Contents

1 Frank Oseen energy

In $\mathbb{R}^3$:

$$E_{\text{FO}} = \frac{1}{2} \int_\Omega K_1 \left(\nabla \cdot \mathbf{p} \right)^2 + K_2 \left(\mathbf{p} \cdot \left[\nabla \times \mathbf{p} \right] \right)^2 + K_3 \left\| \mathbf{p} \times \left[\nabla \times \mathbf{p} \right] \right\|^2 dV \tag{1}$$

With the Langrange identity for the K_3-term, we cann rewrite (1) to

$$E_{\text{FO}} = \frac{1}{2} \int_\Omega K_1 \left(\nabla \cdot \mathbf{p} \right)^2 + \left(K_2 - K_3 \right) \left(\mathbf{p} \cdot \left[\nabla \times \mathbf{p} \right] \right)^2 + K_3 \left\| \mathbf{p} \right\|^2 \left\| \nabla \times \mathbf{p} \right\|^2 dV \tag{2}$$

If we restrict (2) to a 2-dimensional Manifold $M \subset \Omega$ and postulate that $\mathbf{p} \in T_X M$ is a normalized tangential vector in $X \in M$, we get

$$E_{\text{FO}} = \frac{1}{2} \int_M K_1 \left(\text{Div} \mathbf{p} \right)^2 + K_3 \left(\text{Rot} \mathbf{p} \right)^2 dA \tag{3}$$

In terms of exterior calculus with the corresponding 1-form $\mathbf{p}^\flat \in \Lambda^1(M)$, ,i.e. $\left(\mathbf{p}^\flat \right)^\sharp = \mathbf{p}$, we obtain

$$E_{\text{FO}} = \frac{1}{2} \int_M K_1 \left(\mathbf{d}^* \mathbf{p}^\flat \right)^2 + K_3 \left(* \mathbf{d} \mathbf{p}^\flat \right)^2 dA \tag{4}$$

where the exterior coderivative $\mathbf{d}^* := - * \mathbf{d} *$ is the L^2-orthogonal operator of the exterior derivative $\mathbf{d}$. (Note $\text{Div} \mathbf{p} = -\mathbf{d}^* \mathbf{p}^\flat$ and $\text{Rot} \mathbf{p} = * \mathbf{d} \mathbf{p}^\flat$)

1.1 Functional derivative

With the L^2-orthogonality of the exterior derivative and coderivative ($\langle \mathbf{d}\bullet, \bullet \rangle_{L^2} = \langle \bullet, \mathbf{d}^*\bullet \rangle_{L^2}$) and a arbitrary $\alpha \in \Lambda^1(M)$ we get

$$\int_M \left\langle \frac{\delta E_{\text{FO}}}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(E_{\text{FO}} \left[\mathbf{p}^\flat + \epsilon \alpha \right] - E_{\text{FO}} \left[\mathbf{p}^\flat \right] \right) \tag{5}$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_M K_1 \left(2\epsilon \left(\mathbf{d}^* \mathbf{p}^\flat \right) \left(\mathbf{d}^* \alpha \right) + \epsilon^2 \left(\mathbf{d}^* \alpha \right)^2 \right) \tag{6}$$

$$+ K_3 \left(2\epsilon \left\langle \mathbf{dp}^\flat, \mathbf{d}\alpha \right\rangle + \epsilon^2 \| \mathbf{d}\alpha \|^2 \right) dA \tag{7}$$

$$= - \int_M K_1 \left\langle \mathbf{\Delta}^{\text{GD}} \mathbf{p}^\flat, \alpha \right\rangle + K_3 \left\langle \mathbf{\Delta}^{\text{RR}} \mathbf{p}^\flat, \alpha \right\rangle dA \tag{8}$$

$$= \int_M \left\langle - \left(K_1 \mathbf{\Delta}^{\text{GD}} + K_3 \mathbf{\Delta}^{\text{RR}} \right) \mathbf{p}^\flat, \alpha \right\rangle dA \tag{9}$$

where $\mathbf{\Delta}^{\text{RR}} = -\mathbf{d}^*\mathbf{d} = *\mathbf{d}*\mathbf{d}$ is the Vector-Laplace-Beltrami-Operator or Rot-Rot-Laplace and $\mathbf{\Delta}^{\text{GD}} = -\mathbf{dd}^* = \mathbf{d}*\mathbf{d}*$ is the Vector-Laplace-CoBeltrami-Operator or Grad-Div-Laplace. Hence, for a One-Constant-Approximation $K_1 = K_3 =: K_0$, we obtain

$$\int_M \left\langle \frac{\delta E_{\text{FO}}}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \int_M \left\langle K_0 \mathbf{\Delta}^{\text{dR}} \mathbf{p}^\flat, \alpha \right\rangle dA \tag{10}$$

where $\mathbf{\Delta}^{\text{dR}} = -\mathbf{\Delta}^{\text{RR}} - \mathbf{\Delta}^{\text{GD}} = \mathbf{d}^*\mathbf{d} + \mathbf{dd}^*$ is the Laplace-de Rham operator.

