $u + t(v - u) \in K, \forall\, t \in [0, 1]$ and we have that $G(t) := J(u + \lambda(v - u)) \geq J(u) = G(0)$ and a calculation gives

$$ta(u, v - u) + \frac{t^2}{2} a(u - v, u - v) - tl(v - u) \geq 0 \quad t \in (0, 1)$$

and dividing by t and taking the limit we obtain the variational inequality.

Uniqueness/Well posedness/Stability

From the variational inequalities for two solutions u_1 and u_2 we hve

$$a(u_i, u_j - u_i) \geq l_i(u_j - u_i) \quad i \neq j$$

and adding

$$\alpha||u_1 - u_2||_V^2 \leq a(u_1 - u_2, u_1 - u_2) \leq l_1(u_1 - u_2) - l_2(u_2 - u_1) \leq ||l_1 - l_2||_{V^*} ||u_1 - u_2||_V.$$

General case

We consider the general case without assuming symmetry of the bilinear form. By the Riesz representation theorem we have the existence of $\mathcal{A} \in \mathcal{L}(V, V)$ and $L \in V$ such that

$$\langle \mathcal{A}u, v\rangle = a(u, v) \ \ \forall u, v \in V \ \ \text{and} \ \ \langle L, v\rangle = l(v) \ \forall v \in V$$

where $\mathcal{A}^* = \mathcal{A}$ if $a(\cdot, \cdot)$ is symmetric.

Fix any ρ strictly positive. Then the variational inequality is equivalent to

$$\rho\langle \mathcal{A}u - L, v - u\rangle \geq 0 \ \forall v \in K$$

which may be rewritten as

$$\langle u, v - u\rangle \geq \langle u - \rho(\mathcal{A}u - L), v - u\rangle \ \forall v \in K$$

which is equivalent to finding $u \in K$ such that

$$u = \mathbb{P}_K(u - \rho(\mathcal{A}u - L)). \tag{4.4}$$

Here $\mathbb{P}_K$ is the projection operator from V to K in the Hilbert space V. Recall that it is non-expansive.

For convenience set

$$W_\rho(v) := \mathbb{P}_K(v - \rho(\mathcal{A}v - L)) \ , \forall v \in V.$$

We will show for suitable ρ that $W_\rho(\cdot) : V \to K$ is a strict contraction. Since $\mathbb{P}_K$ is nonexapansive we have

$$||W_\rho(v_1) - W_\rho(v_2)||^2 \leq ||v_1 - v_2||^2 + \rho^2||\mathcal{A}(v_1 - v_2)||^2 - 2\rho a(v_1 - v_2, v_1 - v_2)$$

and

$$||W_\rho(v_1) - W_\rho(v_2)||^2 \leq ||v_1 - v_2||^2 + \rho^2||\mathcal{A}||^2||v_1 - v_|||^2 - 2\rho\alpha||v_1 - v_2||^2$$

and

$$||W_\rho(v_1) - |W_\rho(v_2)|| \leq (1 - \rho||\mathcal{A}||^2(\rho^* - \rho))^{1/2}||v_1 - v_2||.$$

Thus $W_K(\cdot)$ is a strict contraction provided

$$0 < \rho < \frac{2\alpha}{||A||^2} := \rho^*.$$

By taking ρ in this range we have that there is a unique fixed point $u = W_\rho(u) = \mathbb{P}_K(u - \rho(\mathcal{A}u - l)) \in K$ and so the variational inequality has a unique solution.

□

4.3 Truncation in L^2

Let $\Omega \subset \mathbb{R}^n$ be measurable and choose $\varphi \in L^2(\Omega)$. Set

$$K := \{v \in L^2(\Omega) : v \geq \varphi \text{ a.e. in } \Omega\}.$$

Clearly K is non-empty, convex and closed. Set

$$a(u, v) := (u, v)$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product. The problem: find $u \in K$ such that

$$(u - f, v - u) \geq 0 \;\forall v \in K$$

has a unique solution. A calculation reveals that

$$u = max(\varphi, f) := \begin{cases} \varphi(x) \text{ if } f(x) \leq \varphi(x) \\ f(x) \text{ if } \varphi(x) \leq f(x) \end{cases}$$

is satisfies the variational inequality and hence is the unique solution.

4.4 Obstacle problem

Let $V := H_0^1(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^d, d = 1, 2, 3$. Set

$$a(w, v) := \int_\Omega \nabla w \cdot \nabla v \quad , \quad l(v) := \int_\Omega f v$$

where $f \in L^2(\Omega)$ is given. Let $\psi \in H^1(\Omega) \cap C^0(\bar{\Omega})$ and $\psi|_{\partial\Omega} \leq 0$ and set

$$K := \{v \in H_0^1(\Omega) : v \geq \psi a.e. \in \Omega\}.$$

It follows that

1. K is non-empty.

   Set $\psi^+ := \frac{1}{2}(\psi + |\psi|) = \max(\psi, 0))$. Recall the following lemma

   Lemma 4.4.1. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, i.e.,*

   $$|\theta(t_1) - \theta(t_2)| \leq \lambda_\theta |t_1 - t_2| \quad \forall \quad t_1, t_2 \in \mathbb{R}.$$

   Suppose $\theta^{'}$ has a finite number of points of discontinuity. Then $\theta : H^1(\Omega) \to H^1(\Omega)$ is continuous and $\theta : H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous in the case $\theta(0) = 0$.

   Hence $\psi^+ \in H_0^1(\Omega)$ and since $\psi^+ \geq \psi$ it follows that K is non-empty.

2. K is convex because for $t \in [0,1]$, $t\eta + (1-t)v \geq \psi \forall \eta, v \in K$.

3. K is closed.

   This follows from the fact that convergence in $H_0^1(\Omega)$ implies convergence in $L^2(\Omega)$ and hence convergence almost everywhere for a sub-sequence. From which we find that $v_n \to v$ in V implies $v_{n_i} \to v$ *a.e.* in *in* Ω and if $v_n \in K$ then $v_{n_i} \geq \psi$ *a.e. in* Ω which implies by the convergence of v_n that $v \geq \psi$ *a.e. in* Ω and so $v \in K$.

Theorem 4.4.2. *There exists a unique solution to the obstacle problem: Find* $u \in K := \{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$ *such that*

$$\int_\Omega \nabla u \cdot (\nabla v - \nabla u) \geq \int_\Omega f(v-u) \quad \forall v \in K.$$

Linear Complementarity Problem

Suppose we have the regularity result that the unique solution satisfies $u \in H^2(\Omega)$. Then integration by parts in the variational inequality yields

$$\int_\Omega (-\Delta u - f)(v-u) \geq 0 \quad \forall v \in K. \tag{4.5}$$

Choosing $v = u + \eta$ where $\eta \geq 0$ and $\eta \in C_0^\infty(\Omega)$ yields

$$\int_\Omega (-\Delta u - f)\eta \geq 0$$

from which we obtain

$$-\Delta u - f \geq 0 \ \ a.e. \ \ \Omega.$$

Suppose u is continuous. (This is true automatically for $u \in H^2(\Omega)$ when $d = 1, 2$.) Then the set

$$\Omega^+ := \{x \in \Omega : u(x) > \psi(x)\} \tag{4.6}$$

is open. For any $\eta \in C_0^\infty(\Omega^+)$ the function $v = u \pm \epsilon\eta \in K$ provided $|\epsilon|$ is small enough. For such an η we have

$$\int_{\Omega^+} (-\Delta u - f)\eta = 0$$

which gives

$$-\Delta u - f = 0 \quad \text{in} \quad \Omega^+.$$

Thus we have shown that if the solution $u \in H_0^1(\Omega)$ also satisfies the regularity $u \in H^2(\Omega) \cap C(\Omega)$ then it satisfies the *linear complementarity system*:

$$-\Delta u - f \geq 0, \quad u \geq \psi \ \ a.e. \ \ \Omega \tag{4.7}$$

$$(-\Delta u - f)(u - \psi) = 0 \ \ a.e. \ \ \Omega \tag{4.8}$$

$$\tag{4.9}$$

The set

$$\Omega^0 := \{x \in \Omega : u(x) = \psi(x)\} \tag{4.10}$$

is called the *coincidence set* wheres ω^+ is called the non-coincidence set. The boundary of the non-coincidence set in Ω

$$\Gamma := \partial\Omega^+ \cap \Omega \tag{4.11}$$

is called the *free boundary.*

Free boundary problem

It can be shown that (for suitable smooth f, ψ and $\partial\Omega$) that the solution of the above obstacle problem satisfies $u \in C^1(\Omega)$ or $u \in W^{2,p}(\Omega)$. It follows that

$$u - \psi = 0, \quad \nabla(u - \psi) = 0 \quad \text{on} \quad \Gamma.$$

Thus we may view u as being the solution of the following *free boundary problem*: Find u, Γ, Ω^+ such that

$$-\Delta u = f \quad \text{in} \quad \Omega^+ \tag{4.12}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{4.13}$$

$$u = \psi, \ \frac{\partial u}{\partial \nu} = g \qquad \text{on} \quad \Gamma + \partial\Omega^+ \cap \Omega \tag{4.14}$$

where we have set $g := \frac{\partial \psi}{\partial \nu}$ and denoted by ν the normal to Γ.

Note that two conditions hold on Γ. This is because Γ is *unknown.* Such free boundary problems arise in many applications and are formulated as boundary value problems for PDEs rather than in variational form. Often they have a structure which enables them to be formulated as a variational inequality. Accounts of free boundary problems may be found in[21, 26, 44].

4.5 Obstacle problem for a membrane

Consider the situation of a membrane stretched over a rigid obstacle. Here suppose that the membrane is a hyper-surface described by the graph $x_3 = u(x), x = (x_1, x_2) \in \Omega$ where Ω is an open bounded planar domain with boundary $\partial\Omega$. Suppose that the surface of the rigid obstacle is also a graph $x_3 = \psi(x), x = (x_1, x_2) \in \Omega$. The membrane lies over the obstacle so

$$u \geq \psi \quad \text{in} \quad \Omega.$$

We have that the domain is decomposed into two domains $\bar{\Omega} = \bar{\Omega}^+ \cup \bar{\Omega}_I$ where

$$u > \psi \quad \text{in} \quad \Omega^+, \quad u = \psi \quad \text{in} \quad \Omega_I.$$

In equilibrium away from the obstacle the membrane satisfies Laplace's equation and on the obstacle the vertical force on the membrane is non-positive so that

$$-\Delta u = 0 \text{ in } \Omega^+, \quad -\Delta u \geq 0 \text{ in } \Omega_I.$$

At the contact interface $\Gamma := \partial\Omega^+ \cap \partial\Omega_I$ the smoothness conditions

$$u = \psi, \quad \nabla u \cdot \nu = \nabla\psi \cdot \nu$$

hold where the second condition is continuity of the tension within the membrane.

Suppose $u = 0$ and $\psi \leq 0$ on the boundary $\partial\Omega$. This problem may be posed as the obstacle variational inequality described in the previous section with $f = 0$. This is achieved by using integration by parts

$$\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq 0$$

for $v \in K$ and u satisfying the above conditions.

Note that the energy for this problem

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$$

is an approximation to the area functional

$$A(u) := \int_\Omega (1 + |\nabla u|^2)^{1/2}$$

when the gradient of the displacement is small.

4.6 Hele-Shaw problem

A free boundary problems for a PDE is boundary value problems in which not only the solution of the PDE is to be determined but also the domain in which the PDE holds! The *Hele-Shaw* free boundary problem is a famous example which has been widely studied in many different contexts and with widely varying mathematical ideas. It concerns a model for the flow of incompressible viscous fluid in a narrow gap of width h between two parallel plates. The fluid then occupies the region $\{(x_1, x_2, x_3) : x = (x_1, x_2) \in \Omega(t), x_3 \in (0, h)\}$. Because the gap is narrow so called *lubrication* approximation methods may be applied to the three dimensional Navier-Stokes equations to yield a problem in two space dimensions. In short only the velocity components parallel to the plates are considered and they are averaged over the thickness of the gap. The upshot is that the average velocity field $\mathbf{q} = \mathbf{q}(x, t)$ is related to the pressure $p = p(x, t)$ by the equation

$$\mathbf{q} = -\frac{h^2}{12\mu}\nabla p \tag{4.15}$$

where μ is the fluid viscosity. Since the fluid is incompressible $\nabla \cdot \mathbf{q} = \mathbf{0}$ we are led to the equation

$$-\nabla \cdot \frac{h^2}{12\mu} \nabla p = 0, \quad \text{in} \quad \Omega(t). \tag{4.16}$$

The fluid blob $\Omega(t)$ is separated from the part of the cell not occupied by fluid by an *interface* $\Gamma(t)$. Conservation of mass then yields that V_ν, the normal velocity of $\Gamma(t)$, i.e. the velocity of $\Gamma(t)$ in the outward pointing normal direction, ν, to $\Omega(t)$, is given by the fluid velocity yielding

$$V_\nu = -\frac{h^2}{12\mu} \nabla p \cdot \nu. \tag{4.17}$$

In certain physical circumstances the balance of momentum at the fluid/void interface yields that the pressure is constant on that interface. Since the pressure is undetermined up to an additive constant then we may take

$$p = 0, \quad \text{on} \quad \Gamma(t). \tag{4.18}$$

Let us suppose that the fluid initially occupies the domain $\Omega(0) = \Omega_0$.

In order to *drive* this process we need to specify some way of injecting or sucking fluid out of the cell. There are fundamental mathematical differences between injection and suction. For our purpose we consider only the simpler situation of injection. We denote by $Q(t) \geq 0$ the injection rate.

1. **Point source injection**

   Let $\Omega(t) \subset \mathbb{R}^2$ be an open bounded domain with boundary $\partial\Omega(t) = \Gamma(t)$. Let $0 \in \Omega(t)$. Consider the problem of given $\Omega(0) = \Omega_0$ and $Q : (0, T) \to \mathbb{R}_+$, finding:$\{p, \Omega(t), \Gamma(t)\}$ such that

$$-\nabla \cdot \frac{h^2}{12\mu} \nabla p = Q(t)\delta, \quad \text{in} \quad \Omega(t) \tag{4.19}$$

$$V_\nu = -\frac{h^2}{12\mu} \nabla p, \quad p = 0 \quad \text{on} \quad \Gamma(t) \tag{4.20}$$

   where δ is the Dirac delta measure.

2. **Surface injection**

   Let $\Omega(t) \subset \mathbb{R}^2$ be an open bounded annular domain with inner boundary Γ_I and outer moving boundary $\Gamma(t)$ so that $\partial\Omega(t) = \Gamma_I \cup \Gamma(t)$. Consider the *moving boundary problem* of given $\Omega(0) = \Omega_0$ and $Q : (0, T) \to \mathbb{R}_+$, finding:$\{p, \Omega(t), \Gamma(t)\}$ such that

$$-\nabla \cdot \frac{h^2}{12\mu} \nabla p = 0, \quad \text{in} \quad \Omega(t) \tag{4.21}$$

$$V_\nu = -\frac{h^2}{12\mu} \nabla p, \quad p = 0 \quad \text{on} \quad \Gamma(t) \tag{4.22}$$

$$\frac{h^2}{12\mu} \nabla p \cdot \nu = Q(t) \quad \text{on} \quad \Gamma_I. \tag{4.23}$$

Note that we may rescale so for convenience in the following we take $\frac{h^2}{12\mu} = 1$.

The above situations may be complicated further by supposing that the fluid lies in a container D bounded by container walls ∂D which are impervious to the flow. It follows that the above systems should be supplemented by the equation

$$\frac{\partial p}{\partial \nu} = 0 \ \text{ on } \ \Gamma_C(t) := \partial\Omega(t) \cap \partial D. \tag{4.24}$$

Here we are supposing that $\partial\Omega(t) = \Gamma(t) \cup \Gamma_I \cup \Gamma_C(t)$.

Assume that fluid is being injected. As a consequence the pressure in $\Omega(t)$ is positive and the normal derivative of p on $\Gamma(t)$ is non-positive which implies that the expanding fluid blob $\Omega(t))$ has a non-negative normal velocity. Thus we may describe the fluid interface in the following way: there exists a function $\omega(x)$ such that for each $t \geq 0$

$$\Omega(t) = \{x \in D : t > \omega(x)\}, \quad \Gamma(t) = \{x \in D : t = \omega(x)\}, \quad \omega(x) = 0 \ \ x \in \Omega_0.$$

The evolution equation for $\Gamma(t)$ may now be written as

$$\nabla p \cdot \nabla \omega = -1. \tag{4.25}$$

Remark 4.6.1. The function

$$\phi(x,t) := t - \omega(x)$$

may be considered as a *level set function* whose zero level set defines the evolving hypersurface $\Gamma(t)$. The outward pointing unit normal to $\Omega(t)$ and the normal velocity of $\Gamma(t)$ are

$$\nu = -\frac{\nabla\phi}{|\nabla\phi|} = \frac{\nabla\omega}{|\nabla\omega|}, \quad V_\nu = \frac{\phi_t}{|\nabla\phi|} = 1/|\nabla\omega|$$

Remark 4.6.2. Let us observe that

$$\frac{d|\Omega(t)|}{dt} = \int_{\Gamma(t)} V_\nu = -\int_{\Gamma(t)} p_\nu = \int_{\Gamma_I} Q(t)$$

from which we deduce that

$$|\Omega(t)| = |\Omega_0| + \mathcal{Q}(t)|\Gamma_I|.$$

Since when the cell is filled with fluid we can no longer inject fluid there can only be a solution as long as

$$\mathcal{Q}(t) \leq \frac{|D| - |\Omega_0|}{|\Gamma_I|}.$$

Since the injection rate is non-negative we have that there can only be a solution for $t \in [0,T]$ where T is the smallest solution of

$$\mathcal{Q}(T) = \frac{|D| - |\Omega_0|}{|\Gamma_I|}.$$

This should be reflected in the mathematical formulation of the problem.

The Hele-Shaw problem for injection of fluid into a bounded cell D across a portion $\Gamma_I \subset \partial D$ may be formulated as an elliptic variational inequality, see [20], for

$$u(x,t) = \int_0^t p(x,\tau)d\tau, \quad x \in D \tag{4.26}$$

where we have extended the definition of p to all of D using the constant zero extension, i.e.

$$p(x,t) = 0 \ \ x \in D \setminus \Omega(t).$$

Calculations reveal that, for each t, u solves the following *free boundary problem*

$$-\Delta u = f \ \text{ in } \ \Omega(t) \tag{4.27}$$
$$u = 0 \ \text{ in } \ D \setminus \Omega(t) \tag{4.28}$$
$$u \geq 0 \ \text{ in } \ D \tag{4.29}$$
$$u = u_\nu = 0 \ \text{ on } \ \Gamma(t) \tag{4.30}$$
$$u_\nu = \mathcal{Q}(t) \ \text{ on } \ \Gamma_I \tag{4.31}$$

where

$$f = \chi_{\Omega_0} - 1.$$

By consideration of

$$\int_D (-\Delta u - f)(v - u)dx$$

for $v \in K$ we may derive a variational inequality using integration by parts and assuming regularity of $\Gamma(t)$ and u

Set

$$a(u,v) := \int_D \nabla u \cdot \nabla v dx, \quad \langle v, w \rangle = \int_{\Gamma_I} vw, \quad .$$

Then the *elliptic variational inequality* is: Find $u \in K := \{v \in H^1(D) : v \geq 0 \ \ a.e. \ \text{ in } \ D\}$ such that

$$a(u, v-u) \geq (f, v-u) + \langle \mathcal{Q}(t), v-u \rangle \ \ \forall v \in K \tag{4.32}$$

where

$$\mathcal{Q}(t) = \int_0^t Q(\tau)d\tau.$$

Note that in this setting the bilinear form is not coercive on K. However by considering for $v \in K$

$$v = \bar{v} + Pv$$

where

$$Pv := \frac{\int_D v}{|D|}$$

and the fact that for $v \in K$, (using the Poincare inequality for functions with zero mean),

$$||v||_{H^1(\Omega)} \to \infty \iff ||\nabla v||_{L^2(\Omega)} \to \infty \quad \text{or} \quad Pv \to \infty$$

we may show that the functional

$$J(v) := \frac{1}{2}a(v,v) - (f,v) - \langle \mathcal{Q}(t), v \rangle$$

is coercive on K provided

$$0 \leq t < T.$$

This shows existence. Uniqueness of u in this time interval follows by noting that the standard argument yields that the difference of two solutions is a positive constant c and that this constant satisfies

$$(f,c) + \langle \mathcal{Q}(t), c \rangle.$$

Chapter 5

Optimisation

5.1 Introduction

Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $J : X \to \bar{\mathbb{R}}$ where X is a reflexive Banach space. We suppose throughout that $J(\cdot)$ is *proper*:

Definition 5.1.1. A function $J : X \to \bar{\mathbb{R}}$ is said to be proper if it takes the value $-\infty$ nowhere and is not identically $+\infty$. Thus $J : X \to (-\infty, +\infty]$ and $dom(J) := \{x \in X : J(x) < \infty\} \neq \{\emptyset\}$.

Definition 5.1.2. A function $J : X \to \bar{\mathbb{R}}$ is called *Gateaux differentiable* at $u \in X$ provided

$$J'(u; v) := \lim_{\lambda \to 0_+} \frac{J(u + \lambda v) - J(u)}{\lambda}$$

exists for all v. If there exists $J'(u) \in X^*$ such that

$$J'(u; v) = \langle J'(u), v \rangle, \quad \forall\, v \in X$$

then $J'(u)$ is called the *Gateaux-derivative* of $J(\cdot)$ at $u \in X$.

Let C be a nonempty subset of X. (C might be X.) We consider the problem of finding $u \in C$ such that

$$J(u) = \inf_C J(v). \tag{5.1}$$

If such a u exists then we say that u is a minimum.

Definition 5.1.3. The sequence $\{v_n\} \in C$ is said to be a *minimizing sequence* provided

$$J(v_n) \to \inf_C J(v).$$

Suppose $J(\cdot)$ is bounded below on C i.e.

$$J(v) \geq b > -\infty \; \forall v \in C$$

then the set of numbers $S := \{r : \exists v \in C : J(v) = r\}$ has an infimum d for which there is a sequence of numbers $\{r_n\} \in S$ such that $r_n \to d$. Since there exist $v_n \in C$ such that $r_n = J(v_n)$ we have constructed a minimizing sequence. Having constructed a minimising sequence the question now is whether this sequence converges to a limit in C.

Definition 5.1.4.

- $J(\cdot) : X \to \bar{\mathbb{R}}$ is called *lower semi-continuous* provided for all convergent sequences $x_n \to_X x$ we have

$$\liminf_{x_n \to x} J(x_n) \geq J(x).$$

- $J(\cdot) : X \to \bar{\mathbb{R}}$ is called *weakly lower semi-continuous* provided for all convergent sequences $x_n \rightharpoonup_X x$ we have

$$\liminf_{x_n \rightharpoonup x} J(x_n) \geq J(x).$$

Remark 5.1.5. $d = \liminf\{d_n\}$ provided that

$$\forall \epsilon \;\; \exists N \;\; \text{such that} \;\; n > N \to d_n > d - \epsilon$$

and

$$\text{given} \;\; \epsilon > 0 \;\; \text{and} \;\; m > 0, \;\; \exists n > m \;\; \text{such that} \;\; d_n < d + \epsilon.$$

Recall that

Definition 5.1.6. The set $K \subset X$ is said to be *convex* provided

$$\mu v + (1 - \mu)w \in K \;\; \forall v, w \in K \;\; \text{and} \;\; \mu \in [0, 1].$$

Definition 5.1.7. $J(\cdot) : X \to \bar{\mathbb{R}}$ is said to be *convex* if

$$J(\mu v + (1 - \lambda)w) \leq \mu J(v) + (1 - \mu)J(w)$$

for all $v, w \in X$ and $\mu \in (0, 1)$.

Definition 5.1.8. $J(\cdot) : X \to \bar{\mathbb{R}}$ is said to be *strictly convex* if

$$J(\mu v + (1 - \mu)w) < \mu J(v) + (1 - \mu)J(w)$$

for all $v, w \in X, w \neq v$ and $\mu \in (0, 1)$.

Example 5.1.9. *The functional $||\cdot|| : X \to \mathbb{R}$ is convex. This is may be shown by observing from the triangle inequality*

$$||\lambda v + (1 - \lambda)w|| \leq \lambda||v|| + (1 - \lambda)||w|| \;\; \forall \; \lambda \in [0, 1], \;\; \forall \; v, w \in X.$$

Example 5.1.10. *Let H be a Hilbert space. The functional $||\cdot||_H^2 : H \to \mathbb{R}$ is strictly convex. This may be shown by observing that*

$$2(y_1, y_2)_H < (y_1, y_1)_H + (y_2, y_2)_H \quad \textit{for} \quad y_1 \neq y_2.$$

The following lemma may be found in [53].

Theorem 5.1.11 (Mazur)**.** *Let $\{x_n\}$ be a weakly convergent sequence in the Banach space X with weak limit x. Then there exists for each n a convex combination $y_n := \sum_{j=n}^{K(n)} \alpha_j^n x_j$ ($\alpha_j^{K(n)} \geq 0, \sum_{j=n}^{K(n)} \alpha_j^n = 1$) such that the sequence $\{y_n\}$ converges strongly to x.*

This may be used to prove the following theorems in convex analysis.

Theorem 5.1.12. *Let $K \subset X$ be closed and convex. Then*

- *K is weakly sequentially closed.*
- *If K is bounded and X is reflexive (in particular if X is a Hilbert space) then K is weakly sequentially compact.*

Theorem 5.1.13. *Let $J(\cdot) : X \to \bar{\mathbb{R}}$ be convex. Then $J(\cdot)$ is weakly lower semi-continuous if and only if $J(\cdot)$ is lower semi-continuous.*

5.2 Optimisation

5.2.1 Unconstrained optimisation problem

Definition 5.2.1. $J(\cdot) : X \to \bar{\mathbb{R}}$ is said to be *coercive* provided

$$J(x_n) \to \infty \quad \text{for} \quad x_n \to \infty.$$

The proof of the following is an example of the *direct method of the calculus of variations.*

Theorem 5.2.2. *Let $J : X \to \bar{\mathbb{R}}$ be a coercive weakly lower semi-continuous functional. Then there exists a solution to the following unconstrained minimisation problem: Find $u \in X$ such that*

$$J(u) = \inf_{v \in X} J(v). \tag{5.2}$$

Proof. We claim that $J(\cdot)$ is bounded from below. Suppose not then since $J(\cdot)$ is proper (*i.e.* $J(v) > -\infty \ \ \forall v \in X$) either there exists a sequence $\{v_n\}$ such that $||v_n|| \to \infty$ and $J(v_n) \to -\infty$ which contradicts the coercivity of J or there exists a sequence $\{v_n\}$ such that $v_n \to v$ and $J(v_n) \to -\infty$ and $J(v) = \alpha > -\infty$ and by weak lower semi-continuity of $J(\cdot)$

$$-\infty = \liminf_{v_n \to v} J(v_n) \geq \alpha > -\infty$$

which again is a contradiction.

We now prove the existence of a minimiser. Let $d := \inf_{v \in X} J(v) > -\infty$ and let $\{x_n\} \in X$ be a minimising sequence (we showed the existence of such a sequence for functions bounded from below) so that

$$J(x_n) \to \inf_X J(x).$$

Since $J(\cdot)$ is coercive there exists M such that $\{x_n\} \in C = \{v : ||v|| \leq M\}$. Since C is weakly sequentially compact there is a weakly convergent subsequence $\{x_{n'}\}$ with limit $x^* \in C$ and by weak lower semicontinuity

$$d \leq J(x^*) \leq \liminf J(x_{n'}) = \lim J(x_{n'}) = d.$$

Hence the existence of a minimiser x^* is proved.

□

Theorem 5.2.3. *Let $J : X \to \bar{\mathbb{R}}$ be a coercive convex continuous or lower semi-continuous functional. Then there exists a solution to the following unconstrained minimisation problem: Find $u \in X$ such that*

$$J(u) = \inf_{v \in X} J(v). \tag{5.3}$$

If $J(\cdot)$ is convex then the solution is unique.

Proof. Existence follows from the observation that contiunuity implies lower semi continuity and for a convex functional weak lower semi-continuity is equivalent to lower semi-continuity and we can now use the preceding theorem.

If there are two solutions $u_1 \neq u_2$ then with $\mu \in (0, 1)$

$$J(\mu u_1 + (1 - \mu)u_2) < \mu J(u_1) + (1 - \mu)J(u_2) = \inf_X J(v)$$

which is a contradiction. □

Theorem 5.2.4 (Necessary optimality condition)**.** *Let $J : X \to \bar{\mathbb{R}}$ be Gateaux-differentiable in $u \in X$ with Gateaux derivative $J'(u) \in X^*$. Then the variational equation*

$$\langle J'(u), v\rangle = 0 \quad \forall v \in X \tag{5.4}$$

is a necessary condition for $u \in X$ to be a minimiser of $J(\cdot)$.

If $J(\cdot)$ is convex then this condition is also sufficient.

Proof. Let $u \in X$ be a minimiser of $J(\cdot)$ in X. It follows that given $v \in X$ then

$$J(u \pm tv) \geq J(u), \quad \forall t \geq 0$$

Since $J(\cdot)$ is Gateau differentiable this implies that

$$\langle J'(u), v\rangle = 0, \quad \forall v \in X.$$

On the other hand if $J(\cdot)$ is convex and the variational equation

$$\langle J'(u), v\rangle = 0, \quad \forall v \in X$$

holds, then for $t \in (0,1)$ and $v \in K$

$$J(u + t(v-u)) \leq tJ(v) + (1-t)J(u)$$

implies

$$\frac{J(u+t(v-u)) - J(u)}{t} \leq J(v) - J(u)$$

and taking the limit $t \to 0$

$$0 = \langle J'(u), v-u\rangle \leq J(v) - J(u)$$

and we infer that u is a minimum. □

5.2.2 Optimisation of convex functionals over convex sets

Theorem 5.2.5. *Let K be a non-empty convex subset of the Banach space X. Let $J(\cdot) : X \to \bar{\mathbb{R}}$ be Gateaux differentiable. If*

$$J(u) = \inf_{v\in K} J(v)$$

then u solves the variational inequality

$$\langle J'(u), v-u\rangle \geq 0 \quad \forall v \in K \tag{5.5}$$

Conversely if $u \in K$ solves the variational inequality (5.5) and $J(\cdot)$ is convex then u is a solution of the minimization problem (5.1).

