Bonus Problem:

Let $X = x - x_0$, $Y = y - y_0$, $Z = z - z_0$.

Let X', Y', Z' be the corresponding quantities along the principal axes. Clearly, $Z = Z'$.

We have:

$$Z' = a(X')^2 + b(Y')^2$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\Rightarrow \begin{aligned} X' &= X\cos\theta - Y\sin\theta \\ Y' &= X\sin\theta + Y\cos\theta \end{aligned}$$

$$\Rightarrow Z' = Z = a\left[(X\cos\theta - Y\sin\theta)^2\right] + b\left[(X\sin\theta + Y\cos\theta)^2\right]$$

$$= (a\cos^2\theta + b\sin^2\theta)X^2 + (a\sin^2\theta + b\cos^2\theta)Y^2 + 2\sin\theta\cos\theta(b-a)XY$$

$$= AX^2 + BY^2 + CXY$$

$$\Rightarrow z - z_0 = A(x-x_0)^2 + B(y-y_0)^2 + C(x-x_0)(y-y_0)$$

$$\Rightarrow z - z_0 = A\left[x^2 + x_0^2 - 2x_0 x\right] + B\left[y^2 + y_0^2 - 2y_0 y\right) + C\left[xy - y_0 x - x_0 y + x_0 y_0\right]$$

$$\Rightarrow \quad z = Ax^2 + By^2 + Cxy - (2Ax_0 + Cy_0)x - (2By_0 + Cx_0)y + \left(Ax_0^2 + By_0 + Cx_0 y_0 + z_0\right)$$

$$\Rightarrow \quad z = F + Ex + Dy + Cxy + By^2 + Ax^2$$

We are interested in finding a, b (and perhaps θ):

$$a\cos^2\theta + b\sin^2\theta = A$$

$$a\sin^2\theta + b\cos^2\theta = B$$

$$2\sin\theta\cos\theta\,(b-a) = C$$

This needs to be solved for a, b and θ in terms of A, B, C.

1 + 2

$$a + b = A + B$$

2 - 1

$$(b-a)\cos(2\theta) = B - A$$

From 3

$$(b-a)\sin 2\theta = C$$

$$\Rightarrow \tan(2\theta) = \frac{c}{B-A} \Rightarrow 2\theta = \arctan\left[\frac{c}{B-A}\right]$$

$$\Rightarrow b - a = \frac{c}{\sin 2\theta}$$

$$b + a = B + A$$

$$b = \frac{1}{2}\left\{ B + A + \frac{c}{\sin(2\theta)} \right\}$$

$$a = \frac{1}{2}\left\{ B + A - \frac{c}{\sin(2\theta)} \right\}$$