1.2 Unit vector invariance

If $\mathbf{p} \in T_X M$ is a unit vector on M, we can describe all unit vectors in $X \in M$ as a rotation in the tangential space with angle $\phi \in \mathbb{R}$:

$$\mathbf{q} = \cos\phi \mathbf{p} + \sin\phi \left(*\mathbf{p} \right) \tag{11}$$

$*\mathbf{p} = \left(*\mathbf{p}^\flat \right)^\sharp$ is the Hodge dual of $\mathbf{p}$, i.e. a quarter rotation of $\mathbf{p}$. For a space independent angle ϕ, i.e. $\mathbf{d}\phi = 0$, straight forward calculations implies

$$\| \text{Rot}(*\mathbf{p}) \| = \left\| *\mathbf{d} * \mathbf{p}^\flat \right\| = \| \text{Div}\mathbf{p} \| \tag{12}$$

$$\| \text{Div}(*\mathbf{p}) \| = \left\| *\mathbf{d} * *\mathbf{p}^\flat \right\| = \left\| *\mathbf{dp}^\flat \right\| = \| \text{Rot}\mathbf{p} \| \tag{13}$$

$$\| \text{Rot}\mathbf{q} \|^2 = \left\| *\mathbf{dq}^\flat \right\|^2 = \left\| \mathbf{dq}^\flat \right\|^2 \tag{14}$$

$$= \cos^2\phi \| \text{Rot}\mathbf{p} \|^2 + \sin^2\phi \| \text{Div}\mathbf{p} \|^2 + 2\cos\phi\sin\phi \left\langle \mathbf{dp}^\flat, \mathbf{d} * \mathbf{p}^\flat \right\rangle \tag{15}$$

$$\| \text{Div}\mathbf{q} \|^2 = \left\| *\mathbf{d} * \mathbf{q}^\flat \right\|^2 = \left\| \mathbf{d} * \mathbf{q}^\flat \right\|^2 \tag{16}$$

$$= \cos^2\phi \| \text{Div}\mathbf{p} \|^2 + \sin^2\phi \| \text{Rot}\mathbf{p} \|^2 - 2\cos\phi\sin\phi \left\langle \mathbf{dp}^\flat, \mathbf{d} * \mathbf{p}^\flat \right\rangle \tag{17}$$

Finally, we get for the One-Constant-Approximation of the Frank-Oseen-Energy

$$E_{\text{FO}}[\mathbf{q}] = E_{\text{FO}}[\mathbf{p}] \tag{18}$$

2 Normalizing energy

To constrain $\mathbf{p}$ is normalized, we add

$$E_n = \int_M \frac{K_n}{4} \left(\| \mathbf{p} \|^2 - 1 \right)^2 dA \tag{19}$$

to the Frank Oseen energy. Note that the norm defined by the metric g on the manifold M is invariant regarding lowering or rising the indices, i.e.

$$\|\mathbf{p}\|^2 = p^i g_{ij} p^j = p_i g^{ij} p_j = \left\| \mathbf{p}^\flat \right\|^2 \tag{20}$$

2.1 Functional derivative

By variating $\mathbf{p}^\flat$ under the norm with an arbitrary $\alpha \in \Lambda^1(M)$, we obtain

$$\left\| \mathbf{p}^\flat + \epsilon \alpha \right\|^2 = \left\| \mathbf{p}^\flat \right\|^2 + 2\epsilon \left\langle \mathbf{p}^\flat, \alpha \right\rangle + \epsilon^2 \|\alpha\|^2 \tag{21}$$

If we are only interesting in linear terms (in ϵ), this leads to

$$\left(\left\| \mathbf{p}^\flat + \epsilon \alpha \right\|^2 - 1 \right)^2 = \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 + 2\epsilon \left\langle \mathbf{p}^\flat, \alpha \right\rangle + \mathcal{O}\left(\epsilon^2\right) \right)^2 \tag{22}$$

$$= \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right)^2 + 4\epsilon \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \left\langle \mathbf{p}^\flat, \alpha \right\rangle + \mathcal{O}\left(\epsilon^2\right) \tag{23}$$

Hence, we get for the functional derivative of E_n

$$\int_M \left\langle \frac{\delta E_n}{\delta \mathbf{p}^\flat}, \alpha \right\rangle dA = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(E_n \left[\mathbf{p}^\flat + \epsilon \alpha \right] - E_n \left[\mathbf{p}^\flat \right] \right) \tag{24}$$

$$= \int_M \left\langle K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat, \alpha \right\rangle dA \tag{25}$$

3 Model equations

To minimize the energy $E := E_{\mathrm{FO}} + E_n$ we choose a time evolving approach. Hence, with the fundamental lemma of calculus of variations, we will use the time depended differential equation in terms of exterior calculus

$$\partial_t \mathbf{p}^\flat = -\frac{\delta E}{\delta \mathbf{p}^\flat} = -K_0 \mathbf{\Delta}^{\mathrm{dR}} \mathbf{p}^\flat - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat \tag{26}$$

or in general, if we don't want to use the One-Constant-Approximation,

$$\partial_t \mathbf{p}^\flat = \left(K_1 \mathbf{\Delta}^{\mathrm{GD}} + K_3 \mathbf{\Delta}^{\mathrm{RR}} \right) \mathbf{p}^\flat - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) \mathbf{p}^\flat \tag{27}$$

Note that if we apply the Hodge operator on the whole equations, we get the Hodge dual equations

$$\partial_t (*\mathbf{p}^\flat) = -K_0 \mathbf{\Delta}^{\mathrm{dR}} (*\mathbf{p}^\flat) - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) (*\mathbf{p}^\flat) \tag{28}$$

$$= \left(K_1 \mathbf{\Delta}^{\mathrm{RR}} + K_3 \mathbf{\Delta}^{\mathrm{GD}} \right) (*\mathbf{p}^\flat) - K_n \left(\left\| \mathbf{p}^\flat \right\|^2 - 1 \right) (*\mathbf{p}^\flat) \tag{29}$$

which are very useful for the DEC discretization later in context. But this leads to pay attention, because only in the first line (One-Constant-Approximation) we see, that the Hodge dual equations in $*\mathbf{p}^\flat$ is the same as the primal equation in $\mathbf{p}^\flat$. In the general case (second line), we must "swap" the Laplace operators.

4 Needed DEC stuff

For further information see for example [Whi57, Hir03].

4.1 Surface Mesh

...wellcentered manifoldlike simplicial complex, bla, bla, blub...

