DELHI TECHNOLOGICAL UNIVERSITY

MATHEMATICS –III (MC-203)

PRACTICAL FILE

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EXPERIMENT 8

AIM

Write a program for obtaining Laurent's series by the use of partial fractions. Using the program, verify calculations for:

$$\frac{-2z+3}{z^2-3z+2}$$

THEORY

If $f(z)$ is analytic throughout the annular region between and on the concentric circles K_1 and K_2 centered at $z=a$ and of radii r_1 and $r_2 < r_1$ respectively, then there exists a unique series expansion in terms of positive and negative powers of $(z-a)$,

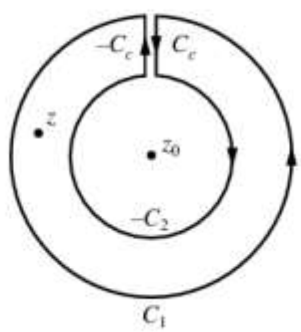
$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k + \sum_{k=1}^{\infty} b_k (z-a)^{-k}, \tag{1}$$

where

$$a_k = \frac{1}{2\pi i} \oint_{K_1} \frac{f(\zeta)\, d\zeta}{(\zeta-a)^{k+1}} \tag{2}$$

$$b_k = \frac{1}{2\pi i} \oint_{K_2} (\zeta-a)^{k-1} f(\zeta)\, d\zeta \tag{3}$$

(Korn and Korn 1968, pp. 197-198).



Let there be two circular contours C_2 and C_1, with the radius of C_1 larger than that of C_2. Let z_0 be at the center of C_1 and C_2, and z be between C_1 and C_2. Now create a cut line C_c between C_1 and C_2, and integrate around the path $C \equiv C_1 + C_c - C_2 - C_c$, so that the plus and minus contributions of C_c cancel one another, as illustrated above. From the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' \tag{4}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{z'-z}\,dz' + \frac{1}{2\pi i}\int_{C_c} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_c} \frac{f(z')}{z'-z}\,dz' \tag{5}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{z'-z}\,dz'. \tag{6}$$

Now, since contributions from the cut line in opposite directions cancel out,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)-(z-z_0)}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z'-z_0)-(z-z_0)}\,dz' \tag{7}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z-z_0)\left(\frac{z'-z_0}{z-z_0}-1\right)}\,dz' \tag{8}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}\,dz' + \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z-z_0)\left(1-\frac{z'-z_0}{z-z_0}\right)}\,dz'. \tag{9}$$

For the first integral, $|z'-z_0| > |z-z_0|$. For the second, $|z'-z_0| < |z-z_0|$. Now use the Taylor series (valid for $|t| < 1$)

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \tag{10}$$

to obtain

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_1} \frac{f(z')}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz' + \int_{C_2} \frac{f(z')}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n dz'\right] \tag{11}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^{-n-1}\int_{C_2}(z'-z_0)^n f(z')\,dz' \tag{12}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\int_{C_2}(z'-z_0)^{n-1} f(z')\,dz', \tag{13}$$

where the second term has been re-indexed. Re-indexing again,

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n \int_{C_2} \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'. \tag{14}$$

Since the integrands, including the function $f(z)$, are analytic in the annular region defined by C_1 and C_2, the integrals are independent of the path of integration in that region. If we replace paths of integration C_1 and C_2 by a circle C of radius r with $r_1 \le r \le r_2$, then

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n \int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' \tag{15}$$

$$= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}(z-z_0)^n \int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' \tag{16}$$

$$\equiv \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n. \tag{17}$$

Generally, the path of integration can be any path γ that lies in the annular region and encircles z_0 once in the positive (counterclockwise) direction.

The complex residues a_n are therefore defined by

$$a_n \equiv \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'. \tag{18}$$

Note that the annular region itself can be expanded by increasing r_1 and decreasing r_2 until singularities of $f(z)$ that lie just outside C_1 or just inside C_2 are reached. If $f(z)$ has no singularities inside C_2, then all the b_k terms in (◇) equal zero and the Laurent series of (◇) reduces to a Taylor series with coefficients a_k.

SOURCE CODE

```
syms z f(z) g(z)


f(z) = (-2*z + 3)/(z^2 - 3*z + 2);
```

```
disp("The function:")
disp(f(z))


g(z) = partfrac(f(z));
disp("The partial fraction:")
disp(g(z))


%g(z) = 1/(1-z) + 1/(2-z)
%binomial expansions of the function
f1=0;
for i=0:6
    f1 = f1 + z^i;
end


f2=0;
for i=0:6
    f2 = f2 + (z/2)^i;
end


lr=f1+(2*f2);


disp("The two binomial expansions:")
disp(f1)
disp(f2)
disp("Laurent's Series: ")
disp(lr)
```

OUTPUT

```
>> laurent_residue
The function:
-(2*z - 3)/(z^2 - 3*z + 2)

The partial fraction:
- 1/(z - 1) - 1/(z - 2)

The two binomial expansions:
z^6 + z^5 + z^4 + z^3 + z^2 + z + 1

z^6/64 + z^5/32 + z^4/16 + z^3/8 + z^2/4 + z/2 + 1

Laurent's Series:
(33*z^6)/32 + (17*z^5)/16 + (9*z^4)/8 + (5*z^3)/4 + (3*z^2)/2 + 2*z + 3
```