Proof. Let $u \in K$ be a minimiser of $J(\cdot)$ in K. It follows that given $v \in K$ then

$$J(u + t(v-u)) \geq J(u), \quad \forall t \in [0,1].$$

(Since $u + t(v-u) \in K,\ t \in [0,1]$.)

It follows that

$$\langle J'(u), v-u\rangle = \lim_{t\to 0_+} \frac{J(u+t(v-u)) - J(u)}{t} \geq 0.$$

Conversely if $u \in K$ solves the variational inequality (5.5) and $J(\cdot)$ is convex then for $t \in (0,1)$ and $v \in K$

$$J(u + t(v-u)) \leq tJ(v) + (1-t)J(u)$$

implies

$$\frac{J(u+t(v-u))-J(u)}{t} \leq J(v)-J(u)$$

and taking the limit $t \to 0_0$

$$0 \leq \langle J'(u), v-u\rangle \leq J(v)-J(u)$$

and we infer that u is a minimum.

□

5.3 Adjoint operators

Definition 5.3.1. Let $\mathcal{A} \in \mathcal{L}(X,Y)$ where X and Y are Banach spaces. The mapping $\mathcal{A}^* : Y^* \to X^*$ defined by

$$\langle \mathcal{A}^* y^*, x\rangle_{X^*,X} := \langle y^*, \mathcal{A}x\rangle_{Y^*,Y}, \quad y^* \in Y^*, x \in X$$

is called the *adjoint* or *dual* operator of $\mathcal{A}$.

Definition 5.3.2. Let $\mathcal{A} \in \mathcal{L}(U,H)$ where U and H are Hilbert spaces. The mapping $\mathcal{A}^* : H \to U$ defined by

$$\langle \mathcal{A}^* y, u\rangle_U := \langle y, \mathcal{A}u\rangle_H, \quad y \in H, u \in U$$

is called the Hilbert space *adjoint* of $\mathcal{A}$.

5.4 Optimal control

We refer to [50] for a text on optimal control for PDEs.

Let U be a Hilbert space. Let $U_{ad} \subset U$ be nonnempty closed and convex. We call $\mathcal{U}_{ad}$ to be the set of *admissable controls* in the control space U. Let H be a Hilbert space of *states*. We suppose there is a continuous linear operator $\mathcal{S} : U \to H$ i.e. $\mathcal{S} \in \mathcal{L}(U,H)$.Let $y_d \in H$ be a desired state. Fix $\lambda \in \mathbb{R}_+$ and set

$$J(v) := \frac{1}{2}||\mathcal{S}(v)-y_d||_H^2 + \frac{\lambda}{2}||v||_U^2. \tag{5.6}$$

Definition 5.4.1. Given the *objective functional* (5.6) we define the *optimal control problem*: Find $u \in U_{ad}$ such that

$$J(u) = \min_{v \in U_{ad}} J(v). \tag{5.7}$$

Lemma 5.4.2. $J(\cdot)$ *is strictly convex provided* λ *is positive or* $\mathcal{S}$ *is injective.*

Proof. Let $\mu \in (0,1)$. Observe that by the strict convexity of $||\cdot||^2 : U \to \mathbb{R}$ that

$$||\mathcal{S}(\mu v_1 + (1-\mu)v_2) - y_d||_H^2 = ||\mu(\mathcal{S}(v_1) - y_d) + (1-\mu)(\mathcal{S}(v_2) - y_d)||_H^2 \tag{5.8}$$
$$\leq \mu||\mathcal{S}(v_1) - y_d||_H^2 + (1-\mu)||\mathcal{S}(v_2) - y_d||_H^2 \tag{5.9}$$

with equality holding only if $\mathcal{S}(v_1) = \mathcal{S}(v_2)$. If $\lambda > 0$ then the strict convexity of $||\cdot||_U^2 : U \to \mathbb{R}$ implies the result. On the other hand if $\lambda = 0$ and $\mathcal{S}$ is injective then

$$J(\mu v_1 + (1-\mu)v_2) \leq \mu J(v_1) + (1-\mu)J(v_2)$$

with equality if and only if $\mathcal{S}(v_1) = \mathcal{S}(v_2)$ if and only if $v_1 = v_2$ which implies the result. □

Remark 5.4.3. The functional

$$E(v) := \frac{1}{2}||\mathcal{S}(v) - y_d||_H^2$$

is Gateaux differentiable as we can see by considering

$$E(u+tv) = \frac{1}{2}\big(\langle Su - y_d, Su - y_d\rangle_H + 2t\langle Su - y_d, Sv\rangle_H + t^2\langle v, v\rangle_H\big)$$

so that

$$\langle E'(u), v\rangle = \langle S^*(Su - y_d), v\rangle_U.$$

Theorem 5.4.4. *The optimal control problem: Find $u \in U_{ad}$ such that*

$$J(u) = \min_{v \in U_{ad}} J(v)$$

has a solution if one of the following holds:

1. $$\lambda > 0.$$

2. *There exists $C_S > 0$ such that*
$$C_S||v|| \leq ||Sv|| \quad \forall v \in H.$$

3. *The set U_{ad} is bounded.*

Furthermore there is at most one solution if one of the following holds:

- $$\lambda > 0$$

- *$\mathcal{S}$ is injective.*

Proof. We settle existence first.

It is clear that $J(\cdot)$ is continuous and convex. Since $J(\cdot)$ is bounded below by 0 there is a minimizing sequence $u_n \in U_{ad}$ i.e. $J(u_n) \to d := \inf_{v \in U_{ad}} J(v))$.

Suppose $\lambda > 0$. It follows easily that $J(\cdot)$ is coercive. Let $d := \inf_{v \in U_{ad}} J(v) > -\infty$ and let $\{u_n\} \in U_{ad}$ be a minimising sequence (we showed the existence of such a sequence for functions bounded from below) so that

$$J(u_n) \to \inf_{v \in U_{ad}} J(v).$$

Since $J(\cdot)$ is coercive there exists M such that $\{u_n\} \in C = \{v : ||v|| \leq M\}$. Since C is weakly sequentially compact there is a weakly convergent subsequence $\{u_{n'}\}$ with limit $u \in C$. Furthermore $u \in U_{ad}$ because U_{ad} is closed. (Every convex and closed subset of a Banach space is weakly sequentially closed.)

By weak lower semicontinuity (since $J(\cdot)$ is convex and continuous)

$$d \leq J(u) \leq \liminf J(u_{n'}) = \lim J(u_{n'}) = d.$$

Hence the existence of a minimiser u is proved.

Suppose $\lambda = 0$.

First suppose that

$$C_S||v|| \leq ||S(v)|| \quad \forall v \in U_{ad}.$$

It follows that $J(\cdot)$ is coercive:-

$$J(v) \geq C_0||v||^2 + C_1||v|| + C_2||$$

where $C_0 > 0$.

The result follows from the previous argument.

Now suppose that U_{ad} is bounded. It follows that the minimising sequence has a weak limit in U_{ad}. (Every convex. closed and bounded set of a reflexive Banach space is weakly sequentially compact.)

The result follows from the previous argument. We now turn to uniqueness.

Strict convexity implies that there can be at most one solution since for any two distinct solutions $u_1 \neq u_2$ with $J(u_i) = d$ we have $w := \mu u_1 + (1-\mu)u_2 \in U_{ad}$ and

$$J(w) = J(\mu u_1 + (1-\mu)u_2) < \mu J(u_1) + (1-\mu)J(u_2) = d$$

which is a contradiction.

Either of the two stated conditions implies strict convexity.

$\square$

The following is an immediate consequence of the general result for optimisation of a Gateux differentiable functional over a convex set.

Theorem 5.4.5. *u is a solution to the optimal control problem if and only if*

$$\langle \mathcal{S}^*(\mathcal{S}u - y_d), v - u\rangle_U + \lambda\langle u, v-u\rangle_U \geq 0 \quad \forall \quad v \in U_{ad}. \tag{5.10}$$

5.5 Optimal control of elliptic PDEs

5.5.1 Source control

State equation

Let Ω be a bounded domain in $\mathbb{R}^m,\ m = 1,2,3$ with a Lipschitz boundary $\partial\Omega$. Let c_0 be non-negative $L^\infty(\Omega)$ function. Let $\beta \in L^\infty(\Omega)$ be given. For $u \in L^2(\Omega)$ consider the following *state problem*: Find $y \in H^1(\Omega)$ such that

$$-\Delta y + c_0 y = \beta u \quad \text{in} \quad \Omega \tag{5.11}$$
$$y = 0 \quad \text{on} \quad \partial\Omega. \tag{5.12}$$

It is convenient to introduce the variational formulation

$$\text{Find } y \in V : a(y,v) = \langle f, v\rangle_{L^2(\Omega)} \forall v \in V$$

where $V := H^1(\Omega)$ and $a(y,v) := \int_\Omega (\nabla y \nabla v + c_0 yv)$. We may also write this as

$$a(\mathcal{G}f, v) = \langle f, v\rangle_{L^2(\Omega)} \forall v \in V.$$

Standard application of Lax-Milgram yields the existence of a unique solution $y = \mathcal{G}(\beta u)$ to the *state problem* which we write as

$$y := \mathcal{S}(u)$$

where $\mathcal{S} : L^2(\Omega) \to L^2(\Omega)$ is the continuous linear *solution operator* satisfying

$$||\mathcal{S}(u_1) - \mathcal{S}(u_2)||_{H^1(\Omega)} \leq C_s ||u_1 - u_2||_{L^2(\Omega)} \quad \forall\ u_1, u_2 \ \in L^2(\Omega) \tag{5.13}$$

and we observe that $\mathcal{S}$ is compact with $Range(S) \subset H^1_0(\Omega)$.

Note that if $\beta \neq 0$ *a.e.* in Ω then $S(\cdot)$ is injective. (S is linear and $S(0) = 0 \iff v = 0$.) On the other hand if there is an non-empty open set ω_β in Ω in which $\beta = 0$ then S is not injective. Since $S(u) = 0$ for any u which vanishes outside ω_β.

Objective functional

Let $y_d \in L^2(\Omega)$ be given. Given u_a, u_b such that $u_a(x) \leq u_b(x)$ *a.e.* in Ω we set

$$U_{ad} := \{v \in L^2(\Omega) : u_a(x) \leq v(x) \leq u_b(x) \quad \text{for a.e. } x \text{ in } \Omega\} \tag{5.14}$$

and assume that U_{ad} is non-empty. Note that if u_a and u_b are in $L^\infty(\Omega)$ then U_{ad} is bounded.

We wish to find $u \in U_{ad}$ such that the solution of the state equation $y = \mathcal{S}(u)$ is close to y_d. This yields the objective functional

$$\tilde{J}(u, y) := \frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{\lambda}{2}||u||^2_{L^2(\Omega)} \tag{5.15}$$

where $y = \mathcal{S}(u)$. Setting $J(u) := \tilde{J}(u, \mathcal{S}(u))$ we obtain the optimal control problem (5.7).

Adjoint equation and adjoint operator

By definition the adjoint operator $\mathcal{S}^*$ is defined by the relation

$$\langle \mathcal{S}^* z, v\rangle_{L^2(\Omega)} = \langle z, \mathcal{S}(v)\rangle_{L^2(\Omega)} \quad \forall z, v \in L^2(\Omega).$$

Let $q = \mathcal{G}z$. It follows that

$$\langle \beta q, v\rangle_{L^2(\Omega)} = \langle \beta v, q\rangle_{L^2(\Omega)} = a(\mathcal{S}(v), q) = a(q, \mathcal{S}(v)) = \langle z, \mathcal{S}(v)\rangle_{L^2(\Omega)}$$

from which we infer that

$$\mathcal{S}^* z = \beta q = \beta \mathcal{G} z \tag{5.16}$$

Theorem 5.5.1. *If $\lambda > 0$ then there is a unique optimal control u. If $\lambda = 0$ and U_{ad} is bounded then there is a unique optimal control.*

Let u be an optimal control and let $y^* = \mathcal{S}(u)$ denote its associated state. Then there exists a unique $p = \mathcal{G}(y^* - y_d)$ satisfying the variational inequality

$$\langle \beta p, v - u\rangle_{L^2(\Omega)} + \lambda\langle u, v - u\rangle_{L^2(\Omega)} \geq 0 \forall v \in U_{ad}. \tag{5.17}$$

Conversely any control $u \in U_{ad}$ with associated state $y = \mathcal{S}u$ and adjoint state $p = \mathcal{G}(y^* - y_d)$ which satisfies the variational inequality (5.17) is optimal.

In case $\lambda > 0$ the solution of (5.17) may be written as

$$u = \mathcal{P}_{U_{ad}}(-\frac{1}{\lambda}\beta p). \tag{5.18}$$

We make some observations:-

- If $\lambda = 0$ and $U_{ad} = L^2(\Omega)$ then there will only be a solution for the special case that y_d satisfies $y_d = \mathcal{G}(\beta u)$ and this will not be true if $y_d \notin H^1_0(\Omega)$ for example. The problem is ill-posed in this case.
- If $\lambda > 0$ and $U_{ad} = L^2(\Omega)$ then the unique solution satisfies the optimality system

$$-\Delta y + c_0 y = \beta u \quad \text{in} \quad \Omega \tag{5.19}$$
$$y = 0 \quad \text{on} \quad \partial\Omega, \tag{5.20}$$

$$-\Delta p + c_0 = y - y_d \quad \text{in} \quad \Omega \tag{5.21}$$

$$p = 0 \quad \text{on} \quad \partial\Omega, \tag{5.22}$$

$$u = -\frac{1}{\lambda}\beta p. \tag{5.23}$$

- In the general case $\lambda > 0$ then the last equation above is replaced by the linear complementarity system

$$u(x) = u_a(x) \qquad if \ \beta(x)p(x) + \lambda u_a(x) > 0 \tag{5.24}$$

$$u(x) = -\beta(x)p(x)/\lambda \qquad if \ -\beta(x)p(x)/\lambda \in [u_a(x), u_b(x)] \tag{5.25}$$

$$u(x) = u_b(x) \qquad if \ \beta(x)p(x) + \lambda u_b(x) < 0. \tag{5.26}$$

- In the case $\lambda = 0$ and $u_a, u_b \in L^\infty(\Omega)$ then we have the bang-bang control solution

$$u(x) = u_a(x) \qquad if \quad \beta(x)p(x) > 0 \tag{5.27}$$

$$u(x) = u_b(x) \qquad if \quad \beta(x)p(x) < 0. \tag{5.28}$$

  In the case β vanishes in ω_β we have that the optimal control is not unique but the optimal state is.

5.6 Inverse problems

Let U be a Banach space which we will call the *input space*. Let Y denote a Banach space which we will call the *output space* or *state space*. We suppose that there is a continuous operator $\mathcal{S} : U \to Y$ which we call the *forward operator*. Given *observations* $y_d \in Y$ of *data* $y_T \in Y$ it is our goal to determine approximations u to an input u_T for which

$$y_T = \mathcal{S}(u_T). \tag{5.29}$$

Note that y_d may be an approximation to y_T. For example it may contain *noise* so that $y_d = y_T + \xi$ where ξ is the unknown noise. If this is the case then we will certainly not be able to recover u_T exactly. Typically *inverse problems* are *ill posed* in that there may not exist a solution, there may exist more than one solution or a solution depends sensitively on the data.

Rather than seeking a solution of (5.29) we may formulate the optimisation (*least squares*) problem

$$\inf_{v \in U} \frac{1}{2}||y_d - \mathcal{S}(v)||_Y^2. \tag{5.30}$$

Again as formulated this problem may have no solution or several solutions which might depend sensitively on the data. It may be that y_d is not in the range of $\mathcal{S}(\cdot)$ and $Range(\mathcal{S})$ is not all of Y. Typically one can add a *regularising functional* of the form

$$\lambda \mathcal{R}(v)$$

where $\lambda > 0$. One desired effect may be to ensure weak lower semi-continuity of the resulting functional and the boundedness of a minimizing sequence and hence weak convergence of a sequence.

It may also be the case that there are natural constraints on the desired input which may be expressed by requiring the input to lie in a convex set U_{ad}. Thus we are lead to the:-

Inverse problem

Find $u \in U_{ad}$ such that

$$J(u) := \inf_{v \in U_{ad}} J(v) \tag{5.31}$$

$$J(v) := \frac{1}{2}||y - \mathcal{S}(v)||_Y^2 + \lambda \mathcal{R}(v) \tag{5.32}$$

Example 5.6.1 (Tikhonov regularisation)**.** *An example might be*

$$\mathcal{R}(u) = \frac{1}{2}||v||_V^2 \tag{5.33}$$

and to look for $u \in V$ where V is compactly embedded in U.

Example 5.6.2 (Finite dimensional parameterisations)**.** *Another possibility is that an appropriate model for the input is to take u in a finite dimensional subset U_{ad}^K of U_{ad}. For example*

$$U_{ad}^K := \{v \in U_{ad} : v = \sum_{j=1}^{K} \alpha_j \phi_j\} \tag{5.34}$$

where $\phi_j \in U$ are linearly independent. The inverse problem (5.31) would then have U_{ad}^K replacing U_{ad} and λ might be 0.

Remark 5.6.3 (Finite number of observations)*.* Now we suppose that $\mathcal{S}(u)$ is not measured completely in Y but we have a number of known *observations* $m_i, i = 1, 2..N$ in the form of linear functionals $l_i(y), i = 1, 2, ..N$. We now formulate the optimization problem: Find $u \in U_{ad}$ such that

$$J(u) := \inf_{v \in U_{ad}} J(v) \tag{5.35}$$

$$J(v) := \frac{1}{2}\sum_{j=1}^{N} |l_i(\mathcal{S}(u)) - m_i|^2 + \frac{\lambda}{2}||v - u_d||_C^2. \tag{5.36}$$

Again one may replace U_{ad} by U_{ad}^K.

5.7 Functions of Bounded Variation

We refer to [55] for an account of functions of bounded variation.

5.7.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. Set

$$\int_\Omega |Du| := \sup \int_\Omega u\mathrm{div}\varphi dx; \varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in C_0^1(\Omega)^n, ||\varphi||_{L^\infty(\Omega)} \leq 1. \tag{5.37}$$

Definition 5.7.1. The linear space of functions of *bounded variation*, $BV(\Omega)$, is defined as

$$BV(\Omega) = \{v \in L^1(\Omega) : \int_\Omega |Dv| < \infty\}.$$

Lemma 5.7.2. *$\int_\Omega |Du| : BV(\Omega) \to \mathbb{R}$ is lower semi-continuous, that is for any sequence $u_j \in BV(\Omega)$ converging to $u \in L^1(\Omega)$*

$$\liminf \int_\Omega |Du_j| \geq \int_\Omega |Du|.$$

Furthermore $BV(\Omega)$ is a Banach space.

Lemma 5.7.3. *Let $u \in BV(\Omega)$. Then there exists a sequence $\{u_j\} \in C^\infty(\Omega)$ such that*

$$\lim_{j\to\infty} ||u_j - u||_{L^1(\Omega)} = 0 \quad \textit{and} \quad \lim_{j\to\infty} \int_\Omega |Du_j| \to \int_\Omega |Du|. \tag{5.38}$$

Lemma 5.7.4. *The Poincare-Wirtinger inequality holds:*

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u dx \right\|_1 \leq C \int_\Omega |\nabla u| \quad \forall u \in BV\,(\Omega) \tag{5.39}$$

where $C = C(\Omega)$.

Lemma 5.7.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then a bounded set in $BV(\Omega)$ is compact in $L^1(\Omega)$.*

5.7.2 Finite perimeter sets

Example 5.7.6. *Let $E \subset \Omega$ have a C^2 boundary Γ. Let χ_E be the characteristic function of E defined by*

$$\chi_E(x) = 1 \; x \in E \quad \textit{and} \quad \chi_E(x) = 0 \; x \in \Omega \setminus \bar{E}.$$

It holds that $\chi_E \in BV(\Omega)$ and

$$\int_\Omega |D\chi_E| = |\Gamma|.$$

Definition 5.7.7 (Finite perimeter sets(Caccioppoli))**.** An n-dimensional Lebesgue measurable set $E \subset \Omega$ is a finite perimeter set if $\chi_E \in BV(\Omega)$ and

$$Per_\Omega(E) := |\chi_E|_{BV(\Omega)} < \infty.$$

5.8 Inverse problem in imaging science

We refer to [3] for material on mathematical problems in image processing.

Observations, input and model

We consider grey scale images as functions $u \in L^2(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^N$. Often $N = 2$ and Ω is a rectangular domain.

Suppose we observe an image as y_o everywhere in Ω. Our model is that

$$y_o = \mathcal{S}(u_T) + \xi \tag{5.40}$$

where

- u_T is the true image.
- $\mathcal{S}(\cdot) : L^2(\Omega) \to L^2(\Omega)$ is a linear operator modelling the process by which the image is obtained and is usually called a *blurring* operator.
- ξ is the unknown *random noise* occurring, for example, in signal transmission.

Our goal is to determine a *good* approximation $u \in L^2(\Omega)$ of the input image u_T which satisfies

$$y_T = \mathcal{S}(u_T).$$

Our first model is of the least squares form

$$\inf_u J(u) := \frac{1}{2}||\mathcal{S}(u) - y_o||^2_{L^2(\Omega)}. \tag{5.41}$$

It is based on supposing that ξ is white Gaussian noise and using arguments relating to the maximium likelihood principle. Note that this may be written as

$$\mathcal{S}^*(\mathcal{S}(u)) = \mathcal{S}^*(y_o). \tag{5.42}$$

Some difficulties with this model are:

- In the case that $\mathcal{S}(\cdot)$ is the identity then we have that the recovered image is

  $$u = u_T + \xi$$

  which is simply the noisy image. We may wish to *denoise* the image.

- Suppose that $Range(\mathcal{S}(\cdot))$ is compact in $L^2(\Omega)$ then we may view the blurring opereator as being smoothing. An issue is that the noisy image y_o is not in $Range(\mathcal{S}(\cdot))$. Thus, for example when $\mathcal{S}(\cdot)$ is self-adjoint the above equation is solvable only for $y_0 \in Range(\mathcal{S}(\cdot))$.
- Suppose that there is no noise. Then it may be case that $\mathcal{S}(\cdot)$ can be inverted in principle but that this inversion is ill-posed in the sense that it does not depend continuously on the data. For example, the blurring operator may be a Gaussian convolution
$$\Omega = \mathbb{R}, \ \mathcal{S}(u) = G \star u, \quad G(x) := \frac{1}{\mu\sqrt(2\pi)} \exp(\frac{-x}{2\mu^2})$$
and inverting is analogous to solving the diffusion equation backwards which is ill posed.

A simple Tikhonov regularisation is to take

$$R(v) := \frac{\lambda}{2} ||\nabla v||^2_{L^2(\Omega)}$$

which leads to the optimisation problem

$$J(u) := \inf_{v \in L^2(\Omega)} \frac{1}{2} ||\mathcal{S}(v) - y_d||^2_{L^2(\Omega)} + \frac{\lambda}{2} ||\nabla v|||^2_{L^2(\Omega)} \tag{5.43}$$

which we observe is an example of an *optimal control problem.* We may view this objective functional as the sum of a *fidelity* term which contains the information from the original image and a second *regularising* term which contributes to the denoising of the image and to ensure that the image belongs to a natural class of images.

5.8.1 Determination of a binary image

Here we are concerned with the image being a characteristic function of a domain $E \subset \Omega$ which we choose to be of the form

$$u(x) = 1 \quad x \quad \in E \quad \text{and} \quad u(x) = -1 \quad x \quad \in \Omega \setminus \bar{E}. \tag{5.44}$$

This is the case for *barcodes.*

Example 5.8.1 (Perimeter/TV regularisation)**.** *Here we regularise by using the perimeter of the set E. The model becomes*

$$J(u) := \inf_v \frac{1}{2} ||\mathcal{S}(v) - y_o||^2_{L^2(\Omega)} + \sigma Per_\Omega(E) \tag{5.45}$$

where $v = 2\chi_E - 1$. Using the variation we may write the optimisation as

$$J(u) := \inf_{v \in \mathcal{B}} \frac{1}{2}||\mathcal{S}(v) - y_o||^2_{L^2(\Omega)} + \int_\Omega |Dv| \tag{5.46}$$

where $\mathcal{B} := \{v \in BV(\Omega); v(x) \in \{-1, 1\} \ \ a.e.\}$*. The existence of a solution may be proved using the direct method of the calculus of variations.*

This is a regularisation because:-

- The restored image u is binary.
- The blurring of an image will give rise to images without *edges* and the perimeter term is a *penalisation* of the length of the restored binary image.
- Noise can introduce a lot of interface if the noisy image is simply projected to be a binary image. Penalising the perimeter has a *denoising* effect.

Example 5.8.2 (Phase field regularisation)**.** *This will be relaxed by taking the input space to be*

$$K := \{v \in H^1(\Omega) : |v(x)| \leq 1 \ \ a.e. \ \ x \in \Omega\} \tag{5.47}$$

and the regularisation to be

$$\mathcal{R}(v) := \frac{c_{do}\sigma}{2} \int_\Omega \{\epsilon |\nabla v|^2 + \frac{1}{\epsilon}(1 - v^2)\} \tag{5.48}$$

Thus our model for the input space consists of diffuse phase field surfaces arising from

$$J(u) := \inf_{v \in K} \frac{1}{2}||\mathcal{S}(v) - y_d||^2_{L^2(\Omega)} + \frac{c_W \sigma}{2} \int_\Omega \{\epsilon |\nabla v|^2 + \frac{1}{\epsilon}(1 - v^2)\}. \tag{5.49}$$

5.9 Inverse problems for PDEs

5.9.1 Determination of a source

Forward problem

Let Ω be a bounded domain in $\mathbb{R}^m, \ m = 1, 2, 3$ with a Lipschitz boundary $\partial\Omega$. Let c_0 be an $L^\infty(\Omega)$ function bounded below by a positive constant α. Let β and $g \in L^\infty(\Omega)$ be given. For $u \in L^2(\Omega)$ consider the following *forward problem*: Find $y \in H^1(\Omega)$ such that

$$-\Delta y + c_0 y = \beta u + g \ \ \text{in} \ \ \Omega \tag{5.50}$$

$$\partial y / \partial \nu = 0 \ \ \text{on} \ \ \partial\Omega. \tag{5.51}$$

It is convenient to introduce the variational formulation

$$\text{Find } y \in V : a(y, v) = \langle f, v \rangle_{L^2(\Omega)} \forall v \in V$$

where $V := H^1(\Omega)$ and $a(y,v) := \int_\Omega (\nabla y \nabla v + c_0 yv)$. We may also write this as

$$a(\mathcal{G}f, v) = \langle f, v\rangle_{L^2(\Omega)} \forall v \in V.$$

Standard application of Lax-Milgram yields the existence of a unique solution $y = \mathcal{G}(\beta u) + y_0$ (where $y_0 = \mathcal{G}(g)$) to the *forward problem.*

Note that $\mathcal{G} : L^2(\Omega) \to H^1(\Omega)$ is a continuous linear *solution operator* satisfying

$$||\mathcal{G}(u_1) - \mathcal{G}(u_2)||_{H^1(\Omega)} \le C_s ||u_1 - u_2||_{L^2(\Omega)} \ \ \forall\ u_1, u_2 \ \in L^2(\Omega). \tag{5.52}$$

Observations, input and model

Suppose we observe the state everywhere in a spatial domain Ω. Since y_0 is known we may suppose that we have observations y_d in $L^2(\Omega)$ of $\mathcal{G}(\beta u)$. Our goal is to determine a *good* approximation $u \in L^2(\Omega)$ of the input data. Our model may be of the form

$$J(u) := \frac{1}{2}||\mathcal{G}(u) - y_d||^2_{L^2(\Omega)} + \lambda \mathcal{R}(u). \tag{5.53}$$

Note that if $\lambda = 0$ and $U = L^2(\Omega)$ then this will not be a well posed problem because in general $y_d \notin H^1(\Omega)$. This is reflected in the observation that a minimising sequence (which exists because the functional is bounded below) is not necessarily bounded and so will not necessarily contain a weakly convergent subsequence.

A simple Tikhonov regularisation is to take

$$R(v) := \frac{\lambda}{2}||v||^2_{L^2(\Omega)}$$

which leads to the optimisation problem

$$J(u) := \inf_{v \in L^2(\Omega)} \frac{1}{2}||\mathcal{G}(v) - y_d||^2_{L^2(\Omega)} + \frac{\lambda}{2}||v|||^2_{L^2(\Omega)} \tag{5.54}$$

which we observe is an example of an *optimal control problem.*

5.9.2 Determination of a binary source

Here we are concerned with the input being a characteristic function of a domain $E \subset \Omega$ which we choose to be of the form

$$u(x) = 1 \ \ x \ \ \in E \ \text{ and } \ u(x) = -1 \ \ x \ \ \in \Omega \setminus \bar{E}. \tag{5.55}$$

We can now place ourselves in the context of the binary image restoration problem by setting

$$\mathcal{S}(\cdot) = \mathcal{G}(\cdot).$$

Chapter 6

Parabolic problems

6.1 Abstract parabolic variational problem

Definition 6.1.1. Let V and H be two separable Hilbert spaces satisfying $V \subset H$, V is dense in H and the injection of V in H is continuous. Let $V^{'}$ and $H^{'}$ be the dual spaces of V and H. Identifying H and $H^{'}$ we find that H is a dense subspace of $V^{'}$ and we call

$$V \subset H \subset V^{'}$$

a *Hilbert triple*. We use $\langle \cdot, \cdot \rangle_V$, $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle$ to denote the inner product for V, the inner products for H and the duality pairing for V' and V so that for $f \in H$

$$\langle f, v \rangle = \langle f, v \rangle_H \ \forall v \in V.$$

Let $\{V, H, V'\}$ be a Hilbert space triple and $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bounded bilinear form which satisfies the relaxed coercivity condition

$$a(v, v) + \beta ||v||_H^2 \geq \alpha ||v||_V^2. \tag{6.1}$$

Consider the following initial value problem:

Definition 6.1.2. *Weak solution*

Given $u_0 \in H$ and $f \in L^2(0, T; V')$, find $u \in L^2(0, T; V)$ with $u' \in L^2(0, T : V')$ such that

$$\langle u', v \rangle + a(u, v) = \langle f, v \rangle \quad \forall v \in V \text{ and a.e. } \ t \in (0, T)$$

and

$$u(0) := u_0.$$

Lemma 6.1.3. *Let $\{V, H, V^{'}\}$ be a Hilbert space triple. If $\eta \in L^2(0,T;V)$ and it has a weak derivative in $L^2(0,T;V^{'})$ then η is almost everywhere equal to a function continuous from $[0,T]$ into H (we write $\eta \in C([0,T];H)$ after a possible modification on a set of measure zero) and the following equation holds in the sense of (scalar) distributions*

$$\frac{d}{dt}||\eta||_H^2 = 2\langle \eta', \eta \rangle.$$

Furthermore

$$\max_{[0,T]} ||\eta(t)||_H \leq C(||\eta||_{L^2(0,T;V)} + ||\eta'||_{L^2(0,T;V')})$$

where the constant C depends only on T.