4.2 Discrete 1-forms

The main concept to represent a discrete 1-form $\mathbf{p}_h^\flat \in \Lambda_h^1(K)$ is to approximate the contraction of the continuous 1-Form $\mathbf{p}^\flat \in \Lambda^1(M)$ on all edges $e \in \mathcal{E}$

$$\mathbf{p}_h^\flat(e) := \int_{\pi(e)} \mathbf{p}^\flat \approx \int_0^1 \mathbf{p}_{X_e(\tau)}^\flat \left(\dot{X}_e(t)\right) dt = \mathbf{p}_{X_e(\tau)}^\flat(\mathbf{e}) \tag{30}$$

where $\pi : K \to M$ is the glueing map, who project the elements of the surface mesh to the manifold. $X_e(t) = t\mathbf{v}_2 + (1-t)\mathbf{v}_1$ is the linear barycentric parametrisation of the edge $e = [v_1, v_2]$. The existence of a intermediate value $\tau \in [0, 1]$, so that $\mathbf{e} \in T_{X_e(\tau)}M$, is ensured by the mean value theorem. Other discrete forms of arbitrary degree and theirs hodge duals can be interpreted in a similarly way.

4.3 Discrete Laplace operators

In the discrete exterior calculus discrete Operators are defined by successively interpretation of the basic operations on the forms, like the Hodge operator $*$ or the exterior derivative $\mathbf{d}$, as geometric operators on the simplices, like the Voronoi dual operator $\star$ or the boundary operator ∂ (see [Hir03]). This results for example to a discrete definition of $\boldsymbol{\Delta}^{\mathrm{RR}}$ for a discrete 1-form $\mathbf{p}_h^\flat \in \Lambda_h^1(M)$ on a edge $e \in \mathcal{E}$

$$\boldsymbol{\Delta}_h^{\mathrm{RR}}\mathbf{p}_h^\flat(e) := \left(*\mathbf{d}*\mathbf{d}\mathbf{p}_h^\flat\right)(e) = -\frac{|e|}{|\star e|}\left(\mathbf{d}*\mathbf{d}\mathbf{p}_h^\flat\right)(\star e) \tag{31}$$

$$= -\frac{|e|}{|\star e|}\left(*\mathbf{d}\mathbf{p}_h^\flat\right)(\partial \star e) = -\frac{|e|}{|\star e|}\sum_{f \succ e} s_{f,e}\left(*\mathbf{d}\mathbf{p}_h^\flat\right)(\star f) \tag{32}$$

$$= -\frac{|e|}{|\star e|}\sum_{f \succ e}\frac{s_{f,e}}{|f|}\left(\mathbf{d}\mathbf{p}_h^\flat\right)(f) = -\frac{|e|}{|\star e|}\sum_{f \succ e}\frac{s_{f,e}}{|f|}\mathbf{p}_h^\flat(\partial f) \tag{33}$$

$$= -\frac{|e|}{|\star e|}\sum_{f \succ e}\frac{s_{f,e}}{|f|}\sum_{\tilde{e} \prec f} s_{f,\tilde{e}}\mathbf{p}_h^\flat(\tilde{e}) \tag{34}$$

or for $\boldsymbol{\Delta}^{\mathrm{GD}}$

$$\boldsymbol{\Delta}_h^{\mathrm{GD}}\mathbf{p}_h^\flat(e) := \left(\mathbf{d}*\mathbf{d}*\mathbf{p}_h^\flat\right)(e) = \left(*\mathbf{d}*\mathbf{p}_h^\flat\right)(\partial e) \tag{35}$$

$$= \sum_{v \prec e} s_{v,e}\left(*\mathbf{d}*\mathbf{p}_h^\flat\right)(v) = \sum_{v \prec e}\frac{s_{v,e}}{|\star v|}\left(\mathbf{d}*\mathbf{p}_h^\flat\right)(\star v) \tag{36}$$

$$= \sum_{v \prec e}\frac{s_{v,e}}{|\star v|}\left(*\mathbf{p}_h^\flat\right)(\partial \star v) = -\sum_{v \prec e}\frac{s_{v,e}}{|\star v|}\sum_{\tilde{e} \succ v} s_{v,\tilde{e}}\left(*\mathbf{p}_h^\flat\right)(\star \tilde{e}) \tag{37}$$

$$= -\sum_{v \prec e}\frac{s_{v,e}}{|\star v|}\sum_{\tilde{e} \succ v} s_{v,\tilde{e}}\frac{|\star \tilde{e}|}{|\tilde{e}|}\mathbf{p}_h^\flat(\tilde{e}) \tag{38}$$

where the sign $s_{f,e}$ is $+1$ if the face $f \succ e$ is in the left of the edge e (-1 otherwise) and $s_{v,e}$ is $+1$ if the edge $e \succ v$ points to the vertex v and -1 if e points away from v (see figure 1). We see $\boldsymbol{\Delta}_h^{\mathrm{RR}}$, $\boldsymbol{\Delta}_h^{\mathrm{GD}}$ and also $\boldsymbol{\Delta}_h^{\mathrm{dR}} := -\boldsymbol{\Delta}_h^{\mathrm{RR}} - \boldsymbol{\Delta}_h^{\mathrm{GD}}$ are linear operators in $\mathbf{p}_h^\flat(\tilde{e})$ and therefor results in sparse matrices if the $\mathbf{p}_h^\flat(\tilde{e})$ are our degree of freedoms.

