Chapter Three

TRANSPORTATION MODEL

Transportation Problems

- The Transportation Model
- Solution of a Transportation Problem

Transportation Problems Overview

- ✔ Part of a larger class of linear programming problems known as network flow models.
- ✔ Possess special mathematical features that enabled development of very efficient, unique solution methods.
- ✔ Methods are variations of traditional simplex procedure.

The Transportation Model
Characteristics

- ✔ A product is transported from a number of sources to a number of destinations at the minimum possible cost.
- ✔ Each source is able to supply a fixed number of units of the product, and each destination has a fixed demand for the product.
- ✔ The linear programming model has constraints for supply at each source and demand at each destination.
- ✔ All constraints are equalities in a balanced transportation model where supply equals demand.
- ✔ Constraints contain inequalities in unbalanced models where supply does not equal demand.

Transportation Model Example, Problem Definition and Data

Problem: How many tons of wheat to transport from each grain elevator to each mill on a monthly basis in order to minimize the total cost of transportation ?

Data:

Grain Elevator	Supply	Mill	Demand
1. Asala	150	A. Mekelle	200
2. Desie	175	B. Nekempte	100
3. Hawassa	275	C. Harar	300
Total	600 tons	Total	600 tons

	Transport cost from Grain Elevator to Mill ($/ton)		
Grain Elevator	A. Mekelle	B. Nekempte	C. Harar
1. Asala	$6	8	10
2. Desie	7	11	11
3. Hawassa	4	5	12

Transportation Model Example - Model Formulation

$$\text{Minimize } Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$$

subject to

$$x_{1A} + x_{1B} + x_{1C} = 150$$
$$x_{2A} + x_{2B} + x_{2C} = 175$$
$$x_{3A} + x_{3B} + x_{3C} = 275$$
$$x_{1A} + x_{2A} + x_{3A} = 200$$
$$x_{1B} + x_{2B} + x_{3B} = 100$$
$$x_{1C} + x_{2C} + x_{3C} = 300$$
$$xij \geq 0$$

where x_{ij} = tons of wheat from each grain elevator, i, i = 1, 2, 3, to each mill j, j = A,B,C

Solution of the Transportation Model
Tableau Format

- ✔ Transportation problems are solved manually within a *tableau* format.
- ✔ Each cell in a transportation tableau is analogous to a decision variable that indicates the amount allocated from a source to a destination.

The Transportation Tableau

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

Solution of the Transportation Model Solution Methods

- Transportation models do not start at the origin where all decision values are zero; they must instead be given an *initial feasible solution*.
- Initial feasible solution determination methods include:
  - northwest corner method
  - minimum cell cost method
  - Vogel's Approximation Method
- Methods for solving the transportation problem itself include:
  - stepping-stone method and
  - modified distribution method.

The Northwest Corner Method

In the northwest corner method the largest possible allocation is made to the cell in the upper left-hand corner of the tableau , followed by allocations to adjacent feasible cells.

The Initial NW Corner Solution

From \ To	A	B	C	Supply
1	[6] 150	[8]	[10]	150
2	[7] 50	[11] 100	[11] 25	175
3	[4]	[5]	[12] 275	275
Demand	200	100	300	600

- The initial solution is complete when all rim requirements are satisfied.

- Transportation cost is computed by evaluating the objective function:

$Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C} = 6(150) + 8(0) + 10(0) + 7(50) + 11(100) + 11(25) + 4(0) + 5(0) + !2(275) = \$5,925$

The Northwest Corner Method
Summary of Steps

1. Allocate as much as possible to the cell in the upper left-hand corner, subject to the supply and demand conditions.
2. Allocate as much as possible to the next adjacent feasible cell.
3. Repeat step 2 until all rim requirements are met.

The Minimum Cell Cost Method (1 of 3)

- In the minimum cell cost method as much as possible is allocated to the cell with the minimum cost followed by allocation to the feasible cell with minimum cost.

From \ To	A	B	C	Supply
1	(6)	(8)	(10)	150
2	(7)	(11)	(11)	175
3	(4) 200	(5)	(12)	275
Demand	200	100	300	600

The Initial Minimum Cell Cost Allocation

From \ To	A	B	C	Supply
1	(6)	(8)	(10)	150
2	(7)	(11)	(11)	175
3	(4) 200	(5) 75	(12)	275
Demand	200	100	300	600

The Second Minimum Cell Cost Allocation

The Minimum Cell Cost Method (2 of 3)

- The complete initial minimum cell cost solution; total cost = $4,550.