Remark 6.1.4. We observe that from the definition and from Lemma 6.1.3 that a weak solution satisfies

$$u \in C(0,T;H).$$

Galerkin approximation

Let V_m be a sequence of finite dimensional subspaces of V with basis functions $\{z_j^m\}_{j=1}^m$ with the property that for each $v \in V$ there exists a sequence $v_m \in V_m$ such that

$$v_m \to v \ \text{ in } \ V.$$

We seek $u_m : [0,T] \to V$ of the form

$$u_m(t) := \sum_{j=1}^m U_j^m(t) z_j \tag{6.2}$$

where $u_m(0) = \mathcal{P}_m u_0$, the H projection of u_0 onto V_m, defined by

$$\langle u_m(0) - u_0, v_m \rangle_H = 0 \ \forall v_m \in V_m$$

which is equivalent to

$$U_j^m(0) := \langle u_0, z_j \rangle_H \ , \ j = 1, 2, ..m. \tag{6.3}$$

By definition this projection satisfies

$$||\mathcal{P}_m v||_H \leq ||v||_H \ \ \forall v \in H.$$

We also assume that

$$||\mathcal{P}_m||_V \leq C||v||_V \ \ \forall v \in V. \tag{6.4}$$

This assumption is satisfied by spaces V_m defined by eigenfunctions of appropriate elliptic operators and also by finite element spaces under certain conditions on the triangulations and finite element spaces.

Theorem 6.1.5. *For each integer m there exists a unique function $u_m(t)$ of the form (6.2) such that (6.3) and*

$$\langle u_m', v_m \rangle + a(u_m, v_m) = \langle f, v_m \rangle \quad \forall v_m \in V_m \text{ and } t \in (0,T) \tag{6.5}$$

hold.

Proof. It is clear that one can take $v_m = z_j, j = 1, 2, ..m$ and obtain the following equations for the coefficients $U_j^m(t)$

$$(U_j^m)'(t) + \sum_{k=1}^{m} S_{jk}^m U_k^m(t) = F_j(t) \tag{6.6}$$

where $S_{jk}^m = a(z_k, z_j)$ and $F_j(t) := \langle f(t), z_j \rangle$. Note that we have used the fact that by definition $u_m"(t) \in V$ so $\langle u_m', v_m \rangle = \langle u_m', v_m \rangle_H$. Since $F_j(\cdot) \in L^2(0,T)$ standard ODE theory yields the existence of a unique solution to this system of equations for every $T > 0$. Note that $U_j^m \in H^1(0,T)$ and so is Holder continuous. $\square$

Energy estimates

We prove a priori bounds on norms of the Galerkin approximation. These are called *energy estimates.*

Theorem 6.1.6. *There exists a constant C=C(T) independent of m, u_0 and f such that*

$$\max_{0 \le t \le T} ||u_m(t)||_H + ||u_m||_{L^2(0,T;V)} \le C(||f||_{L^2(0,T;V')} + ||u_0||_H), \tag{6.7}$$

$$||u_m'||_{L^2(0,T;V')} \le C(||f||_{L^2(0,T;V')} + ||u_0||_H) \tag{6.8}$$

Proof. Using the relaxed coercivity of $a(\cdot,\cdot)$ we find

$$\frac{1}{2}\frac{d}{dt}||u_m||_H^2 + \alpha||u_m||_V^2 \le ||f||_{V'}||u_m||_V + \beta||u_m||_H^2$$

from which we have

$$\frac{d}{dt}||u_m||_H^2 + \alpha||u_m||_V^2 \le \frac{1}{\alpha}||f||_{V'}^2 + 2\beta||u_m||_H^2.$$

A standard Gronwall argument yields the desired estimates, (6.7). (In this instance we multiply the above inequality by the integrating factor $\exp(-2\beta t)$ and then integrate.) The estimate on u_m' is a consequence of the following

$$\langle u_m', v \rangle = \langle \mathcal{P}_m u_m', v \rangle_H = \langle u_m', \mathcal{P}_m v \rangle_H = \langle f, \mathcal{P}_m v \rangle - a(u_m, \mathcal{P}_m v) \quad \forall v \in V.$$

It follows that taking $v \in L^2(0,T;V)$ and $v_m = \mathcal{P}_m v \in L^2(0,T;V_m)$ with

$$\int_0^T \langle u_m', v \rangle dt \le \int_0^T ||f||_{V'}||\mathcal{P}_m v||_V + \gamma||u_m||_V||\mathcal{P}_m v||_V$$

from which we have that

$$|\int_0^T \langle u_m', v\rangle dt| \le C(||f||_{L^2(0,T;V')} + ||u_m||_{L^2(0,T;V)})||v||_{L^2(0,T;V)}.$$

This yields the desired estimate (6.8). □

Existence and uniqueness

Theorem 6.1.7. *There exists a weak solution satisfying definition (6.1.2).*

Proof. 1. The energy estimates (6.7) imply that the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;V)$ and $\{u_m'\}_{m=1}^\infty$ is bounded in $L^2(0,T;V')$. Thus there exist subsequences $\{u_{m_j}\}_{m=1}^\infty$ and $\{u_{m_j}'\}_{m=1}^\infty$ and $u \in L^2(0,T;V)$ with $u' \in L^2(0,T;V)$ such that

$$\begin{cases} u_{m_j} \to u \quad \text{weakly in} \quad L^2(0,T;V) \\ u_{m_j}' \to u' \quad \text{weakly in} \quad L^2(0,T;V') \end{cases} \tag{6.9}$$

2. Given $v \in V$ there exists $v_m \in V_m$ such that $v_m \to v$ in V. Therefore for $\psi \in C^1[0,T]$ and the subsequence labelled m_j the following hold

$$\begin{cases} \int_0^T \psi(t)\langle f, v_{m_j}\rangle \to \int_0^T \psi(t)\langle f, v\rangle \\ \int_0^T \psi(t)\langle u_{m_j}', v_{m_j}\rangle \to \int_0^T \psi(t)\langle u', v\rangle \\ \int_0^T \psi(t)a(u_{m_j}, v_{m_j}) \to \int_0^T \psi(t)a(u,v). \end{cases} \tag{6.10}$$

Since

$$\int_0^T \psi(t)\{\langle u_m', v_m\rangle + a(u_m, v_m)\} = \int_0^T \psi(t)\langle f, v_m\rangle$$

we obtain in the limit for any $v \in V$

$$\int_0^T \psi(t)\langle u', v\rangle + a(u,v) = \int_0^T \psi(t)\langle f, v\rangle \tag{6.11}$$

which implies that

$$\langle u', v\rangle + a(u,v) = \langle f, v\rangle \quad a.e.\ t \in (0,T).$$

3. We observe that for $\psi \in C^1[0,T]$ such that $\psi(T) \equiv 0$ that the discrete equation yields

$$\int_0^T -\langle u_m, v_m\rangle \psi'(t) + a(u_m, v_m)\psi(t) = \int_0^T \psi(t)\langle f, v_m\rangle + \psi(0)\langle u_m(0), v_m\rangle$$

and since

$$\langle u_{m_j}(0), v_{m_j})\rangle \to \langle u_0, v\rangle,$$

passing to the limit for a subsequence we find

$$\int_0^T \psi(t)\{\langle f, v\rangle - a(u,v)\}dt = -\int_0^T \psi'(t)\langle u, v\rangle dt + \psi(0)\langle u_0, v\rangle$$

On the other hand from (6.11)

$$\int_0^T \psi(t)\{\langle f, v\rangle - a(u,v)\}dt = \int_0^T \psi(t)\langle u', v\rangle dt$$

and since

$$\int_0^T \psi(t)\langle u', v\rangle dt = \int_0^T \psi(t)\frac{d}{dt}\langle u', v\rangle dt = -\int_0^T \psi'(t)\langle u, v\rangle dt + \psi(0)\langle u(0), v\rangle$$

we deduce that

$$u(0) = u_0.$$

□

Theorem 6.1.8. *A weak solution satisfying definition (6.1.2) is unique.*

Proof. By linearity it suffices to show that the only weak solution when $f \equiv u_0 \equiv 0$ is

$$u \equiv 0.$$

In this case, recall Lemma 6.1.3,

$$0 = \langle u', u\rangle + a(u,u) = \frac{1}{2}\frac{d}{dt}||u||_H^2 + a(u,u) \geq \frac{1}{2}\frac{d}{dt}||u||_H^2 + \alpha||u||_V^2 - \beta||u||_H^2$$

and, using $u(0) = 0$, an application of Gronwall's inequality yields the result. □

6.2 An example of a parabolic equation

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 1, 2$. Let $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$, $H = L^2(\Omega)$ and $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product. Let f, p, q and u_0 be given functions in $C(\bar{\Omega})$ with $p(x) \geq p_m > 0, q(x) \geq 0$ for all $x \in \Omega$. Consider the following initial value problem:

$$u_t = \nabla(p(x)\nabla u) - q(x)u + f(x), \quad x \in \Omega, \quad 0 < t \leq T \quad ,$$
$$u = 0, \ x \in \partial\Omega, \quad u(x,0) = u_0(x) \ x \in \Omega.$$

This may be formulated as a variational problem of the form:

Find $u \in L^2(0,T;V), u_t \in L^2(0,T;V')$ such that

$$(u_t, v) + a(u,v) = (f, v) \quad \forall v \in V \text{ and a.e. } t \in (0,T)$$

and

$$u(0) := u_0.$$

It is convenient to recall the Poincare inequality

$$||v||_H \leq C_* ||v||_V, \ \forall v \in V$$

where $||v||_H := ||v||_{L^2(\Omega)}$ and for $H_0^1(\Omega)$ we take the norm $||v||_V := ||\nabla v||_{L^2(\Omega)}$.

Here $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the bounded, coercive symmetric bilinear form

$$a(w,v) := \int_\Omega p \nabla w \cdot \nabla v + qwv.$$

This follows multiplying by a test function $v \in H_0^1(\Omega)$, integrating by parts and using the boundary conditions on v.

Exponential decay

The following *exponential decay* result in $L^2(\Omega)$ holds:

Lemma 6.2.1. *There is a positive K independent of u, u_0 and f such that*

$$||u(t)||_H \leq e^{-Kt}||u_0||_H + \frac{1}{K}(1 - e^{-Kt})||f||_H.$$

Proof. Taking $v = u$ in the variational formulation yields

$$\langle u_t, u \rangle + a(u,u) = (f,u)$$

and applying the Cauchy Schwarz inequality on the RHS and the Poincare inequality on the LHS we find

$$\frac{1}{2}\frac{d}{dt}||u||_H^2 + \frac{p_m}{(C_*)^2}||u||_H^2 \leq ||f||_H ||u||_H$$

which implies that ($K := \frac{p_m}{(C_*)^2}$) (since $\frac{1}{2}\frac{d}{dt}||u||_H^2 = ||u||_H \frac{d}{dt}||u||_H$)

$$\frac{d}{dt}||u||_H + K||u||_H \leq ||f||_H$$

and using an integrating factor

$$\frac{d}{dt}(e^{Kt}||u||_H) \leq e^{Kt}||f||_H.$$

Integrating yields the desired inequality.

□

6.2.1 Semi-discrete finite element approximation

Let V_h be a finite element subspace of V. Here we may consider piecewise linear functions on a partition of Ω ($d = 1$) or on a triangulation of a polygonal Ω ($d = 2$). Find $u(t) \in V_h, t \in [0, T]$ such that

$$((u_h)_t, v_h) + a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h \text{ and a.e. } \ t \in (0, T)$$

and

$$u_h(0) := u_0^h$$

where $u_0^h \in V_h$ is an approximation to u_0.

Lemma 6.2.2. *There is a positive K independent of u_h, u_0^h and f such that*

$$||u_h(t)||_H \le e^{-Kt}||u_0^h||_H + \frac{1}{K}(1 - e^{-Kt})||f||_H.$$

Proof. This is the same as the proof for the continuous problem. □

Let $\{\phi_j\}_{j=1}^J$ be a basis for V_h. Let A, M, b be defined by

$$A_{ij} = a(\phi_j, \phi_i), M_{ij} = (\phi_i, \phi_j), F_i = (f, \phi_i).$$

Then writing $u_h(t) := \sum_{j=1}^J U(t)_j \phi_j$ we have that

$$M\frac{dU}{dt} + AU = F \quad , \quad U(0) = U_0$$

where $u_0^h = \sum_{j=1}^J U(t)_0 \phi_j$.

6.2.2 Fully discrete finite element approximation

A fully discrete numerical scheme for the initial value problem may be obtained by using difference quotients to approximate the time derivative.

The idea is to construct a sequence $\{u_h^n\}_{n=0}^N$ with $\Delta t := T/N$ and $u_h^n \in V_h$ approximating $u(n\Delta t)$.

The backward Euler scheme in time is:-

$$(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h) + a(u_h^{n+1}, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

with u_h^0 given.

The system of linear algebraic equations resulting from the backward Euler scheme is easily seen to be

$$LU^{n+1} = MU^n + b$$

where $L = M + \Delta t A$ and $b_i = \Delta t(f, \phi_i)$.

Lemma 6.2.3. Gronwall inequality

Let $z_k \geq 0, k = 0, 1, 2, ...n, ..N$ *satisfy*

$$z_{n+1} \leq \lambda z_n + \lambda G$$

where $\lambda > 0$ *and* $G \geq 0$. *Then*

$$z_n \leq \lambda^n z_0 + \frac{1-\lambda^n}{1-\lambda}\lambda G \ , \lambda \neq 1$$
$$z_n \leq z_0 + nG \ , \ \ \lambda = 1.$$

Proof. This follows by induction. We consider the case $\lambda \neq 1$. For a particular k, if

$$z_{k+1} \leq \lambda z_k + \lambda G$$

and

$$z_k \leq \lambda^k z_0 + \frac{1-\lambda^k}{1-\lambda}\lambda G$$

it follows that

$$z_{k+1} \leq \lambda(\lambda^k z_0 + \frac{1-\lambda^k}{1-\lambda}\lambda G) + \lambda G$$

and rearranging we find

$$z_{k+1} \leq \lambda^{k+1} z_0 + \frac{1-\lambda^k}{1-\lambda}\lambda G.$$

□

Lemma 6.2.4. *For the backward Euler scheme, the following stability bound holds*

$$||u_h^n||_H^2 \leq (1 + K\Delta t)^{-n}||u_0||_H^2 + \frac{1}{K^2}(1 - (1 + K\Delta t)^{-n})||f||_H^2 \ \forall n \geq 0.$$

Proof. Taking $v_h = u_h^{n+1}$ in the backward Euler scheme yields

$$(\frac{u_h^{n+1} - u_h^n}{\Delta t}, u_h^{n+1}) + a(u_h^{n+1}, u_h^{n+1}) = (f, u_h^{n+1})$$

and rearranging we find

$$\frac{1}{2}(||u_h^{n+1}||_H^2 - ||u_h^n||_H^2 + ||u_h^{n+1} - u_h^n||_H^2) + \Delta t K||u_h^{n+1}||_H^2 \leq \frac{1}{2K}\Delta t||f||_H^2 + \frac{K}{2}\Delta t||u_h^{n+1}||_H^2$$

which yields

$$(1+K\Delta t)||u_h^{n+1}||_H^2 \leq ||u_h^n||_H^2 + \Delta t \frac{1}{K}||f||_H^2$$

from which using the Gronwall inequality we obtain the desired result. $\square$

Lemma 6.2.5. *In the case $f = 0$ for the backward Euler scheme, the following stability bound holds*

$$||u_h^n||_H^2 \leq (1+2K\Delta t)^{-n}||u_0||_H^2 \;\; \forall n \geq 0.$$

Proof. Exercise $\square$

The forward Euler scheme is:

$$(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h) + a(u_h^n, v_h) = (f, v_h) \;\; \forall v_h \in V_h$$

with u_h^0 given.

In order to show stability results for this scheme we require an *inverse inequality*:-

$$||v_h||_V \leq S(h)||v_h||_H \;\; \forall v_h \in V_h.$$

That such an inequality holds follows from the fact that V_h is finite dimensional and all norms are equivalent in finite dimensions. However such an inequality does not hold for $v \in V$. For V_h being the space of piecewise linear functions it can be shown that $S(h) \leq \frac{c}{h}$.

Lemma 6.2.6. *In the case $f = 0$ for the forward Euler scheme, the following stability bound holds*

$$||u_h^n||_H^2 \leq (1+K\Delta t)^{-n}||u_0||_H^2 \;\; \forall n \geq 0$$

provided the stability condition

$$\Delta t \leq \frac{p_m}{\gamma^2 S(h)^2}$$

holds

Proof. Taking $v_h = u_h^{n+1}$ in the forward Euler scheme yields

$$(\frac{u_h^{n+1}-u_h^n}{\Delta t}, u_h^{n+1}) + a(u_h^n, u_h^{n+1}) = 0$$

and rearranging we find

$$\frac{1}{2}(||u_h^{n+1}||_H^2 - ||u_h^n||_H^2 + ||u_h^{n+1}-u_h^n||_H^2) + \Delta t p_m ||u_h^{n+1}||_V^2 \leq \Delta t a(u_h^{n+1}, u_h^{n+1}-u_h^n)$$

and using

$$\begin{aligned}\Delta t a(u_h^{n+1}, u_h^{n+1} - u_h^n) &\leq \gamma \Delta t ||u_h^{n+1}||_V ||u_h^{n+1} - u_h^n||_V \\ &\leq \Delta t \frac{p_m}{2} ||u_h^{n+1}||_V^2 + \frac{\Delta t \gamma^2}{2p_m} S(h)^2 ||u_h^{n+1} - u_h^n||_H^2\end{aligned}$$

which yields

$$(1 + K\Delta t)||u_h^{n+1}||_H^2 \leq ||u_h^n||_H^2$$

from which we obtain the desired inequality.

□

This is typical for *explicit* schemes for evolutionary PDEs in which in order to have stability the time step must satisfy a condition which restricts its size in terms of the spatial mesh size. In this case

$$\Delta t \leq Ch^2$$

for stability.

6.3 Turing instability

Let $\mathbf{f} : \mathbb{R}^{\mathbf{m}} \to \mathbb{R}^{\mathbf{m}}$ be a suffuciently smooth map satisfying

$$\mathbf{f}(\mathbf{0}) = \mathbf{0}. \tag{6.12}$$

It follows that 0 is a steady state solution of the initial value problem

$$\begin{aligned}\mathbf{u}_t - \mathcal{A}\Delta\mathbf{u} &= \mathbf{f}(\mathbf{u}) \text{ in } \Omega \times (0, \infty) &(6.13)\\ \mathbf{u} &= 0 \text{ on } \partial\Omega \times (0, \infty). &(6.14)\end{aligned}$$

In the case of *small* initial data we may linearize about 0 and obtain the linear system

$$\begin{aligned}\mathbf{v}_t - \mathcal{A}\Delta\mathbf{v} &= D\mathbf{f}(0)\mathbf{v} \text{ in } \Omega \times (0, \infty) &(6.15)\\ \mathbf{v} &= 0 \text{ on } \partial\Omega \times (0, \infty). &(6.16)\end{aligned}$$

We may find a formula for the solution of this linear system using the eigensolution (z_j, λ_j) for the Laplacian on Ω with homogeneous Dirichlet boundary conditions and separation of variables. We obtain

$$\mathbf{v}(x.t) = \sum_{j=1}^{\infty} \boldsymbol{\alpha}_j(t) z_j(x) \tag{6.17}$$

where

$$\boldsymbol{\alpha}_j(t)' = \mathcal{S}_j \boldsymbol{\alpha}_j \tag{6.18}$$

and

$$\mathcal{S}_j = D\mathbf{f}(0) - \lambda_j \mathcal{A}. \tag{6.19}$$

The solution $\boldsymbol{v}(x,t) = 0$ is *stable* if and only if the coeffficients $\boldsymbol{\alpha}_j$ decay to zero as $t \to \infty$. Thus a sufficient condition for stability is that the eigenvalues of $\mathcal{A}_j$ have a negative real part for all j.

It is interesting to consider the comparison with the O.D.E. systems

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) \quad (t \geq 0) \tag{6.20}$$

$$\boldsymbol{y}' = D\boldsymbol{f}(0)\boldsymbol{y} \quad (t \geq 0) \tag{6.21}$$

whose stability is governed by the eigenvalues of $D\boldsymbol{f}(0)$. Consider the eigenvalue problem for

$$\mathcal{S}(\lambda) := D\mathbf{f}(0) - \lambda\mathcal{A}$$

where $\lambda \geq 0$. Thus we are concerned in finding $\sigma(\lambda)$ such that

$$Q_\lambda(\sigma) := det(\mathcal{S}(\lambda) - \sigma I) = 0.$$

To fix ideas we take $m = 2$ and write $\mathbf{f} = (f^1, f^2)$ and

$$D\mathbf{f}(0) = \begin{pmatrix} \partial_1 f^1 & \partial_2 f^1 \\ \partial_1 f^2 & \partial_2 f^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

For the diffusion tensor we take

$$\mathcal{A} = diag(a_1.a_2)$$

where $a_i \geq 0$. It follows that

$$Q_\lambda(\sigma) = \sigma^2 - \sigma(\alpha + \delta - \lambda(a_1 + a_2)) + p(\lambda)$$

where

$$p(\lambda) = \lambda^2 a_1 a_2 - \lambda(a_1\delta + a_2\alpha) + \alpha\delta - \beta\gamma.$$

Let us suppose that 0 is asymptotically stable for the O.D.E. system so that

$$\begin{cases} \alpha + \delta = \sigma_1(0) + \sigma_2(0) < 0 \\ \alpha\delta - \gamma\beta = \sigma_1(0)\sigma_2(0) > 0. \end{cases} \tag{6.22}$$

It follows that least one of α and δ must be negative.

A

We assume without loss of generality that

$$\delta < 0.$$

Observe that (1)

$$\sigma_1(\lambda) + \sigma_2(\lambda) = \alpha + \delta - \lambda(a_1 + a_2) < 0 \tag{6.23}$$

where we use the nonnegativity of the diffusion coefficients and λ.

(2)

$$\sigma_1(\lambda)\sigma_2(\lambda) = p(\lambda) \tag{6.24}$$

and

$$p(\lambda) > -\lambda(a_1\delta + a_2\alpha) \tag{6.25}$$

where we use the nonegativity of the diffusion coefficients and the stability assumption $\alpha\delta - \gamma\beta = \sigma_1(0)\sigma_2(0) > 0$.

Case 1: Complex roots and no loss of stability

We may write

$$\sigma_1(\lambda) = b + ic \quad \sigma_2(\lambda) = b - ic$$

so that

$$2b = \sigma_1(\lambda) + \sigma_2(\lambda) < 0$$

from which we see that the eigenvalues have negative real part.

Case 2: Real roots and loss of stability

For the case of real roots we label $\sigma_2(\lambda) \leq \sigma_1(\lambda)$. From (1) we see that there can be at most one root $\sigma_1(\lambda)$ which is positive. It follows by looking at the graph of $Q_\lambda(\sigma)$ that the existence of a positive root is equivalent to $Q_\lambda(0) = p(\lambda) < 0$. It follows from (2) above that if $a_2 = 0$ then $\sigma_1(\lambda)\sigma_2(\lambda) = p(\lambda) > 0$. Also if $\alpha \leq 0$ then by (2) again we see that $p(\lambda) > 0$. Thus a *necessary* condition for loss of stability is

$$a_2 > 0 \text{ and } \alpha > 0.$$

It is easy to derive *sufficient* conditions. Given λ choose a_1 so that

$$\lambda a_1 - \alpha < 0.$$

Rearranging we obtain

$$p(\lambda) = \lambda(\lambda a_1 - \alpha)[a_2 - \frac{\alpha\delta - \beta\gamma - \lambda a_1\delta}{\lambda(\alpha - \lambda a_1)}]$$

from which we see that choosing

$$a_2 > \frac{\alpha\delta - \beta\gamma - \lambda a_1\delta}{\lambda(\alpha - \lambda a_1)}$$

we obtain $p(\lambda) > 0$.

Summarising we see that given Ω and $\mathbf{f}$ such that

$$\alpha > 0 > \delta$$

we may change the asympotic stability of the zero solution of the O.D.E. system to being unstable for the PDE system by adding diffusion which satisfies

$$\begin{cases} a_1 < \frac{\alpha}{\lambda} \\ a_2 > 0 \quad \text{and} \quad a_2 > \frac{\alpha\delta - \beta\gamma - \lambda a_1 \delta}{\lambda(\alpha - \lambda a_1)} \end{cases} \tag{6.26}$$

for at least one eigenvalue λ.

6.3.1 Reaction diffusion equations

The system (6.13) is a *reaction diffusion* system. It is used to model chemical reactions where $\mathbf{u}$ is a vector of concentrations and $\mathbf{f}$ defines the kinetics. The reaction takes place in a spatial domain Ω and the chemicals diffuse with a diffusive flux $-\mathcal{A}\nabla\mathbf{u}$ where $\mathcal{A}$ is a matrix of diffusion coefficients. In the last subsection it was observed that adding diffusion to a system of ODEs may destabilise a steady state. This is known as *Turing instability*, see the classical paper of Turing [51]. Such equations exhibit diffusion-driven instability of spatially uniform structures leading to spatially non-uniform patterns. Many mathematical models based upon this mechanism for the formation of patterns work were later developed and analysed motivated by applications such as pattern formation in hydra, sequential pattern formation in vertebrates, coat markings of animals and pigmentation patterns on butterfly wings, amongst others. For an overview we refer the reader to [36].

6.4 The porous medium equation

The *porous medium* equation is

$$u_t - \Delta u^\gamma \tag{6.27}$$

where $\gamma > 0$ and we seek $u \geq 0$. Note that this is a diffusion equation

$$u_t - \gamma\nabla(u^{\gamma-1}\nabla u)$$

with a nonlinear diffusivity $u^{\gamma-1}$ which is degenerate in the sense that if $\gamma > 1$ then then the diffusion coefficient vanishes at $u = 0$ and if $\gamma < 1$ then the diffusion coefficient blows up at $u = 0$.

Remark 6.4.1.
- This is a simple model for flow through a porous medium. Let u denote the density of a compressible fluid with velocity $\mathbf{v}$ in a porous medium subject to D'Arcy's law that

$$\mathbf{v} = -\frac{K}{\mu}\nabla p$$

where K is permeability and μ is the fluid viscosity. The continuity equation (conservation of mass) is

$$u_t + \nabla \cdot (u\mathbf{v})$$

and assuming the equation of state

$$p = S(u)$$

we obtain

$$u_t - \nabla \cdot (\frac{K}{\mu} \nabla S(u)).$$

Thus if $S(u) = Cu^{\gamma}$ rescaling space and time yields the desired equation.

- The equation turns up in other settings. For example,

  $$h_t = \nabla \cdot (h^3 \nabla h)$$

  models the height, $h = h(x,t)$, of a thin viscous drop spreading under gravity on a horizontal surface.

A beautiful theory has been developed for this equation, [52]. Here we limit ourselves to finding a special solution formula. We assume that

$$\gamma > 1.$$

Consider the *ansatz*

$$u(x,t) = \frac{1}{t^{\alpha}} v(\frac{x}{t^{\beta}}) \tag{6.28}$$

where the constants α and β and the function v must be determined for u to solve the equation (6.27).

Substitution into the porous medium equation yields

$$\alpha t^{-(1+\alpha)} v(y) + \beta t^{-1+\alpha)} y \cdot Dv(y) + t^{-(\alpha\gamma+2\beta)} \Delta v^{\gamma}(y) = 0$$

for $y = x/t^{\beta}$. In order for this equation to hold for all t we require

$$\alpha + 1 = \alpha\gamma + 2\beta$$

yielding

$$\alpha v(y) + \beta y \cdot Dv(y) + \Delta v^{\gamma}(y) = 0.$$

Now imposing radial symmetry we set $v(y) := w(|y|)$ and obtain

$$\alpha w + \beta r w' + (w^{\gamma})'' + \frac{n-1}{r}(w^{\gamma})' = 0$$

where $r = |y|$ and $' = \frac{d}{dr}$. Again simplifying by setting

$$\alpha = n\beta$$

we obtain the integrable equation

$$\beta(r^n w)' + (r^{n-1}(w^\gamma)')' = 0.$$

Integration yields

$$r^{n-1}(w^\gamma)' + \beta r^n w = a$$

for some constant a. Fixing w to be bounded and $w' = 0$ as $r \to 0$ we obtain that $a = 0$. This leads to the equation

$$(w^{\gamma-1})' = -\frac{\gamma - 1}{\gamma}\beta r$$

which has the solution

$$w^{\gamma-1} = b - \frac{\gamma - 1}{2\gamma}\beta r^2$$

where b is a constant.Finally we find that

$$w(r) = (b - \frac{\gamma - 1}{2\gamma}\beta r^2)_+^{\frac{1}{\gamma-1}}$$

where the positive part ensures that w is non-negative.

