MATH3952 Talk: Symmetry and Conservation

Nitya Nigam

February 27, 2025

1 A Crash Course in Lagrangian Mechanics

1.1 Principle of Least Action (Hamilton's Principle)

We define the Lagrangian $L = T - V$ where T is the total kinetic energy of a system and V is the total potential energy. L therefore depends on generalized coordinates q_i of the system's constituents, the first time derivatives of these coordinates $\frac{dq_i}{dt} = \dot{q}_i$, and time itself, i.e. $L = L(q, \dot{q})$.

The Lagrangian is a useful quantity to consider when solving problems in physics, and will allow us to quantify the notions of symmetry and conservation in the context of physics.

We further define the following quantity:

$$S = \int_{t_1}^{t_2} L\, dt$$

where L is the Lagrangian and S is the **action**. Hamilton's principle, or the principle of least action, tells us that among all conceivable trajectories that could connect the given end points, the true trajectories are those that make S stationary. Put quantitatively:

$$\int_{t_1}^{t_2} \delta L\, dt = \delta S = 0$$

This corresponds to particles picking out the path with a stationary action, with the end points of the path in configuration space held fixed at the initial and final times, instead of thinking about particles accelerating in response to applied forces. It therefore makes intuitive sense that such a principle can give rise to equations of motion, and we will show that this is the case.

We can write:

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q})dt - \int_{t_1}^{t_2} L(q, \dot{q})dt \\
&= \int_{t_1}^{t_2} \left[L(q, \dot{q}) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right] dt - \int_{t_1}^{t_2} L(q, \dot{q})dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right] dt
\end{aligned}$$

Since $\dot{q} = \frac{dq}{dt}$ we have $\delta\dot{q} = \frac{d(\delta q)}{dt}$, so the second term above becomes:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d(\delta q)}{dt} dt = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt$$

where we integrated by parts. The first term above goes to zero since the states at t_1 and t_2 are fixed, so δq vanishes. Putting it all together:

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q\, dt = 0$$

Since the equation must be true for any arbitrary δq, the quantity in square brackets must vanish. This finally yields:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

This is called the **Euler-Lagrange equation**. It is a second-order ODE in time, and can be applied to each particle i in a system to solve for its motion $q_i(t)$.

1.2 Why Lagrangians are great: block sliding down sliding wedge

A block of mass m starts from rest at the top of a frictionless wedge of mass M which is on a frictionless horizontal floor. The block slides down the wedge, while the wedge slides across the floor. We want to find the equations of motion for the block and the wedge.

Let d be the distance the block moves down the wedge, and let x be the distance the wedge moves across the floor. These are our generalized coordinates q_i.

We first construct the system's kinetic energy, T. The velocity of the wedge is $\dot{x}$. In the inertial frame of the floor, the block's horizontal and vertical velocity components are $v_x = \dot{x} - \dot{d}\cos\theta$ and $v_y = -\dot{d}\sin\theta$. So the combined kinetic energy is:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{d}^2 - 2\dot{x}\dot{d}\cos\theta)$$

The potential energy (assuming $U = 0$ at the top of the wedge) is:

$$U = -mgd\sin\theta$$

So the Lagrangian is:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{d}^2 - 2\dot{x}\dot{d}\cos\theta) + mgd\sin\theta$$

Computing the relevant partial derivatives gives:

$$\frac{\partial L}{\partial x} = 0,\ \frac{\partial L}{\partial d} = mg\sin\theta,\ \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - m\dot{d}\cos\theta,\ \frac{\partial L}{\partial \dot{d}} = m(\dot{d} - \dot{x}\cos\theta)$$

Taking time derivatives where necessary, the Lagrange equations are hence:

$$(M+m)\ddot{x} = m\ddot{d}\cos\theta$$
$$m(\ddot{d} - \ddot{x}\cos\theta) = mg\sin\theta$$

Solving and rearranging gives:

$$\ddot{x} = \frac{m}{M + m\sin^2\theta}\cdot g\sin\theta\cos\theta$$
$$\ddot{d} = \frac{M+m}{M + m\sin^2\theta}\cdot g\sin\theta$$

These are both constant accelerations, so it is easy to find both x and d at any given time given initial conditions. Note also that as $M \to \infty$, $\ddot{x} \to 0$ and $\ddot{d} \to g\sin\theta$, since the wedge will not move the block will simply slide down due to gravity.

To solve this problem using Newton's laws, we would need to account for the normal force between the wedge and the block, which is unknown. We would also have to account for the fact that the wedge is accelerating when making a free-body diagram for the block, which is a non-inertial frame. All this is very complicated, making the Lagrangian method comparatively much easier.