4.4 Discrete norm

Approximating the norm $\left\|\mathbf{p}^\flat\right\|$ on a edge $e \in \mathcal{E}$ is not so easy like the development of discrete linear operators. We only know how $\mathbf{p}_h^\flat$ "act" on a single edge, so $\mathbf{p}_h^\flat$ gives us only one dimensional informations. In other words, we only know the proportion $\mathbf{p}_h \cdot \mathbf{e} = \mathbf{p}_h^\flat(e)$ of the discrete contravariant vector field $\mathbf{p}_h = \left(\mathbf{p}_h^\flat\right)^\sharp$ in the $\mathbf{e}$ direction, but we don't know the length of $\mathbf{p}_h$ defined on this edge.
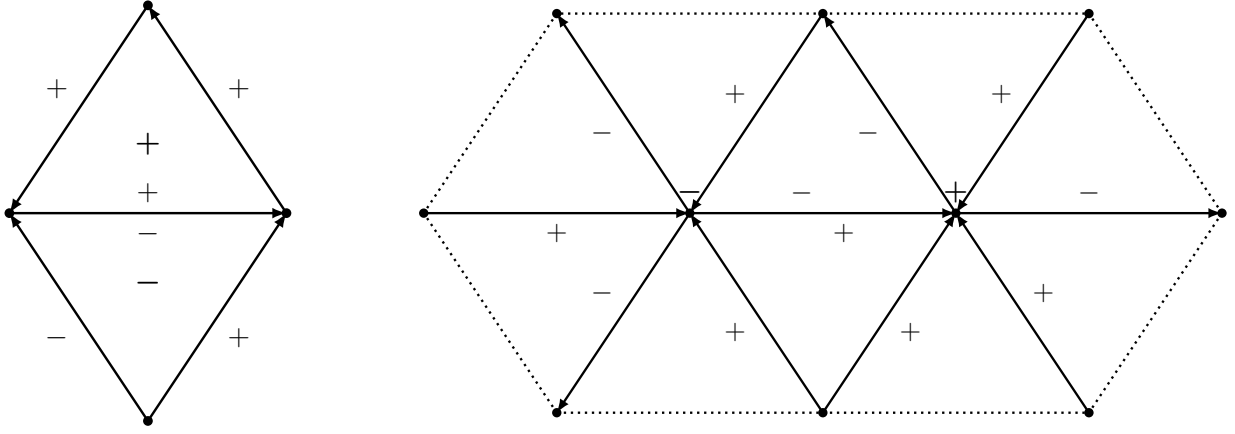
Figure 1: Signs s on example mesh extracts, which affected the discretization of the discrete Laplace operators. Left ($\boldsymbol{\Delta}_h^{\mathrm{RR}}$): The edge of interest e is the middle edge. The bold signs in the middle of the faces indicate $s_{f,e}$. The other signs at the local edges $\tilde{e}$ (regarding the two faces f) indicate $s_{f,\tilde{e}}$. Right ($\boldsymbol{\Delta}_h^{\mathrm{GD}}$): The edge of interest e is also the middle edge. The bold signs above the two inner vertices v indicate $s_{v,e}$. The other signs at the local edges $\tilde{e}$ (regarding the two vertices v) indicate $s_{v,\tilde{e}}$.

...maybe some averaging techniques, which I tried and why these sucks...

One way out is to rise the dimension of the discrete 1-forms, therefor we introduce some bases at the intersection $c(e) = e \cap (\star e)$ of a edge and its dual edge to describe discrete contra- and covariant vector fields in a local (piecewise flat) coordinate system.

The basis for contravariant vectors at $c(e)$ is composed of

$$\partial_e X := \mathbf{e} = \mathbf{v}_2 - \mathbf{v}_1 \tag{39}$$

$$\partial_{\star e} X := *\mathbf{e} = c(f_2) - c(f_1) \tag{40}$$

if $e = [v_1, v_2]$ and the face f_1 lay right and f_2 left of e (...figure...). This definitions are consistent with the canonical basis, if the position X is a barycentric parametrisation of the edge e resp. its dual, i.e. for

$$X_e(t_e) = t_e \mathbf{v}_2 + (1 - t_e) \, \mathbf{v}_2 \text{ for } t \in [0, 1] \tag{41}$$

$$X_{\star e}(t_{\star e}) = \begin{cases} 2 t_{\star e} c(e) + (1 - 2 t_{\star e}) \, c(f_1) & \text{if } t_{\star e} \in \left[0, \frac{1}{2}\right] \\ (1 - t_{\star e}) \, c(e) + (2 t_{\star e} - 1) \, c(f_1) & \text{if } t_{\star e} \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{42}$$

holds $\partial_e X = \partial_{t_e} X_e$ and $\partial_{\star e} X = \frac{1}{2}\left(\partial_{t_{\star e}} X_{\star e}|_{[0,\frac{1}{2}]} + \partial_{t_{\star e}} X_{\star e}|_{[\frac{1}{2},1]}\right)$. Therefore we get the local metric tensor

$$\mathbf{g}_h(e) = |e|^2 \, (dx^e)^2 + |\star e|^2 \, (dx^{\star e})^2 \tag{43}$$

where $\{dx^e, dx^{\star e}\}$ are the dual base of $\{\partial_e X, \partial_{\star e} X\}$, i.e. $dx^i \, (\partial_j X) = \delta_j^i$ for $i, j \in \{e, \star e\}$. This gives us the great possibility to define

$$\underline{\mathbf{p}}_h^\flat(e) := \mathbf{p}_h^\flat(e) dx^e + \mathbf{p}_h^\flat(\star e) dx^{\star e} = \mathbf{p}_h^\flat(e) dx^e - \frac{|\star e|}{|e|} \left(*\mathbf{p}_h^\flat\right)(e) dx^{\star e} \tag{44}$$

$$= \mathbf{p}_h^\flat(e) \boldsymbol{\xi}^e + \left(*\mathbf{p}_h^\flat\right)(e) \boldsymbol{\xi}^{\star e} =: \begin{bmatrix} \mathbf{p}_h^\flat \\ *\mathbf{p}_h^\flat \end{bmatrix}_{\mathrm{PD}} (e) \tag{45}$$