Thus we have derived the following solution to the porous medium equation

$$u(x,t) = \frac{1}{t^\alpha}(b - \frac{\gamma - 1}{2\gamma}\beta\frac{|x|^2}{t^{2\beta}})_+^{\frac{1}{\gamma-1}} \tag{6.29}$$

which in general is known as the *Barenblatt-Kompaneetz-Zeldovich* solution and for $n = 1, \gamma = 2$ as the *Barenblatt-Pattle* solution,

$$u(x,t) = \frac{1}{6t^{1/3}}(s^2 - \frac{x^2}{t^{2/3}})_+.$$

Note that the solution has spreading compact support and that it is singular as $t \to 0_+$.

Chapter 7

Parametrised surfaces and hypersurfaces

In this section we introduce the elementary geometric analysis which is necessary to treat partial differential equations on surfaces. This material is taken from the article [19].

We begin with recalling some facts from elementary differential geometry concerning parametrised surfaces in Section 7.1. We then continue with hypersurfaces in $\mathbb{R}^{n+1}$ and the basic analysis concepts on such hypersurfaces in Section 7.2. We introduce the necessary geometric concepts, for example the notion of curvature. The formula for integration by parts is proved and we formulate the coarea formula. We introduce global coordinates in a neighbourhood of a hypersurface, the Fermi coordinates, in Section 7.2.1. They will be quite useful for the numerical analysis of PDEs on surfaces. For theoretical reasons we will introduce the oriented distance function. For the treatment of surface PDEs the Poincaré inequality on surfaces is central. We prove it in Section 7.2.2.

7.1 Parametrised surfaces

Let $n \in \mathbb{N}$. We call $\Gamma \subset \mathbb{R}^{n+1}$ an *m-dimensional parametrised C^k-surface* ($k \in \mathbb{N} \cup \{\infty\}$), if for every point $x_0 \in \Gamma$ there exists an open set $U \subset \mathbb{R}^{n+1}$ with $x_0 \in U$, an open connected set $V \subset \mathbb{R}^n$ and a map $X : V \to U \cap \Gamma$ with the properties $X \in C^k(V, \mathbb{R}^{n+1})$, X is bijective and rank $\nabla X = n$ on V.

The map X is called a local *parametrization* of Γ while X^{-1} is called a local *chart*. A collection $(X_i)_{i \in I}, X_i \in C^k(V_i, \mathbb{R}^{n+1})$ of local parametrizations such that $\cup_{i \in I} X_i(V_i) = \Gamma$ is called a *C^k–atlas*. If $X_i(V_i) \cap X_j(V_j) \neq \emptyset$, then the map $X_i^{-1} \circ X_j$ by assumption is a C^k-diffeomorphism.

A function $f : \Gamma \to \mathbb{R}$ is k *times differentiable* if all the functions $f \circ X_i : V_i \to \mathbb{R}$ are k times differentiable. A function $f : \Gamma \to \mathbb{R}$ is k times differentiable if all the functions $f \circ X_i : V_i \to \mathbb{R}$ are k times differentiable.

Let $X \in C^2(V, \mathbb{R}^{n+1})$ be a local parametrization of Γ, $\theta \in V$. We define the *first fundamental form* $G(\theta) = (g_{ij}(\theta))_{ij=1,\ldots,n}$, $\theta \in V$ by

$$g_{ij}(\theta) = \frac{\partial X}{\partial \theta_i}(\theta) \cdot \frac{\partial X}{\partial \theta_j}(\theta), \quad i, j = 1, \ldots, n.$$

Superscript indices denote the inversion of the matrix G so that $(g^{ij})_{ij=1,\ldots,n} = G^{-1}$, and by $g = \det(G)$ we denote the determinant of the matrix G.

The *Laplace-Beltrami operator* on Γ is defined for a twice differentiable function $f : \Gamma \to \mathbb{R}$ as follows. Let $F(\theta) = f(X(\theta))$, $\theta \in V$. Then

$$\left(\Delta_\Gamma f\right)(X(\theta)) = \frac{1}{\sqrt{g(\theta)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial \theta_j}\Big(g^{ij}(\theta)\sqrt{g(\theta)}\frac{\partial F}{\partial \theta_i}(\theta)\Big). \tag{7.1}$$

The *tangential gradient* is given by

$$\left(\nabla_\Gamma f\right)(X(\theta)) = \sum_{i,j=1}^{n} g^{ij}(\theta)\frac{\partial F}{\partial \theta_j}(\theta)\frac{\partial X}{\partial \theta_i}(\theta). \tag{7.2}$$

7.2 Hypersurfaces

Definition 7.2.1. Let $k \in \mathbb{N} \cup \{\infty\}$. $\Gamma \subset \mathbb{R}^{n+1}$ is called a C^k–hypersurface, if for each point $x_0 \in \Gamma$ there exists an open set $U \subset \mathbb{R}^{n+1}$ containing x_0 and a function $\phi \in C^k(U)$ with the property that $\nabla\phi \neq 0$ on $\Gamma \cap U$ and such that

$$U \cap \Gamma = \{x \in U \mid \phi(x) = 0\}. \tag{7.3}$$

The linear space

$$\begin{aligned} T_x\Gamma = \big\{\tau \in \mathbb{R}^{n+1} \mid \exists\, \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1} \text{ differentiable },\\ \gamma((-\epsilon,\epsilon)) \subset \Gamma, \gamma(0) = x \text{ and } \gamma'(0) = \tau\} \end{aligned}$$

is called the *tangent space* to Γ at $x \in \Gamma$. It is easy to show that $T_x\Gamma = [\nabla\phi(x)]^\perp$, the set of all vectors that are orthogonal to $\nabla\phi(x)$, where ϕ is as in (7.3). In particular, $T_x\Gamma$ is an n–dimensional subspace of $\mathbb{R}^{n+1}$.
A vector $\nu(x) \in \mathbb{R}^{n+1}$ is called a *unit normal vector* at $x \in \Gamma$ if $\nu(x) \perp T_x\Gamma$ and $|\nu(x)| = 1$. In view of the above characterisation of $T_x\Gamma$ we then have

$$\nu(x) = \frac{\nabla\phi(x)}{|\nabla\phi(x)|} \quad \text{or} \quad \nu(x) = -\frac{\nabla\phi(x)}{|\nabla\phi(x)|}. \tag{7.4}$$

A C^1–hypersurface is called *orientable* if there exists a continuous vector field $\nu : \Gamma \to \mathbb{R}^{n+1}$ such that $\nu(x)$ is a unit normal vector to Γ for all $x \in \Gamma$.

The connection between parametrised surfaces from Section 7.1 and hypersurfaces is given by the following well known little lemma.

Lemma 7.2.2. *Assume that Γ is a C^k-hypersurface in $\mathbb{R}^{n+1}$. Then for every $x \in \Gamma$ there exists an open set $U \subset \mathbb{R}^{n+1}$ with $x \in U$ and a parametrised C^k-surface $X : V \to U \cap \Gamma$ such that X is a bijective map from V onto $U \cap \Gamma$. If $X : V \to U \cap \Gamma$ is a parametrised C^k-surface and $\theta \in V$, then there is an open set $\tilde{V} \subset V$ with $\theta \in \tilde{V}$ such that $X(\tilde{V})$ is a C^k-hypersurface.*

This means that locally we always can work with hypersurfaces. And we may use all the definitions from Section 7.1 for hypersurfaces.

Definition 7.2.3. Let $\Gamma \subset \mathbb{R}^{n+1}$ be a C^1–hypersurface and $f : \Gamma \to \mathbb{R}$ be differentiable at $x \in \Gamma$. We define the tangential gradient of f at $x \in \Gamma$ by

$$\nabla_\Gamma f(x) = \nabla \overline{f}(x) - \nabla \overline{f}(x) \cdot \nu(x)\, \nu(x) = P(x) \nabla \overline{f}(x)$$

where $P(x)_{ij} = \delta_{ij} - \nu_i(x)\nu_j(x)$ $(i, j = 1, \ldots, n+1)$. Here $\overline{f}$ is a smooth extension of $f : \Gamma \to \mathbb{R}$ to an $n+1$-dimensional neighbourhood U of the surface Γ, so that $\overline{f}\big|_\Gamma = f$. ∇ denotes the gradient in $\mathbb{R}^{n+1}$ and $\nu(x)$ is a unit normal at x.
The *Laplace-Beltrami operator* applied to a twice differentiable function $f \in C^2(\Gamma)$ is given by

$$\Delta_\Gamma f = \nabla_\Gamma \cdot \nabla_\Gamma f = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i f. \tag{7.5}$$

See the proof of Theorem 7.2.10 in Section 7.2.1 for the construction of an extension $\overline{f}$. We shall use the notation, (as in the above definition),

$$\nabla_\Gamma f(x) = (\underline{D}_1 f(x), \ldots, \underline{D}_{n+1} f(x))$$

for the $n+1$ components of the tangential gradient. Note that $\nabla_\Gamma f(x) \cdot \nu(x) = 0$ and hence $\nabla_\Gamma f(x) \in T_x\Gamma$.

Let us show, that (7.1) and (7.2) are equivalent to the settings in Definition 7.2.3. Since the tangential gradient is a tangent vector, $\nabla_\Gamma f \circ X = \sum_{i=1}^n \alpha_i X_{\theta_i}$ with certain scalars α_j. We solve this equation for $\alpha_1, \ldots, \alpha_n$ by multiplying it by X_{θ_k} to get

$$F_{\theta_k} = \nabla_\Gamma f \circ X \cdot X_{\theta_k} = \sum_{i=1}^n \alpha_i X_{\theta_i} \cdot X_{\theta_k} = \sum_{i=1}^n \alpha_i g_{ik}. \tag{7.6}$$

For the first equality on the left we have used that by the chain rule we have from $F(\theta) = f(X(\theta))$ that $F_{\theta_k} = \sum_{l=1}^n \underline{D}_l f \circ X X_{l\theta_k}$ since X_{θ_k} is a tangent vector. From (7.6) we infer that $\alpha_l = \sum_{k=1}^n F_{\theta_k} g^{kl}$, and this finally gives (7.2). Now it is easy to derive (7.1).

Lemma 7.2.4. *$\nabla_\Gamma f(x)$ only depends on the values of f on $\Gamma \cap U$, where $U \subset \mathbb{R}^{n+1}$ is a neighbourhood of x.*

Proof. It is sufficient to show that $f \equiv 0$ on $\Gamma \cap U$ implies that $\nabla_\Gamma f(x) = 0$. Choose $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ such that $\gamma(0) = x, \gamma((-\epsilon,\epsilon)) \subset \Gamma \cap U$ and $\gamma'(0) = \nabla_\Gamma f(x)$. Since $\tilde{f}(\gamma(t)) = f(\gamma(t)) = 0$ for all $|t| < \epsilon$ we have

$$0 = \nabla \tilde{f}(x) \cdot \gamma'(0) = \big(\nabla_\Gamma f(x) + \nabla \tilde{f}(x) \cdot \nu(x)\, \nu(x)\big) \cdot \nabla_\Gamma f(x) = |\nabla_\Gamma f(x)|^2,$$

which implies the result. ☐

We denote by $C^1(\Gamma)$ the set of functions $f : \Gamma \to \mathbb{R}$, which are differentiable at every point $x \in \Gamma$ and for which $\underline{D}_j f : \Gamma \to \mathbb{R}, j = 1, \ldots, n+1$ are continuous. Similarly one can define $C^l(\Gamma)\, (l \in \mathbb{N})$ provided that Γ is a C^k–hypersurface with $k \geq l$.

Definition 7.2.5. For $\Gamma \in C^2$ we define

$$\mathcal{H}_{ij} = \underline{D}_i \nu_j \qquad (i, j = 1, \ldots, n+1). \tag{7.7}$$

It is easily shown that the matrix $\mathcal{H}$ is symmetric and that it possesses an eigenvalue 0 in the normal direction: $\mathcal{H}\nu = 0$. $\mathcal{H}$ is called the *extended Weingarten map*. The restriction of $\mathcal{H}$ to the tangent space is called the Weingarten map.

For $x \in \Gamma$ the quantity

$$H(x) = \mathrm{trace}\mathcal{H}(x) = \sum_{i=1}^{n+1} \mathcal{H}_{ii}(x) \tag{7.8}$$

is the *mean curvature* of Γ at the point x. It differs from the common definition by a factor n. We note that the eigenvalues $\kappa_1, \ldots, \kappa_n$ of $\mathcal{H}$ (apart from the trivial eigenvalue 0 in ν-direction) are the *principal curvatures* of Γ.

Let us have a look at the most simple example. The sphere of radius $R > 0$, $\Gamma = \{x \in \mathbb{R}^{n+1} |\, |x| = R\}$, is given by the level set function $\phi(x) = |x| - R$ for $0 < |x| < \infty$. We may choose $\nu = \frac{\nabla \phi}{|\nabla \phi|} = \frac{x}{|x|}$ and get for $x \in \Gamma$

$$\mathcal{H}_{ij}(x) = \underline{D}_i \nu_j(x) = \underline{D}_i \frac{x_j}{|x|} = \frac{1}{R} \underline{D}_i x_j = \frac{1}{R}\left(\delta_{ij} - \nu_i \nu_j\right) = \frac{1}{R}\left(\delta_{ij} - \frac{x_i x_j}{R^2}\right).$$

This matrix has an eigenvalue 0 with eigenvector $\frac{x}{R}$ and n eigenvalues $\kappa_j = \frac{1}{R}$ $(j = 1, \ldots, n)$. The mean curvature of Γ is then given as $H = \frac{n}{R}$.

The following result on the exchange of tangential derivatives is easily proved.

Lemma 7.2.6. *For $\Gamma \in C^2$ and $u \in C^2(\Gamma)$ we have*

$$\underline{D}_i \underline{D}_j u - \underline{D}_j \underline{D}_i u = \left(\mathcal{H}\nabla_\Gamma u\right)_j \nu_i - \left(\mathcal{H}\nabla_\Gamma u\right)_i \nu_j. \tag{7.9}$$

for $i, j = 1, \ldots, n+1$.

7.2.1 Global coordinates

It is quite convenient to use global coordinates in a neighbourhood of a hypersurface, the so called Fermi coordinates. This avoids working with charts and atlases (see Section 7.1) when proving results and carrying out the numerical analysis. For this one introduces the *oriented distance function* for Γ.

Remark 7.2.7. In the context of surface finite elements we will use the oriented distance function only for our analysis and numerical analysis. We will not use it in defining the computational methods. We will not need the oriented distance function for the implementation of our algorithms. It may be of use in *implicit surface* methods

Assume in the following that $G \subset \mathbb{R}^{n+1}$ is bounded and open with exterior normal ν and assume that $\Gamma = \partial G$ is a C^k-hypersurface ($k \geq 2$). The oriented distance function for Γ is defined by

$$d(x) = \begin{cases} \inf_{y \in \Gamma} |x - y|, & x \in \mathbb{R}^{n+1} \setminus \bar{G} \\ -\inf_{y \in \Gamma} |x - y|, & x \in G. \end{cases}$$

One easily verifies that d is globally Lipschitz–continuous with Lipschitz constant 1. Since ∂G is a C^2–hypersurface, it satisfies both a uniform interior and a uniform exerior sphere condition, which means that for each point $x_0 \in \partial\Omega$ there are balls B and B' such that

$$\bar{B} \cap (\mathbb{R}^{n+1} \setminus \Omega) = \{x_0\}, \quad \bar{B}' \cap \bar{\Omega} = \{x_0\}$$

and the radii of B, B' are bounded from below by a positive constant δ uniformly in x_0. With this observation the following Lemma is easily proved.

Lemma 7.2.8. *We define*

$$U_\delta = \left\{ x \in \mathbb{R}^{n+1} \,|\, |d(x)| < \delta \right\}.$$

Then $d \in C^k(U_\delta)$ and for every point $x \in U_\delta$ there exists a unique point $a(x) \in \Gamma$, such that

$$x = a(x) + d(x)\nu(a(x)). \tag{7.10}$$

Additionally we have that

$$\nabla d(x) = \nu(a(x)), \quad |\nabla d(x)| = 1 \qquad \text{for } x \in U_\delta.$$

We also extend the normal constantly in normal direction: $\nu(x) = \nu(a(x))$ for $x \in U_\delta$. Thus we have introduced a global coordinate system around Γ. Every point $x \in U_\delta$ can be described by its *Fermi coordinates* (normal coordinates) $d(x)$ and $a(x)$ according to (7.10).

The introduction of global coordinates allows to work with the well known coarea formula [22].

Theorem 7.2.9. *Let $\Gamma(r) = \left\{ x \in \mathbb{R}^{n+1} \,|\, d(x) = r \right\}$ be the parallel surface to $\Gamma = \Gamma(0)$ for $|r| < \delta$. Then*

$$\int_{U_\varepsilon} f(x)\, dx = \int_{-\epsilon}^{\epsilon} \int_{\Gamma(r)} f(x)\, dA(x) dr \tag{7.11}$$

for $f \in C^0(U_\delta)$ and $0 < \epsilon < \delta$.

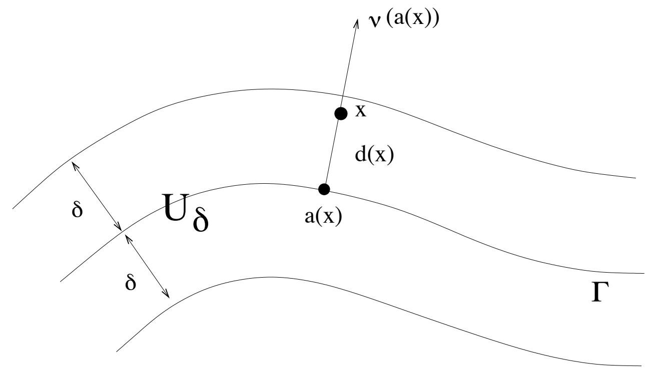


Figure 7.1: Strip U_δ around the hypersurface Γ and normal coordinates $x = a(x) + d(x)\nu(a(x))$.

We note that this formula changes to

$$\int_{U_\varepsilon} f(x)\, dx = \int_{-\epsilon}^{\epsilon} \int_{\Gamma(r)} f(x)|\nabla\phi(x)|\, dA(x)dr \tag{7.12}$$

if the surfaces are given by an arbitrary level set function ϕ as in (7.3), $\Gamma(r) = \big\{x \in \mathbb{R}^{n+1} |\, \phi(x) = r\big\}$, and the strip around Γ is taken to be $U_\delta = \big\{x \in \mathbb{R}^{n+1} |\, |\phi(x)| < \delta\big\}$. In this case one does not work with parallel surfaces to Γ.

With the coarea formula one can prove the formula for *integration by parts* on surfaces Γ.

Theorem 7.2.10. *Assume that Γ is a hypersurface in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Gamma$ and that $f \in C^1(\overline{\Gamma})$. Then*

$$\int_\Gamma \nabla_\Gamma f\, dA = \int_\Gamma f H \nu\, dA + \int_{\partial\Gamma} f \mu\, dA. \tag{7.13}$$

Here, μ denotes the conormal vector which is normal to $\partial\Gamma$ and tangent to Γ. A compact hypersurface Γ does not have a boundary, $\partial\Gamma = \emptyset$, and the last term on the right hand side vanishes.

Note that in (7.13) dA in connection with an integral over Γ denotes the n-dimensional surface measure, while dA in connection with an integral over $\partial\Gamma$ is the $n-1$-dimensional surface measure.

Proof. We extend f to the tubular neighbourhood U_ϵ of Γ by

$$\overline{f}(x) = f(a(x)) \quad (x \in U_\epsilon).$$

Then the chain rule gives

$$\frac{\partial \overline{f}}{\partial x_j}(x) = \sum_{k=1}^{n+1} \underline{D}_k f(a(x)) \frac{\partial a_k}{\partial x_j}(x).$$

The tangential derivative $\underline{D}_k f$ appears because we have that

$$\frac{\partial a_k}{\partial x_j}(x) = \frac{\partial}{\partial x_j}\left(x_k - d(x)\nu_k(x)\right) = \delta_{jk} - \nu_j(x)\nu_k(x) - d(x)\mathcal{H}_{jk}(x)$$

and the matrix $(\tilde{a}_{kj})_{j,k=1,\dots,n+1}$ ($\tilde{a}_{kj} = a_{k,x_j}$) maps any vector into a tangent vector. Thus we have

$$\nabla \overline{f}(x) = (I - d(x)\mathcal{H}(x))\nabla_\Gamma f(a(x)). \tag{7.14}$$

Specially one obtains $\nabla \overline{f}(x) = \nabla_\Gamma f(x)$ for $x \in \Gamma$. We apply Gauß' theorem to $\overline{f}$ on U_ϵ and get

$$\int_{U_\epsilon} \nabla \overline{f}(x)\, dx = \int_{\partial U_\epsilon} \overline{f}(x)\nu_{\partial U_\epsilon}(x)\, dA(x).$$

We have that $\partial U_\epsilon = \Gamma(\epsilon) \cup \Gamma(-\epsilon) \cup M(\epsilon)$ where $M(\epsilon) = \{x + r\nu_\Gamma(x) |\, x \in \partial\Gamma, r \in [-\epsilon, \epsilon]\}$ (see Figure 7.2). So,

$$\begin{aligned}
&\frac{1}{2\epsilon}\int_{U_\epsilon} (I - d(x)\mathcal{H}(x))\nabla_\Gamma f(a(x))\, dx \qquad (7.15)\\
&\quad = \frac{1}{2\epsilon}\Bigg(\int_{\Gamma(\epsilon)} \overline{f}(x)\nu_\Gamma(x)\, dA(x) - \int_{\Gamma(-\epsilon)} \overline{f}(x)\nu_\Gamma(x)\, dA(x)\\
&\qquad + \int_{M(\epsilon)} \overline{f}(x)\mu_\Gamma(x)\, dA(x)\Bigg)
\end{aligned}$$

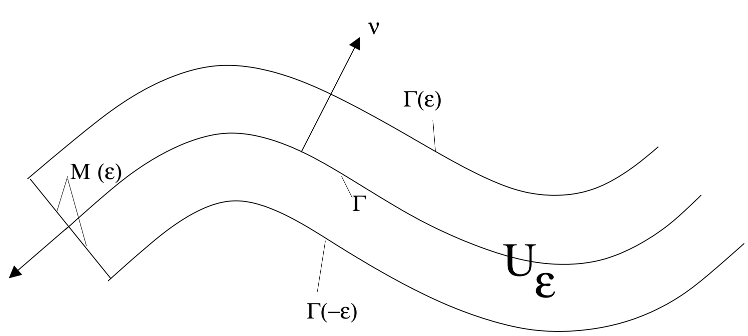


Figure 7.2: Geometric situation around the given surface Γ. Parallel surfaces $\Gamma(\epsilon)$, $\Gamma(-\epsilon)$ and normal ν, conormal μ.

with the normal ν_Γ and the conormal μ_Γ to Γ, which do not depend on ϵ. We take the limit $\epsilon \to 0$ on both sides of this equation. Obviously for the left side

$$\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{U_\epsilon} (I - d(x)\mathcal{H}(x))\nabla_\Gamma f(a(x))\, dx = \int_\Gamma \nabla_\Gamma f(x)\, dA(x).$$

The limit of the first two terms of the right hand side of (7.15) is given by

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{\Gamma(\epsilon)} \overline{f}(x)\nu_\Gamma(x)\, dA(x) = \int_\Gamma f(x)H(x)\nu_\Gamma(x)\, dA(x).$$

The last equality from the transport theorem (Leibniz formula) in Theorem 8.2.1. In the proof of that Theorem we will not use integration by parts.

For the last term on the right hand side of (7.15) we have that

$$\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{M(\epsilon)} \overline{f}(x)\mu_\Gamma(x)\, dA(x) = \int_{\partial\Gamma} f(x)\mu_\Gamma(x)\, dA(x),$$

because the integrand does not depend on ϵ. $\square$

The formula for integration by parts on Γ leads to the notion of a weak derivative and to the concept of Sobolev spaces on surfaces. Sobolev spaces are the natural spaces for solutions of elliptic partial differential equations. Let $\Gamma \in C^2$ for the following.

For $p \in [1, \infty]$ we denote by $L^p(\Gamma)$ the space of functions $f : \Gamma \to \mathbb{R}$ which are measurable with repsect to the surface measure dA (the n-dimensional Hausdorff measure) and have finite norm where

$$\|f\|_{L^p(\Gamma)} = \left(\int_\Gamma |f|^p\, dA\right)^{\frac{1}{p}}$$

for $p < \infty$ and for $p = \infty$ we mean the essential supremum norm.

$L^p(\Gamma)$ is a Banach space and for $p = 2$ a Hilbert space. For $1 \le p < \infty$ the spaces $C^0(\Gamma)$ and $C^1(\Gamma)$ are dense in $L^p(\Gamma)$.

Definition 7.2.11. A function $f \in L^1(\Gamma)$ has the weak derivative $\underline{v_i = \underline{D}_i f \in L^1(\Gamma)}$ ($i \in \{1, \dots, n+1\}$), if for every function $\varphi \in C^1(\Gamma)$ with compact support $\overline{\{x \in \Gamma | \varphi(x) \neq 0\}} \subset \Gamma$ we have the relation

$$\int_\Gamma f \underline{D}_i \varphi\, dA = -\int_\Gamma \varphi v_i\, dA + \int_\Gamma f\varphi H\nu_i\, dA$$

The Sobolev space $H^{1,p}(\Gamma)$ is defined by

$$H^{1,p}(\Gamma) = \{f \in L^p(\Gamma) | \, \underline{D}_i f \in L^p(\Gamma), i = 1, \dots, n+1\}$$

with norm

$$\|f\|_{H^{1,p}(\Gamma)} = \left(\|f\|_{L^p(\Gamma)} + \|\nabla_\Gamma f\|_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

For $k \in \mathbb{N}$ we define

$$H^{k,p}(\Gamma) = \left\{ f \in H^{k-1,p}(\Gamma) | \, \underline{D}_i v^{(k-1)} \in L^p(\Gamma), i = 1, \dots, n+1 \right\}$$

where $H^{0,p}(\Gamma) = L^p(\Gamma)$. For $p = 2$ we use the notation $H^k(\Gamma) = H^{k,2}(\Gamma)$. If we denote by $v^{(l)}$ *all* weak derivatives of order l, then

$$\|v\|_{H^{k,p}(\Gamma)} = \left(\sum_{l=0}^{k} \|v^{(l)}\|^p_{L^p(\Gamma)} \right)^{\frac{1}{p}}.$$

Note that for the previous definition we only have assumed that $\Gamma \in C^2$. This was done because in the formulation of the weak derivative we used the mean curvature of Γ.

7.2.2 Poincaré's inequality

For the convenience of the reader we show how Poincaré's inequality for a function with mean value zero on a compact n-dimensional hypersurface can be deduced from the Poincaré inequality in $\mathbb{R}^{n+1}$ with the use of global coordinates.

Theorem 7.2.12. *Assume that $\Gamma \in C^3$ and $1 \leq p < \infty$. Then there is a constant c such that for every function $u \in H^{1,p}(\Gamma)$ with $\int_\Gamma u \, dA = 0$ one has the inequality*

$$\|u\|_{L^p(\Gamma)} \leq c \|\nabla_\Gamma u\|_{L^p(\Gamma)}. \tag{7.16}$$

Proof. Clearly it is sufficient to prove the inequality for L^1 instead of L^p and it is sufficient to work with smooth functions. Assume that $u \in C^1(\Gamma)$ with $\int_\Gamma u \, dA = 0$. We extend this function to the tubular neighbourhood U_δ of the surface Γ, δ being sufficiently small, by

$$\overline{f}(x) = f(a(x)), \, x \in U_\delta.$$

We state the following intermediate Lemma.

Lemma 7.2.13. *Let $\Gamma \in C^2$ be a compact hypersurface. Then there is a constant c, such that for every $0 < \epsilon < \delta$ we have*

$$\left| \frac{1}{2\epsilon} \int_{U_\epsilon} \overline{f}(x) \, dx - \int_\Gamma f(x) \, dA(x) \right| \leq c\epsilon \int_\Gamma |f(x)| \, dA(x). \tag{7.17}$$

The *proof* of this lemma is left to the reader. We remark, that by the coarea formula from Theorem 7.2.9 we have

$$\frac{1}{2\epsilon} \int_{U_\epsilon} \overline{f}(x) \, dx = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\Gamma(r)} f(a(x)) \, dA(x) dr$$

with an integrand which does not depend on ϵ.

