This homework is based on the thermodynamics section of the lecture notes and the calculations done therein. I am sure that atleast some of you may have just glossed over these sections without fully absorbing them. They are admittedly difficult, but unfortunately, they happen to be extremely useful. It is therefore important that you get your hands dirty and really learn how to do these types of calculations. There is only one problem in this assignment, and it is essentially going to ask you to reproduce the steps of the lecture notes with some extra twists. Good luck!

Remember that $\frac{\partial}{\partial x}F(x,y)$ is the same as the usual differentiation with respect to x, but now you must treat y (and everything that isn't x) as though it were a constant.

$\log(x)$ here means the natural log ($\ln(x)$).

Problem 1: Blackbody Fluctuations (15 points + 2 Bonus points)
Following the derivation in class, it was concluded that a blackbody may be thought of as exchanging energy with the thermal reservoir in discrete quanta. The average energy of each quantum was then computed using the partition function. In this problem, you will calculate the fluctuation in the energy of each quantum (photon) and then for the entire blackbody. It was argued that the probability that a photon with frequency ω has energy $\epsilon(n) = n\hbar\omega$ was given by the Boltzmann rule:

$$P(n) = \frac{e^{-\frac{n\hbar\omega}{\tau}}}{\sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{\tau}}} = \frac{e^{-\frac{n\hbar\omega}{\tau}}}{Z}$$

$$Z = \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{\tau}}$$

Let's define a new variable $\beta = \frac{1}{\tau}$, and the notation $\langle x_n \rangle$ to mean the average of the function $x(n)$ with respect to the distribution $P(n)$ above, i.e.:

$$\langle x_n \rangle \equiv \sum_{n=0}^{\infty} x(n)P(n) = \frac{\sum_{n=0}^{\infty} x(n)e^{-\beta n\hbar\omega}}{\sum_{n=0}^{\infty} e^{-\beta n\hbar\omega}}$$

It is to be remembered that $\langle x_n \rangle$ is actually independent of n since during the averaging process, n has been summed over.

(a) (**2 points**) With these new definitions, prove that the average energy per photon with frequency ω ($\varepsilon(\omega, \tau) \equiv \langle \epsilon_n \rangle$) is given by

$$\varepsilon(\omega, \tau) = -\frac{\partial}{\partial \beta} \log(Z)$$

Solution:

$$\begin{aligned}
-\frac{\partial}{\partial \beta} \log(Z) &= -\frac{1}{Z}\frac{\partial}{\partial \beta} Z \\
&= -\left[\sum_{n=0}^{\infty} \frac{\partial}{\partial \beta} e^{-\beta n \hbar \omega}\right] \Big/ Z \\
&= +\left[\sum_{n=0}^{\infty} n\hbar\omega e^{-\beta n \hbar \omega}\right] \Big/ Z \\
&= \sum_{n=0}^{\infty} \epsilon(n) P(n) = \langle \epsilon_n \rangle = \varepsilon(\omega, \tau)
\end{aligned}$$

(b) (**2 points**) We define as a measure of the fluctuation (for the statistics experts, the technical term for this quantity is the *variance*)

$$\sigma_\varepsilon^2 \equiv \langle(\epsilon_n - \varepsilon)^2\rangle = \langle(\epsilon_n - \langle\epsilon_n\rangle)^2\rangle$$

Show that this may alternatively be expressed as:

$$\sigma_\varepsilon^2 = \langle\epsilon_n^2\rangle - \langle\epsilon_n\rangle^2 = \langle\epsilon_n^2\rangle - \varepsilon^2$$

(**Hint:** It is easy to see that the process of averaging is *linear*, i.e., $\langle a_n + b_n\rangle = \langle a_n\rangle + \langle b_n\rangle$, and $\langle\gamma a_n\rangle = \gamma\langle a_n\rangle$ for any constant γ. However, in general $\langle a_n^2\rangle \neq \langle a_n\rangle^2$.)

