DELHI TECHNOLOGICAL UNIVERSITY

MATHEMATICS –III (MC-203)

PRACTICAL FILE

SUBMITTED TO:
PROF. V P KAUSHIK

SUBMITTED BY:
NITYA MITTAL
(2K19/MC/089)

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EXPERIMENT 9

AIM

Write a program for calculating the residue at pole of any order. Use it for solving i.e. getting residue of the given functions:

i. $4/(1+z^4)$
ii. $\sin(2z)/(z^6)$
iii. $(z^4)/(z^2 - iz + 2)$

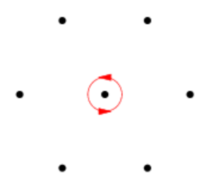
THEORY

The constant a_{-1} in the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{1}$$

of $f(z)$ about a point z_0 is called the residue of $f(z)$. If f is analytic at z_0, its residue is zero, but the converse is not always true (for example, $1/z^2$ has residue of 0 at $z = 0$ but is not analytic at $z = 0$). The residue of a function f at a point z_0 may be denoted $\operatorname{Res}_{z=z_0}(f(z))$. The residue is implemented in the Wolfram Language as `Residue[f, {z, z0}]`.

Two basic examples of residues are given by $\operatorname{Res}_{z=0} 1/z = 1$ and $\operatorname{Res}_{z=0} 1/z^n = 0$ for $n > 1$.



The residue of a function f around a point z_0 is also defined by

$$\operatorname{Res}_{z_0} f = \frac{1}{2\pi i} \oint_\gamma f\, dz, \tag{2}$$

where γ is counterclockwise simple closed contour, small enough to avoid any other poles of f. In fact, any counterclockwise path with contour winding number 1 which does not contain any other poles gives the same result by the Cauchy integral formula. The above diagram shows a suitable contour for which to define the residue of function, where the poles are indicated as black dots.

It is more natural to consider the residue of a meromorphic one-form because it is independent of the choice of coordinate. On a Riemann surface, the residue is defined for a meromorphic one-form α at a point p by writing $\alpha = f\, dz$ in a coordinate z around p. Then

$$\operatorname{Res}_p \alpha = \operatorname{Res}_{z=p} f. \tag{3}$$

The sum of the residues of $\int f\, dz$ is zero on the Riemann sphere. More generally, the sum of the residues of a meromorphic one-form on a compact Riemann surface must be zero.

The residues of a function $f(z)$ may be found without explicitly expanding into a Laurent series as follows. If $f(z)$ has a pole of order m at z_0, then $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Therefore, The sum of the residues of $\int f \, d\, z$ is zero on the Riemann sphere. More generally, the sum of the residues of a meromorphic one-form on a compact Riemann surface must be zero.

The residues of a function $f(z)$ may be found without explicitly expanding into a Laurent series as follows. If $f(z)$ has a pole of order m at z_0, then $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Therefore,

$$
\begin{aligned}
f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n &= \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^{-m+n} && (4)\\
(z - z_0)^m f(z) &= \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^n && (5)\\
\frac{d}{d\,z} \left[(z - z_0)^m f(z) \right] &= \sum_{n=0}^{\infty} n\, a_{-m+n} (z - z_0)^{n-1} && (6)\\
&= \sum_{n=1}^{\infty} n\, a_{-m+n} (z - z_0)^{n-1} && (7)\\
&= \sum_{n=0}^{\infty} (n+1)\, a_{-m+n+1} (z - z_0)^n && (8)\\
\frac{d^2}{d\,z^2} \left[(z - z_0)^m f(z) \right] &= \sum_{n=0}^{\infty} n\,(n+1)\, a_{-m+n+1} (z - z_0)^{n-1} && (9)\\
&= \sum_{n=1}^{\infty} n\,(n+1)\, a_{-m+n+1} (z - z_0)^{n-1} && (10)\\
&= \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{-m+n+2} (z - z_0)^n. && (11)
\end{aligned}
$$

Iterating,

$$
\begin{aligned}
&\frac{d^{m-1}}{d\,z^{m-1}} \left[(z - z_0)^m f(z) \right] = \sum_{n=0}^{\infty} (n+1)(n+2) \cdots (n+m-1)\, a_{n-1} (z - z_0)^n \\
&= (m-1)!\, a_{-1} + \sum_{n=1}^{\infty} (n+1)(n+2) \cdots (n+m-1)\, a_{n-1} (z - z_0)^{n-1}. \qquad (12)
\end{aligned}
$$

So

$$
\begin{aligned}
\lim_{z \to z_0} \frac{d^{m-1}}{d\,z^{m-1}} \left[(z - z_0)^m f(z) \right] &= \lim_{z \to z_0} (m-1)!\, a_{-1} + 0 && (13)\\
&= (m-1)!\, a_{-1}, && (14)
\end{aligned}
$$

and the residue is

$$
a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{d\,z^{m-1}} \left[(z - z_0)^m f(z) \right]_{z = z_0}. \qquad (15)
$$

The residues of a holomorphic function at its poles characterize a great deal of the structure of a function, appearing for example in the amazing residue theorem of contour integration.

SOURCE CODE

```
syms z f(z)


% 1) function : 4/(1+z^4)
f(z)= 4/(1+(z^4));
disp("For function: f(z) = 4/(1+(z^4)")


%finding the residue and pole
[r,p,k]=poles(f(z),z);


for i=1:4
    disp(i)
    disp("Pole : ");
    disp(r(i))
    disp('Residue : ')
    disp(k(i))
```

```
end


% 2) function : sin(2*z)/(z^6)
f(z)= sin(2*z)/(z^6);
disp("For function: f(z) = sin(2*z)/(z^6)")


%finding the residue and pole
[r,p,k]=poles(f(z),z);


disp(1)
disp("Pole : ");
disp(r(1))
disp('Residue : ')
disp(k(1))




% 3) function : (z^4)/(z^2 - i*z + 2)
f(z)= (z^4)/(z^2 - i*z + 2);
disp("For function: f(z) = (z^4)/(z^2 - i*z + 2)")


%finding the residue and pole
[r,p,k]=poles(f(z),z);


for i=1:2
    disp(i)
    disp("Pole : ");
    disp(r(i))
    disp('Residue : ')
    disp(k(i))
end
```

OUTPUT

```
>> residue_any_order_pole
For function: f(z) = 4/(1+(z^4)
     1

Pole :
2^(1/2)*(1/2 - 1i/2)

Residue :
2^(1/2)*(- 1/2 + 1i/2)

     2

Pole :
2^(1/2)*(- 1/2 - 1i/2)

Residue :
2^(1/2)*(1/2 + 1i/2)

     3

Pole :
2^(1/2)*(- 1/2 + 1i/2)

Residue :
2^(1/2)*(1/2 - 1i/2)

     4
```

```
     4

Pole :
2^(1/2)*(1/2 + 1i/2)

Residue :
2^(1/2)*(- 1/2 - 1i/2)

For function: f(z) = sin(2*z)/(z^6)
     1

Pole :
0

Residue :
4/15

For function: f(z) = (z^4)/(z^2 - i*z + 2)
     1

Pole :
2^(1/2) + 2

Residue :
(2^(1/2)*(2^(1/2) + 2)^4)/4

     2

Pole :
```

```
For function: f(z) = (z^4)/(z^2 - i*z + 2)
     1

Pole :
2^(1/2) + 2

Residue :
(2^(1/2)*(2^(1/2) + 2)^4)/4

     2

Pole :
2 - 2^(1/2)

Residue :
-(2^(1/2)*(2^(1/2) - 2)^4)/4
```