- The minimum cell cost method will provide a solution with a lower cost than the northwest corner solution because it considers cost in the allocation process.

The Initial Solution

From \ To	A	B	C	Supply
1	6	8 25	10 125	150
2	7	11	11 175	175
3	4 200	5 75	12	275
Demand	200	100	300	600

The Minimum Cell Cost Method
Summary of Steps
(3 of 3)

1. Allocate as much as possible to the feasible cell with the minimum transportation cost, and adjust the rim requirements.

2. Repeat step 1 until all rim requirements have been met.

Vogel's Approximation Method (VAM) (1 of 5)

Method is based on the concept of *penalty cost* or *regret*.

A penalty cost is the difference between the largest and the next largest cell cost in a row (or column).

In VAM the first step is to develop a penalty cost for each source and destination.

Penalty cost is calculated by subtracting the minimum cell cost from the next higher cell cost in each row and column.

To / From	A	B	C	Supply	
1	6	8	10	150	2
2	7	11	11	175	4
3	4	5	12	275	1
Demand	200	100	300	600	
	2	3	1		

The VAM Penalty Costs

Vogel's Approximation Method (VAM) (2 of 5)

the largest penalty cost.

The Initial VAM Allocation

From \ To	A	B	C	Supply	
1	6	8	10	150	2
2	7 175	11	11	175	
3	4	5	12	275	1
Demand	200	100	300	600	
	2	3	2		

Vogel's Approximation Method (VAM) (3 of 5)

From \ To	A	B	C	Supply	
1	6	8	10	150	4
2	7 175	11	11	175	
3	4	5 100	12	275	8
Demand	200	100	300	600	
	2		2		

The Second
M Allocation

Vogel's Approximation Method (VAM) (4 of 5)

The Third VAM Allocation

From \ To	A	B	C	Supply
1	6	8	10 150	150
2	7 175	11	11	175
3	4 25	5 100	12 150	275
Demand	200	100	300	600

Vogel's Approximation Method (VAM) (5 of 5)

VAM solution; total cost = $5,125

VAM and minimum cell cost methods both provide better initial solutions than does the northwest corner method.

The Initial VAM Solution

From \ To	A	B	C	Supply
1	[6]	[8]	[10] 150	150
2	[7] 175	[11]	[11]	175
3	[4] 25	[5] 100	[12] 150	275
Demand	200	100	300	600

Vogel's Approximation Method (VAM) Summary of Steps

1. Determine the penalty cost for each row and column.

2. Select the row or column with the highest penalty cost.

3. Allocate as much as possible to the feasible cell with the lowest transportation cost in the row or column with the highest penalty cost.

4. Repeat steps 1, 2, and 3 until all rim requirements have been met.

The Stepping-Stone Solution Method (1 of 12)

Once an initial solution is derived, the problem must be solved using either the stepping-stone method or the modified distribution method (MODI).

The initial solution used as a starting point in this problem is the minimum cell cost method solution because it had the minimum total cost of the three methods used.

From \ To	A	B	C	Supply
1	(6)	(8) 25	(10) 125	150
2	(7)	(11)	(11) 175	175
3	(4) 200	(5) 75	(12)	275
Demand	200	100	300	600

The Minimum Cell Cost Solution

The Stepping-Stone Solution Method (2 of 12)

The stepping-stone method determines if there is a cell with no allocation that would reduce cost if used.

From \ To	A	B	C	Supply	
1	6 +1	8 25	10 125	150	151
2	7	11	11 175	175	
3	4 200	5 75	12	275	
Demand	200	100	300	600	

The Allocation of One Ton to Cell 1A

The Stepping-Stone Solution Method (3 of 12)

From \ To	A	B	C	Supply
1	+1 [6]	−1 [8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100	300	600

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The Subtraction of One Ton from Cell 1B

The Stepping-Stone Solution Method (4 of 12)

- A requirement of this solution method is that units can only be added to and subtracted from cells that already have allocations, thus one ton must be added to a cell as shown.

The Addition of One Ton to Cell 3B and the Subtraction of One Ton from Cell 3A

From \ To	A	B	C	Supply
1	+1 (6)	−1 (8) 25	(10) 125	150
2	(7)	(11)	(11) 175	175
3	−1 (4) 200	+1 (5) 75	(12)	275
Demand	200	100	300	600

The Stepping-Stone Solution Method(5 of 12)

- An empty cell that will reduce cost is a potential entering variable.

- To evaluate the cost reduction potential of an empty cell, a closed path connecting used cells to the empty cells is identified.