Solution:
This is a very general result in probability theory.

$$\begin{aligned}\sigma_\varepsilon^2 &= \langle(\epsilon_n - \langle\epsilon_n\rangle)^2\rangle \\ &= \langle\epsilon_n^2 + \langle\epsilon_n\rangle^2 - 2\epsilon_n\langle\epsilon_n\rangle\rangle\end{aligned}$$

Now, $\langle\epsilon_n\rangle = \varepsilon$ is already an average quantity, it does not depend on n at all. Same holds for $\langle\epsilon_n\rangle^2 = \varepsilon^2$. In particular, $\langle\epsilon_n\langle\epsilon_n\rangle\rangle = \langle\epsilon_n\rangle\langle\epsilon_n\rangle$, and $\langle\langle\epsilon_n\rangle^2\rangle = \langle\varepsilon^2\rangle = \varepsilon^2 = \langle\epsilon_n\rangle^2$. So it follows that:

$$\begin{aligned}\sigma_\varepsilon^2 &= \langle\epsilon_n^2 + \langle\epsilon_n\rangle^2 - 2\epsilon_n\langle\epsilon_n\rangle\rangle \\ &= \langle\epsilon_n^2\rangle + \langle\epsilon_n\rangle^2 - 2\langle\epsilon_n\rangle^2 \\ &= \langle\epsilon_n^2\rangle - \langle\epsilon_n\rangle^2 \\ &= \langle\epsilon_n^2\rangle - \varepsilon^2\end{aligned}$$

As an aside, note that since $P(n) \geq 0\ \forall n$, $\sigma_\varepsilon^2 = \langle(\epsilon_n - \varepsilon)^2\rangle \geq 0$, which tells us that always $\langle\epsilon_n^2\rangle \geq \langle\epsilon_n\rangle^2 = \varepsilon^2$.

(c) (**3 points**) Prove the relationship

$$\sigma_\varepsilon^2 = \frac{\partial^2}{\partial \beta^2} \log(Z)$$

Solution:

$$\begin{aligned}
\frac{\partial^2}{\partial \beta^2} \log(Z) &= \frac{\partial}{\partial \beta} \left[\frac{\partial}{\partial \beta} \log(Z) \right] \\
&= \frac{\partial}{\partial \beta} \left[\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right] \\
&= \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \left(\frac{\partial Z}{\partial \beta} \Big/ Z \right)^2
\end{aligned}$$

The second term in the last equation is just $\langle \epsilon_n \rangle^2$ from (a). The first term is:

$$\begin{aligned}
\frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} &= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{\partial^2}{\partial \beta^2} e^{-\beta n \hbar \omega} \\
&= \frac{1}{Z} \sum_{n=0}^{\infty} (n \hbar \omega)^2 e^{-\beta n \hbar \omega} \\
&= \sum_{n=0}^{\infty} \epsilon(n)^2 P(n) \\
&= \langle \epsilon_n^2 \rangle
\end{aligned}$$

Putting these together gives:

$$\frac{\partial^2}{\partial \beta^2} \log(Z) = \langle \epsilon_n^2 \rangle - \langle \epsilon_n \rangle^2 = \sigma_\varepsilon^2$$

Sidebar: This result is sometimes called the "fluctuation-dissipation" theorem, and is worth looking up if you are interested in the very very many situations where it is useful. You will undoubtedbly encounter it multiple times in your scientific careers. As one example, note that:

$$\begin{aligned}
&\frac{\partial}{\partial \beta} \log(Z) = -\varepsilon \\
&\frac{\partial^2}{\partial \beta^2} \log(Z) = -\frac{\partial}{\partial \beta} \varepsilon = \tau^2 \frac{\partial \varepsilon}{\partial \tau} = k_B T^2 c_V
\end{aligned}$$

where c_V is the specific heat per photon (at constant volume in this case). So this means that the same parameter, c_V, that tells us about the "dissipation", i.e., the change in energy when temperature changes[1], also tells us about the *intrinsic* statistical "fluctuations" of the energy at a *fixed* temperature.