where $\boldsymbol{\xi}^e := dx^e$ and $\boldsymbol{\xi}^{\star e} := -\frac{|\star e|}{|e|} dx^{\star e}$ are the contravariant Primal-Dual-basis (PD-basis), which we want to use. $\underline{\mathbf{p}}_h^\flat(e)$ is called the discrete Primal-Dual-1-form (PD-1-form). With this scaling and

the condition $\boldsymbol{\xi}_i(\boldsymbol{\xi}^j) = \delta_i^j$, it is also possible to define the covariant PD-basis as $\boldsymbol{\xi}_e := \partial_e X$ and $\boldsymbol{\xi}_{\star e} := -\frac{|e|}{|\star e|}\partial_{\star e}X$. Hence, we obtain for the discrete Primal-Dual-vector field (PD-vector field) $\underline{\mathbf{p}}_h$

$$\underline{\mathbf{p}}_h(e) := \left(\underline{\mathbf{p}}_h^\flat(e)\right)^\sharp = \frac{1}{|e|^2}\mathbf{p}_h^\flat(e)\partial_e X + \frac{1}{|\star e|^2}\mathbf{p}_h^\flat(\star e)\partial_{\star e}X \tag{46}$$

$$= \frac{1}{|e|^2}\mathbf{p}_h^\flat(e)\partial_e X - \frac{1}{|e|\,|\star e|}\left(*\mathbf{p}_h^\flat\right)(e)\partial_{\star e}X \tag{47}$$

$$= \frac{1}{|e|^2}\left[\mathbf{p}_h^\flat(e)\boldsymbol{\xi}_e + \left(*\mathbf{p}_h^\flat\right)(e)\boldsymbol{\xi}_{\star e}\right] =: \begin{bmatrix}\mathbf{p}_h^\flat \\ *\mathbf{p}_h^\flat\end{bmatrix}^{\mathrm{PD}}(e) \tag{48}$$

Hence, we get for the square of the norm on $c(e)$ (and also with a constant interpolation on the whole edge) by contract the PD-1-Form with its corresponding PD-Vector

$$\left\|\underline{\mathbf{p}}_h^\flat\right\|_h^2(e) := \left(\underline{\mathbf{p}}_h^\flat\left(\underline{\mathbf{p}}_h\right)\right)(e) = \frac{1}{|e|^2}\left(\left[\mathbf{p}_h^\flat(e)\right]^2 + \left[\left(*\mathbf{p}_h^\flat\right)(e)\right]^2\right) \tag{49}$$

In general, we also get a local discrete inner product between $\underline{\mathbf{p}}_h^\flat$ and another PD-1-form $\underline{\mathbf{q}}_h^\flat$ by

$$\left\langle \underline{\mathbf{q}}_h^\flat, \underline{\mathbf{p}}_h^\flat\right\rangle_h(e) := \left(\underline{\mathbf{q}}_h^\flat\left(\underline{\mathbf{p}}_h\right)\right)(e) = \frac{1}{|e|^2}\left(\left[\mathbf{p}_h^\flat\mathbf{q}_h^\flat\right](e) + \left[\left(*\mathbf{p}_h^\flat\right)\left(*\mathbf{q}_h^\flat\right)\right](e)\right) \tag{50}$$

or a outer product between the PD-1-form $\underline{\mathbf{p}}_h^\flat$ and the PD-vector $\underline{\mathbf{q}}_h$ to produce the PD-(1,1)-Tensor

$$\left(\underline{\mathbf{p}}_h^\flat \otimes \underline{\mathbf{q}}_h\right) := \frac{1}{|e|^2}\left(\mathbf{p}_h^\flat\mathbf{q}_h^\flat\boldsymbol{\xi}^e \otimes \boldsymbol{\xi}_e + \mathbf{p}_h^\flat(*\mathbf{q}_h^\flat)\boldsymbol{\xi}^e \otimes \boldsymbol{\xi}_{\star e} + (*\mathbf{p}_h^\flat)\mathbf{q}_h^\flat\boldsymbol{\xi}^{\star e} \otimes \boldsymbol{\xi}_e + (*\mathbf{p}_h^\flat)(*\mathbf{q}_h^\flat)\boldsymbol{\xi}^{\star e} \otimes \boldsymbol{\xi}_{\star e}\right) \tag{51}$$

$$=: \frac{1}{|e|^2}\begin{bmatrix}\mathbf{p}_h^\flat\mathbf{q}_h^\flat & \mathbf{p}_h^\flat(*\mathbf{q}_h^\flat) \\ (*\mathbf{p}_h^\flat)\mathbf{q}_h^\flat & (*\mathbf{p}_h^\flat)(*\mathbf{q}_h^\flat)\end{bmatrix}_P^D \tag{52}$$

(Arguments (e) are omitted)

5 Discrete PD-Problem, a DEC approach

To use a PD-1-form solution, we must also determine the dual part $*\mathbf{p}_h^\flat$, therefore we discretize the Hodge dual equation (28) (resp. (29)) simultaneous to the primal equation (26) (resp. (27)). This leads to the DEC-discretized Primal-Dual-problem (DEC-PD-problem)

$$\left[\partial_t + K_0\boldsymbol{\Delta}_h^{\mathrm{dR}} + K_n\left(\left\|\underline{\mathbf{p}}_h^\flat\right\|_h^2 - 1\right)\right]\underline{\mathbf{p}}_h^\flat = 0 \tag{53}$$

respective, without the One-Constant-Approximation,

$$\left[\partial_t - \begin{bmatrix}K_1 \\ K_3\end{bmatrix}\boldsymbol{\Delta}_h^{\mathrm{GD}} - \begin{bmatrix}K_3 \\ K_1\end{bmatrix}\boldsymbol{\Delta}_h^{\mathrm{RR}} + K_n\left(\left\|\underline{\mathbf{p}}_h^\flat\right\|_h^2 - 1\right)\right]\underline{\mathbf{p}}_h^\flat = 0 \tag{54}$$

