Contents

1 Frank Oseen energy

In $\mathbb{R}^3$:

$$E_{\text{OS}} = \frac{1}{2}\int_\Omega K_1 \left(\nabla\cdot\mathbf{p}\right)^2 + K_2\left(\mathbf{p}\cdot[\nabla\times\mathbf{p}]\right)^2 + K_3\left\|\mathbf{p}\times[\nabla\times\mathbf{p}]\right\|^2 dV \tag{1}$$

With the Langrange identity for the K_3-term, we cann rewrite (1) to

$$E_{\text{OS}} = \frac{1}{2}\int_\Omega K_1 \left(\nabla\cdot\mathbf{p}\right)^2 + \left(K_2 - K_3\right)\left(\mathbf{p}\cdot[\nabla\times\mathbf{p}]\right)^2 + K_3\left\|\mathbf{p}\right\|^2\left\|\nabla\times\mathbf{p}\right\|^2 dV \tag{2}$$

If we restrict (2) to a 2-dimensional Manifold $M\subset\Omega$ and postulate that $\mathbf{p}\in T_X M$ is a normalized tangential vector in $X\in M$, we get

$$E_{\text{OS}} = \frac{1}{2}\int_M K_1\left(\text{Div}\mathbf{p}\right)^2 + K_3\left(\text{Rot}\mathbf{p}\right)^2 dA \tag{3}$$

In terms of exterior calculus with the corresponding 1-form $\mathbf{p}^\flat\in\Lambda^1(M)$, ,i.e. $\left(\mathbf{p}^\flat\right)^\sharp = \mathbf{p}$, we obtain

$$E_{\text{OS}} = \frac{1}{2}\int_M K_1\left(\mathbf{d}^*\mathbf{p}^\flat\right)^2 + K_3\left(*\mathbf{d}\mathbf{p}^\flat\right)^2 dA \tag{4}$$

where the exterior coderivative $\mathbf{d}^* := -*\mathbf{d}*$ is the L^2-orthogonal operator of the exterior derivative $\mathbf{d}$. (Note $\text{Div}\mathbf{p} = -\mathbf{d}^*\mathbf{p}^\flat$ and $\text{Rot}\mathbf{p} = *\mathbf{d}\mathbf{p}^\flat$)

1.1 Functional derivative

With the L^2-orthogonality of the exterior derivative and coderivative and a arbitrary $\alpha \in \Lambda^1(M)$ we get

$$\int_M \left\langle \frac{\delta E_{\mathrm{OS}}}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left(E_{\mathrm{OS}}\left[\mathbf{p}^\flat + \epsilon\alpha\right] - E_{\mathrm{OS}}\left[\mathbf{p}^\flat\right]\right) \tag{5}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_M K_1 \left(2\epsilon \left(\mathbf{d}^*\mathbf{p}^\flat\right)(\mathbf{d}^*\alpha) + \epsilon^2 (\mathbf{d}^*\alpha)^2\right) \tag{6}$$

$$+ K_3 \left(2\epsilon \left\langle \mathbf{d}\mathbf{p}^\flat, \mathbf{d}\alpha \right\rangle + \epsilon^2 \left\|\mathbf{d}\alpha\right\|^2\right) dA \tag{7}$$

$$= -\int_M K_1 \left\langle \Delta^{\mathrm{GD}}\mathbf{p}^\flat, \alpha \right\rangle + K_3 \left\langle \Delta^{\mathrm{RR}}\mathbf{p}^\flat, \alpha \right\rangle dA \tag{8}$$

$$= \int_M \left\langle -\left(K_1\Delta^{\mathrm{GD}} + K_3\Delta^{\mathrm{RR}}\right)\mathbf{p}^\flat, \alpha \right\rangle dA \tag{9}$$

where $\Delta^{\mathrm{RR}} = -\mathbf{d}^*\mathbf{d} = *\mathbf{d} * \mathbf{d}$ is the Vector-Laplace-Beltrami-Operator or Rot-Rot-Laplace and $\Delta^{\mathrm{GD}} = -\mathbf{d}\mathbf{d}^* = \mathbf{d} * \mathbf{d} *$ is the Vector-Laplace-CoBeltrami-Operator or Grad-Div-Laplace. Hence, for a One-Constant-Approximation $K_1 = K_3 =: K_0$, we obtain

$$\int_M \left\langle \frac{\delta E_{\mathrm{OS}}}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \int_M \left\langle K_0 \Delta^{\mathrm{dR}}\mathbf{p}^\flat, \alpha \right\rangle dA \tag{10}$$

where $\Delta^{\mathrm{dR}} = -\Delta^{\mathrm{RR}} - \Delta^{\mathrm{GD}} = \mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*$ is the Laplace-de Rham operator.