Since by definition $\sigma_\varepsilon^2 \geq 0$ and $\tau^2 \geq 0$, it better be that the specific heat $c_V \geq 0$. In the previous assignment you got a glimpse of the general rule that $c_P > c_V$, so even $c_P \geq 0$.

[1] Actually, the term "dissipation" comes from applying something very similar to non-equilibrium situations.

(d) (**4 points**) Now, using the following expression for the sum of a geometric series (assume $|r| < 1$):

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

derive a closed form expression expression for $Z(\omega, \tau)$. Use this, combined with (c), to then obtain a closed form expression for $\sigma_\varepsilon^2(\omega, \tau)$.

Solution:

First, observe that $e^{-n\beta\hbar\omega} = \left(e^{-\beta\hbar\omega}\right)^n$, so that, with $a = 1$ and $r = e^{-\beta\hbar\omega} \leq 1$ (the equality only occurs when $\omega = 0$, but that is of course just the absence of any photons and hence makes no contribution):

$$\begin{aligned} Z(\omega, \tau) &= \frac{1}{1 - e^{-\beta\hbar\omega}} \\ \log(Z(\omega, \tau)) &= -\log(1 - e^{-\beta\hbar\omega}) \end{aligned}$$

From this, and using (c):

$$\begin{aligned} \frac{\partial}{\partial\beta} \log(Z) &= -\frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = -\frac{\hbar\omega}{e^{+\beta\hbar\omega} - 1} \quad \text{(compare this with lecture notes)} \\ \frac{\partial^2}{\partial\beta^2} \log(Z) &= +\frac{(\hbar\omega)^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2} = \sigma_\varepsilon^2(\omega, \tau) \end{aligned}$$

(e) (**4 points**) With a little bit of work, it can be shown (see bonus question) that the fluctuation in the *total* energy of the system is given by:

$$(\Delta E)^2 = \int_0^\infty N(\omega)\sigma_\varepsilon^2(\omega,\tau)d\omega$$

where $N(\omega)$ has the same meaning as in the lecture notes. Use this to write an integral expression for $(\Delta E)^2$. By making a change of variables in the integral such that all the factors of temperature appear *outside* the integral, find m such that $(\Delta E)^2 \propto \tau^m$ or equivalently $(\Delta E)^2 \propto \beta^{-m}$. **Don't actually evaluate the integral**.

Solution:
From the lecture notes

$$\begin{aligned}
N(\omega) &= \pi\left(\frac{L}{c\pi}\right)^3\omega^2 \\
\implies (\Delta E)^2 &= \int_0^\infty \pi\left(\frac{L}{c\pi}\right)^3\omega^2\left[\frac{(\hbar\omega)^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}\right]d\omega \\
\implies (\Delta E)^2 &= \frac{\hbar^2 L^3}{\pi^2 c^3}\times\int_0^\infty\left[\frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}\right]\omega^4 d\omega
\end{aligned}$$

Now make the change of variables $x = \beta\hbar\omega \implies \omega = \frac{x}{\beta\hbar} \implies d\omega = \frac{dx}{\beta\hbar}$. Making this substitution gives:

$$(\Delta E)^2 = \frac{1}{\beta^5}\times\left[\frac{L^3}{\hbar^3\pi^2c^3}\times\int_0^\infty\frac{x^4e^x}{(e^x-1)^2}\,dx\right] = \tau^5\times\left[\frac{L^3}{\hbar^3\pi^2c^3}\times\int_0^\infty\frac{x^4e^x}{(e^x-1)^2}\,dx\right]$$

The terms in the braces are now independent of $\tau = \frac{1}{\beta}$. From this we can read off:

$$\begin{aligned}
&(\Delta E)^2 \propto \tau^5 \text{ (or } \beta^{-5}) \\
\implies &m = 5
\end{aligned}$$

(f) (**Bonus: 3 points**) The way we went from the fluctuation of *one* photon to the fluctuation of the entire system using the integral in (e) may seem sketchy. It makes sense if we were trying to calculate the total Energy, since Energy is additive, but why would it work for the variance, which looks like it is non-linear? In order to see why, you will now try and prove it.