5.1 Time discretization

The simplest way to discretize the DEC-PD-problem in time is to use a implicit Euler scheme, where we handle the norm $\left\|\underline{\mathbf{p}}_h^\flat\right\|_h^2$ explicit. For one Euler step, we have to solve

$$\left[\frac{1}{\tau} + K_0\boldsymbol{\Delta}_h^{\mathrm{dR}} + K_n\left(\left\|\widehat{\underline{\mathbf{p}}}_h^\flat\right\|_h^2(e) - 1\right)\right]\underline{\mathbf{p}}_h^\flat(e) = \frac{1}{\tau}\widehat{\underline{\mathbf{p}}}_h^\flat(e) \tag{55}$$

resp.

$$\left[\frac{1}{\tau} - \begin{bmatrix}K_1 \\ K_3\end{bmatrix}\boldsymbol{\Delta}_h^{\mathrm{GD}} - \begin{bmatrix}K_3 \\ K_1\end{bmatrix}\boldsymbol{\Delta}_h^{\mathrm{RR}} + K_n\left(\left\|\widehat{\underline{\mathbf{p}}}_h^\flat\right\|_h^2(e) - 1\right)\right]\underline{\mathbf{p}}_h^\flat(e) = \frac{1}{\tau}\widehat{\underline{\mathbf{p}}}_h^\flat(e) \tag{56}$$

for all $e \in \mathcal{E}$. $\widehat{\underline{\mathbf{p}}}_h^\flat$ is the solution of the last time step or the initial condition, if this is the first Euler step. $\tau = t - \hat{t}$ is the time step wide.

The drawback of this semi-implicit Euler scheme is, that we need very small τ. Therefore, it is better to use a Taylor-linearisation for $\left\| \underline{\mathbf{p}}_h^\flat \right\|_h^2 \underline{\mathbf{p}}_h^\flat$. First we calculate the partially component derivative on a edge $e \in \mathcal{E}$ of

$$\Phi \left(\underline{\mathbf{p}}_h^\flat \right) = \Phi \left(\mathbf{p}_h^\flat, *\mathbf{p}_h^\flat \right) := \left\| \underline{\mathbf{p}}_h^\flat \right\|_h^2 \underline{\mathbf{p}}_h^\flat = \frac{1}{|e|^2} \left(\left(\mathbf{p}_h^\flat \right)^2 + \left(*\mathbf{p}_h^\flat \right)^2 \right) \begin{bmatrix} \mathbf{p}_h^\flat \\ *\mathbf{p}_h^\flat \end{bmatrix} \tag{57}$$

(Henceforward, for a better readability, the arguments e are omitted). Hence, we obtain

$$\partial_{\mathbf{p}_h^\flat} \Phi = \frac{1}{|e|^2} \begin{bmatrix} 3 \left(\mathbf{p}_h^\flat \right)^2 + \left(*\mathbf{p}_h^\flat \right)^2 \\ 2\mathbf{p}_h^\flat \left(*\mathbf{p}_h^\flat \right) \end{bmatrix} \tag{58}$$

$$\partial_{*\mathbf{p}_h^\flat} \Phi = \frac{1}{|e|^2} \begin{bmatrix} 2\mathbf{p}_h^\flat \left(*\mathbf{p}_h^\flat \right) \\ \left(\mathbf{p}_h^\flat \right)^2 + 3 \left(*\mathbf{p}_h^\flat \right)^2 \end{bmatrix} \tag{59}$$

For one step Taylor at $\widehat{\mathbf{p}}_h^\flat$, we get

$$\Phi(\underline{\mathbf{p}}_h^\flat) \approx \Phi(\widehat{\underline{\mathbf{p}}}_h^\flat) + \left(\mathbf{p}_h^\flat - \widehat{\mathbf{p}}_h^\flat \right) \partial_{\mathbf{p}_h^\flat} \Phi \left(\widehat{\underline{\mathbf{p}}}_h^\flat \right) + \left(*\mathbf{p}_h^\flat - *\widehat{\mathbf{p}}_h^\flat \right) \partial_{*\mathbf{p}_h^\flat} \Phi \left(\widehat{\underline{\mathbf{p}}}_h^\flat \right) \tag{60}$$

$$= \left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 \widehat{\underline{\mathbf{p}}}_h^\flat + \frac{1}{|e|^2} \begin{bmatrix} \left(\mathbf{p}_h^\flat - \widehat{\mathbf{p}}_h^\flat \right) \left(3 \left(\mathbf{p}_h^\flat \right)^2 + \left(*\mathbf{p}_h^\flat \right)^2 \right) + 2 \left(*\mathbf{p}_h^\flat - *\widehat{\mathbf{p}}_h^\flat \right) \mathbf{p}_h^\flat \left(*\mathbf{p}_h^\flat \right) \\ 2 \left(\mathbf{p}_h^\flat - \widehat{\mathbf{p}}_h^\flat \right) \mathbf{p}_h^\flat \left(*\mathbf{p}_h^\flat \right) + \left(*\mathbf{p}_h^\flat - *\widehat{\mathbf{p}}_h^\flat \right) \left(3 \left(\mathbf{p}_h^\flat \right)^2 + \left(*\mathbf{p}_h^\flat \right)^2 \right) \end{bmatrix} \tag{61}$$

$$= \left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 \widehat{\underline{\mathbf{p}}}_h^\flat - \frac{3}{|e|^2} \left(\left(\widehat{\mathbf{p}}_h^\flat \right)^2 + \left(*\widehat{\mathbf{p}}_h^\flat \right)^2 \right) \widehat{\underline{\mathbf{p}}}_h^\flat + \frac{2}{|e|^2} \begin{bmatrix} \left(*\widehat{\mathbf{p}}_h^\flat \right)^2 \\ \left(\widehat{\mathbf{p}}_h^\flat \right)^2 \end{bmatrix} \widehat{\underline{\mathbf{p}}}_h^\flat \tag{62}$$