1.2 Unit vector invariance

If $\mathbf{p} \in T_X M$ is a unit vector on M, we can describe all unit vectors in $X \in M$ as a rotation in the tangential space with angle $\phi \in \mathbb{R}$:

$$\mathbf{q} = \cos\phi\mathbf{p} + \sin\phi\,(*\mathbf{p}) \tag{11}$$

$*\mathbf{p} = \left(*\mathbf{p}^\flat\right)^\sharp$ is the Hodge dual of $\mathbf{p}$, i.e. a quarter rotation of $\mathbf{p}$. For a space independent angle ϕ, i.e. $\mathbf{d}\phi = 0$, straight forward calculations implies

$$\left\|\mathrm{Rot}(*\mathbf{p})\right\| = \left\| *\mathbf{d} * \mathbf{p}^\flat \right\| = \left\|\mathrm{Div}\mathbf{p}\right\| \tag{12}$$

$$\left\|\mathrm{Div}(*\mathbf{p})\right\| = \left\| *\mathbf{d} * *\mathbf{p}^\flat \right\| = \left\| *\mathbf{d}\mathbf{p}^\flat \right\| = \left\|\mathrm{Rot}\mathbf{p}\right\| \tag{13}$$

$$\left\|\mathrm{Rot}\mathbf{q}\right\|^2 = \left\| *\mathbf{d}\mathbf{q}^\flat \right\|^2 = \left\| \mathbf{d}\mathbf{q}^\flat \right\|^2 \tag{14}$$

$$= \cos^2\phi \left\|\mathrm{Rot}\mathbf{p}\right\|^2 + \sin^2\phi \left\|\mathrm{Div}\mathbf{p}\right\|^2 + 2\cos\phi\sin\phi \left\langle \mathbf{d}\mathbf{p}^\flat, \mathbf{d} * \mathbf{p}^\flat \right\rangle \tag{15}$$

$$\left\|\mathrm{Div}\mathbf{q}\right\|^2 = \left\| *\mathbf{d} * \mathbf{q}^\flat \right\|^2 = \left\| \mathbf{d} * \mathbf{q}^\flat \right\|^2 \tag{16}$$

$$= \cos^2\phi \left\|\mathrm{Div}\mathbf{p}\right\|^2 + \sin^2\phi \left\|\mathrm{Rot}\mathbf{p}\right\|^2 - 2\cos\phi\sin\phi \left\langle \mathbf{d}\mathbf{p}^\flat, \mathbf{d} * \mathbf{p}^\flat \right\rangle \tag{17}$$

Finally, we get for the One-Constant-Approximation of the Frank-Oseen-Energy

$$E_{\mathrm{OS}}[\mathbf{q}] = E_{\mathrm{OS}}[\mathbf{p}] \tag{18}$$

2 Normalizing energy

To constrain $\mathbf{p}$ is normalized, we add

$$E_n = \int_M \frac{K_n}{4} \left(\|\mathbf{p}\|^2 - 1 \right)^2 dA \tag{19}$$

to the Frank Oseen energy. Note that the norm defined by the metric g on the manifold M is invariant regarding lowering or rising the indices, i.e.

$$\|\mathbf{p}\|^2 = p^i g_{ij} p^j = p_i g^{ij} p_j = \left\| \mathbf{p}^\flat \right\|^2 \tag{20}$$

2.1 Functional derivative

By variating $\mathbf{p}^\flat$ under the norm with an arbitrary $\alpha \in \Lambda^1(M)$, we obtain

$$\left\| \mathbf{p}^\flat + \epsilon\alpha \right\|^2 = \left\| \mathbf{p}^\flat \right\|^2 + 2\epsilon \left\langle \mathbf{p}^\flat, \alpha \right\rangle + \epsilon^2 \|\alpha\|^2 \tag{21}$$