Suppose we have N random variables, $x_1, x_2, ..., x_N$, with corresponding probability distributions $p_1(x_1), p_2(x_2), ..., p_N(x_N)$. Assume the random variables $\{x_i\}$ are *independent*, i.e., the joint probability distribution $p(x_1, x_2, ..., x_N) = p_1(x_1)p_2(x_2)....p_N(x_N)$. If we now define $X = \sum_{i=1}^{N} x_i$, show that:

$$(\Delta X)^2 \equiv \langle X^2 \rangle - \langle X \rangle^2 = \sum_{i=1}^{N} (\Delta x_i)^2 = \sum_{i=1}^{N} \left[\langle x_i^2 \rangle - \langle x_i \rangle^2 \right]$$

This is precisely what the formula in (e) is. You take a bunch of independent photons with various frequencies, such that there are $N(\omega)$ of them with a given frequency ω, and each of these has energy $\varepsilon(\omega)$. Then the total energy is $E = \int d\omega N(\omega)\varepsilon(\omega)$, which is the analogue of X above. We can therefore apply the rule above with $(\Delta X)^2 \to (\Delta E)^2$, $(\Delta x_i)^2 \to \sigma_\varepsilon^2(\omega)$ and $\sum_i \to \int d\omega N(\omega)$ which gives us the same expression as the one used in (e). The same rule also applies if we wanted to find the fluctuations of the spectral energy density $E(\omega)$, i.e., the total energy with a given frequency ω. The result is then simply $(\Delta E_\omega)^2 = N(\omega)\sigma_\varepsilon^2(\omega)$. The function $(\Delta E_\omega)^2$ is usually called the "power spectrum".

Solution:

$$\langle X^2 \rangle = \sum_i \sum_j \langle x_i x_j \rangle$$

Now, for $i \neq j$:

$$\begin{aligned}
\langle x_i x_j \rangle &= \sum_{\{x_1\}} \sum_{\{x_2\}} ... \sum_{\{x_i\}} ... \sum_{\{x_j\}} ... \sum_{\{x_N\}} p(x_1, x_2, ..., x_i, ..., x_j, ..., x_N)(x_i x_j) \\
&= \sum_{\{x_1\}} \sum_{\{x_2\}} ... \sum_{\{x_i\}} ... \sum_{\{x_j\}} ... \sum_{\{x_N\}} [p_1(x_1)][p_2(x_2)][....][x_i p_i(x_i)]...[x_j p_j(x_j)]....[p_N(x_N)] \\
&= \langle x_i \rangle \langle x_j \rangle
\end{aligned}$$

where the sums are taken over all the possible values of each x_i. By the same logic, when $i = j$, it is clear that $\langle x_i x_i \rangle = \langle x_i^2 \rangle$. Hence

$$\langle X^2 \rangle = \sum_i \langle x_i^2 \rangle + \sum_{k,j,k \neq j} \langle x_k \rangle \langle x_j \rangle$$

From the linearity of the average, $\langle X \rangle = \sum_i \langle x_i \rangle$, which implies:

$$\begin{aligned}
\langle X \rangle^2 &= \sum_i \sum_j \langle x_i \rangle \langle x_j \rangle \\
&= \sum_i \langle x_i \rangle^2 + \sum_{k,j,k \neq j} \langle x_k \rangle \langle x_j \rangle
\end{aligned}$$

If we subtract these two expression to get $(\Delta X)^2$, we see that all the terms where $i \neq j$ mutually cancel. When $i = j$, we get a term $\langle x_i^2 \rangle - \langle x_i \rangle^2$ for each i. Hence:

$$(\Delta X)^2 = \sum_i [\langle x_i^2 \rangle - \langle x_i \rangle^2]$$

which completes the proof.