We continue with the proof of Theorem 7.2.12. From (7.17) we get the inequality

$$
\begin{aligned}
(1-c_1\epsilon)\int_\Gamma |f(x)|\,dA(x) &\le \frac{1}{2\epsilon}\int_{U_\epsilon} |\overline{f}(x)|\,dx \\
&\le \frac{1}{2\epsilon}\int_{U_\epsilon}\left|\overline{f}(x)-\frac{1}{|U_\epsilon|}\int_{U_\epsilon}\overline{f}(y)\,dy\right|\,dx + c_2\left|\frac{1}{|U_\epsilon|}\int_{U_\epsilon}\overline{f}(y)\,dy\right| \\
&\le c_3(\epsilon)\int_{U_\epsilon}|\nabla\overline{f}(x)|\,dx + c_2\left|\frac{1}{|U_\epsilon|}\int_{U_\epsilon}\overline{f}(y)\,dy\right|
\end{aligned}
$$

by using the Poincaré inequality for $\overline{f}$ on U_ϵ. We also have that

$$
\begin{aligned}
\left|\frac{1}{|U_\epsilon|}\int_{U_\epsilon}\overline{f}(x)\,dx\right| = \frac{2\epsilon}{|U_\epsilon|}\left|\frac{1}{2\epsilon}\int_{U_\epsilon}\overline{f}(x)\,dx - \int_\Gamma f(x)\,dA(x)\right| \\
\le c_4\epsilon\int_\Gamma |\nabla_\Gamma f(x)|\,dA(x)
\end{aligned}
$$

Thus we have the estimate

$$
\begin{aligned}
(1-c_1\epsilon-c_4\epsilon)\int_\Gamma |f(x)|\,dA(x) \le c_3(\epsilon)\int_{U_\epsilon}|\nabla\overline{f}(x)|\,dx \\
\le c_5(\epsilon)\int_\Gamma |\nabla_\Gamma f(x)|\,dA(x).
\end{aligned}
$$

For the last estimate we have used that by definition $\frac{\partial\overline{f}}{\partial\nu}=0$. A suitable choice of $\epsilon>0$ gives the estimate

$$
\int_\Gamma |f(x)|\,dA(x) \le c\int_\Gamma |\nabla_\Gamma f(x)|\,dA(x).
$$

This is Poincar'e's inequality in $L^1(\Gamma)$. For $p>1$ we apply this result to $|f|^p$ instead of f, use that $|\nabla_\Gamma|f|^p| = p|f|^{p-1}|\nabla_\Gamma f|$ and the Hölder inequality and the theorem is proved. $\square$

The formula for integration by parts on surfaces directly implies Green's formula. One has from Theorem 7.2.10, using the summation convention that we sum over doubly appearing indices,

$$
\begin{aligned}
\int_\Gamma \nabla_\Gamma f\cdot\nabla_\Gamma g\,dA = \int_\Gamma \underline{D}_i f\underline{D}_i g\,dA = \int_\Gamma \underline{D}_i\left(f\underline{D}_i g\right)\,dA - \int_\Gamma f\underline{D}_i\underline{D}_i g\,dA \\
= \int_\Gamma f\underline{D}_i g H\nu_i\,dA + \int_\Gamma f\underline{D}_i g\mu_i\,dA - \int_\Gamma f\Delta_\Gamma g\,dA.
\end{aligned}
$$

Since $\underline{D}_i g\nu_i = \nabla_\Gamma g\cdot\nu = 0$ we have the following Theorem.

Theorem 7.2.14.

$$
\int_\Gamma \nabla_\Gamma f\cdot\nabla_\Gamma g\,dA = -\int_\Gamma f\Delta_\Gamma g\,dA + \int_\Gamma f\nabla_\Gamma g\cdot\mu\,dA \tag{7.18}
$$

Chapter 8

Partial differential equations on surfaces

This material is taken from [19] and [15]

8.1 Elliptic equations on hypersurfaces

In this section we shortly give the basic ideas for the analysis of elliptic PDEs on hypersurfaces. We assume that Γ is a compact and connected hypersurface.

8.1.1 The Poisson equation

We begin with the model case of the Poisson equation

$$-\Delta_\Gamma u = f \tag{8.1}$$

on a compact hypersurface Γ in $\mathbb{R}^{n+1}$ and thus without boundary. Here f is a given right hand side or source term which is taken to be from $L^2(\Gamma)$ or more generally from $H^{-1}(\Gamma)$.

A *weak solution* of (8.1) is a function $u \in H^1(\Gamma)$ which satisfies the relation

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \, dA = \int_\Gamma f \varphi \, dA \tag{8.2}$$

for every test function $\varphi \in H^1(\Gamma)$. Since $\varphi = 1$ is allowed as a test function we have to impose the condition $\int_\Gamma f = 0$ on the right hand side. If the right hand side f is a functional from $H^{-1}(\Gamma)$ only then the weak form of the equation reads

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \, dA = \langle \varphi, f \rangle$$

where the brackets stand for the evaluation of the functional f at the function φ.

Obviously there is no uniqueness of weak solutions in this case, since every constant is a solution. We will fix the free constant by choosing the mean value of u to vanish. The following theorem can easily be proved.

Theorem 8.1.1. *Let $\Gamma \in C^2$ be a compact hypersurface in $\mathbb{R}^{n+1}$ and assume that $f \in H^{-1}(\Gamma)$ with the property $\langle 1, f \rangle = 0$. Then there exists a unique solution $u \in H^1(\Gamma)$ of (8.2) with $\int_\Gamma u \, dA = 0$.*

The *proof* is an application of the Lax-Milgram theorem or of the Riesz representation theorem. The bilinear form

$$a(u, v) = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, dA$$

is a scalar product on the Hilbert space $X = \{u \in H^1(\Gamma) | \int_\Gamma u \, dA = 0\}$ because of Poincaré's inequality (7.16). The right hand side f was chosen to be in the space $H^{-1}(\Gamma)$ of linear functionals.

Besides the existence of weak solutions the most important ingredient for suitable numerics is the proof of regularity and of a priori estimates for solutions of the Poisson equation. We shortly show how one proves an a priori estimate in the $H^2(\Gamma)$ norm. For this we use the following little lemma.

Lemma 8.1.2. *Let $\Gamma \in C^2$ and $u \in H^2(\Gamma)$. Then*

$$|u|_{H^2(\Gamma)} \leq \|\Delta_\Gamma u\|_{L^2(\Gamma)} + c|u|_{H^1(\Gamma)} \tag{8.3}$$

with the constant $c = \sqrt{\|H\mathcal{H}\|_{L^\infty(\Gamma)}}$.

Proof. By approximation arguments we can assume that $\Gamma \in C^3$ and $u \in C^3(\Gamma)$. We have

$$|u|^2_{H^2(\Gamma)} = \sum_{i,j=1}^{n+1} \int_\Gamma \left(\underline{D}_i \underline{D}_j u\right)^2 \, dA,$$

and with the formula for integration by parts on Γ (7.13) in combination with Lemma 7.2.6

we obtain (using the summation convention)

$$\begin{aligned}
\int_\Gamma \underline{D}_i\underline{D}_j u\underline{D}_i\underline{D}_j u\,dA &= \int_\Gamma \underbrace{\underline{D}_i\underline{D}_j u\underline{D}_j u H\nu_i}_{=0}\,dA - \int_\Gamma \underline{D}_i\underline{D}_i\underline{D}_j u\underline{D}_j u\,dA \\
&= -\int_\Gamma \underline{D}_i\left(\underline{D}_j\underline{D}_i u + \underline{D}_k u\left(\nu_i\mathcal{H}_{jk} - \nu_j\mathcal{H}_{ik}\right)\right)\underline{D}_j u\,dA \\
&= -\int_\Gamma \underline{D}_i\underline{D}_j\underline{D}_i u\underline{D}_j u\,dA - \int_\Gamma \underbrace{\underline{D}_i\underline{D}_k u\left(\nu_i\mathcal{H}_{jk} - \nu_j\mathcal{H}_{ik}\right)\underline{D}_j u}_{=0}\,dA \\
&\quad -\int_\Gamma \underline{D}_k u\underline{D}_j u\underline{D}_i\left(\nu_i\mathcal{H}_{jk} - \nu_j\mathcal{H}_{ik}\right)\,dA \\
&= -\int_\Gamma \underline{D}_i\underline{D}_j\underline{D}_i u\underline{D}_j u\,dA - \int_\Gamma \left(H\mathcal{H} - \mathcal{H}^2\right)_{jk}\underline{D}_j u\underline{D}_k u\,dA.
\end{aligned}$$

For the remaining third order term we observe that

$$\begin{aligned}
\int_\Gamma \underline{D}_i\underline{D}_j\underline{D}_i u\underline{D}_j u\,dA &= \int_\Gamma \underline{D}_j\underline{D}_i\underline{D}_i u\underline{D}_j u + \nu_i\mathcal{H}_{jk}\underline{D}_k\underline{D}_i u\underline{D}_j u\,dA \\
&= \int_\Gamma \underline{D}_j\Delta_\Gamma u\underline{D}_j u\,dA - \int_\Gamma \left(\mathcal{H}^2\right)_{ij}\underline{D}_i u\underline{D}_j u\,dA \\
&= -\int_\Gamma \left(\Delta_\Gamma u\right)^2\,dA + \int_\Gamma \left(\mathcal{H}^2\right)_{ij}\underline{D}_i u\underline{D}_j u\,dA.
\end{aligned}$$

Here we have used the fact that

$$\nu_i\underline{D}_k\underline{D}_i u = \underline{D}_k\left(\nu_i\underline{D}_i u\right) - \underline{D}_k\nu_i\underline{D}_i u = -\mathcal{H}_{ik}\underline{D}_i u.$$

Altogether we have shown that

$$\begin{aligned}
|u|^2_{H^2(\Gamma)} &= \|\Delta_\Gamma u\|^2_{L^2(\Gamma)} - \int_\Gamma H\mathcal{H}\nabla_\Gamma u\cdot\nabla_\Gamma u\,dA \\
&\le \|\Delta_\Gamma u\|^2_{L^2(\Gamma)} + \|H\mathcal{H}\|_{L^\infty(\Gamma)}\|\nabla_\Gamma u\|^2_{L^2(\Gamma)}
\end{aligned}$$

and this finally proves the estimate (8.3). □

With the help of the previous Lemma and standard arguments we arrive at the regularity estimate for the solution of the Poisson equation on a compact hypersurface.

Theorem 8.1.3. *Assume that $\Gamma \in C^2$ and that $f \in L^2(\Gamma)$ with $\int_\Gamma f\,dA = 0$. Then the weak solution from Theorem 8.1.1 satisfies $u \in H^2(\Gamma)$ and*

$$\|u\|_{H^2(\Gamma)} \le c\|f\|_{L^2(\Gamma)}.$$

For the *proof* we use the basic estimate (choose $\varphi = u$ in (8.2))

$$|u|_{H^1(\Gamma)} \le c\|f\|_{L^2(\Gamma)}$$

– for which we use Poincaré's inequality again – together with the PDE pointwise almost everywhere to obtain

$$|u|_{H^2(\Gamma)} \leq c\|f\|_{L^2(\Gamma)},$$

if the solution has square integrable second derivatives.

The $H^2(\Gamma)$-regularity of u is taken from the theory of linear PDEs on Cartesian domains in $\mathbb{R}^n$. Here the arguments are purely local. For this we parametrize the C^2 surface Γ according to Lemma 7.2.2 locally by $X \in C^2(\Omega, \Gamma)$, $X = X(\theta)$ with some open domain $\Omega \subset \mathbb{R}^n$. If we set $U(\theta) = u(X(\theta)$, then U is a weak solution of the linear PDE

$$-\left(g^{kj} U_{\theta_j} \sqrt{g}\right)_{\theta_k} = f \circ X \sqrt{g}$$

on Ω. For the notation see Section 7.1. The coefficients of this PDE are in $C^1(\Omega)$ and the right hand side is in $L^2(\Omega)$ because by assumption $\Gamma \in C^2$. The well known regularity result from Cartesian PDEs (see for example [28]) then gives $U \in H^2(\Omega')$ for any $\Omega' \subset\subset \Omega$ and this in turn gives $u \in H^2(\Gamma)$.

General elliptic PDEs

In the previous section we have shown how the Poisson equation is solved on a compact surface. The methods are easily extended to general linear elliptic PDEs in divergence form and to boundary value problems (on surfaces with a boundary).

$$-\sum_{i,j=1}^{n+1} \underline{D}_i \left(a_{ij} \underline{D}_j u\right) - \sum_{i=1}^{n+1} \underline{D}_i \left(a_i u\right) + \sum_{i=1}^{n+1} b_i \underline{D}_i u + cu = f - \sum_{i=1}^{n+1} \underline{D}_i g_i. \tag{8.4}$$

We assume for the given coefficients that

$$a_{ij}, a_i, b_i, c \in L^\infty(\Gamma), g_i \in L^2(\Gamma) \quad (i, j = 1, \ldots, n+1).$$

We also assume that the coefficient vectors $a(x) = (a_1(x), \ldots, a_{n+1}(x))$ and $g(x) = (g_1(x), \ldots, g_{n+1}(x))$ are tangent vectors at $x \in \Gamma$, i. e. lie in $T_x\Gamma$, and that the matrix $\mathcal{A}(x) = \left(a_{ij}(x)\right)_{i,j=1,\ldots,n+1}$ is symmetric and maps the tangent space $T_x\Gamma$ into itself. We emphasize that the latter condition implies that in general constant coefficients a_{ij} are not admissible. They have to depend on the x-variable. Nevertheless $a_{ij} = \delta_{ij}$ obviously is allowed.

As ellipticity condition we assume a so called Ladyzenskaja condition, which says that there exists a number $c_0 > 0$, such that

$$\sum_{i,j=1}^{n+1} a_{ij}\xi_i\xi_j + \sum_{i=1}^{n+1} a_i\xi_0\xi_i + \sum_{i=1}^{n+1} b_i\xi_i\xi_0 + c\xi_0^2 \geq c_0 \sum_{i=1}^{n+1} \xi_i^2$$

almost everywhere on Γ for all $\xi = (\xi_1, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1}$ with $\xi \cdot \nu = 0$ and all $\xi_0 \in \mathbb{R}$.

With the PDE (8.4) we associate the bilinear form a,

$$a(u,\varphi) = \int_\Gamma \sum_{i,j=1}^{n+1} a_{ij}\underline{D}_j u \underline{D}_i \varphi + \sum_{i=1}^{n+1} a_i \underline{D}_i \varphi + \sum_{i=1}^{n+1} b_i \underline{D}_i u \varphi + cu\varphi \, dA,$$

and the functional F,

$$\langle \varphi, F \rangle = \int_\Gamma f\varphi + \sum_{i=1}^{n+1} g_i \underline{D}_i \varphi \, dA,$$

for $u, \varphi \in H^1(\Gamma)$.

Theorem 8.1.4. *Let $\Gamma \in C^2$ be a compact hypersurface. Assume that the coefficients satisfy the above conditions. Then there exists a unique weak solution of (8.4) with $\int_\Gamma u \, dA = 0$, i. e. there exists a unique $u \in H^1(\Gamma)$ such that*

$$a(u,\varphi) = \langle \varphi, F \rangle$$

for every $\varphi \in H^1(\Gamma)$.

Proof. The proof of this theorem is a direct application of the Lax-Milgram theorem. $\square$

8.2 Partial differential equations on moving surfaces

Quite often one has to solve PDEs which live on a moving surface or interface. In this section we will treat the most basic linear PDE on an evolving surface. The motion of the surface will be prescribed. The geometry will be described in Section 8.2.1. It will be important to use the space-time structure of the given geometry. We will describe in Section 8.2.2 how the standard conservation law can be derived. We will work with moving triangulated surfaces and use them to discretize the heat equation on a moving surface in Section **??**.

8.2.1 The geometry of moving surfaces

For each $t \in [0,T]$ let $\Gamma(t)$ be a compact hypersurface oriented by the normal vector field $\nu(\cdot,t)$ and $\Gamma_0 = \Gamma(0)$. We assume that there exists a map $G(\cdot,t) : \Gamma_0 \to \Gamma(t)$, $G \in C^1([0,T], C^2(\Gamma_0))$, such that $G(\cdot,t)$ is a diffeomorphism from Γ_0 to $\Gamma(t)$ and we define the velocity of $\Gamma(t)$ by

$$v(G(\cdot,t),t) = \frac{\partial G}{\partial t}(\cdot,t),$$

$G(\cdot,0) = Id$. We assume that $v(\cdot,t) \in C^2(\Gamma(t))$. The normal velocity of Γ then is defined by $v_\nu = v \cdot \nu\nu$.

We use the appropriate time derivative, that is

$$\partial^\bullet f = \frac{\partial f}{\partial t} + v \cdot \nabla f. \tag{8.5}$$

Obviously this derivative only depends on values of the function f on $\mathcal{G}_T$. It is quite often convenient to work with the space-time surface (see Figure 8.1)

$$\mathcal{G}_T = \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}. \tag{8.6}$$

Note that for a function $f : \mathcal{G}_T \to \mathbb{R}$ the time derivative $\frac{\partial f}{\partial t}$ and the spatial derivatives ∇f do not make sense separately.

Leibniz formulae, transport theorems

The following formulae for the differentiation of time dependent surface integrals are called *transport formulae* and are proved in [17], [18].

Theorem 8.2.1. *Let $\mathcal{M}(t)$ be an evolving surface with normal velocity v_ν. Let v_τ be a tangential velocity field on $\mathcal{M}(t)$. Let the boundary $\partial\mathcal{M}(t)$ evolve with the velocity $v = v_\nu + v_\tau$. Assume that f is a function such that all the following quantities exist. Then*

$$\frac{d}{dt}\int_{\mathcal{M}(t)} f\,dA = \int_{\mathcal{M}(t)} \partial^\bullet f + f\,\nabla_\Gamma \cdot v\,dA. \tag{8.7}$$

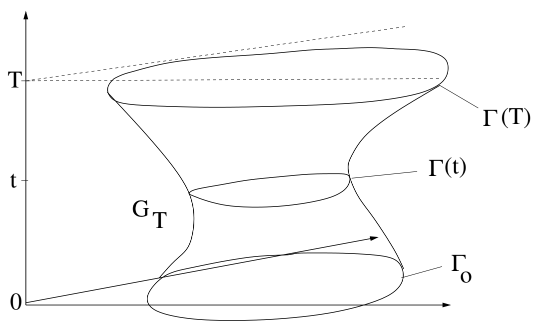


Figure 8.1: Space-time surface $\mathcal{G}_T$ for the dimension $n = 1$. Here each $\Gamma(t)$ is a curve.

Proof. Let $\Omega \subset \mathbb{R}^n$ be open and $X = X(\theta, t)$, $\theta \in \Omega$, $X(\cdot, t) : \Omega \to U \cap \Gamma$ be a local regular parametrization of the open portion $U \cap \Gamma$ of the surface Γ which evolves so that $X_t = v(X(\theta, t), t)$. The induced metric $(g_{ij})_{i,j=1,\dots,n}$ is given by $g_{ij} = X_{\theta_i} \cdot X_{\theta_j}$ with determinant $g = det(g_{ij})$. Let $(g^{ij}) = (g_{ij})^{-1}$. See also Section 7.1. Define

$$F(\theta, t) = f(X(\theta, t), t) \quad \text{and} \quad V(\theta, t) = v(X(\theta, t), t).$$

Then with the Euler relation for the derivative of the determinant,

$$\frac{\partial}{\partial t}\sqrt{g} = \sqrt{g} \sum_{i,j=1}^{n} g^{ij} X_{\theta_i} \cdot V_{\theta_j},$$

we have the following proof of (8.7):

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma \cap U} f \, dA = \frac{d}{dt}\int_{\Omega} F\sqrt{g}\, d\theta &= \int_{\Omega} \frac{\partial F}{\partial t}\sqrt{g} + F\frac{\partial \sqrt{g}}{\partial t}\, d\theta \\
= \quad \int_{\Omega} \left(\frac{\partial f}{\partial t} + \nabla f(X, \cdot) \cdot X_t\right)\sqrt{g} &+ f(X, \cdot))\sqrt{g} \sum_{i,j=1}^{n} g^{ij} X_{\theta_i} \cdot V_{\theta_j}\, d\theta \\
= \quad \int_{\Gamma \cap U} \dot{f} + f \nabla_\Gamma \cdot v \, dA.
\end{aligned}$$

where in the last step we have used that $V = X_t$ and that the tangential divergence of v is given by

$$\left(\nabla_\Gamma \cdot v\right)(X, \cdot) = \sum_{i,j=1}^{n} g^{ij} X_{\theta_i} \cdot V_{\theta_j}.$$

The theorem is proved. □

We give transport formulae for the time derivative of the most important bilinear forms m,

a and g given by

$$m(\phi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \phi(x,t)\psi(x,t)\,dA(x)$$

$$a(\phi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \mathcal{A}(\cdot,t)\nabla_{\Gamma(t)}\phi(\cdot,t)\cdot\nabla_{\Gamma(t)}\psi(\cdot,t)\,dA(x)$$

$$g(v;\phi(\cdot,t),\psi(\cdot,t)) = \int_{\Gamma(t)} \phi(x,t)\psi(x,t)\nabla_\Gamma\cdot v(x,t)\,dA(x).$$

Note, that now these bilinear forms explicitly depend on time too. But instead of writing e.g. $m(t,\phi(\cdot,t),\psi(\cdot,t))$ we suppress the explicit dependence on t. It will always be clear from the arguments ϕ and ψ at which time the bilinear form has to be evaluated.

Lemma 8.2.2. *We have for $\varphi,\psi \in H^1(\mathcal{G}_T)$*

$$\frac{d}{dt}m(\varphi,\psi) = m(\partial^\bullet\varphi,\psi) + m(\varphi,\partial^\bullet\psi) + g(v;\varphi,\psi). \tag{8.8}$$

and

$$\frac{d}{dt}a(\varphi,\psi) = a(\partial^\bullet\varphi,\psi) + a(\varphi,\partial^\bullet\psi) + b(v;\varphi,\psi) \tag{8.9}$$

with the bilinear form

$$b(v;\varphi,\psi) = \int_\Gamma \mathcal{B}(v)\nabla_\Gamma\varphi\cdot\nabla_\Gamma\psi\,dA. \tag{8.10}$$

With the deformation tensor

$$D(v)_{ij} = \frac{1}{2}\sum_{k=1}^{n+1}\left(\mathcal{A}_{ik}\left(\nabla_\Gamma\right)_k v_j + \mathcal{A}_{jk}\left(\nabla_\Gamma\right)_k v_i\right)(i,j=1,\ldots,n+1)$$

and the tensor

$$\mathcal{B}(v) = \partial^\bullet\mathcal{A} + \nabla_\Gamma\cdot v\,\mathcal{A} - 2D(v) \tag{8.11}$$

we have the formula

$$\frac{d}{dt}\int_{\mathcal{M}(t)} \mathcal{A}\nabla_\Gamma f\cdot\nabla_\Gamma g\,dA = \tag{8.12}$$

$$\int_{\mathcal{M}(t)} \mathcal{A}\nabla_\Gamma\partial^\bullet f\cdot\nabla_\Gamma g + \mathcal{A}\nabla_\Gamma f\cdot\nabla_\Gamma\partial^\bullet g\,dA + \int_{\mathcal{M}(t)} \mathcal{B}(v)\nabla_\Gamma f\cdot\nabla_\Gamma g\,dA.$$

For the convenience of the reader we derive the transport formula for Dirichlet's integral

$$\int_\Gamma |\nabla_\Gamma f|^2\,dA$$

for a time dependent surface. We continue to use the notation of the previous proof. The generalization to the more general case in Lemma 8.2.2 then follows easily. We first observe that we have

$$|\left(\nabla_\Gamma f\right)(X,\cdot)|^2 = \sum_{i,j=1}^{n} g^{ij}F_{\theta_i}F_{\theta_j}, \tag{8.13}$$

so that

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma\cap U}|\nabla_\Gamma f|^2\,dA = \int_\Omega \sqrt{g}\sum_{i,j=1}^n g^{ij}F_{\theta_i}F_{\theta_j t}\,d\theta$$
$$+\frac{1}{2}\int_\Omega \sqrt{g}\sum_{i,j=1}^n g_t^{ij}F_{\theta_i}F_{\theta_j}\,d\theta + \frac{1}{2}\int_\Omega \sqrt{g}\sum_{i,j,k,l=1}^n g^{ij}g^{kl}X_{\theta_k}\cdot\mathcal{V}_{\theta_l}F_{\theta_i}F_{\theta_j}\,d\theta.$$

An easy calculation shows that

$$\begin{aligned} g_t^{ij} &= -\sum_{k,l=1}^n g^{ik}g^{jl}g_{kl,t} = -\sum_{k,l=1}^n g^{ik}g^{jl}\left(X_{\theta_k}\cdot X_{\theta_l}\right)_t \\ &= -\sum_{k,l=1}^n g^{ik}g^{jl}\left(\mathcal{V}_{\theta_k}\cdot X_{\theta_l} + X_{\theta_k}\cdot\mathcal{V}_{\theta_l}\right) \end{aligned}$$

and we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma\cap U}|\nabla_\Gamma f|^2\,dA =$$
$$\int_{\Gamma\cap U}\nabla_\Gamma f\cdot\nabla_\Gamma\partial^\bullet f\,dA - \int_{\Gamma\cap U}\sum_{i,j=1}^n \underline{D}_i v_j\underline{D}_i f\underline{D}_j f\,dA + \frac{1}{2}\int_{\Gamma\cap U}|\nabla_\Gamma f|^2\nabla_\Gamma\cdot v\,dA.$$

The formula (8.12) for $\mathcal{A}=I$ then follows by polarization.

8.2.2 Conservation and diffusion on moving surfaces

Conservation law

Let u be the density of a scalar quantity on $\Gamma(t)$ (for example mass per unit area $n=2$ or mass per unit length $n=1$). We suppose there is a surface flux q. The basic conservation law we wish to consider can be formulated for an arbitrary portion $\mathcal{M}(t)$ of $\Gamma(t)$, which is the image of a portion $\mathcal{M}(0)$ of $\Gamma(0)$ evolving with the prescribed velocity $v=v_\nu$. In the following we write $\partial^\circ u$ for the material time derivative of u with respect to this purely normal velocity:

$$\partial^\circ u = u_t + v_\nu\cdot\nabla u.$$

$\partial^\circ u$ is sometimes known as the *normal time derivative*, [8].

The conservation law is that, for every $\mathcal{M}(t)$,

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u\,dA = -\int_{\partial\mathcal{M}(t)} q\cdot\mu\,dA, \tag{8.14}$$

where, $\partial\mathcal{M}(t)$ is the boundary of $\mathcal{M}(t)$ (a curve if $n=2$ and the end points of a curve if $n=1$) and μ is the conormal on $\partial\mathcal{M}(t)$. Thus μ is the unit normal to $\partial\mathcal{M}(t)$ pointing out of

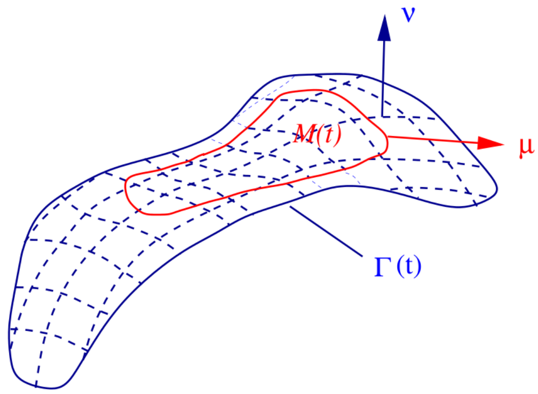


Figure 8.2: Conservation on a moving surface. Moving surface $\Gamma(t)$ and subset $M(t)$ with conormal μ.

$\mathcal{M}(t)$ and tangential to $\Gamma(t)$. The surface flux is denoted by q. Observe that components of q normal to $\mathcal{M}$ do not contribute to the flux, so we may assume that q is a tangent vector.

With the use of integration by parts, (7.13), we obtain

$$\int_{\partial\mathcal{M}(t)} q\cdot\mu\,dA = \int_{\mathcal{M}(t)} \nabla_\Gamma\cdot q\,dA - \int_{\mathcal{M}(t)} q\cdot\nu H\,dA = \int_{\mathcal{M}(t)} \nabla_\Gamma\cdot q\,dA.$$

On the other hand by the transport formula (8.7) we have

$$\frac{d}{dt}\int_{\mathcal{M}(t)} u\,dA = \int_{\mathcal{M}(t)} \partial^\circ u + u\nabla_\Gamma\cdot v_\nu\,dA,$$

so that

$$\int_{\mathcal{M}(t)} \partial^\circ u + u\,\nabla_\Gamma\cdot v_\nu + \nabla_\Gamma\cdot q\,dA = 0,$$

which implies the pointwise conservation law

$$\partial^\circ u + u\,\nabla_\Gamma\cdot v_\nu + \nabla_\Gamma\cdot q = 0 \tag{8.15}$$

This may also be written as

$$u_t + V\frac{\partial u}{\partial\nu} - HVu + \nabla_\Gamma\cdot q = 0 \tag{8.16}$$

where $V = v_\nu\cdot\nu$, see also [47].

We wish to consider a diffusive flux $q_d = -\mathcal{A}\nabla_\Gamma u$ and an advective flux $q_a = uv_\tau$ where v_τ is an advective tangential velocity field, i.e. $v_\tau\cdot\nu = 0$, so that

$$q = q_d + q_a = -\mathcal{A}\nabla_\Gamma u + uv_\tau$$

then we arrive at the PDE

$$\partial^\bullet u + u\,\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathcal{A}\nabla_\Gamma u) = 0. \tag{8.17}$$

In the following we assume that $\mathcal{A}$ is a sufficiently smooth symmetric $(n+1)\times(n+1)$ matrix which maps the tangent space of Γ at each point into itself and is positive definite on the tangent space, i. e.

$$\mathcal{A}\xi \cdot \xi \geq c_0|\xi|^2 \quad \forall \xi \in \mathbb{R}^{m+1}, \xi \cdot \nu = 0 \tag{8.18}$$

with some constant $c_0 > 0$. For the definition of a solution we assume that the elements of $\mathcal{A}$ belong to $L^\infty(\mathcal{G}_T)$. A weak solution of the PDE 8.17 is a function $u \in H^1(\mathcal{G}_T)$ which satisfies the equation

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi\, dA + \int_{\Gamma(t)} \mathcal{A}\nabla_\Gamma u \cdot \nabla_\Gamma \varphi\, dA = \int_{\Gamma(t)} u\partial^\bullet \varphi\, dA \tag{8.19}$$

almost everywhere on $(0, T)$, where φ is an arbitrary test function defined on the space-time surface $\mathcal{G}_T$.

In [17] we proved the existence of a weak solution.

Theorem 8.2.3. *Assume that the initial data $u_0 \in H^1(\Gamma_0)$, where $\Gamma_0 = \Gamma(0)$. Then there exists a unique weak solution $u \in H^1(\mathcal{G}_T)$ of the PDE (8.17), i. e. equation (8.19) is satisfied for almost every $t \in (0, T)$, which satisfies the initial condition $u(\cdot, 0) = u_0$ on Γ_0. Furthermore if $\mathcal{A}$ and $v \in C^1(\mathcal{G}_T)$ the solution satisfies the energy estimates*

$$\sup_{(0,T)} \|u\|^2_{L^2(\Gamma)} + \int_0^T \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} dt \leq c\|u_0\|^2_{L^2(\Gamma_0)}, \tag{8.20}$$

$$\int_0^T \|\partial^\bullet u\|^2_{L^2(\Gamma)} dt + \sup_{(0,T)} \|\nabla_\Gamma u\|^2_{L^2(\Gamma)} \leq c\|u_0\|^2_{H^1(\Gamma_0)}, \tag{8.21}$$

where $c = c(\mathcal{A}, v, \mathcal{G}_T, T)$.