If we are only interesting in linear terms (in ϵ), this leads to

$$\left(\left\| \mathbf{p}^\flat + \epsilon\alpha \right\|^2 - 1 \right)^2 = \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 + 2\epsilon \left\langle \mathbf{p}^\flat, \alpha \right\rangle + \mathcal{O}\left(\epsilon^2\right) \right)^2 \tag{22}$$

$$= \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right)^2 + 4\epsilon \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \left\langle \mathbf{p}^\flat, \alpha \right\rangle + \mathcal{O}\left(\epsilon^2\right) \tag{23}$$

Hence, we get for the functional derivative of E_n

$$\int_M \left\langle \frac{\delta E_n}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(E_n \left[\mathbf{p}^\flat + \epsilon\alpha \right] - E_n \left[\mathbf{p}^\flat \right] \right) \tag{24}$$

$$= \int_M \left\langle K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat, \alpha \right\rangle dA \tag{25}$$

3 Model equations

To minimize the energy $E := E_{\mathrm{OS}} + E_n$ we choose a time evolving approach. Hence, with the fundamental lemma of calculus of variations, we will use the time depended differential equation in terms of exterior calculus

$$\partial_t \mathbf{p}^\flat = -\frac{\delta E}{\delta \mathbf{p}^\flat} = -K_0 \Delta^{\mathrm{dR}} \mathbf{p}^\flat - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat \tag{26}$$

or in general, if we don't want to use the One-Constant-Approximation,

$$\partial_t \mathbf{p}^\flat = \left(K_1 \Delta^{\mathrm{GD}} + K_3 \Delta^{\mathrm{RR}}\right) \mathbf{p}^\flat - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat \tag{27}$$

Note that if we apply the Hodge operator on the whole equations, we get the Hodge dual equations

$$\partial_t (*\mathbf{p}^\flat) = -K_0 \Delta^{\mathrm{dR}} (*\mathbf{p}^\flat) - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) (*\mathbf{p}^\flat) \tag{28}$$

$$= \left(K_1 \Delta^{\mathrm{RR}} + K_3 \Delta^{\mathrm{GD}}\right) (*\mathbf{p}^\flat) - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) (*\mathbf{p}^\flat) \tag{29}$$

which are very useful for the DEC discretization later in context. But this leads to pay attention, because only in the first line (One-Constant-Approximation) we see, that the Hodge dual equations in $*\mathbf{p}^\flat$ is the same as the primal equation in $\mathbf{p}^\flat$. In the general case (second line), we must "swap" the Laplace operators.

4 A DEC approach

For further information see for example [Whi57, Hir03].

4.1 Surface Mesh

...wellcentered manifoldlike simplicial complex, bla, bla, blub...

4.2 Discrete 1-forms

The main concept to represent a discrete 1-form $\mathbf{p}_h^\flat \in \Lambda_h(K)$ is to approximate the contraction of the continuous 1-Form $\mathbf{p}^\flat \in \Lambda(M)$ on all edges $e \in \mathcal{E}$

$$\mathbf{p}_h^\flat(e) := \int_{\pi(e)} \mathbf{p}^\flat \approx \int_0^1 \mathbf{p}^\flat_{X_e(\tau)} \left(\dot{X}_e(t) \right) dt = \mathbf{p}^\flat_{X_e(\tau)}(\mathbf{e}) \tag{30}$$

where $\pi : K \to M$ is the glueing map, who project the elements of the surface mesh to the manifold. $X_e(t) = t\mathbf{v}_2 + (1-t)\mathbf{v}_1$ is the linear barycentric parametrisation of the edge $e = [v_1, v_2]$. The existence of a intermediate value $\tau \in [0,1]$, so that $\mathbf{e} \in T_{X_e(\tau)}M$, is ensured by the mean value theorem. Other discrete forms of arbitrary degree and theirs hodge duals can be interpreted in a similarly way.