$$+ \frac{1}{|e|^2} \left(\left(\widehat{\mathbf{p}}_h^\flat \right)^2 + \left(*\widehat{\mathbf{p}}_h^\flat \right)^2 \right) \underline{\mathbf{p}}_h^\flat + \frac{2}{|e|^2} \begin{bmatrix} \mathbf{p}_h^\flat \widehat{\mathbf{p}}_h^\flat \\ \left(*\mathbf{p}_h^\flat \right) \left(*\widehat{\mathbf{p}}_h^\flat \right) \end{bmatrix} \widehat{\underline{\mathbf{p}}}_h^\flat$$

$$+ \frac{2}{|e|^2} \begin{bmatrix} \left(*\mathbf{p}_h^\flat - *\widehat{\mathbf{p}}_h^\flat \right) \left(*\mathbf{p}_h^\flat \right) \\ \left(\mathbf{p}_h^\flat - \widehat{\mathbf{p}}_h^\flat \right) \mathbf{p}_h^\flat \end{bmatrix} \widehat{\underline{\mathbf{p}}}_h^\flat$$

$$= -2 \left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 \widehat{\underline{\mathbf{p}}}_h^\flat + \left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 \underline{\mathbf{p}}_h^\flat + 2 \left\langle \underline{\mathbf{p}}_h^\flat, \widehat{\underline{\mathbf{p}}}_h^\flat \right\rangle_h \widehat{\underline{\mathbf{p}}}_h^\flat \tag{63}$$

The inner product term can be also expressed as matrix-vector multiplication:

$$\left\langle \underline{\mathbf{p}}_h^\flat, \widehat{\underline{\mathbf{p}}}_h^\flat \right\rangle_h \widehat{\underline{\mathbf{p}}}_h^\flat = \frac{1}{|e|^2} \begin{bmatrix} \left(\widehat{\mathbf{p}}_h^\flat \right)^2 & \widehat{\mathbf{p}}_h^\flat \left(*\widehat{\mathbf{p}}_h^\flat \right) \\ \widehat{\mathbf{p}}_h^\flat \left(*\widehat{\mathbf{p}}_h^\flat \right) & \left(*\widehat{\mathbf{p}}_h^\flat \right)^2 \end{bmatrix}_P^D \cdot \underline{\mathbf{p}}_h^\flat = \left(\widehat{\underline{\mathbf{p}}}_h^\flat \otimes \underline{\widehat{\mathbf{p}}}_h^\flat \right) \cdot \underline{\mathbf{p}}_h^\flat \tag{64}$$

With the Taylor linearization and the implicit Euler scheme, we have to solve the following PD-DEC-Problem in every time steps and all $e \in \mathcal{E}$

$$\left[\frac{1}{\tau} + K_0 \boldsymbol{\Delta}_h^{\mathrm{dR}} + K_n \left(\left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 - 1 \right) \right] \underline{\mathbf{p}}_h^\flat + 2K_n \left(\widehat{\underline{\mathbf{p}}}_h^\flat \otimes \widehat{\underline{\mathbf{p}}}_h^\flat \right) \cdot \underline{\mathbf{p}}_h^\flat = \left[\frac{1}{\tau} + 2K_n \left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 \right] \widehat{\underline{\mathbf{p}}}_h^\flat \tag{65}$$

resp.

$$\left[\frac{1}{\tau} - \begin{bmatrix} K_1 \\ K_3 \end{bmatrix} \boldsymbol{\Delta}_h^{\mathrm{GD}} - \begin{bmatrix} K_3 \\ K_1 \end{bmatrix} \boldsymbol{\Delta}_h^{\mathrm{RR}} + K_n \left(\left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 - 1 \right) \right] \underline{\mathbf{p}}_h^\flat + 2K_n \left(\widehat{\underline{\mathbf{p}}}_h^\flat \otimes \widehat{\underline{\mathbf{p}}}_h^\flat \right) \cdot \underline{\mathbf{p}}_h^\flat = \left[\frac{1}{\tau} + 2K_n \left\| \widehat{\underline{\mathbf{p}}}_h^\flat \right\|_h^2 \right] \widehat{\underline{\mathbf{p}}}_h^\flat \tag{66}$$

6 Notes on Implementation

6.1 Edge mesh

6.1.1 Iterators

6.2 Matrix assembling

6.3 Sharp and flat interpolations

7 Experiments

7.1 Nonic Surface

We are starting with the standard parametrization ($\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$)

$$X_{\mathbb{S}}(\theta, \phi) = \sin\theta\cos\phi\boldsymbol{\xi}^x + \sin\theta\sin\phi\boldsymbol{\xi}^y + \cos\theta\boldsymbol{\xi}^z \tag{67}$$

of the unit sphere and make a displacement in the $\boldsymbol{\xi}^x$-direction depending on the z-position:

$$X(\theta, \phi) := X_{\mathbb{S}}(\theta, \phi) + f(\cos\theta)\boldsymbol{\xi}^x \tag{68}$$

This results also in an implicit description of this surface in standard Euclidean (x, y, z)-coordinates:

$$\varphi(x, y, z) := (x - f(z))^2 + y^2 + z^2 - 1 = 0 \tag{69}$$

We choose for the displacement function $f : [-1, 1] \to \mathbb{R}$ a double well function, so that the north pole ($z = 1$) of the initial sphere is shifting right in $\boldsymbol{\xi}^x$-direction by $C > 0$ and the south pole ($z = -1$) by $r \cdot C$ with the proportion factor $0 < r < 1$. This implies

$$f_{C,r}(z) = \frac{1}{4}Cz^2 \left[(z+1)^2(4-3z) + r(z-1)^2(4+3z) \right] \tag{70}$$

and the double well conditions $f(1) = C$, $f(-1) = rC$ and $f'(1) = f'(0) = f'(-1) = 0$ are fulfilled. We call this subsequent manifold a nonic surface, because the left-hand side of the resulting implicit description (69) is a polynomial of degree 10. The reason why we don't choose $r = 1$ to get a less complex surface is to break the symmetry in the x-y-plane, this is necessary for preventing meta stable states in the dynamic of the orientation fields.