Proof. For the convenience of the reader we include a proof of the a priori estimates. For (8.20) we set $\varphi = u$ in (8.19) and get with Lemma 8.2.2

$$\frac{d}{dt}m(u, u) + a(u, u) = m(u, \partial^\bullet u) = \frac{1}{2}\frac{d}{dt}m(u, u) - \frac{1}{2}g(v; u, u).$$

This gives

$$\frac{1}{2}\frac{d}{dt}m(u, u) + a(u, u) + \frac{1}{2}g(v; u, u) = 0$$

and with a Gronwall argument this implies (8.20).

For (8.21) we use (8.8). The weak equation (8.19) implies

$$m(\partial^\bullet u, \varphi) + g(v; u, \varphi) + a(u, \varphi) = 0.$$

We insert $\varphi = \partial^\bullet u$ and get from (8.9)

$$m(\partial^\bullet u, \partial^\bullet u) + g(v; u, \partial^\bullet u) + \frac{1}{2}\frac{d}{dt}a(u,u) - \frac{1}{2}b(v; u, u) = 0.$$

Standard arguments then lead to (8.21). $\square$

For $\varphi, \psi \in H^1(\Gamma)$ we use the bilinear forms

$$a(\varphi(\cdot,t), \psi(\cdot,t)) = \int_{\Gamma(t)} \mathcal{A}(\cdot,t)\nabla_\Gamma \varphi(\cdot,t) \cdot \nabla_\Gamma \psi(\cdot,t)\, dA, \tag{8.22}$$

$$m(\varphi(\cdot,t), \psi(\cdot,t)) = \int_{\Gamma(t)} \varphi(\cdot,t)\psi(\cdot,t)\, dA, \tag{8.23}$$

$$g(v(\cdot,t); \varphi(\cdot,t), \psi(\cdot,t)) = \int_{\Gamma(t)} \varphi(\cdot,t)\psi(\cdot,t)\nabla_\Gamma \cdot v(\cdot,t)\, dA. \tag{8.24}$$

Using this notation the weak form (8.19) of the PDE (8.17) becomes,

$$\frac{d}{dt}m(u,\varphi) + a(u,\varphi) = m(u, \partial^\bullet \varphi). \tag{8.25}$$

Of course the variational problem may be posed on the initial surface Γ_0. This would lead to non-constant coefficients even with $\mathcal{A} = \mathcal{I}$. It would also require knowledge of the map G in Section 8.2.1 which we choose to avoid.

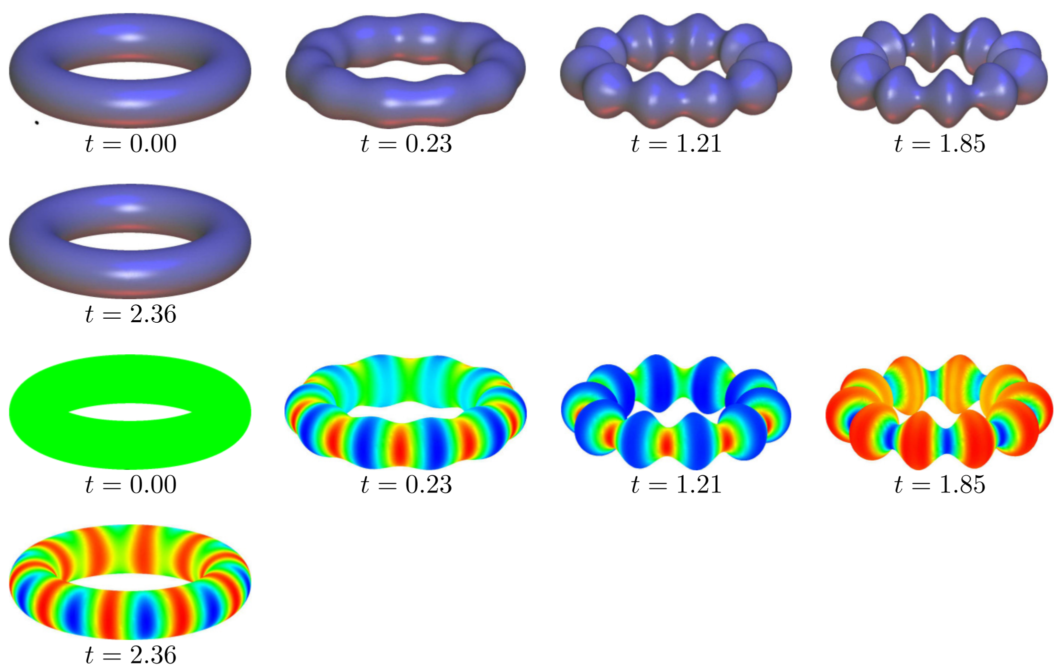


Figure 8.3: Deformation of a torus (upper rows) and solution of equation (8.17) (lower rows). The colors indicate the magnitude of the solution. There was no source term. The surface is deformed and reaches the initial form of a round torus again at time $t = 2.36$. The initial value was constant: $u_0 = 10.0$ (green at time $t = 0.00$). The solution at final time $t = 2.36$ is due purely to geometric motion.

8.3 Surface evolution

8.3.1 Motion by mean curvature

An evolving surface $\Gamma(t), \quad t \in (0, T)$ is said to evolve by *mean curvature flow* if the normal velocity of the surface V is given by the mean curvature H so that

$$V = -H. \tag{8.26}$$

As an example we consider ball of radius $R(t)$ in $\mathbb{R}^{n+1}$ evolving by mean curvature which gives the equation

$$\dot{R} = -nR^{-1}, \quad R(0) = R_0$$

which has the solution

$$R(t) = \sqrt{(}R_0^2 - nt), \quad t \in ([0, R_0^2/n]$$

where $T_e = R_0^2/n$ is the finite extinction time at which the ball shrinks to zero.

8.3.2 Curve evolution

Let $X(\theta, t) \in \mathbb{R}^2, \theta \in \mathbb{R}, \ t \in [0, T]$ be an evolving planar closed curve $(X(\theta + 1, t) = X(\theta, t))$ with velocity X_t and curvature vector $H\nu = -\Delta_\Gamma X$. Suppose $u(X(\theta, t), t) =: U(\theta, t)$ is a scalar field on the curve. We may consider the forced mean curvature evolution equation

$$V = -H + u$$

which may be rewritten as

$$X_t = \frac{1}{|X_\theta|}\frac{\partial}{\partial\theta}\frac{X_\theta}{|X_\theta|} + u\frac{X_\theta^\perp}{|X_\theta|}. \tag{8.27}$$

Multiplying by a test function $\phi(\theta), \ \theta \in \mathbb{R}, \quad \phi(\theta + 1) = \phi(\theta)$ and integrating by parts we obtain the variational form

$$\int_0^1 X_t(\theta, t)\phi(\theta)|X_\theta(\theta, t)|d\theta + \int_0^1 \frac{X_\theta(\theta, t)}{|X_\theta(\theta, t)|}\phi_\theta(\theta)d\theta = \int_0^1 U(\theta, t)X_\theta^\perp(\theta, t)d\theta. \tag{8.28}$$

8.3.3 Mean curvature flow of graphs

We turn our attention to the mean curvature evolution of surfaces $\Gamma(t)$, which can be written as graphs over some base domain $\Omega \subset \mathbb{R}^n$, i.e.

$$\Gamma(t) = \{(x, u(x,t)) \mid x \in \Omega\}.$$

In order to find the differential equation to be satisfied by the height function u, we recall (**??**) and (**??**) to see that the mean curvature H and the velocity V in the direction of $\nu = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}$ are given by

$$H = \nabla \cdot \big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\big), \quad V = -\frac{u_t}{\sqrt{1+|\nabla u|^2}}. \tag{8.29}$$

Thus, the evolution law $V = -H$ on $\Gamma(t)$ translates into the nonlinear parabolic partial differential equation

$$u_t - \sqrt{1+|\nabla u|^2}\ \nabla \cdot \big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\big) = 0 \qquad \text{in } \Omega \times (0,T), \tag{8.30}$$

to which we add the following boundary and initial conditions

$$u = g \quad \text{on } \partial\Omega \times (0,T) \tag{8.31}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{8.32}$$

where $g : \partial\Omega \to \mathbb{R}$ and $u_0 : \bar{\Omega} \to \mathbb{R}$ are given functions. The boundary condition (8.31) implies that the boundaries of the surfaces $\Gamma(t)$ are kept fixed during the evolution. It would also be possible to replace (8.31) by

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{8.33}$$

in which case the surfaces $\Gamma(t)$ would meet the boundary of the cylinder $\Omega \times \mathbb{R}$ at a right angle.

Analytical results

The main difficulties for the mathematical analysis are due to the fact that the operator

$$A(u) = \sqrt{1+|\nabla u|^2}\ \nabla \cdot \big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\big)$$

is not strictly parabolic and not in divergence form. Only in one space dimension the equation is in divergence form, since $A(u) = (\arctan u_x)_x$.

Theorem 8.3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{2+\alpha}$ and $u_0 \in C^{2,\alpha}(\bar{\Omega})$.*
a) Suppose that $g \in C^{2,\alpha}(\bar{\Omega})$ and that the compatibility conditions

$$u_0 = g \quad \text{and} \ \sqrt{1+|\nabla u_0|^2}\ \nabla \cdot \Big(\frac{\nabla u_0}{\sqrt{1+|\nabla u_0|^2}}\Big) = 0 \qquad \text{on } \partial\Omega$$

are satisfied. If $\partial\Omega$ has nonnegative mean curvature, the initial–boundary value problem (8.30), (8.31), (8.32) has a unique smooth solution which converges to the solution of the mimimal surface equation with boundary data g as $t \to \infty$.

b) Suppose that the compatibility condition $\frac{\partial u_0}{\partial n} = 0$ on $\partial\Omega$ holds. Then the initial–boundary value problem (8.30), (8.33), (8.32) has a unique smooth solution which converges to a constant function as $t \to \infty$.

Proof. see [33] and also [30] for a); b) is proved in [30]. □

The assumption that the boundary of the domain has nonnegative mean curvature is a necessary condition. If it is dropped, the gradient of the solution will become infinite on the boundary, see [41]. The main tool in the proof of the previous theorem is the derivation of an evolution equation for the surface element. Our numerical algorithms will be based on a variational formulation of (8.30), (8.31). To derive it, divide (8.30) by

$$Q := \sqrt{1+|\nabla u|^2}, \tag{8.34}$$

multiply by a test function $\phi \in H_0^1(\Omega)$ and integrate. Integration by parts implies

$$\int_\Omega \frac{u_t \phi}{Q} + \int_\Omega \frac{\nabla u \cdot \nabla \phi}{Q} = 0, \qquad \phi \in H_0^1(\Omega),\ 0 < t < T. \tag{8.35}$$

It is straightforward to derive from (8.35) the decrease in area.

Lemma 8.3.2. *Suppose that u is a smooth solution of (8.30). Then*

$$\int_\Omega \frac{u_t^2}{Q} + \frac{d}{dt}\int_\Omega Q = 0. \tag{8.36}$$

Proof. Since $u(\cdot,t) = g$ on $\partial\Omega \times (0,T)$ we have $u_t(\cdot,t) = 0$ on $\partial\Omega$ for $0 < t < T$. The relation (8.36) now follows by inserting $\phi = u_t(\cdot,t)$ in (8.35) and observing that $Q_t = \frac{\nabla u \cdot \nabla u_t}{Q}$. □

Recalling that $V = -\frac{u_t}{Q}$ we may rewrite the relation (8.36) in the following, more geometric form,

$$\int_{\Gamma(t)} V^2\, dA + \frac{d}{dt}|\Gamma(t)| = 0, \tag{8.37}$$

where $|\Gamma(t)|$ denotes the surface area of $\Gamma(t)$.

8.3.4 Mean curvature flow of level sets

If we want to compute topological changes of free boundaries then it is necessary to leave the parametric world which fixes the topological type of the interface. One method to do this is to define the interface as the level set of a scalar function:

$$\Gamma(t) = \left\{ x \in \mathbb{R}^{n+1} | \, u(x,t) = 0 \right\}.$$

Let us assume for the moment that $u \in C^{2,1}(\mathbb{R}^{n+1} \times (0,T))$ with $\nabla u \neq 0$ in a neighbourhood of $\bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. Recalling (**??**) and (**??**), the relation $V = -H$ on $\Gamma(t)$ would hold if

$$u_t - \sum_{i,j=1}^{n+1} \big(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\big) u_{x_i x_j} = 0 \tag{8.38}$$

in a neighbourhood of $\bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. This partial differential equation is highly nonlinear, degenerate parabolic and not defined where the gradient of u vanishes. Therefore, standard methods for parabolic equations fail, but it is possible to develop an existence and uniqueness theory for (8.38) within the framework of viscosity solutions. The corresponding notion involves pointwise relations and the analysis mainly relies on the maximum principle. It is therefore not straightforward to use finite element methods which typically are L^2–methods and normally do not allow a maximum principle. This difficulty will be reflected in any numerical analysis.

Analytical results

Starting from (8.38), we are interested in the following problem:

$$u_t - \sum_{i,j=1}^{n+1} \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j} = 0 \quad \text{in } \mathbb{R}^{n+1} \times (0,\infty) \tag{8.39}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^{n+1}. \tag{8.40}$$

As mentioned above, an existence and uniqueness theory for (8.39), (8.40) can be carried out within the framework of viscosity solutions.

Definition 8.3.3. A function $u \in C^0(\mathbb{R}^{n+1} \times [0,\infty))$ is called a viscosity subsolution of (8.39) provided that for each $\phi \in C^\infty(\mathbb{R}^{n+2})$, if $u - \phi$ has a local maximum at $(x_0, t_0) \in \mathbb{R}^{n+1} \times (0,\infty)$, then

$$\begin{aligned} &\phi_t - \sum_{i,j=1}^{n+1} \left(\delta_{ij} - \frac{\phi_{x_i} \phi_{x_j}}{|\nabla \phi|^2}\right) \phi_{x_i x_j} \leq 0 \quad \text{at } (x_0,t_0), \text{ if } \nabla\phi(x_0,t_0) \neq 0 \\ &\phi_t - \sum_{i,j=1}^{n+1} (\delta_{ij} - p_i p_j)\, \phi_{x_i x_j} \leq 0 \qquad \text{at } (x_0,t_0) \text{ for some } |p| \leq 1, \text{ if } \nabla\phi(x_0,t_0) = 0. \end{aligned} \tag{8.41}$$

A viscosity supersolution is defined analogously: maximum is replaced by minimum and $\leq$ by $\geq$. A viscosity solution of (8.39) is a function $u \in C^0(\mathbb{R}^n \times [0,\infty))$ that is both a subsolution and a supersolution.

We shall assume that the initial function u_0 is smooth and satisfies

$$u_0(x) = 1, \qquad \text{for } |x| \geq S \tag{8.42}$$

for some $S > 0$. The following existence and uniqueness theorem is a special case of results proved independently in [25] and [11].

Theorem 8.3.4. *Assume $u_0 : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies (8.42). Then there exists a unique viscosity solution of (8.39), (8.40), such that*

$$u(x,t) = 1 \qquad \textit{for } |x| + t \geq R$$

for some $R > 0$ depending only on S.

The level set approach can now be described as follows: given a compact hypersurface Γ_0, choose a continuous function $u_0 : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\Gamma_0 = \{x \in \mathbb{R}^{n+1} \,|\, u_0(x) = 0\}$. If $u : \mathbb{R}^{n+1} \times [0,\infty) \to \mathbb{R}$ is the unique viscosity solution of (8.39, 8.40), we then call

$$\Gamma(t) = \{x \in \mathbb{R}^{n+1} \,|\, u(x,t) = 0\}, \quad t \geq 0$$

a generalized solution of the mean curvature flow problem. We remark that the authors in [25] and [11] also established that the sets $\Gamma(t), t > 0$ are independent of the particular choice of u_0 which has Γ_0 as its zero level set, so that the generalized evolution $(\Gamma(t))_{t\geq 0}$ is well defined for a given Γ_0. As $\Gamma(t)$ exists for all times, it provides a notion of solution beyond singularities in the flow. Note however that it is possible that the set $\Gamma(t)$ may develop an interior for $t > 0$, even if Γ_0 had none, a phenomenon which is referred to as *fattening*.

For later purposes it is convenient to analyze a regularized version of (8.39, 8.40) on a bounded domain. Let $\Omega = B_{\tilde{S}}(0)$, $\tilde{S} > S$ (S as in (8.42)) and consider for $\epsilon > 0$

$$\begin{aligned} u^\epsilon_t &= \sum_{i,j=1}^{n+1} \Big(\delta_{ij} - \frac{u^\epsilon_{x_i} u^\epsilon_{x_j}}{\epsilon^2 + |\nabla u^\epsilon|^2}\Big) u^\epsilon_{x_i x_j} && \text{in } \Omega \times (0,T) && (8.43)\\ u^\epsilon &= 1 && \text{on } \partial\Omega \times (0,T) && (8.44)\\ u^\epsilon(.,0) &= u_{0|\Omega} && \text{in } \Omega. && (8.45) \end{aligned}$$

To simplify matters we assume that

$$-1 \leq u_0(x) \leq 1, \qquad x \in \mathbb{R}^n. \tag{8.46}$$

Let us start by discussing the existence and a–priori estimates for u^ϵ.

Theorem 8.3.5. *The problem (8.43)–(8.45) has a unique smooth solution u^ϵ. Furthermore,*

$$\|(u^\epsilon, \nabla u^\epsilon, u^\epsilon_t)\|_{L^\infty(\Omega\times(0,T))} \le C \tag{8.47}$$

uniformly in $0 < T < \infty$ and $0 < \epsilon \le 1$.

Proof. Note first that (8.43)–(8.45) is equivalent to the following initial–boundary value problem for the function $U^\epsilon := \frac{u^\epsilon}{\epsilon}$:

$$\begin{aligned} U^\epsilon_t &= \sum_{i,j=1}^{n+1}\Big(\delta_{ij} - \frac{U^\epsilon_{x_i}U^\epsilon_{x_j}}{1+|\nabla U^\epsilon|^2}\Big)U^\epsilon_{x_ix_j} && \text{in } \Omega\times(0,T) \\ U^\epsilon &= \frac{1}{\epsilon} && \text{on } \partial\Omega\times(0,T) \\ U^\epsilon(.,0) &= \frac{u_{0|\Omega}}{\epsilon} && \text{in } \Omega. \end{aligned}$$

Hence, existence and uniqueness of u^ϵ follow from Theorem 8.3.1. Our next aim is to derive estimates on u^ϵ that are uniform in ϵ. The maximum principle implies that

$$\max_{\bar\Omega\times[0,T]}|u^\epsilon| = \max_{\bar\Omega}|u_0| = 1 \tag{8.48}$$

by (8.46) and (8.44). In order to estimate $\max_{\bar\Omega\times[0,T]}|\nabla u^\epsilon|$ we introduce

$$Q^\epsilon := \sqrt{\epsilon^2 + |\nabla u^\epsilon|^2}$$

as well as

$$a^\epsilon_{ij} := \delta_{ij} - \frac{u^\epsilon_{x_i}u^\epsilon_{x_j}}{\epsilon^2+|\nabla u^\epsilon|^2}, \quad i,j = 1,\dots,n+1.$$

Following Huisken [30] a calculation shows that

$$Q^\epsilon_t - \sum_{i,j=1}^{n+1} a^\epsilon_{ij}Q^\epsilon_{x_ix_j} = -\frac{1}{Q^\epsilon}\sum_{i,j=1}^{n+1} a^\epsilon_{ij}Q^\epsilon_{x_i}Q^\epsilon_{x_j} - \frac{1}{Q^\epsilon}\sum_{i,j=1}^{n+1} a^\epsilon_{ij}u^\epsilon_{x_ix_k}u^\epsilon_{x_jx_k} \le 0. \tag{8.49}$$

Our next aim is to derive a boundary condition for Q^ϵ. Since $u^\epsilon = 1$ on $\partial\Omega$, we have $\nabla_\Gamma u^\epsilon = 0$ on $\partial\Omega$ and hence

$$\nabla u^\epsilon = u^\epsilon_n n, \qquad \text{where } u^\epsilon_n = \nabla u^\epsilon\cdot n \tag{8.50}$$

and n denotes the unit outward normal to $\partial\Omega$. Furthermore,

$$u^\epsilon_{x_ix_j} = \underline{D}_j(u^\epsilon_{x_i}) + (\nabla u^\epsilon_{x_i}\cdot n)n_j = \underline{D}_j(u^\epsilon_{x_i}) + \sum_{k=1}^{n+1} u^\epsilon_{x_ix_k}n_kn_j \quad \text{on } \partial\Omega. \tag{8.51}$$

Combining (8.51) with (8.50) we infer

$$\begin{aligned} \sum_{i,j=1}^{n+1} u^\epsilon_{x_i}u^\epsilon_{x_j}u^\epsilon_{x_ix_j} &= \sum_{i,j=1}^{n+1}(u^\epsilon_n)^2 n_in_j\underline{D}_j(u^\epsilon_{x_i}) + \sum_{i,j,k=1}^{n+1}(u^\epsilon_n)^2u^\epsilon_{x_ix_k}n_in_k \\ &= (u^\epsilon_n)^2u^\epsilon_{nn}, \end{aligned}$$

while

$$\Delta u^\epsilon = \sum_{i=1}^{n+1} \underline{D}_i(u_n^\epsilon n_i) + u_{nn}^\epsilon = u_n^\epsilon H_{\partial\Omega} + u_{nn}^\epsilon.$$

Here, $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$. If we evaluate (8.43) on $\partial\Omega$ and observe that $u_t^\epsilon = 0$ on $\partial\Omega$ we obtain

$$0 = \Delta u^\epsilon - \sum_{i,j=1}^{n+1} \frac{u_{x_i}^\epsilon u_{x_j}^\epsilon u_{x_i x_j}^\epsilon}{\epsilon^2 + |\nabla u^\epsilon|^2} = u_n^\epsilon H_{\partial\Omega} + \frac{\epsilon^2}{\epsilon^2 + |\nabla u^\epsilon|^2} u_{nn}^\epsilon,$$

and hence

$$u_{nn}^\epsilon = -\frac{(Q^\epsilon)^2}{\epsilon^2} u_n^\epsilon H_{\partial\Omega} \qquad \text{on } \partial\Omega.$$

Using once more (8.51) and (8.50) we finally obtain

$$\begin{aligned} \sum_{i,j=1}^{n+1} a_{ij}^\epsilon Q_{x_i}^\epsilon n_j &= Q_n^\epsilon - \frac{(u_{\epsilon,n)}^2}{\epsilon^2 + |\nabla u^\epsilon|^2} Q_n^\epsilon == \frac{\epsilon^2}{\epsilon^2 + |\nabla u^\epsilon|^2} \sum_{i,j=1}^{n+1} \frac{u_{x_i x_j}^\epsilon u_{x_i}^\epsilon n_j}{Q^\epsilon} \\ &= \frac{\epsilon^2}{\epsilon^2 + |\nabla u^\epsilon|^2} \frac{u_n^\epsilon u_{nn}^\epsilon}{Q^\epsilon} = -\frac{(u_n^\epsilon)^2}{Q^\epsilon} H_{\partial\Omega} \le 0. \end{aligned}$$

The maximum principle then implies that $Q^\epsilon \le \max_{\bar\Omega} \sqrt{\epsilon^2 + |\nabla u_0|^2}$ on $\bar\Omega \times [0,T]$ and hence

$$\max_{\Omega\times[0,T]} |\nabla u^\epsilon| \le C \quad \text{uniformly in } 0 < T < \infty \text{ and } 0 < \epsilon \le 1. \tag{8.52}$$

Differentiating (8.43) with respect to time and noting that $u_t^\epsilon = 0$ on $\partial\Omega \times (0,T]$ the maximum principle yields a uniform bound on u_t^ϵ. $\square$

The bounds (8.47) enable us to apply the Arzela–Ascoli theorem to $(\tilde u_\epsilon)_{0<\epsilon\le 1}$. Thus, there exists a sequence $\epsilon_j \searrow 0$ and a function $\tilde u \in C^0(\bar\Omega \times [0,\infty))$ such that

$$\tilde u^{\epsilon_j} \to \tilde u \qquad \text{uniformly on } \bar\Omega \times [0,T], \quad \text{for all } T < \infty \text{ as } j \to \infty. \tag{8.53}$$

Let us extend $\tilde u$ to $\mathbb{R}^n \times [0,\infty)$ by defining $\tilde u(x,t) := 1, x \in \mathbb{R}^n \setminus \bar\Omega, t > 0$. Clearly, $\tilde u \in C^0(\mathbb{R}^n \times [0,\infty))$; we claim

Lemma 8.3.6. *$\tilde u$ is a viscosity solution of (8.39), (8.40).*

Proof. Clearly, (8.45), (8.53) and (8.42) imply that $\tilde u(\cdot,0) = u_0$. We prove that $\tilde u$ is a viscosity subsolution of (8.39). Let $\phi \in C^\infty(\mathbb{R}^{n+1})$ and assume that $\tilde u - \phi$ has a local maximum at $(x_0,t_0) \in (0,\infty)$. If $x_0 \in \Omega$ we can use (8.53) and the argument in the proof of Theorem 4.2 of [25] to derive (8.41). Next, if $x_0 \in \partial\Omega = \partial B_{\tilde S}(0)$ we have

$$1 - \phi(x_0,t_0) \ge \tilde u(x,t) - \phi(x,t) \tag{8.54}$$

for all (x,t) in a neighborhood of (x_0,t_0). Since $\tilde u \equiv 1$ on $\partial\Omega \times (0,\infty)$ we deduce

$$\phi(x_0,t_0) \le \phi\big(\tilde S \frac{x_0 + \epsilon v}{|x_0 + \epsilon v|}, t\big)$$

for all $v \in \mathbb{R}^{n+1}, |\epsilon| \leq \epsilon_0(v)$ and all t close to t_0. This implies

$$\nabla\phi(x_0,t_0)\cdot(v-(v\cdot\tilde{n})\tilde{n}) = 0, \quad \phi_t(x_0,t_0) \;=\; 0, \tag{8.55}$$

$$(v-(v\cdot\tilde{n})\tilde{n})^t D^2\phi(x_0,t_0)(v-(v\cdot\tilde{n})\tilde{n})$$
$$+\frac{1}{\tilde{S}}\nabla\phi(x_0,t_0)\cdot\left(-2(v\cdot\tilde{n})v+3(v\cdot\tilde{n})^2\tilde{n}-|v|^2\tilde{n}\right) \;\geq\; 0 \tag{8.56}$$

where $\tilde{n} = \frac{x_0}{\tilde{S}}$. The first relation in (8.55) implies that $\nabla\phi(x_0,t_0) = \alpha\tilde{n}$ for some $\alpha \in \mathbb{R}$. Recalling that $\tilde{u}(x,t) = 1$ for $x \in \mathbb{R}^{n+1}\setminus\bar{\Omega}$ we infer from (8.54) that $\alpha = \nabla\phi(x_0,t_0)\cdot\tilde{n} \geq 0$. Inserting $v = e_i, 1 \leq i \leq n+1$ into (8.56) and summing from 1 to $n+1$ we deduce with the help of a short calculation

$$\sum_{i=1}^{n+1}(\delta_{ij}-\tilde{n}_i\tilde{n}_j)\phi_{x_ix_j}(x_0,t_0) - \alpha\frac{n}{\tilde{S}} \geq 0. \tag{8.57}$$

If $\nabla\phi(x_0,t_0) = \alpha\tilde{n} \neq 0$, (8.55) and (8.57) imply at (x_0,t_0)

$$\phi_t - \sum_{i,j=1}^{n+1}\Big(\delta_{ij}-\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\Big)\phi_{x_ix_j} = -\sum_{i,j=1}^{n+1}\Big(\delta_{ij}-\tilde{n}_i\tilde{n}_j\Big)\phi_{x_ix_j} \leq -\alpha\frac{n}{\tilde{S}} \leq 0,$$

since $\alpha \geq 0$. On the other hand, if $\nabla\phi(x_0,t_0) = 0$ we have as above for the choice $p = \tilde{n}$

$$\phi_t - \Big(\delta_{ij}-p_ip_j\Big)\phi_{x_ix_j} \leq 0 \qquad \text{at } (x_0,t_0)$$

proving (8.41) for the case $x_0 \in \partial\Omega$. Finally, if $x_0 \in \mathbb{R}^{n+1}\setminus\bar{\Omega}$, we immediately deduce that ϕ has a local minimum at (x_0,t_0) and (8.41) then follows in a similar way as before. Thus $\tilde{u}$ is a viscosity subsolution and in an analogous way we prove that it is also a supersolution. This completes the proof of the lemma. □

Remark 2 We immediately infer from Lemma 8.3.6 that $\tilde{u}$ agrees with the solution of u of Theorem 8.3.4 and that

$$u^\epsilon \to u \qquad \text{uniformly on } \bar{\Omega}\times[0,T] \text{ as } \epsilon \to 0 \tag{8.58}$$

for all $T < \infty$.

Chapter 9

Phase field equations

9.1 Introduction

Phase field equations provide a diffuse interface approach to modelling interfaces. A principal feature is a particular kind of *energy functional*. These models depend on Van der Waals -Cahn-Hilliard (or Ginzburg-Landau) gradient energy functionals, the prototypical example of which is

$$\mathcal{E}^\epsilon(\varphi) = \int_\Omega \big(\frac{\epsilon}{2}|\nabla\varphi|^2 + \frac{1}{\epsilon}W(\varphi)\big)dx \tag{9.1}$$

where $W(\cdot) : \mathbb{R} \to \mathbb{R}$ is a double well bulk free energy function and ϵ is a small positive constant which is commensurate with the thickness of the diffuse interface.

