Question 1

(a) I claim that the minimum number of points to guarantee that any new test point is within 0.01 of an old point is 50.

Lemma 1. *At least 50 points are required.*

Proof. This problem reduces to finding a collection S of points such that the union of the closed intervals contains $[0, 1]$. That is, we seek $S = \{p_1, \ldots, p_k\}$ such that

$$[0,1] \subseteq \bigcup_{p \in S} [p - 0.01, p + 0.01]$$

From this, we must certainly have that the total lengths of the intervals $[p_i + 0.01, p_i - 0.01]$ is at least 1. This gives us a lower bound on k as follows:

$$\begin{aligned} 1 &\leq \sum_{i=1}^{k} 0.02 \\ 1 &\leq 0.02k \\ 50 &\leq k \end{aligned}$$

□

Lemma 2. *50 points are sufficient.*

Proof. Take $S = \{0.01 \cdot (2n + 1) : n \in \mathbb{N}, 0 \leq n \leq 49\}$. Then $|S| = 50$. Given $x \in [0, 1]$, choose the largest $n \in \mathbb{N}$ such that $0.01 \cdot (2n + 1) < x$. Since $x \in [0, 1]$, we must have that $n < 50$. Since n was the largest such integer, we must have that

$$0.01 \cdot (2n + 1) < x \leq 0.01 \cdot (2n + 3)$$

Aiming for a contradiction, let us suppose x was further than 0.01 from both $0.01 \cdot (2n + 1)$ and $0.01 \cdot (2n + 3)$. Then we have that

$$\begin{aligned} x - 0.01 \cdot (2n + 1) &> 0.01 \\ 0.01 \cdot (2n + 3) - x &> 0.01 \end{aligned}$$

then we get

$$0.01 \cdot (2n + 3 - 2n - 1) = 0.02 > 0.02$$

which is a contradiction. Thus, we must have that $x \in [0.01 \cdot (2n + 1) - 0.01, 0.01 \cdot (2n + 1) + 0.01]$ or $x \in [0.01 \cdot (2n + 3) - 0.01, 0.01 \cdot (2n + 3) + 0.01]$. Thus, we have that x is within 0.01 of some point in S. □

(b) Such a guarantee when $d = 10$ can be formulated by a cover of $[0, 1]^{10}$ by closed balls of radius 0.01. From Euclidean geometry, we know that volumes of balls in $\mathbb{R}^d$ are given by

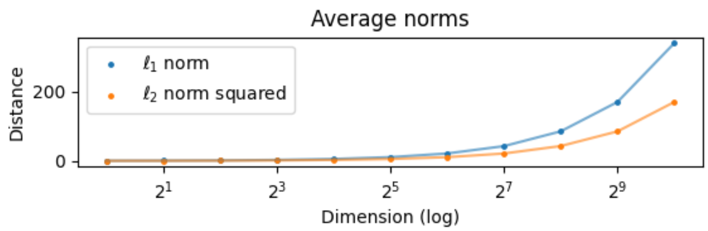
$$B(r) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} r^d$$

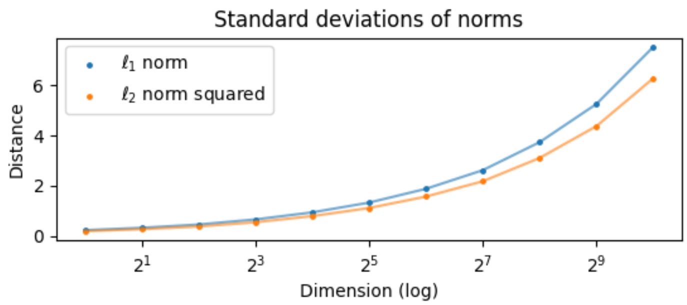
where $B(r)$ is the volume of a ball of radius r, and Γ refers to the gamma function. It is known that

$$\frac{\pi^{10/2}}{\Gamma(10/2 + 1)} \geq 0.4$$

showing that the minimum number of balls required to cover $[0, 1]^{10}$ is at least $0.4/0.01^{10} \approx 4 \cdot 100^{10}$. This number is much larger than 50. As d grows, this number will grow exponentially with d.

(c) The plot is attached below:





(d) Since expectation is linear:

$$\mathbb{E}(R) = \mathbb{E}(Z_1 + \cdots + Z_d) = \sum_{i=1}^{d} \mathbb{E}(Z_i) = \frac{d}{6}$$

Since Z_i are independent and identically distributed, and that variance is additive for independent random variables:

$$\mathrm{Var}(R) = \mathrm{Var}(Z_1 + \cdots + Z_d) = \sum_{i=1}^{d} \mathrm{Var}(Z_i) = d \cdot \frac{7}{180}$$

(e) We proceed in steps as justified in the handout.

i. The event "R is at least k away from its expectation" can be written as

$$E = |R - \mathbb{E}(R)| \geq k$$

ii. Using Markov's inequality, we have that

$$\Pr(E) \leq \frac{\mathrm{Var}(R)}{k^2} = \frac{d}{k^2} \cdot \frac{7}{180}$$

iii. If $k = cd$, then we have that

$$\Pr(E) \leq \frac{1}{c^2 d} \cdot \frac{7}{180}$$

With $d \to \infty$, we see that $\Pr(E) \to 0$.

This supports the claim that most pairs of points are approximately the same distance. Given fixed k proportional to d, we see that the probability of R being at least k away from its expectation goes to 0 as d grows. That is, it is very likely that R is k-close to its expectation as d grows. More formally, given fixed c, we can find d large enough such that for $k = cd$, we have $P(E) \leq \varepsilon$, or $P(|R - \mathbb{E}(R)| < k) > 1 - \varepsilon$.