The Stepping-Stone Path for Cell 2A

From \ To	A	B	C	Supply
1	6	− 8 25	+ 10 125	150
2	+ 7	11	− 11 175	175
3	− 4 200	+ 5 75	12	275
Demand	200	100	300	600

2A → 2C → 1C → 1B → 3B → 3A

$+ \$7 - 11 + 10 - 8 + 5 - 4 = -\$1$

The Stepping-Stone Solution Method (6 of 12)

The remaining stepping-stone paths and resulting computations for cells 2B and 3C.

From \ To	A	B	C	Supply
1	6	+ 8 25	− 10 125	150
2	7	− 11	+ 11 175	175
3	4 200	5 75	12	275
Demand	200	100	300	600

2B → 2C → 1C → 1B
+ $11 – 11 + 10 – 8 = +$2

The Stepping-Stone Path for Cell 2B

From \ To	A	B	C	Supply
1	6	+ 8 25	− 10 125	150
2	7	11	11 175	175
3	4 200	− 5 75	+ 12	275
Demand	200	100	300	600

3C → 1C → 1B → 3B
+ $12 – 10 + 8 – 5 = +$5

The Stepping-Stone Path for Cell 3C

The Stepping-Stone Solution Method (7 of 12)

- After all empty cells are evaluated, the one with the greatest cost reduction potential is the entering variable.

- A tie can be broken arbitrarily.

The Stepping-Stone Path for Cell 1A

From \ To	A	B	C	Supply
1	+ (6)	− (8) 25	(10) 125	150
2	(7)	(11)	(11) 175	175
3	− (4) 200	+ (5) 75	(12)	275
Demand	200	100	300	600

The Stepping-Stone Solution Method (8 of 12)

When reallocating units to the entering variable (cell), the amount is the minimum amount subtracted on the stepping-stone path.

At each iteration one variable enters and one leaves (just as in the simplex method).

From \ To	A	B	C	Supply
1	6 25	8	10 125	150
2	7	11	11 175	175
3	4 175	5 100	12	275
Demand	200	100	300	600

The Second Iteration of the Stepping-Stone Method

The Stepping-Stone Solution Method (9 of 12)

From \ To	A	B	C	Supply
1	− [6] 25	[8]	+ [10] 125	150
2	+ [7]	[11]	− [11] 175	175
3	[4] 175	[5] 100	[12]	275
Demand	200	100	300	600

2A → 2C → 1C → 1A
+ \$7 – 11 + 10 – 6 = \$0

The Stepping-Stone Path for Cell 2A

From \ To	A	B	C	Supply
1	− [6] 25	+ [8]	[10] 125	150
2	[7]	[11]	[11] 175	175
3	+ [4] 175	− [5] 100	[12]	275
Demand	200	100	300	600

1B → 3B → 3A → 1A
+ \$8 – 5 + 4 – 6 = +\$1

The Stepping-Stone Path for Cell 1B

The Stepping-Stone Solution Method(10 of 12)

- Continuing check for optimality.

From \ To	A	B	C	Supply
1	− [6] 25	[8] +	[10] 125	150
2	[7]	+ [11]	− [11] 175	175
3	+ [4] 175	− [5] 100	[12]	275
Demand	200	100	300	600

2B → 3B → 3A → 1A → 1C → 2C
+ \$11 – 5 + 4 – 6 + 10 – 11 = +\$3

The Stepping-Stone Path for Cell 2B

From \ To	A	B	C	Supply
1	+ [6] 25	[8]	− [10] 125	150
2	[7]	[11]	[11] 175	175
3	− [4] 175	[5] 100	+ [12]	275
Demand	200	100	300	600

3C → 3A → 1A → 1C
+ \$12 – 4 + 6 – 10 = +\$4

The Stepping-Stone Path for Cell 3C

The Stepping-Stone Solution Method (11 of 12)

- The stepping-stone process is repeated until none of the empty cells will reduce costs

  (i.e., an optimal solution).

- In example, evaluation of four paths indicates no cost reductions, therefore Table in first iterations solution is optimal.

- Solution and total minimum cost :

x_{1A} = 25 tons, x_{2C} = 175 tons, x_{3A} = 175 tons, x_{1C} = 125 tons, x_{3B} = 100 tons

$$Z = \$6(25) + 8(0) + 10(125) + 7(0) + 11(0) + 11(175) + 4(175) + 5(100) + 12(0) = \$4,525$$

The Stepping-Stone Solution Method (12 of 12)

A multiple optimal solution occurs when an empty cell has a cost change of zero and all other empty cells are positive. An alternate optimal solution is determined by allocating to the empty cell with a zero cost change. Alternate optimal total minimum cost also equals $4,525.