The terminoloogy *double well* indicates that there are exactly two local minima separated by exactly one local maximum. Often the energy density $W(\cdot)$ is assumed to be symmetric $W(r) = W(-r)$ and to have exactly two local minima at $r = \pm 1$ where the global minimum of $W(\cdot)$ is attained. A simple, widely used, example is the quartic potential

$$W(r) := \frac{1}{4}(r^2 - 1)^2. \tag{9.2}$$

9.2 Semilinear second order parabolic equation

Here we wish to consider an initial value problem for the diffusion equation with a nonlinear right hand side forcing. Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary. We set Ω_T to be space-time cylinder $\Omega_T = \Omega \times (0, T)$. Given $u_0 \in L^2(\Omega)$ consider the initial value problem:

$$\begin{cases} u_t - \Delta u + W'(u) = 0 & \text{in } \Omega \\ u(\cdot, 0) = u_0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.3}$$

Here $W : \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies the growth conditions, with $p > 2$,

$$|W'(r)| \leq \alpha_2(|r|^{p-1} + 1) \tag{9.4}$$

$$\alpha_0|r|^p - c_0 \leq W'(r)r \leq \alpha_1|r|^p + c_0 \tag{9.5}$$

$$W''(r) \geq -c_W \tag{9.6}$$

where $\alpha_0, \alpha_1, \alpha_2, c_0$ and c_W are positive constants. We assume that

$$\begin{cases} n \leq 2 & 1 < p < \infty \\ n \geq 3 & p \leq \frac{2n}{n-2}. \end{cases}$$

Remark 9.2.1. For example we may choose the polynomial nonlinearity, with $p = 2k \geq 4$,

$$W'(r) = \sum_{j=1}^{2k-1} a_j r^j, \quad a_{2k-1} > 0. \tag{9.7}$$

An example would be

$$W'(r) = r^3 - r.$$

Remark 9.2.2. It is useful to note that we may write

$$r^p = (r^{p-1})^{\frac{p}{p-1}} = (r^{p-1})^q$$

where $q = \frac{p}{p-1}$ is conjugate to p i.e.

$$\frac{1}{p} + \frac{1}{q} = 1.$$

From this we see that if $\eta_m \in L^p(D)$ is uniformly bounded then $W'(\eta_m)$ is uniformly bounded in $L^q(D)$; viz

$$\int_D |W'(\eta_m)|^q \leq \alpha_2 \int_D (|\eta_m|^{p-1} + 1)^q \leq C \int_D ((|\eta_m|^{q(p-1)} + 1) \leq C(||\eta_m||^p_{L^p(D)} + 1).$$

Remark 9.2.3. 1. Note that by the Sobolev embedding theorem there is an $s \geq 1$ such that

$$H^s(\Omega) \subset L^p(\Omega)$$

from which it follows that

$$L^q(\Omega) \subset (H^s)'(\Omega).$$

Here $1 < q < 2$ is the conjugate Holder exponent ($q^{-1} = 1 - p^{-1}$ to p) and we have used that $L^q(\Omega) = (L^p(\Omega))'$.

2. Observe that we may take $s = 1$ in the case of $n = 1, 2$ because $H^1(\Omega) \subset L^p(\Omega)$ for all $\infty > p \geq 1$.

3. Note that both $L^2(0, T; H^1(\Omega)')$ and $L^q(0, T; L^q(\Omega))$ are continuously included in $L^q(0, T; (H^s(\Omega))')$.

4. Note that the dual space of $L^2(0, T; H^1(\Omega)) \cup L^p(\Omega_T)$ is $L^2(0, T : (H^1(\Omega))') + L^q(\Omega_T)$.

5. Finally we remark that the following result may be shown:

   If $u \in L^2(0, T; H^1(\Omega)) \cup L^p(\Omega_T)$ and $u' \in L^2(0, T : (H^1(\Omega))') + L^q(\Omega_T)$ then $u \in C([0, T]; L^2(\Omega))$.

Galerkin approximation

Let V_m be the finite dimensional subspace of V spanned by the first m elements of a complete set of basis functions $\{z_j\}_{j=1}^{\infty}$for V which are orthonormal in H. For example we may use the eigenfunctions associated with a symmetric coercive bilinear form on $V \times V$. We seek $u_m : [0, T] \to V$ of the form

$$u_m(t) := \sum_{j=1}^{m} U_j^m(t) z_j \tag{9.8}$$

where $u_m(0)$ is defined to be the H projection of u_0 onto V_m defined by

$$\langle u_m(0) - u_0, v_m \rangle_H = 0 \ \forall v_m \in V_m$$

which is equivalent to

$$U_j^m(0) := \langle u_0, z_j \rangle_H \ , \ j = 1, 2, ..m. \tag{9.9}$$

Lemma 9.2.4. *For each integer m there exists a unique function $u_m(t)$ of the form (9.8) such that (9.9) and*

$$\langle u_m', v_m \rangle + a(u_m, v_m) + \langle W'(u_m), v_m \rangle = 0 \quad \forall v_m \in V_m \ \textit{and} \ \ t \in (0, T) \tag{9.10}$$

hold.

Proof. The local existence and uniqueness of u_m is a consequence of ODE theory since the nonlinearity is C^1. Existence for all T is a consequence of a bound which is uniform in the discrete data. The bound is obtained by testing with $v_m = u_m$. Using the assumption on $W'(\cdot)$ this yields

$$\frac{1}{2}\frac{d}{dt}||u_m||_H^2 + \tilde{c}_1||u_m||_V^2 + \alpha_0||u_m||_{L^p(\Omega)}^p \leq \tilde{c}_2||u_m||_H^2 + c_o|\Omega|.$$

It follows from a Gronwall argument and the uniform boundedness of u_m in H that

$$\begin{cases} ||u_m||_{L^\infty(0,T;L^2(\Omega))} \le C(T, u_0) \\ u_m \text{ is uniformly bounded in } \ L^\infty(0,T;L^2(\Omega)) \end{cases} \tag{9.11}$$

$$\begin{cases} ||u_m||_{L^2(0,T;H^1(\Omega))} \le C(T, u_0) \\ u_m \text{ is uniformly bounded in } \ L^2(0,T;H^1(\Omega)) \end{cases} \tag{9.12}$$

$$\begin{cases} ||u_m||_{L^p(0,T;L^p(\Omega))} \le C(T, u_0) \\ u_m \text{ is uniformly bounded in } \ L^p(\Omega_T). \end{cases} \tag{9.13}$$

On the other hand we may test the variational Galerkin equation with u_m' which leads to

$$||u_m||^2_{L^2(\Omega)} + \frac{d}{dt}\mathcal{E}(u_m(t)) = 0. \tag{9.14}$$

Thus we have from the boundedness of the initial data that

$$\begin{cases} ||u_m||_{L^\infty(0,T;H^1(\Omega))} \le C(T, u_0) \\ u_m \text{ is uniformly bounded in } \ L^\infty(0,T;H^1(\Omega)) \end{cases} \tag{9.15}$$

$$\begin{cases} ||u_m'||_{L^2(0,T;L^2(\Omega))} \le C(T, u_0) \\ u_m' \text{ is uniformly bounded in } \ L^2(0,T;L^2(\Omega)) \end{cases} \tag{9.16}$$

$$\begin{cases} ||u_m||_{L^\infty(0,T;L^p(\Omega))} \le C(T, u_0) \\ u_m \text{ is uniformly bounded in } \ L^\infty((0,T);L^p(\Omega)). \end{cases} \tag{9.17}$$

□

We have the following theorem concerning *weak solutions.*

Theorem 9.2.5. • *For $u_0 \in L^2(\Omega)$ there exists a unique solution which satisfies*

$$u \in L^2(0,T;H^1(\Omega)) \cap L^p(0,T;L^p(\Omega)), \quad \forall T > 0 \tag{9.18}$$

$$u' \in L^2(0,T;(H^1(\Omega)'), \quad u \in C(0,T;L^2(\Omega)). \tag{9.19}$$

The mapping $u_0 \to u(t)$ is continuous in $L^2(\Omega)$.

• *For $u_0 \in H^1(\Omega) \cup L^p(\Omega)$ there exists a unique solution which satisfies*

$$u \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^p(\Omega)), \quad \forall T > 0 \tag{9.20}$$

$$u' \in L^2(0,T;L^2(\Omega)), \quad u \in C(0,T;L^2(\Omega)). \tag{9.21}$$

The mapping $u_0 \to u(t)$ is continuous in $L^2(\Omega)$.

Proof. The proof of this follows the line of the linear case. However we have to pass to the limit in the nonlinear term. The case of the initial data $u_0 \in L^2(\Omega)$ is more technical and can be found in [43].

$\square$

9.3 Cahn-Hilliard equation

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary. Consider the initial value problem:

$$\begin{cases} u_t = \Delta w \ \text{ in } \ \Omega \\ w = -\epsilon \Delta u + \frac{1}{\epsilon}\psi'(u) \ \text{ in } \ \Omega \\ u(\cdot, 0) = u_0 \ \text{ in } \ \Omega \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \\ \frac{\partial w}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{9.22}$$

We assume the structural conditions on the potential to be

Potential assumptions

1. $\psi(\cdot) \in C^1(\mathbb{R}, \mathbb{R})$
2. $\psi(r) \geq -C_0 \ \forall r$
3. $|\psi'(r)| \leq C_1 |r|^q + C_2 \ \forall r$
4. $(\psi(r) - \psi(s))(r - s) \geq -C_\psi |r - s|^2$

where C_ψ, C_0, C_1 and C_2 are positive constants and $0 < q \leq \frac{n}{n-1}$ if $n > 3$ and $q > 0$ if $n = 1, 2$.

We set $a(\eta, v) = (\nabla \eta, \nabla v)$ where $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product and we use $\langle \cdot, \cdot \rangle$ to denote the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega)')$.

Theorem 9.3.1. *Suppose $u_0 \in H^1(\Omega)$. Then there exists a pair $\{u, w\}$ such that*

$$u \in L^\infty(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega)) \tag{9.23}$$
$$u_t \in L^2(0, T; (H^1(\Omega))') \tag{9.24}$$
$$u(0) = u_0 \tag{9.25}$$
$$w \in L^2(0, T; H^1(\Omega)) \tag{9.26}$$

such that

$$\int_0^T \langle u_t, \eta \rangle + a(w, \eta) = 0 \quad \forall \eta \in L^2(0, T; H^1(\Omega)) \tag{9.27}$$

and

$$(w,\xi) = \epsilon a(u,\xi) + \frac{1}{\epsilon}(\psi'(u),\xi)\ \forall \xi \in H^1(\Omega) \quad a.e.\ t \in [0,T]. \tag{9.28}$$

Proof. To prove this theorem we use the Galerkin method. We are in the situation of a modified version of section 10.8. The eigenfunctions $\{z_j\}$ of the Laplace operator with homogeneous Neumann boundary conditions

$$a(z_j,\eta) = \lambda_j(z_j,\eta)\ \forall \eta \in H^1(\Omega)$$

orthogonal in $H^1(\Omega)$ and $L^2(\Omega)$ form a basis of $H^1(\Omega)$. Note that z_1 is constant and $\lambda_1 = 0$. We take V^m to be the span of the first m eigenfunctions. We set $\mathcal{P}_m : L^2(\Omega) \to V^m$ to be the Galerkin projection defined by

$$(\mathcal{P}_m v - v, v_m) = 0 \ \ \forall v \in L^2(\Omega) \tag{9.29}$$

for which we have the strong convergence results

$$\forall v \in L^2(\Omega) \ \ \mathcal{P}_m v \to v \ \text{ in } \ L^2(\Omega) \tag{9.30}$$

$$\forall v \in H^1(\Omega) \ \ \mathcal{P}_m v \to v \ \text{ in } \ H^1(\Omega). \tag{9.31}$$

The Galerkin method is

$$\langle u_m', v_m\rangle + a(w_m,v_m) = 0 \tag{9.32}$$

$$(w_m,v_m) = \epsilon a(u_m,v_m) + \frac{1}{\epsilon}(\psi'(u_m),v_m) \tag{9.33}$$

$\forall v_m \in V_m$ and $t \in (0,T)$.

Note that

$$\langle u_m', v_m\rangle = (u_m', v_m)$$

and that taking $v_m = 1$ in (9.32) yields the discrete mass conservation law

$$\frac{d}{dt}(u_m,1) = 0, \quad (u_m(t),1) = (\mathcal{P}_m u_0, 1) \tag{9.34}$$

There exists a local in time unique solution to this system which is equivalent to an initial value problem for a system of ordinary differential equations with locally Lipschitz right hand side. In order to derive an a priori estimate we consider the energy functional and differentiate with respect to time. This yields

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(u_m(t)) &= \frac{d}{dt}\Big(\frac{\epsilon}{2}a(u_m,u_m) + \frac{1}{\epsilon}(\psi(u_m),1)\Big) \\
&= \epsilon a(u_m,(u_m)_t) + (\psi'(u_m),(u_m)_t) \\
&= (w_m,(u_m)_t) = -a(w_m,w_m)
\end{aligned}$$

where we have used $(u_m)_t$ as a test function in the equation (9.33) and w_m as a test function in the equation (9.32). Thus we have derived the following

Discrete energy equation:

$$\mathcal{E}(u_m(t)) + \int_0^t a(w_m(s), w_m(s))ds = \mathcal{E}(u_m(0)). \tag{9.35}$$

Our assumption that the initial data $u_0 \in H^1(\Omega)$ and the structural assumptions on the potential imply that

$$\mathcal{E}(u_m(0)) = \mathcal{E}(\mathcal{P}_m u_0) \leq C \tag{9.36}$$

where $C = C(u_o, \epsilon, \Omega)$ is independent of m. It follows that we have proved the *a priori* bounds:

$$||\nabla u_m(t)||^2_{L^2(\Omega)} + \int_\Omega \psi(u_m(t)) + \int_0^t ||\nabla w_m(s)||^2_{L^2(\Omega)} ds \leq C \tag{9.37}$$

where C is independent of m. Sine the mean value of u_m is constant and bounded independently of m by the conservation of mass equation and the construction of the initial data we have by the Poincare inequality that $u_m(t)$ is uniformly bounded for all $t \in [0,T]$ in $H^1(\Omega)$. Taking $v_m = 1$ in (9.33) yields

$$(w_m, 1) = (\psi'(u_m), 1)$$

and the structural assumption (2) on the potential together with the bound (9.37) implies that w_m is uniformly bounded in $L^2(0,T;H^1(\Omega))$ independently of m. From equation (9.32) we observe that for all $v \in L^2(0,T;H^1(\Omega))$

$$\begin{aligned}
|\int_0^T \langle u_m', v\rangle| &= |\int_0^T \langle u_m', P^m v\rangle| \\
&= |\int_0^T a(w_m, P^m v)| \\
&\leq ||\nabla w_m||_{L^2(0,T;L^2(\Omega))} ||\nabla P^m v||_{L^2(0,T;L^2(\Omega))} \\
&\leq C||\nabla v||_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

We summarise as

$$\begin{aligned}
||u_m||_{L^\infty(0,T;H^1(\Omega)} &\leq C && (9.38)\\
||w_m||_{L^2(0,T;H^1(\Omega))} &\leq C && (9.39)\\
||u_m'||_{L^2(0,T;(H^1(\Omega))')} &\leq C && (9.40)
\end{aligned}$$

where C is independent of m.

It follows that for subsequences which we still denote by u_m and w_m (for convenience and by a standard convention) there exist u^* and w^* in the indicated spaces such that

$$u_m \to u^* \qquad \text{strongly in} \quad C([0,T];L^2(\Omega)) \tag{9.41}$$
$$u_m' \to (u^*)' \qquad \text{weakly in} \quad L^2(0,T;(H^1(\Omega))') \tag{9.42}$$
$$u_m \to u^* \qquad \text{strongly in} \quad L^2(0,T;L^p(\Omega)) \text{ and } a.e. \text{ in } \Omega_T \tag{9.43}$$
$$w_m \to w^* \qquad \text{weakly in} \quad L^2(0,T;H^1(\Omega)) \tag{9.44}$$

where $p < \frac{2n}{2n-2}$. Furthermore since $u_m(0) - u^*(0) \to 0 \in L^2(\Omega)$ it holds that

$$u^*(0) = u_0.$$

Taking $\eta \in C(0,T;H^1(\Omega)), \xi \in H^1(\Omega)$ and $\varphi \in C[0,T]$ we have that

$$\int_0^T \langle u_m', \mathcal{P}_m\eta\rangle + a(w_m, \mathcal{P}_m\eta) = 0 \tag{9.45}$$
$$\int_0^T \varphi[(w_m, \mathcal{P}_m\xi) - \epsilon a(u_m, \mathcal{P}_m\xi) - \frac{1}{\epsilon}(\psi'(u_m), \mathcal{P}_m\xi)] = 0 \tag{9.46}$$

and passing to the limit using the strong convergence properties of $\mathcal{P}_m$ and the weak convergence of the subsequence of u_m and w_m we find

$$\int_0^T \langle (u^*)', \eta\rangle + a(w^*, \eta) = 0$$
$$\int_0^T \varphi[(w^*, \xi) - \epsilon a(u^*, \xi) - \frac{1}{\epsilon}(\psi'(u^*), \xi)] = 0$$

from which we obtain that u^* and w^* satisfy (9.27) and (9.28). Thus we have shown the existence of a pair satisfying the desired properties.

□

Theorem 9.3.2. *There is at most one solution pair.*

Proof. Let (u_u, w_i) $i = 1, 2$ be two solution pairs. Set $\theta^u = u_1 - u_2$ and $\theta^w = w_1 - w_2$. Subtraction of the relevant equations yields

$$\int_0^T \langle \theta_t^u, \eta\rangle + a(\theta^w, \eta) = 0 \quad \forall \eta \in L^2(0,T;H^1(\Omega)) \tag{9.47}$$

and

$$(\theta^w, \xi) = \epsilon a(\theta^u, \xi) + \frac{1}{\epsilon}(\psi'(u_1) - \psi'(u_2), \xi) \; \forall \xi \in H^1(\Omega) \quad a.e. \; t \in [0,T]. \tag{9.48}$$

Using the property (4) of the potential we find that

$$\epsilon||\nabla\theta^u||^2_{L^2(\Omega)} \leq \frac{C_W}{\epsilon}||\theta^u||^2_{L^2(\Omega)} + (\theta^w, \theta^u)$$

and since

$$\theta^w = -\mathcal{G}\theta^u_t + \theta^w_m$$

where

$$f_m := \frac{\int_\Omega f}{|\Omega|}$$

we find that, since $\theta^u_m = 0$,

$$(\mathcal{G}\theta^u_t, \theta^u) + \epsilon||\nabla\theta^u||^2_{L^2(\Omega)} \leq \frac{C_W}{\epsilon}||\theta^u||^2_{L^2(\Omega)}$$

which may be rewritten as

$$\frac{1}{2}\frac{d}{dt}||\theta^u||^2_{-1} + \epsilon||\nabla\theta^u||^2_{L^2(\Omega)} \leq \frac{C_W}{\epsilon}(\nabla\theta^u, \nabla\mathcal{G}\theta^u) \leq \frac{\epsilon}{2}||\nabla\theta^u||^2_{L^2(\Omega)} + C_\epsilon||\theta^u||^2_{-1}.$$

A Gronwall argument implies uniqueness. □

9.4 Phase separation

The Cahn-Hilliard equation provides a model for phase separation in binary alloy. Suppose that the concentrations of each component of the alloy are η_1 and η_2 so that $\eta_1 + \eta_2 = 1$ and set the difference in concentrations to be $u = \eta_2 - \eta_1$ so that $0 \leq u \leq 1$. An alloy at a high temperature will exist in an almost uniform homogeneous mixed state. Phase separation occurs when an alloy in an initially stable homogeneous composition undergoes a rapid reduction in temperature (quenching) to below a critical value where the uniform composition is energetically unstable. The result is the formation of large domains in which the alloy is in a single phase and leads to a complex morphology. In time these single phase domains coarsen and grow.

This may be understood in the following way. The homogeneous free energy is the logarithmic potential $\psi : [-1, 1] \to \mathbb{R}$

$$\psi(r) := \frac{\theta}{2}\left[(1+r)\ln[1+r] + (1-r)\ln[1-r]\right] + \frac{1}{2}(1-r^2). \tag{9.49}$$

For $\theta > \theta_c = 1$ the potential is strictly convex with a single global minimum at $r = 0$. On the other hand for $\theta < \theta_c$ this potential has two equal minima at $\pm u_b$ where u_b is the positive root of

$$2 = \frac{\theta}{u_b} \log\left[(1+u_b)/(1-u_b)\right] \tag{9.50}$$

and, since,

$$\psi''(r) = -1 + \frac{\theta}{1-r^2}$$

the potential is non-convex in the spinodal interval $r \in (-u_s, u_s)$ where

$$u_s = (1-\theta)^{\frac{1}{2}}.$$

For $\theta > 1$ the homogeneous free energy has just one minimum at $r = 0$ which corresponds to a uniform mixed state whereas for $\theta < 1$ minimizers correspond to a decomposition of the material into two phases characterised by the concentrations $\pm u_b$.

Suppose that the alloy occupies the spatial domain Ω and is isolated. Given that the amount of each component is fixed we may consider the state which minimises the energy leading to the problem

$$\min \int_\Omega \psi(u(x))dx \ \text{ subject to } \int_\Omega u(x)dx = u_m|\Omega| \tag{9.51}$$

where u_m is the mean value of the concentration u. Introducing a Lagrange multiplier λ for the prescribed mean value constraint we see that the problem is equivalent to

$$\min \int_\Omega F)\lambda(u(x))dx \ \text{ subject to } \int_\Omega u(x)dx = u_m|\Omega| \tag{9.52}$$

where $F_\lambda(r) := \psi(r) - \lambda(r - u_m)$. Set $\lambda^* = \psi'(u_m)$. It holds that $\int_\Omega F\lambda(u(x))dx$ is independent of λ when u satisfies the mean value constraint.

We now consider the case where $\theta > \theta_c$ so that $\psi(\cdot)$ is strictly convex. It is sufficient to consider $\int_\Omega F_{\lambda^*}(u(x))dx$. Notice that $F_{\lambda^*}(\cdot)$ is strictly convex with a unique minimum at u_m. Thus

$$\int_\Omega F\lambda(u(x))dx \geq F(u_m)|\Omega|$$

and we have a unique minimum which is the constant state

$$u(x) = u_m.$$

We now consider the case of a quench to $\theta < \theta_c$. Now we have the situation that $\psi(\cdot)$ has two absolute minima at $\pm u_b$. So that

$$\int_\Omega \psi(u)dx \geq \psi(u_b)|\Omega).$$

If

$$u_m \in [-u_b, u_b]$$

then we may take as the minimiser any piece-wise constant function

$$u(x) = \begin{cases} -u_b & x \in \Omega^- \\ u_b & x \in \Omega^+ \end{cases}$$

where

$$u_b(|\Omega^+| - |\Omega^-|) = u_m|\Omega|.$$

[35, 31, 32]

Thus we have an arbitrary decomposition of the domain Ω into sets in which the alloy takes the two *phase* values $\pm u_b$. If we denote by Γ the interface between the phase domains we see that Γ may have a large measure.

Remark 9.4.1. If $|u_m| > |u_b|$ then the minimizing state is $u(x) = u_m$.

We can now associate with the interface separating these phase domains an energy by penalising jumps in the gradient using the Dirichlet energy

$$\frac{\epsilon}{2}\int_\Omega |\nabla u|^2.$$

This leads to the consideration of the *Cahn-Hilliard* functional.

9.5 Phase Field Approach to Mean Curvature Flow

The phase field approach to interface evolution is based on physical models for problems involving phase transitions. In this chapter Ω is a bounded domain in $\mathbb{R}^{n+1}$ and $\Gamma(t)$ is a hypersurface moving through Ω which separates Ω into two regions $\Omega_I(t)$ and $\bar{\Omega} \setminus \bar{\Omega}_I$ where $\Omega_I(t)$ is the interior of $\Gamma(t)$. We are concerned with a mathematical model in which there are two phases characterized by two specific values of an order parameter or phase field function $\varphi : \Omega \times (0, T) \to \mathbb{R}$. These values are associated with the the minima of a C^2 double well bulk energy function $W(\cdot) : \mathbb{R} \to \mathbb{R}$. Furthermore there is the notion of an interfacial transition layer with a characteristic small width of ϵ. In these models the domain Ω is divided into three regions $\Omega_\epsilon^+(t)$, Ω_ϵ^- and $\Gamma_\epsilon(t)$, respectively associated with the two phase values and the interfacial layer. These domains are expected to approximate for small ϵ, respectively, the sets $\Omega_I(t)$, $\bar{\Omega} \setminus \bar{\Omega}_I$ and $\Gamma(t)$. Interfacial energy is modeled using a gradient energy function

$$\mathcal{E}^\epsilon(\varphi) = \int_\Omega \big(\frac{\epsilon}{2}|\nabla\varphi|^2 + \frac{W(\varphi)}{\epsilon}\big)dx. \tag{9.53}$$

For simplicity we suppose that the bulk homogeneous free energy $W(\cdot)$ is symmetric, $W(r) = W(-r)$ and has two minima located at ± 1. A typical example is depicted in Figure and a canonical example is

$$W(r) = \frac{1}{4}(r^2-1)^2. \tag{9.54}$$

It is clear that minimizing the bulk energy leads to arbitrary decompositions of Ω into the two phases where φ takes the values ± 1. Interfacial energy is measured by the Dirichlet gradient energy term $\int_\Omega \frac{\epsilon}{2}|\nabla\varphi|^2$.

Evolving from an initial phase function $\varphi_0(\cdot) : \Omega \to \mathbb{R}$, steepest descent or gradient flow for this functional leads to the parabolic Allen-Cahn equation ([2])

$$\epsilon\varphi_t - \epsilon\Delta\varphi + \frac{1}{\epsilon}W'(\varphi) = 0 \qquad \text{in } \Omega\times(0,T) \tag{9.55}$$

with intitial and boundary conditions

$$\varphi(\cdot,0) = \varphi_0(\cdot) \text{ in } \Omega, \quad \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{9.56}$$

That (9.55) is gradient flow for (9.53) is easily shown by testing the equation with φ_t and integrating by parts leading to

$$\epsilon\int_\Omega |\varphi_t|^2 dx + \frac{d\mathcal{E}^\epsilon(\varphi)}{dt} = 0$$

which is the analogue of the energy equation in Lemma **??**.

In order to understand the behavior of this evolution equation, observe that the flow of the ordinary differential equation $\varphi_t = -\frac{W'(\varphi)}{\epsilon^2}$ drives positive values of φ_0 to 1 and negative values to -1. On the other hand the Laplacian term in the equation (9.55) has a smoothing effect which will diffuse large gradients of the solution. Thus, on the basis of these heuristics, after a short time the solution of (9.55) will develop a structure consisting of bulk regions in which φ is smooth and takes the values ± 1 and separating these regions there will be interfacial transition layers across which φ changes rapidly from one bulk value to the other. These transition layers are due to the interaction between the regularizing effect of the gradient energy term and the flow associated with the bi–stable potential term W'. It turns out that the motion of these interfacial transition layers approximate mean curvature flow.

We can argue informally to support this in the following way. Let, for $t \in (0,T)$, $\Gamma(t)$ be a smoothly evolving closed and compact hypersurface satisfying $V = -H$. Suppose that $\Gamma(t)$ is the boundary of an open set $\Omega(t) \subset \Omega$ and denote by $d(\cdot,t)$ the signed distance function to $\Gamma(t)$. We consider the semi–linear parabolic operator

$$P(v) = \epsilon v_t - \epsilon\Delta v + \frac{1}{\epsilon}W'(v).$$

A calculation yields for $v(x,t) = \psi\big(\frac{d(x,t)}{\epsilon}\big)$, where $\psi : \mathbb{R} \to \mathbb{R}$, that

$$P(v) = (d_t - \Delta d)\psi'\big(\frac{d}{\epsilon}\big) - \frac{1}{\epsilon}\Big(\psi''\big(\frac{d}{\epsilon}\big) - W'\big(\psi\big(\frac{d}{\epsilon}\big)\big)\Big).$$

Hence it is natural to define $\psi = \psi(z)$ to be the unique solution of

$$
\begin{aligned}
-\psi''(z) + W'(\psi(z)) &= 0 \quad z \in \mathbb{R} && (9.57)\\
\psi(z) \to \pm 1, \quad z \to \pm\infty, \quad \psi(0) &= 0, \quad \psi'(z) > 0. && (9.58)
\end{aligned}
$$

If W is given by (9.54) we have that $\psi(z) = \tanh(\frac{z}{\sqrt{2}})$ and therefore

$$P(v) = (d_t - \Delta d)\psi'\big(\frac{d}{\epsilon}\big).$$

Recalling (**??**) and (**??**) we obtain $d_t - \Delta d = -V - H = 0$ on $\Gamma(t)$, so that the smoothness of d implies

$$|d_t - \Delta d| \leq C|d|$$

in a neighbourhood U of $\bigcup_{0<t<T} \Gamma(t) \times \{t\}$. Hence

$$|P(v)| \leq C\epsilon|\frac{d}{\epsilon}\,\psi'\big(\frac{d}{\epsilon}\big)| \leq C\epsilon \quad \text{in } U$$

and it follows that $v = \psi\big(\frac{d}{\epsilon}\big)$ is close to being a solution of (9.55) with initial data $\varphi_0 = \psi\big(\frac{d(\cdot,0)}{\epsilon}\big)$.