An simple choice for an initial condition is

$$\mathbf{p}_0^\flat = \frac{\mathbf{d}x}{\|\mathbf{d}x\|} \tag{71}$$

because we can discretize $\mathbf{d}x$ on an edge $e = [v_1, v_2]$ by (stokes theorem)

$$(\mathbf{d}x)_h (e) = (v_2 - v_1) \cdot \boldsymbol{\xi}^x = v_2^x - v_1^x \tag{72}$$

independent in the parametrization of the surface. Hence, after normalizing, there are three defects with a positive charge on the three bulges and one defect with a negative charge on the saddle point. (That are the 4 positions, where the tangential space is orthogonal to $\boldsymbol{\xi}^x$.) If needed, by rising the indices we get the surface gradient of the x-component

$$(\mathbf{d}x)^\sharp = \nabla_M x = \left[g_{C,r}^{ij} \partial_j x(\theta, \phi) \right]^{i=\theta,\phi} = \left(I - \frac{\nabla\varphi_{C,r}}{\|\nabla\varphi_{C,r}\|} \otimes \frac{\nabla\varphi_{C,r}}{\|\nabla\varphi_{C,r}\|} \right) \boldsymbol{\xi}^x \tag{73}$$

For the normalizing procedure, it is also the Hodge dual initial solution to discretized (the rotation of the x-component). If the face $f_1 \succ e$ is right of the edge e and $f_2 \succ e$ located left, so that $\star e = [c(f_1), c(f_2)]$ is the dual edge, than we can approximate

$$(\ast\mathbf{d}x)_h (e) = -\frac{|e|}{|\star e|} (\mathbf{d}x) (\star e) = -\frac{|e|}{|\star e|} \left([c(f_2)]^x - [c(f_1)]^x \right) \tag{74}$$
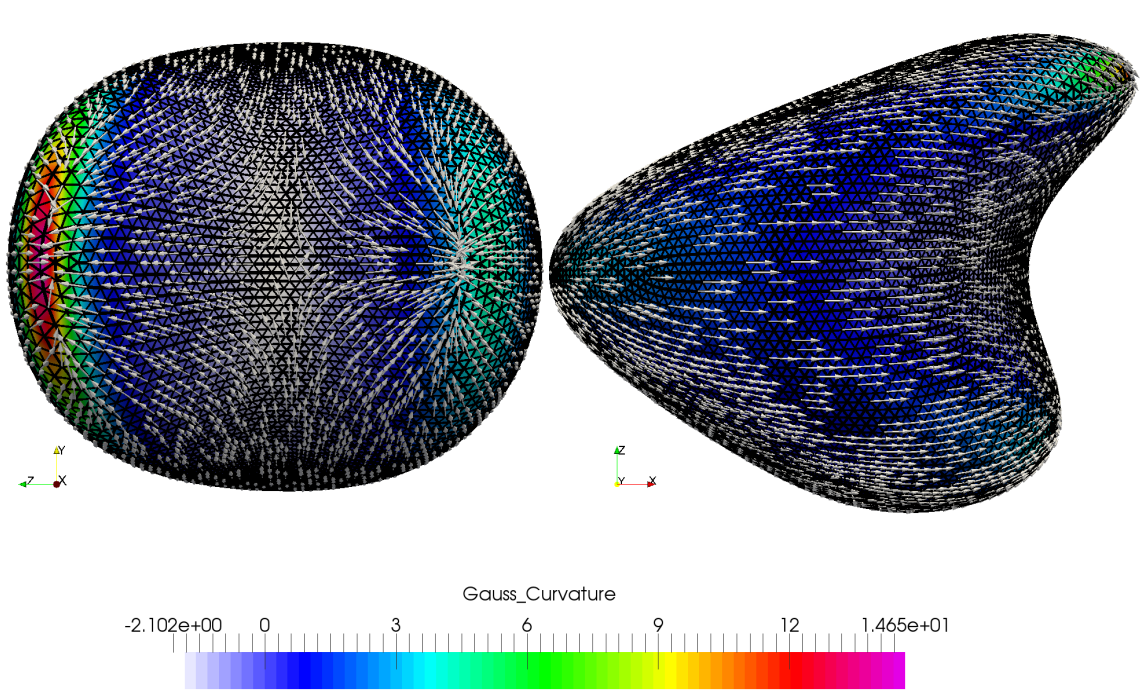
Figure 2: Nonic Surface with shift factor $C = 1$ and proportion factor $r = 0.5$. The mesh is well-centered. Plotted is the Gauss curvature. The arrows show the covariant initial solution $\mathbf{p}_0$ on some vertices.

With the discrete norm (49) of the PD-1-form, we obtain the initial PD-1-form on $e \in \mathcal{E}$ by

$$
\underline{\mathbf{p}}_{h,0}^{\flat}(e) = \frac{\left[\begin{array}{c} v_2^x - v_1^x \\ -\frac{|e|}{|\star e|} \left([c(f_2)]^x - [c(f_1)]^x \right) \end{array} \right]_{PD}}{\sqrt{\frac{1}{|e|^2} \left(v_2^x - v_1^x \right)^2 + \frac{1}{|\star e|^2} \left([c(f_2)]^x - [c(f_1)]^x \right)^2}}
\tag{75}
$$

See figure 2.

8 Appendix

8.1 Nonic Surface

References

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[Whi57] H. Whitney. *Geometric Integration Theory*. Princeton mathematical series. University Press, 1957.