The Alternative Optimal Solution

From \ To	A	B	C	Supply
1	6	8	10 150	150
2	7 25	11	11 150	175
3	4 175	5 100	12	275
Demand	200	100	300	600

The Stepping-Stone Solution Method
Summary of Steps

1. Determine the stepping-stone paths and cost changes for each empty cell in the tableau.

2. Allocate as much as possible to the empty cell with the greatest net decrease in cost.

3. Repeat steps 1 and 2 until all empty cells have positive cost changes that indicate an optimal solution.

The Modified Distribution Method (MODI) (1 of 6)

MODI is a modified version of the stepping-stone method in which math equations replace the stepping-stone paths.

In the table, the extra left-hand column with the u_i symbols and the extra top row with the v_j symbols represent values that must be computed.

- Computed for all cells with allocations :

u_i

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	6	8 25	10 125	150
$u_2 =$	2	7	11	11 175	175
$u_3 =$	3	4 200	5 75	12	275
	Demand	200	100	300	600

The Minimum Cell Cost Initial Solution

The Modified Distribution Method (MODI)

- Formulas for ce

x_{1B}: $u_1 + v_B = 8$

x_{1C}: $u_1 + v_C = 10$

x_{2C}: $u_2 + v_C = 11$

x_{3A}: $u_3 + v_A = 4$

x_{3B}: $u_3 + v_B = 5$

	v_j	$v_A = 7$	$v_B = 8$	$v_C = 10$	
u_i	From \ To	A	B	C	Supply
$u_1 = 0$	1	6	8 25	10 125	150
$u_2 = 1$	2	7	11	11 175	175
$u_3 = -3$	3	4 200	5 75	12	275
	Demand	200	100	300	600

The Initial Solution with All ui and vj Values

- Five equations with 6 unknowns, therefore let $u_1 = 0$ and solve to obtain:

$v_B = 8$, $v_C = 10$, $u_2 = 1$, $u_3 = -3$, $v_A = 7$

The Modified Distribution Method (MODI) (3 of 6)

- Each MODI allocation replicates the stepping-stone allocation.

- Use following to evaluate all empty cells:

$$c_{ij} - u_i - v_j = e_{ij}$$

where k_{ij} equals the cost increase or decrease that would occur by allocating to a cell.

- For the empty cells in the above Table:

x_{1A}: $e_{1A} = c_{1A} - u_1 - v_A = 6 - 0 - 7 = -1$

x_{2A}: $e_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 7 = -1$

x_{2B}: $e_{2B} = c_{2B} - u_2 - v_B = 11 - 1 - 8 = +2$

x_{3C}: $e_{3C} = c_{3C} - u_3 - v_C = 12 - (-3) - 10 = +5$

The Modified Distribution Method (MODI) (4 of 6)

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	6 25	8	10 125	150
$u_2 =$	2	7	11	11 175	175
$u_3 =$	3	4 175	5 100	12	275
	Demand	200	100	300	600

The Second Iteration of the MODI Solution Method

The Modified Distribution Method (MODI) (5 of 6)

- Recomputing u_i and v_j values:

x_{1A}: $u_1 + v_A = 6$, $v_A = 6$ x_{1C}: $u_1 + v_C = 10$, $v_C = 10$ x_{2C}: $u_2 + v_C = 11$, $u_2 = 1$

x_{3A}: $u_3 + v_A = 4$, $u_3 = -2$ x_{3B}: $u_3 + v_B = 5$, $v_B = 7$

	v_j	$v_A = 6$	$v_B = 7$	$v_C = 10$	
u_i	From \ To	A	B	C	Supply
$u_1 = 0$	1	6 25	8	10 125	150
$u_2 = 1$	2	7	11	11 175	175
$u_3 = -2$	3	4 175	5 100	12	275
	Demand	200	100	300	600

The New ui and vj Values for the Second Iteration

The Modified Distribution Method (MODI) (6 of 6)

- Cost changes for the empty cells, $c_{ij} - u_i - v_j = e_{ij}$;

x_{1B}: $e_{1B} = c_{1B} - u_1 - v_B = 8 - 0 - 7 = +1$

x_{2A}: $e_{2A} = c_{2A} - u_2 - v_A = 7 - 1 - 6 = 0$

x_{2B}: $e_{2B} = c_{2B} - u_2 - v_B = 11 - 1 -7 = +3$

x_{3C}: $e_{2B} = c_{2B} - u_3 - v_C = 12 - (-2) - 10 = +4$

- Since none of the values are negative, solution obtained is optimal.

