DELHI TECHNOLOGICAL UNIVERSITY

MATHEMATICS –III (MC-203)

PRACTICAL FILE

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EXPERIMENT 10

AIM

Write a program to evaluate the following function from 0 to 2*pi using residue theorem:

$$\int 1 \frac{\cos x}{3 + \sin x}\, dx$$

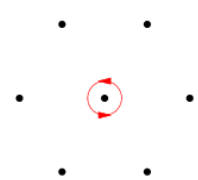
THEORY

The constant a_{-1} in the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad (1)$$

of $f(z)$ about a point z_0 is called the residue of $f(z)$. If f is analytic at z_0, its residue is zero, but the converse is not always true (for example, $1/z^2$ has residue of 0 at $z = 0$ but is not analytic at $z = 0$). The residue of a function f at a point z_0 may be denoted $\operatorname{Res}_{z=z_0}(f(z))$. The residue is implemented in the Wolfram Language as Residue[f, {z, z0}].

Two basic examples of residues are given by $\operatorname{Res}_{z=0} 1/z = 1$ and $\operatorname{Res}_{z=0} 1/z^n = 0$ for $n > 1$.



The residue of a function f around a point z_0 is also defined by

$$\operatorname{Res}_{z_0} f = \frac{1}{2\pi i} \oint_{\gamma} f\, dz, \quad (2)$$

where γ is counterclockwise simple closed contour, small enough to avoid any other poles of f. In fact, any counterclockwise path with contour winding number 1 which does not contain any other poles gives the same result by the Cauchy integral formula. The above diagram shows a suitable contour for which to define the residue of function, where the poles are indicated as black dots.

It is more natural to consider the residue of a meromorphic one-form because it is independent of the choice of coordinate. On a Riemann surface, the residue is defined for a meromorphic one-form α at a point p by writing $\alpha = f\, dz$ in a coordinate z around p. Then

$$\operatorname{Res}_{p} \alpha = \operatorname{Res}_{z=p} f. \quad (3)$$

The sum of the residues of $\int f\, dz$ is zero on the Riemann sphere. More generally, the sum of the residues of a meromorphic one-form on a compact Riemann surface must be zero.

The residues of a function $f(z)$ may be found without explicitly expanding into a Laurent series as follows. If $f(z)$ has a pole of order m at z_0, then $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Therefore, The sum of the residues of $\int f \, d z$ is zero on the Riemann sphere. More generally, the sum of the residues of a meromorphic one-form on a compact Riemann surface must be zero.

The residues of a function $f(z)$ may be found without explicitly expanding into a Laurent series as follows. If $f(z)$ has a pole of order m at z_0, then $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Therefore,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^{-m+n} \quad (4)$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^n \quad (5)$$

$$\frac{d}{d z} [(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} n \, a_{-m+n} (z - z_0)^{n-1} \quad (6)$$

$$= \sum_{n=1}^{\infty} n \, a_{-m+n} (z - z_0)^{n-1} \quad (7)$$

$$= \sum_{n=0}^{\infty} (n+1) \, a_{-m+n+1} (z - z_0)^n \quad (8)$$

$$\frac{d^2}{d z^2} [(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} n (n+1) \, a_{-m+n+1} (z - z_0)^{n-1} \quad (9)$$

$$= \sum_{n=1}^{\infty} n (n+1) \, a_{-m+n+1} (z - z_0)^{n-1} \quad (10)$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) \, a_{-m+n+2} (z - z_0)^n. \quad (11)$$

Iterating,

$$\frac{d^{m-1}}{d z^{m-1}} [(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} (n+1)(n+2) \cdots (n+m-1) \, a_{n-1} (z - z_0)^n$$
$$= (m-1)! \, a_{-1} + \sum_{n=1}^{\infty} (n+1)(n+2) \cdots (n+m-1) \, a_{n-1} (z - z_0)^{n-1}. \quad (12)$$

So

$$\lim_{z \to z_0} \frac{d^{m-1}}{d z^{m-1}} [(z - z_0)^m f(z)] = \lim_{z \to z_0} (m-1)! \, a_{-1} + 0 \quad (13)$$

$$= (m-1)! \, a_{-1}, \quad (14)$$

and the residue is

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{d z^{m-1}} [(z - z_0)^m f(z)]_{z=z_0}. \quad (15)$$

The residues of a holomorphic function at its poles characterize a great deal of the structure of a function, appearing for example in the amazing residue theorem of contour integration.

SOURCE CODE

```
syms x z f(x)


%defining the function
f(x)=cos(x)/(3+sin(x));


disp("The given function: ")
disp(f(x))


g=(z^2+1)/(2*z);
h=(z^2-1)/(2*i*z);


%substituting the functions
fz=subs(f(x),[cos(x) sin(x)],[g h]);
```

```
%calculating the residue
[p,n,r]=poles(fz,z);


ans=0+0*i;
for i=1:length(p)
 if abs(p(i))<=1
 ans=ans+2*i*pi*r(i);
 end
end


disp("The integrated value from 0 to 2pi: ")
disp(ans)
```

OUTPUT

```
>> residue
The given function:
cos(x)/(sin(x) + 3)

The integrated value from 0 to 2pi:
(2*pi*37^(1/2)*((37^(1/2) - 6)^2 + 1))/37

>>
```