4.3 Discrete Laplace operators

In the discrete exterior calculus discrete Operators are defined by successively interpretation of the basic operations on the forms, like the Hodge operator $*$ or the exterior derivative $\mathbf{d}$, as geometric operators on the simplices, like the Voronoi dual operator $\star$ or the boundary operator ∂ (see [Hir03]). This results for example to a discrete definition of Δ^{RR} for a discrete 1-form $\mathbf{p}_h^\flat \in \Lambda_h^1(M)$ on a edge $e \in \mathcal{E}$

$$\Delta_h^{\mathrm{RR}}\mathbf{p}_h^\flat(e) := \left(*\mathbf{d}*\mathbf{d}\mathbf{p}_h^\flat\right)(e) = -\frac{|e|}{|\star e|}\left(\mathbf{d}*\mathbf{d}\mathbf{p}_h^\flat\right)(\star e) \tag{31}$$

$$= -\frac{|e|}{|\star e|}\left(*\mathbf{d}\mathbf{p}_h^\flat\right)(\partial\star e) = -\frac{|e|}{|\star e|}\sum_{f\succ e} s_{f,e}\left(*\mathbf{d}\mathbf{p}_h^\flat\right)(\star f) \tag{32}$$

$$= -\frac{|e|}{|\star e|}\sum_{f\succ e}\frac{s_{f,e}}{|f|}\left(\mathbf{d}\mathbf{p}_h^\flat\right)(f) = -\frac{|e|}{|\star e|}\sum_{f\succ e}\frac{s_{f,e}}{|f|}\mathbf{p}_h^\flat(\partial f) \tag{33}$$

$$= -\frac{|e|}{|\star e|}\sum_{f\succ e}\frac{s_{f,e}}{|f|}\sum_{\tilde{e}\prec f} s_{f,\tilde{e}}\mathbf{p}_h^\flat(\tilde{e}) \tag{34}$$

or for Δ^{GD}

$$\Delta_h^{\mathrm{GD}}\mathbf{p}_h^\flat(e) := \left(\mathbf{d}*\mathbf{d}*\mathbf{p}_h^\flat\right)(e) = \left(*\mathbf{d}*\mathbf{p}_h^\flat\right)(\partial e) \tag{35}$$

$$= \sum_{v\prec e} s_{v,e}\left(*\mathbf{d}*\mathbf{p}_h^\flat\right)(v) = \sum_{v\prec e}\frac{s_{v,e}}{|\star v|}\left(\mathbf{d}*\mathbf{p}_h^\flat\right)(\star v) \tag{36}$$

$$= \sum_{v\prec e}\frac{s_{v,e}}{|\star v|}\left(*\mathbf{p}_h^\flat\right)(\partial\star v) = -\sum_{v\prec e}\frac{s_{v,e}}{|\star v|}\sum_{\tilde{e}\succ v} s_{v,\tilde{e}}\left(*\mathbf{p}_h^\flat\right)(\star\tilde{e}) \tag{37}$$

$$= -\sum_{v\prec e}\frac{s_{v,e}}{|\star v|}\sum_{\tilde{e}\succ v} s_{v,\tilde{e}}\frac{|\star\tilde{e}|}{|\tilde{e}|}\mathbf{p}_h^\flat(\tilde{e}) \tag{38}$$

where the sign $s_{f,e}$ is +1 if the face $f \succ e$ is in the left of the edge e (-1 otherwise) and $s_{v,e}$ is +1 if the edge $e \succ v$ points to the vertex v and -1 if e points away from v. We see Δ_h^{RR}, Δ_h^{GD} and also $\Delta_h^{\mathrm{dR}} := -\Delta_h^{\mathrm{RR}} - \Delta_h^{\mathrm{GD}}$ are linear operators in $\mathbf{p}_h^\flat(\tilde{e})$ and therefor results in sparse matrices if the $\mathbf{p}_h^\flat(\tilde{e})$ are our degree of freedoms. ...explainable figures...

4.4 Discrete norm

Approximating the norm $\left\|\mathbf{p}^\flat\right\|$ on a edge $e \in \mathcal{E}$ is not so easy like the development of discrete linear operators. We only know how $\mathbf{p}_h^\flat$ "act" on a single edge, so $\mathbf{p}_h^\flat$ gives us only one dimensional informations. In other words, we only know the proportion $\mathbf{p}_h \cdot \mathbf{e} = \mathbf{p}_h^\flat(e)$ of the discrete contra vector field $\mathbf{p}_h = \left(\mathbf{p}_h^\flat\right)^\sharp$ in the $\mathbf{e}$ direction, but we don't know the length of $\mathbf{p}_h$ defined on this edge.