- Cell 2A with a zero cost change indicates a multiple optimal solution.

The Modified Distribution Method (MODI)
Summary of Steps

1. Develop an initial solution.
2. Compute the u_i and v_j values for each row and column.
3. Compute the cost change, e_{ij}, for each empty cell.
4. Allocate as much as possible to the empty cell that will result in the greatest net decrease in cost (most negative e_{ij})
5. Repeat steps 2 through 4 until all e_{ij} values are positive or zero.

The Unbalanced Transportation Model (1 of 2)

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Dummy	0	0	0	50
Demand	200	100	350	650

An Unbalanced Model
(Demand . Supply)

The Unbalanced Transportation Model(2 of 2)

- When supply exceeds demand, a dummy column is added to the tableau.
- The dummy column (or dummy row) has no effect on the initial solution methods or the optimal solution methods.

From \ To	A	B	C	Dummy	Supply
1	6	8	10	0	150
2	7	11	11	0	175
3	4	5	12	0	375
Demand	200	100	300	100	700

An Unbalanced Model (Supply . Demand)

Degeneracy
(1 of 3)

- In a transportation tableau with m rows and n columns, there must be $m + n - 1$ cells with allocations; if not, it is *degenerate*.

- The tableau in the figure does not meet the condition since 3 + 3 -1 = 5 cells and there are only 4 cells with allocations.

To / From	A	B	C	Supply
1	6	8 100	10 50	150
2	7	11	11 250	250
3	4 200	5	12	200
Demand	200	100	300	600

The Minimum Cell Cos

Degeneracy (2 of 3)

- In a degenerate tableau, all the stone paths or MODI equations cannot be developed.
- To rectify a degenerate tableau, an empty cell must artificially be treated as an occupied cell.

Cont’d

- Degeneracy may happen at two stages;
  - **When obtaining an initial basic feasible solution**
  - **At any stage while moving towards optimal solution.**

Cont'd

- To resolve degeneracy at the initial solution, we proceed by allocating very small quantity close to zero to one or more unoccupied cells so as to get m+n-1 number of occupied cells.
- In minimization transportation problems, it is better to allocate Δ to unoccupied cells that have the lowest transportation costs whereas in maximization problems it should be allocated to a cell that has a high pay off value.

Example

To / From	A	B	C	Supply
1	6 0	8 100	10 50	150
2	7	11	11 250	250
3	4 200	5	12	200
Demand	200	100	300	600

Degeneracy
(3 of 3)

- The stepping-stone path s and cost changes for this tableau:

2A 2C 1C 1A

x_{2A}: 7 - 11 + 10 - 6 = 0

2B 2C 1C 1B

x_{2B}: 11 - 11 + 10 - 8 = -

3B 1B 1A 3A

x_{3B}: 5 - 8 + 6 - 4 = - 1

3C 1C 1A

x_{3C}: 12 - 10 + 6 - 4 = +

From \ To	A	B	C	Supply
1	(6) 100	(8)	(10) 50	150
2	(7)	(11)	(11) 250	250
3	(4) 100	(5) 100	(12)	200
Demand	200	100	300	600

The Second Stepping-Stone Iteration

Cont’d

- To resolve degeneracy which occurs during optimality test, the quantity can be allocated to one or more cells which have become unoccupied recently to have m+n-1 number of occupied cells in the new solution

Prohibited Routes

- A prohibited route is assigned a large cost such as M.
- When the prohibited cell is evaluated, it will always contain the cost M, which will keep it from being selected as an entering variable.

Multiple optimal solution

- Transportation problems may have multiple optimal solutions.
- This can be useful for the managers since it gives the manager an option of bringing non quantitative considerations in to account.
- The existence of an alternate solution is evidenced by an empty cell evaluation of zero.

Maximization problems

- Some transportation type problems concern profits or revenues rather than costs.
- In such cases, the objective is to maximize rather than to minimize.
- Such problems can be handled by adding one additional step at the start: identify the cell with the largest profit and subtract all the other cell profits from the value.

Cont'd

- Then replace the cell profits with the resulting values.
- These values represent the opportunity costs that would be incurred by using routes with unit profits that are less than the largest unit profit.
- Replace the original unit profits by the newly calculate opportunity costs and solve in the usual way for the minimum opportunity cost solution.

Exercise

Taking the following data, obtain an optimal solution by MODI method

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	10	8	70	20	18
Demand	5	8	7	14	34