A more general isotropic phase field equation is

$$\epsilon\varphi_t = \epsilon\Delta\varphi - \frac{1}{\epsilon}W'(\varphi) + c_W g \tag{9.59}$$

where g is a forcing term. The constant c_W is a scaling constant dependent on the precise definition of the double well potential W and is given by the formula

$$c_W = \frac{1}{\sqrt{2}}\int_{-1}^{1}\sqrt{W(r)}dr. \tag{9.60}$$

The equation of motion that this phase field model approximates is

$$V = H + g. \tag{9.61}$$

We refer to ([46], [14]) for formal and rigorous interface asymptotics relating the Allen-Cahn equation to mean curvature flow. Error bounds for the Hausdorff distance betwen the zero level set of the phase field function and the interface have been derived ([9], [4]). In particular a convergence rate of $O(\epsilon^2|\log\epsilon|^2)$ was established in [5]. These bounds are proved using comparison theorems for the phase field equation and this can be extended to prove convergence to the viscosity solution of the level set equation in the case of non–smooth evolution and without the interface thickening (fattening) [24].

9.6 The double obstacle phase field model

We consider the phase field model

$$\epsilon\varphi_t - \epsilon\Delta\varphi - c_W g \in -\frac{1}{\epsilon}W'(\varphi) \tag{9.62}$$

The potential W is taken to be of double obstacle form

$$W(r) = \frac{1}{2}(1-r^2) + I_{[-1,1]}(r), \tag{9.63}$$

where

$$I_{[-1,1]}(r) = \begin{cases} +\infty & \text{for } \ |r| > 1 \\ 0 & \text{for } \ |r| \le 1, \end{cases} \tag{9.64}$$

introduced in the gradient phase field models by [27], [6],[7], [10], [42].

Properly we should interpret $W'(r)$ in the following way

$$W'(r) = \begin{cases} (-\infty, 1] & \text{if } r = -1, \\ -r & \text{if } |r| < 1, \\ [-1, \infty) & \text{if } r = 1. \end{cases}$$

For this potential, a calculation reveals that the profile of the phase variable in the transition layer given by the solution of (9.57), (9.58) is

$$\psi(r) = \begin{cases} -1 & \text{if } r \le -\frac{\pi}{2} \\ \sin(r) & \text{if } |r| < \frac{\pi}{2} \\ 1 & \text{if } r \ge \frac{\pi}{2}. \end{cases}$$

Furthermore, $c_W = \frac{\pi}{4}$. The double obstacle problem can be written in an equivalent variational inequality formulation. Let $\mathcal{K}$ be the convex set

$$\mathcal{K} = \{\eta \in H^1(\Omega) : |\eta| \le 1\}.$$

Then the problem is to seek $\varphi \in L^\infty(0,T;\mathcal{K}) \cap H^1(0,T;L^2(\Omega))$ such that $\varphi(\cdot,0) = \varphi_0 \in \mathcal{K}$ and

$$\epsilon\int_\Omega \varphi_t(\eta-\varphi) + \epsilon\int_\Omega \nabla\varphi\cdot\nabla(\eta-\varphi) - \frac{1}{\epsilon}\int_\Omega \varphi(\eta-\varphi) \ge \frac{\pi}{4}\int_\Omega g(\eta-\varphi) \tag{9.65}$$

for all $\eta \in \mathcal{K}$ and for a.e. $t \in (0,T)$. It is standard that this problem has a unique solution. The following theorem ([10]) proves convergence to mean curvature flow for smooth surfaces in the case $g = 0$ using the following comparison principle for parabolic obstacle problems.

Lemma 9.6.1. *Let $v \in L^2(0,T;H^1(\Omega))$ and $v_t \in L^2(0,T;L^2(\Omega))$ satisfy*

$$v(x,t) \le 1 \quad in \quad \Omega_T \tag{9.66}$$

$$v(\cdot,0) \le \varphi_0(\cdot) \quad in \quad \Omega \tag{9.67}$$

$$\int_0^T \left\{(v_t,\eta) + (\nabla v, \nabla\eta) - \frac{1}{\epsilon^2}(f,\eta)\right\}dt \le 0, \quad \forall\eta \ge 0 \quad in \quad \Omega_T \tag{9.68}$$

where
$$f(v-\varphi)^+ \leq v(v-\varphi)^+$$
and φ is the unique solution of the Allen-Cahn variational inequality with initial data φ_0. Then
$$v \leq \varphi \quad a.e. \quad \Omega_T.$$

Proof. Take $\eta = (v-\varphi)^+$ in the inequality satisfied by v and $\eta = \varphi + (v-\varphi)^+$ in the parabolic variational inequality for φ. Adding the inequalities yields

$$\frac{1}{2}\frac{d}{dt}\int_0^T ||(v-\varphi)^+||^2 dt + \int_0^T ||\nabla((v-\varphi)^+)||^2 dt \leq \int_0^T \frac{1}{\epsilon^2}||(v-\varphi)^+||^2 dt.$$

The assertion then follows from Gronwall's inequality.

□

Theorem 9.6.2. *Suppose that the smooth hypersurfaces $\Gamma(t) \subset \mathbb{R}^{n+1}$ satisfy*
(i) $\Gamma(t) = \partial\Omega(t)$ for open sets $\Omega(t) \subset \mathbb{R}^{n+1}$;
(ii) There exists $\delta > 0$ such that $dist(\Gamma(t), \partial\Omega) \geq \delta$ for $t \in [0,T]$ and d is smooth in a δ neighbourhood of $\Gamma(t)$
(iii) $|d_t - \Delta d| \leq D_0|d|$ for $|d| \leq \delta, t \in [0,T]$, where $d(\cdot,t)$ is the signed distance function to $\Gamma(t)$;
(iv) $V = -H$ on $\Gamma(t)$ for $t \in [0,T]$.
Let ϵ be sufficiently small such that $\frac{1}{2}\pi\epsilon \leq \delta(1+2e^{2D_0T})^{-1}$ and denote by $\varphi = \varphi_\epsilon$ the unique solution of (9.65) with $g = 0$ and initial data $\varphi_0 = \psi(\frac{d(\cdot,0)}{\epsilon})$. Then for all $t \in [0,T]$,

$$\begin{aligned} d(x,t) &\geq \frac{1}{2}\pi\epsilon(1+2e^{2D_0t}) \Rightarrow \varphi_\epsilon(x,t) = 1, \\ d(x,t) &\leq \frac{1}{2}\pi\epsilon(1+2e^{2D_0t}) \Rightarrow \varphi_\epsilon(x,t) = -1. \end{aligned}$$

Proof. The proof uses comparison arguments concerning barrier functions constructed using the interface profile.

□

A consequence of this theorem is that the diffuse interfacial region $\{(x,t) : |\varphi_\epsilon(x,t)| < 1\}$ is sharply defined with finite width bounded by $c(t)\pi\epsilon$ and that both the zero level set of $\varphi_\epsilon(\cdot,t)$ and $\Gamma(t)$ are in a narrow strip of width $c(t)\pi\epsilon$. Here $c(t) = \frac{1}{2}(1+e^{2D_0t})$ but in practice it is observed that this is pessimistic and the growth of the interface width is not usually an issue. A more refined analysis by Nochetto, Paolini and Verdi in [37, 38] revealed in the case of a smooth evolution of the forced mean curvature flow that the Hausdorff-distance between the zero level set of φ_ϵ and the interface of the flow (9.61) is of order $O(\epsilon^2)$. Furthermore there is convergence to the unique viscosity solution of the level set formulation, [39].

Chapter 10

Appendix

10.1 The spatial domain Ω.

Definition 10.1.1. Let Ω be a subset of $\mathbb{R}^n$ if it is a non-empty and open set then we say that it is a *domain*.

Definition 10.1.2. The boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^n$ is said to be *locally Lipschitz* if for each point $x \in \partial\Omega$ there exists a neighbourhood $U_x \subset \mathbb{R}^n$ such that $U_x \cap \partial\Omega$ is a graph of a Lipschitz continuous function.

An example of a bounded domain Ω whose boundary is not Lipschitz is

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < |x| < 1, 0 < y < 1\}.$$

Actually the boundary of this domain satisfies a *cone* condition. Also this domain Ω is not connected.

Definition 10.1.3. The domain $\Omega \subset \mathbb{R}^n$ is said to be *(path)-connected* if the for any two points in the domain there is curve joining the points whose points lie in the domain.

Definition 10.1.4. The connected domain $\Omega \subset \mathbb{R}^n$ is said to be *simply connected* if any simple closed curve within the domain can be shrunk continuously to a point in the domain.

A connected domain which is not simply connected is said to be *multiply connected*.

If $\Omega \subset \mathbb{R}^n$ is nonempty we denote by $\bar{\Omega}$ the closure of Ω in $\mathbb{R}^n$.

We write $G \Subset \Omega$ if $\bar{G} \subset \Omega$ and $\bar{G}$ is a compact (i.e. closed and bounded) subset of $\mathbb{R}^n$.

If u is a function defined on Ω then we define the *support* of u to be the set

$$\mathrm{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We say that u has *compact support* in Ω if $\text{supp}(u) \Subset \Omega$.

We denote by *bdry* Ω the boundary of Ω in $\mathbb{R}^n$ that is the set $\bar{\Omega} \cap \overline{\Omega}^c$ where Ω^c is the complement of Ω in $\mathbb{R}^n$; $\Omega^c = \mathbb{R}^n - \Omega = \{x \in \mathbb{R}^n : x \notin \Omega\}$.

10.2 Lebesgue spaces

Let Ω be a domain in $\mathbb{R}^n$. We identify in $L^p(\Omega)$ functions that are equal almost everywhere in Ω; the elements of L^p are thus equivalence classes of measurable functions: two functions being equivalent if they are equal *a.e.* in Ω. For convenience we ignore this distinction. We say that $u = 0$ in $L^p(\Omega)$ if $u = 0$ *a.e.* in Ω.

Theorem 10.2.1 (An imbedding theorem for L^p spaces)**.** *Suppose that $vol(\Omega) = \int_\Omega 1\, dx < \infty$ and $1 \le p \le q \le \infty$. If $u \in L^q(\Omega)$ then $u \in L^p(\Omega)$ and*

$$||u||_p \le (vol(\Omega))^{\frac{1}{p} - \frac{1}{q}} ||u||_q.$$

If $u \in L^\infty(\Omega)$ then

$$\lim_{p \to \infty} ||u||_p = ||u||_\infty.$$

If $u \in L^p(\Omega)$ for $\forall 1 \le p < \infty$ and there exists K such that

$$||u||_p \le K, \quad \forall p$$

then $u \in L^\infty(\Omega)$ and

$$||u||_\infty \le K.$$

$L^p(\Omega) \subset L^1_{loc}(\Omega)$ for $1 \le p \le \infty$ and any domain Ω.

10.3 Convergence

Definition 10.3.1. Weak convergence Let X be a real Banach space. We say that the sequence $x_k \in X$ converges weakly to the element $x \in X$ if

$$f(x_k) \to f(x)\ \forall f \in X^*.$$

(Here X^* is the dual space consisting of all continuous linear functionals on X.) We write

$$x_k \rightharpoonup x.$$

Definition 10.3.2. Weakly sequentially continmuous Let X, Y be a real Banach spaces. A mapping $F : X \to Y$ is said to be weakly sequentially continuous if the following holds:- For each weakly convergent sequence x_n converging to x in X it holds that

$$F(x_n) \rightharpoonup F(x) \text{ in } Y.$$

Definition 10.3.3. Weak star convergence Let the Banach space X be the dual of the Banach space Y, i.e. $X = Y'$. If for any $y \in Y$ we have that the sequence $x_k \in X$ and the element $x \in X$ satisfy

$$x_k(y) \to x(y)$$

then we say that x_k *weak-star* converges to x.

Remark 10.3.4. Let $u_k \in L^\infty(\Omega)$ weak star converge in $L^\infty(\Omega)$ to u. Since $L^\infty(\Omega) = (L^1(\Omega))'$ we have that

$$\int_\Omega u_k(x)v(x)dx \to \int_\Omega u(x)v(x)dx \quad \forall v \in L^1(\Omega).$$

Lemma 10.3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded or unbounded domain. Let $\{u_k\}, u \in L^p(\Omega),\ 1 \leq p \leq \infty$ such that u_k converges strongly to u in $L^p(\Omega)$. If $1 \leq p < \infty$ then there exists a subsequence u_{k_m} which converges almost everywhere to u in Ω. If $p = \infty$ then the sequence u_k itself converges almost everywhere to u in Ω.*

Lemma 10.3.6. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\{u_k\}$ be a bounded sequence of functions in $L^p(\Omega), 1 \leq p < \infty$ with $u_k \to\ u$ a.e. Then*

$$u \in L^p(\Omega) \quad \textit{and} \quad u_k \to u \quad \textit{weakly in } L^p(\Omega).$$

If $p = \infty$ then u_k converges weak -star to u.

Lemma 10.3.7. *Let $Q \subset \mathbb{R}^n \times \mathbb{R}$ be an open bounded space-time domain and f_k, f be elements of $L^p(Q),\ p \in [1, \infty)$ such that*

$$||f_k||_{L^p(Q)} \leq C, \quad f_k \to f \quad \textit{a.e. in } Q$$

then

$$f_k \to f \quad \textit{weakly in} \quad L^p(Q).$$

If $p = \infty$ then f_k converges weak -star to f.

10.4 Hilbert space triple

Definition 10.4.1. Let V and H be two separable Hilbert spaces satisfying $V \subset H$, V is dense in H and the injection of V in H is continuous. Let $V^{'}$ and $H^{'}$ be the dual spaces of V and H. Identifying H and $H^{'}$ we find that H is a dense subspace of $V^{'}$ and we call

$$V \subset H \subset V^{'}$$

a *Hilbert triple*. We use $\langle \cdot, \cdot \rangle_V, \langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle$ to denote the inner product for V, the inner products for H and the duality pairing for V' and V so that for $f \in H$

$$\langle f, v \rangle = \langle f, v \rangle_H \ \forall v \in V.$$

10.5 Bochner function spaces

Let X be a Banach space with norm$||\cdot||_X$.

Definition 10.5.1. The space $L^p(0,T;X)$ consists of all measurable functions $\eta : [0,T] \to X$ with

$$||\eta||_{L^p(0,T;X)} := (\int_0^T ||\eta(t)||_X^p dt)^{1/p} < \infty, \quad 1 \le p < \infty$$

and

$$||\eta||_{L^\infty(0,T;X)} := \operatorname{ess} \sup_{0\le t\le T} ||\eta(t)||_X < \infty.$$

Definition 10.5.2. The space $C([0,T];X)$ consists of all continuous functions $\eta : [0,T] \to X$ with

$$||\eta||_{C([0,T];X)} := \max_{0\le t\le T} ||\eta(t)||_X < \infty.$$

Definition 10.5.3. Let $\eta \in L^1(0,T;X)$. We say that ξ is the weak derivative of η written

$$\eta' = \xi$$

provided the following equation holds in X

$$\int_0^T \phi'(t)\eta(t)dt = -\int_0^T \phi(t)\xi(t)dt$$

for all test functions $\phi \in C_0^\infty(0,T)$.

Definition 10.5.4. The space $W^{1,p}(0,T);X)$ consists of all functions $\eta \in L^p(0,T;X)$ which has a weak derivative η' and $\eta' \in L^p(0,T;X)$. Furthermore

1. $$||\eta(t)||_{W^{1,p}(0,T);X)} := (\int_0^T [||\eta(t)||^p + ||\eta'(t)||^p]dt)^{\frac{1}{p}} \quad (1 \le p < \infty)$$

2. $$||\eta(t)||_{W^{1,\infty}(0,T);X)} := \sup_{[0,T]}(||\eta(t)|| + ||\eta'(t)||).$$

We write $H^1(0,T;X) = W^{1,2}(0,T);X)$.

Theorem 10.5.5. *Let $\eta \in W^{1,p}(0,T);X)$ for some $p \in [1,\infty]$. Then*

1. $$\eta \in C[0,T;X].$$

2. $$\eta(t) = \eta(s) + \int_0^T \eta'(\tau)d\tau \;\forall\;\; 0 \le s \le t \le T.$$

3.
$$\max_{0\le t\le T} ||\eta(t)|| \le C(T)||\eta||_{W^{1,p}(0,T);X)}.$$

Lemma 10.5.6. *If $g \in L^p(0,T;V')$ for any $1 \le p \le \infty$ then the following are equivalent:*

- $$g = 0 \quad \textit{in } L^p(0,T;V').$$

- $$\langle g(t), v\rangle_V = 0 \ \forall v \in V' \quad \textit{and} \quad \textit{a.e. } t \in [0,T].$$

Lemma 10.5.7. *If*

$$\eta_k \to \eta \quad \textit{weakly in} \quad L^2(0,T;V) \ , \quad \eta_k' \to \xi \quad \textit{weakly in} \quad L^2(0,T;V')$$

then $\eta' \in L^2(0,T;V')$ and
$$\xi = \eta'.$$

10.6 Sobolev embedding, Sobolev inequalities, compactness

Definition 10.6.1. Let $1 \le p < n$ then the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p.$$

Theorem 10.6.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with a locally Lipschitz boundary.*

- *If $p > n$ and $\gamma = 1 - \frac{n}{p}$ then the space $W^{1,p}(\Omega)$ is continuously embedded in $C^{0,\gamma}(\bar\Omega)$ and*
$$||v||_{C^{0,\gamma}(\bar\Omega)} \le C||v||_{W^{1,p}(\Omega)}, \quad C = C(\Omega,p,n).$$
In the case $n = 1$ the space $W^{1,1}(\Omega)$ is continuously embedded in $C^0(\bar\Omega)$ and
$$||v||_{C^0(\bar\Omega)} \le C||v||_{W^{1,1}(\Omega)}, \quad C = C(\Omega).$$

- *For*
$$p \ge n \quad \textit{and} \quad p \le q$$
the space $W^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ and
$$||v||_{L^q(\Omega)} \le C||v||_{W^{1,p}(\Omega)}, \quad C = C(\Omega,p).$$

- *For*
$$p < n \quad and \quad p < q \le p^* = \frac{np}{n-p}$$
the space $W^{1,p}(\Omega)$ *is continuously embedded in* $L^q(\Omega)$ *and*
$$||v||_{L^q(\Omega)} \le C||v||_{W^{1,p}(\Omega)}, \quad C = C(\Omega, p).$$

Theorem 10.6.3 (Rellich-Kondrachov)**.**

- *If* $p > n$ *and* $\gamma < 1 - \frac{n}{p}$ *then the space* $W^{1,p}(\Omega)$ *is compactly embedded in* $C^{0,\gamma}(\bar{\Omega})$.
- *For*
$$p < n \quad and \quad 1 < q < p^* = \frac{np}{n-p}$$
the space $W^{1,p}(\Omega)$ *is compactly embedded in* $L^q(\Omega)$.
- *For*
$$p \ge n \quad and \quad 1 \le q < \infty$$
the space $W^{1,p}(\Omega)$ *is compactly embedded in* $L^q(\Omega)$.

This theorems are versions of the *Sobolev inequalities* and the embedding theorems (including the *Rellich-Kondrachov* theorem). Many more such results may be found in the book by Adams and Fournier [1].

Theorem 10.6.4. *Let* X, Y, Z *be three Banach spaces where* X *and* Y *are reflexive. Suppose* $X \subset Z \subset Y$ *with continuous embeddings. Suppose the embedding of* X *into* Z *is compact. For any* $1 < p, q < \infty$ *set*
$$W : \{v : v \in L^p(0,T;X), v' \in L^q(0,T;Y)\}.$$
Then the embedding fom W *into* $L^p(0,T:Z)$ *is compact.*

Remark 10.6.5. This compactness result is due to Aubin. A setting might be $p = q = 2$, $X = H^1(\Omega), Z = L^2(\Omega), Y = (H^1(\Omega))'$.

Lemma 10.6.6. *Let* X, Y *be Banach spaces,* $X = Y'$ *and* $1 < p \le \infty$. *Suppose*
$$\begin{cases} u_n \to u \quad weak\ star\ in \quad L^p(0,T;X) \\ u_n' \to u' \quad weak\ star\ in \quad L^p(0,T;X) \end{cases}$$
then
$$u_n(0) \to u(0) \quad weak\ star\ in \quad X.$$
If X *is reflexive then the weak star convergence is equivalent to weak convergence.*

10.7 Elliptic regularity

Theorem 10.7.1. *Let Ω be bounded and convex. Let $\mathcal{A}$ be strictly positive definite and $\mathcal{A}_{ij} \in C^{0,1}(\bar{\Omega})$.*

- *If $u \in H^1_0(\Omega)$ then*
$$(\sum_{|\alpha|=2} |D^\alpha u|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq ||\Delta u||_{L^2(\Omega)}$$

- *If $u \in H^1_0(\Omega)$ and*
$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v dx = (f, v) = 0 \quad \forall v \in H^1_0(\Omega)$$
then
$$(\sum_{|\alpha|=2} |D^\alpha u|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq C(||f||_{L^2(\Omega)} + ||\nabla u||_{L^2(\Omega)})$$
where $C = C(\Omega)$ depends only on the diameter of Ω.

- *If $u \in H^1(\Omega)$ and*
$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v dx = (f, v) = 0 \quad \forall v \in H^1 0(\Omega)$$
then
$$||u||_{H^2(\Omega)} \leq C||f||_{L^2(\Omega)}$$
where $C = C(\lambda, \mathcal{A}, \Omega)$.

Theorem 10.7.2. *Let Ω be bounded and $\partial\Omega \in C^2$. Let $\mathcal{A}$ be strictly positive definite and $\mathcal{A}_{ij} \in C^1(\bar{\Omega})$. Let $u \in H^1_0(\Omega)$ solve*

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla v dx + \lambda \int_\Omega uv = (f, v) = 0 \quad \forall v \in H^1_0(\Omega)$$

then
$$||u||_{H^2(\Omega)} \leq C(||f||_{L^2(\Omega)} + ||u||_{L^2(\Omega)})$$
where $C = C(\Omega, \mathcal{A}, \lambda)$ depends only on Ω, $\mathcal{A}$ and λ.

10.8 Eigenfunction expansions

Let $\{V, H, V'\}$ be a Hilbert space triple and $A(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a symmetric coercive bounded bilinear form. Consider the *eigen-problem*: Find $\{z, \lambda\} \in V \times \mathbb{R}\}$ such that

$$A(z, v) = \lambda\langle z, v\rangle_H \quad \forall v \in V. \tag{10.1}$$

This can be reformulated as : Find $\{z, \lambda\} \in V \times \mathbb{R}\}$ such that

$$\mathcal{G}z = \frac{1}{\lambda}z \tag{10.2}$$

where $\mathcal{G} : V \to V$ is the compact solution operator of

$$\mathcal{G}f \in V : a(\mathcal{G}f, v) = \langle f, v\rangle_H \quad \forall v \in V.$$

It follows that there exist eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and eigenfunctions $\{z_j\}_{j=1}^{\infty}$ such that

$$A(z_j, v) = \lambda_j \langle z, v\rangle_H \quad \forall v \in V \tag{10.3}$$

and

$$0 < \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_k \leq \lambda_{k+1} \leq ...... \text{ with } \lim_{k\to\infty} = \infty \tag{10.4}$$

$$A(z_j, z_i) = \lambda\delta_{ji} \ , \quad \langle z_j, z_i\rangle_H = \delta_{ji}, \quad j, i \geq 1 \tag{10.5}$$

and the eigenfunctions form a complete system in both H and V.

10.9 Finite dimensional approximation

Let $\{V, H, V'\}$ be a Hilbert space triple. Let V_m be a sequence of finite dimensional subspaces of V with basis functions $\{z_j^m\}_{j=1}^m$ with the property that for each $v \in V$ there exists a sequence $v_m \in V_m$ such that

$$v_m \to v \text{ in } V.$$

It follows that there is a sequence $v_m \in V_m$ which also has the approximation property

$$v_m \to v \text{ in } H.$$

In particular let $\mathcal{P}_m : H \to H$ be the $H-$projection defined by $u_m = \mathcal{P}_m u \in V_m$

$$\langle u_m - u, v_m\rangle \quad \forall v_m \in V_m.$$

It follows that $||\mathcal{P}_m u||_H \leq ||u||_H$ and

$$||\mathcal{P}_m u - u||_H \leq ||u - v_m||_H \quad \forall v_m \in V_m$$

and that

$$\mathcal{P}_m u \to u \text{ in } H.$$

Furthermore $\mathcal{P}_m u = \sum_{j=1}^m U_j z_j^m$ where

$$\mathcal{M}\mathbf{U} = \mathbf{b}$$

where

$$\mathcal{M}_{ij} = \langle z_j^m, z_i^m\rangle_H, \quad b_i = \langle u, z_i^m\rangle.$$

The matrix $\mathcal{M}$ is called the *Gram* matrix or in finite element settings it is called the *mass* matrix.

Note that
$$\langle f, \mathcal{P}_m v\rangle_H = \langle \mathcal{P}_m f, \mathcal{P}_m v\rangle_H = \langle \mathcal{P}_m f, v\rangle_H.$$
Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bounded coercive bilinear form. Then we can define the *Ritz* projection $\mathcal{R}_m : V \to V_m$ by

$$a(\mathcal{R}_m u - u, v_m) = 0 \quad \forall v_m \in V_m.$$

It follows that $||\mathcal{R}_m u||_V \leq \frac{\gamma}{\alpha}||u||_V$ and

$$||\mathcal{R}_m u - u||_V \leq \frac{\gamma}{\alpha}||u - v_m||_V \quad \forall v_m \in V_m$$

and that
$$\mathcal{R}_m u \to u \;\text{ in }\; V.$$
Furthermore $\mathcal{R}_m u = \sum_{j=1}^m \alpha_j z_j^m$ where

$$\mathcal{S}\mathbf{U} = \mathbf{b}$$

where
$$\mathcal{S}_{ij} = a(z_j^m, z_i^m), \quad b_i = a(u, z_i^m).$$
The matrix $\mathcal{S}$ is called the *stiffness* matrix in finite element settings.

Lemma 10.9.1. *The space $L^p(0,T;V_m)$ is dense in $L^p(0,T;V)$, $1 \leq p < \infty$.*

10.10 Inverse Laplacian

Let Ω be a bounded domain in $\mathbb{R}^n$. It is often useful to consider rewriting equations using the inverse operator $\mathcal{G}$ as introduced in the previous section. Here we consider the Green operator for the Neumann problem with homogeneous data. Set

$$\mathcal{F} := \{f \in (H^1(\Omega))' : \langle f, 1\rangle = 0\}.$$

Set $V := \{\eta \in H^1(\Omega) : \int_\Omega \eta = 0\}$ and $H = L^2(\Omega)$. Note that $\mathcal{F} \subset V'$. Set $\mathcal{G} : \mathcal{F} \to V$ as the solution operator for

$$\text{Find} \;\; u \in V : \quad (\nabla u, \nabla \eta) = \langle f, \eta\rangle \quad \forall \eta \in V \tag{10.6}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product, by

$$\mathcal{G}f := u.$$

That there is a unique $u \in V$ is a direct consequence of the Lax-Milgram theorem and the Poincare inequality

$$||\eta||_{L^2(\Omega)} \leq C_P(|(\eta,1)| + ||\nabla\eta||_{L^2(\Omega)}) \ \forall \ \eta \in H^1(\Omega) \tag{10.7}$$

where $C_P = C_P(\Omega)$.

Let us note that for $f \in \mathcal{F}$ we may take

$$||f||_{-1} := |\mathcal{G}f|_1$$

and that for for $f \in L^2(\Omega) \cap \mathcal{F}$

$$||f||_{-1} = (\mathcal{G}f, f)^{1/2} \tag{10.8}$$

and using the Poincare inequality

$$||f||_{-1} \leq C_P||f||_{L^2(\Omega)}. \tag{10.9}$$

10.11 Functions of Bounded Variation

10.11.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. Set

$$\int_\Omega |Du| := \sup \int_\Omega u \mathrm{div}\phi dx; \varphi = (\varphi_1, \varphi_2, ..., \varphi_n) \in C_0^1(\Omega)^n, ||\varphi||_{L^\infty(\Omega)} \leq 1. \tag{10.10}$$

Definition 10.11.1. The linear space of functions of *bounded variation*, $BV(\Omega)$, is defined as

$$BV(\Omega) = \{v \in L^1(\Omega) : \int_\Omega |Dv| < \infty\}.$$

Lemma 10.11.2. *$\int_\Omega |Du| : BV(\Omega) \to \mathbb{R}$ is lower semi-continuous, that is for any sequence $u_j \in BV(\Omega)$ converging to $u \in L^1(\Omega)$*

$$\liminf \int_\Omega |Du_j| \geq \int_\Omega |Du|.$$

Furthermore $BV(\Omega)$ is a Banach space.

Example 10.11.3. *Let $E \subset \Omega$ have a C^2 boundary Γ. Let χ_E be the characteristic function of E defined by*

$$\chi_E(x) = 1 \ x \in E \quad \textit{and} \quad \chi_E(x) = 0 \ x \in \Omega \setminus \bar{E}.$$

It holds that $\chi_E \in BV(\Omega)$ and

$$\int_\Omega |D\chi_E| = |\Gamma|.$$

Lemma 10.11.4. *Let $u \in BV(\Omega)$. Then there exists a sequence $\{u_j\} \in C^\infty(\Omega)$ such that*

$$\lim_{j\to\infty} ||u_j - u||_{L^1(\Omega)} = 0 \quad \textit{and} \quad \lim_{j\to\infty} \int_\Omega |Du_j| \to \int_\Omega |Du|. \tag{10.11}$$

Lemma 10.11.5. *The Poincare-Wirtinger inequality holds:*

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u dx \right\|_1 \leq C \int_\Omega |\nabla u| \quad \forall u \in BV\left(\Omega\right) \tag{10.12}$$

where $C = C(\Omega)$.

10.12 Fixed point theorems

Theorem 10.12.1. Brouwer fixed-point theorem

Let $K \subset \mathbb{R}^n$ be a compact convex set and $F : K \to K$ be a continuous mapping. Then there exists a fixed point $u \in K$ of F i.e. $u = F(u)$.

Lemma 10.12.2. *Let a continuous mapping $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ satisfy*

$$\mathbf{F}(\boldsymbol{x}) \cdot \boldsymbol{x} \geq 0 \quad \textit{if} \quad |\boldsymbol{x}| = r \tag{10.13}$$

for some $r > 0$. Then there exists $\boldsymbol{x} \in B(\mathbf{0}, r)$ such that

$$\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}.$$

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