...some averaging techniques and why these sucks...

One way out is to rise the dimension of the discrete 1-forms, therefor we introduce some bases at the intersection $c(e) = e \cap (\star e)$ of a edge and its dual edge to describe discrete contra- and covariant vector fields in a local (flat) coordinate system.

The basis for contravariant vectors at $c(e)$ is composed of

$$\partial_e X := \mathbf{e} = \mathbf{v}_2 - \mathbf{v}_1 \tag{39}$$

$$\partial_{\star e} X := *\mathbf{e} = c(f_2) - c(f_1) \tag{40}$$

if $e = [v_1, v_2]$ and the face f_1 lay right and f_2 left of e (...figure...). This definitions are consistent with the canonical basis, if the position X is a barycentric parametrisation of the edge e resp. its dual, i.e. for

$$X_e(t_e) = t_e \mathbf{v}_2 + (1 - t_e)\,\mathbf{v}_2 \text{ for } t \in [0,1] \tag{41}$$

$$X_{\star e}(t_{\star e}) = \begin{cases} 2t_{\star e} c(e) + (1 - 2t_{\star e})\, c(f_1) & \text{if } t_{\star e} \in \left[0, \frac{1}{2}\right] \\ (1 - t_{\star e})\, c(e) + (2t_{\star e} - 1)\, c(f_1) & \text{if } t_{\star e} \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{42}$$

holds $\partial_e X = \partial_{t_e} X_e$ and $\partial_{\star e} X = \frac{1}{2}\left(\partial_{t_{\star e}} X_{\star e}|_{[0,\frac{1}{2}]} + \partial_{t_{\star e}} X_{\star e}|_{[\frac{1}{2},1]}\right)$. Therefore we get the local metric tensor

$$g_h(e) = |e|^2 \,(dx^e)^2 + |\star e|^2 \,(dx^{\star e})^2 \tag{43}$$

where $\{dx^e, dx^{\star e}\}$ are the dual base of $\{\partial_e X, \partial_{\star e} X\}$, i.e. $dx^i\,(\partial_j X) = \delta^i_j$ for $i, j \in \{e, \star e\}$. This gives us the great possibility to define

$$\begin{aligned} \underline{\mathbf{p}}^\flat_h(e) &:= \mathbf{p}^\flat_h(e) dx^e + \mathbf{p}^\flat_h(\star e) dx^{\star e} && (44) \\ &= \mathbf{p}^\flat_h(e) dx^e - \frac{|\star e|}{|e|}\left(*\mathbf{p}^\flat_h\right)(e) dx^{\star e} && (45) \\ \underline{\mathbf{p}}_h(e) &:= \left(\underline{\mathbf{p}}^\flat_h(e)\right)^\sharp = \frac{1}{|e|^2}\mathbf{p}^\flat_h(e)\partial_e X + \frac{1}{|\star e|^2}\mathbf{p}^\flat_h(\star e)\partial_{\star e} X && (46) \\ &= \frac{1}{|e|}\left(\frac{1}{|e|}\mathbf{p}^\flat_h(e)\partial_e X - \frac{1}{|\star e|}\left(*\mathbf{p}^\flat_h\right)(e)\partial_{\star e} X\right) && (47) \end{aligned}$$

We call $\underline{\mathbf{p}}^\flat_h(e)$ the discrete Primal-Dual-1-form (short PD-1-form) and $\underline{\mathbf{p}}_h$ the discrete Primal-Dual-vector field (short PD-vector field). Hence, we get for the square of the norm on $c(e)$ by contract the PD-1-Form with its corresponding PD-Vector

$$\left\|\underline{\mathbf{p}}^\flat_h\right\|^2(e) := \left(\underline{\mathbf{p}}^\flat_h\left(\underline{\mathbf{p}}_h\right)\right)(e) = \frac{1}{|e|^2}\left(\left[\mathbf{p}^\flat_h(e)\right]^2 + \left[\left(*\mathbf{p}^\flat_h\right)(e)\right]^2\right) \tag{48}$$

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[Whi57] H. Whitney. *Geometric Integration Theory*. Princeton mathematical series. University Press, 1957.