Takeaway: Lagrangians let us learn important things about system evolution without having to concern ourselves with individual interactions

2 Symmetries and Conservation

2.1 Symmetries of the Lagrangian

We're all familiar with symmetry as a concept – you're probably first exposed to it in first grade math, when we learn about 2D shapes. We intuitively understand that a regular pentagon somehow has "more symmetry" than an equilateral triangle (if lots of time, give an aside on dihedral groups **??**). Similarly, physical systems can exhibit rotation and reflection symmetries, amongst many others.

When we talked about symmetries in the 2D geometry sense, we were looking for ways to change the shape (such as rotation and reflection) that left some characteristic (appearance) unchanged, or invariant. This is the same way we think about symmetry in physics.

In physics, a **symmetry** is a way to change the system that leaves the Lagrangian invariant. Some common symmetries are time translation symmetry, spatial translation symmetry, and rotational symmetry.

2.2 Noether's theorem

One of the most important theorems in theoretical physics is **Noether's theorem**, which states that every continuous symmetry of the action of a physical system (with conservative forces [i.e. path independence]) has a corresponding conservation law. This means that studying the symmetries of a system can help us understand what quantities might be conserved, like energy and momentum.

To prove Noether's theorem, we need to define symmetry and conserved quantities more rigorously.

Def. A **continuous symmetry** is a transformation $q \to \tilde{q}(q, \zeta)$ such that the transformation is the identity for $\zeta = 0$ (i.e. $\tilde{q}(q, 0) = q$) and the Lagrangian is invariant under the replacement $q \to \tilde{q}$.

Def. A **conserved quantity** is a function of q, its time derivatives, and time that is constant along all paths of motion, i.e. $f(q, \dot{q}, \ldots, t)$ s.t. $\frac{df}{dt}\Big|_{q(t)} = 0$.

We can now proceed to prove Noether's theorem. For a continuous symmetry $q \to \tilde{q}$, we have:

$$\begin{aligned} 0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} L(q, \dot{q}, t) &= \frac{\partial L}{\partial q}\frac{\partial \tilde{q}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{\tilde{q}}}{\partial \zeta}\bigg|_{\zeta=0} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\frac{\partial \tilde{q}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\left(\frac{\partial \tilde{q}}{\partial \zeta}\right)_{\zeta=0} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\frac{\partial \tilde{q}}{\partial \zeta}\right)_{\zeta=0} \end{aligned}$$

Hence there is a conserved quantity

$$\Lambda = \left(\frac{\partial L}{\partial \dot{q}}\frac{\partial \tilde{q}}{\partial \zeta}\right)_{\zeta=0}$$

2.3 Spatial and time translation symmetry

To find the conserved quantity associated with spatial translation symmetry $q \to q + \varepsilon$, we first note that such a symmetry means the Lagrangian is independent of q, so $L = L(\dot{q}$ only. Then, plugging into the Euler-Lagrange equation, we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \implies \frac{\partial L}{\partial \dot{q}} \text{ conserved}$$

We can also arrive at this conclusion from the formula for conserved quantity we just derived, since:

$$\frac{\partial(q + \zeta)}{\partial \zeta} = 1 \implies \Lambda = \frac{\partial L}{\partial \dot{q}}$$

To interpret this physically, let's set up a standard Lagrangian $L = \frac{1}{2}mv^2 - V(q) = \frac{1}{2}m\dot{q}^2 - V(q)$ which includes a single-particle kinetic energy and a potential that depends only on position, such as a gravitational or electric potential. Then we can see that:

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} = mv$$

which we can easily recognize as **momentum**. Hence, momentum is the conserved quantity associated with spatial translation invariance (or symmetry) of the Lagrangian.

Now for time translation symmetry, we have that $\frac{\partial L}{\partial t} = 0$, i.e. the Lagrangian only depends on time implicitly via q and $\dot{q}$, if at all. Then the total derivative is:

$$\begin{aligned}
\frac{dL}{dt} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) \\
&\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = 0 \\
\Lambda &= \frac{\partial L}{\partial \dot{q}}\dot{q} - L
\end{aligned}$$

Let's interpret this physically, using our same test Lagrangian as in the momentum conservation proof:

$$\frac{\partial L}{\partial \dot{q}}\dot{q} - L = m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + V(q) = \frac{1}{2}m\dot{q}^2 + V(q) = \text{ total energy} = H$$

Hence we have shown that **total energy** is the conserved quantity associated with time translation symmetry.

Note that the groups associated with these transformations are Lie groups: $\mathbb{R}$ and $\mathbb{R}^3$ respectively.

The group associated with rotational symmetry, and hence angular momentum conservation, is $SO(3)$.