PHYS UN1602 Recitation Week 4 Worksheet

TA: Nitya Nigam

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Problem 1

Find the capacitance of a capacitor that consists of two coaxial cylinders of radii a and b and length L. Assume $L \gg b - a$ so that end corrections may be neglected. Check your results by showing that, if the gap between the cylinders $b - a$ is very small compared to the radius, your formula reduces to one that could have been obtained by using the formula for a parallel-plate capacitor.

Solution. The electric field for $a < r < b$ is given by applying Gauss' law:

$$\oint \vec{E} \cdot d\vec{a} = E(2\pi r L) = \frac{Q}{\varepsilon_0} \implies \vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{Q}{rL} \hat{r}$$

We integrate radially to find $\Delta\phi$:

$$\Delta\phi = -\frac{1}{2\pi\varepsilon_0} \frac{Q}{L} \int_a^b \frac{1}{r} dr = -\frac{1}{2\pi\varepsilon_0} \frac{Q}{L} \ln\left(\frac{b}{a}\right)$$

Therefore:

$$C = \frac{Q}{|\Delta\phi|} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

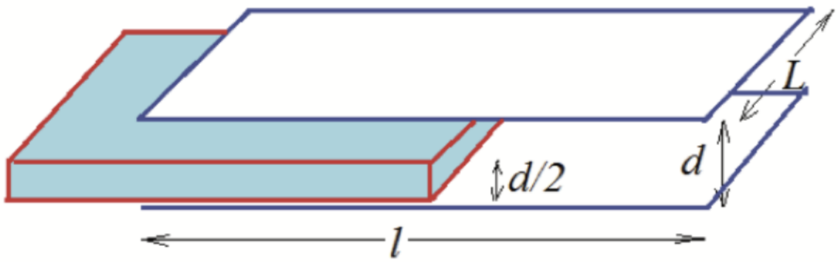
We check our answer by taking the limit $b - a \ll a$:

$$\ln\left(\frac{b}{a}\right) = \ln\left(1 + \frac{b-a}{a}\right) \approx \frac{b-a}{a}$$
$$\implies C = \frac{2\pi\varepsilon_0 La}{b-a} = \frac{\varepsilon_0 A}{d}$$

where $d = b - a$, which is the formula for a parallel-plate capacitor.

Problem 2

Consider two parallel plates separated by a distance d with length L and width l. A conducting slab of thickness $d/2$ and width l protrudes into the gap between the two plates, as shown below.



a) Find the force pulling the slab into the gap between the plates in the case where the plates carry charges $+Q$ and $-Q$.

b) Solve for the force again, now in the case where the two plates are connected to a battery of voltage V. Be careful to consider both the energy stored in the plate-plate-slab system and in the battery.

c) Show that the force is the same for the two cases considered above when the potential difference V between the plates is the same. Explain why.

Solution.

a) We treat the system as two capacitors in parallel, $C = C_1 + C_2$ with $C = \frac{A}{4\pi D}$. We have:

$$C_1 = \frac{Lx}{4\pi(d/2)}, \; C_2 = \frac{L(l-x)}{4\pi d} \implies C = \frac{L(l+x)}{4\pi d}$$

The energy generated by the system is:

$$E = \frac{Q^2}{2C} = \frac{Q^2}{\frac{2L(l+x)}{4\pi d}} = \frac{2\pi d Q^2}{L(l+x)}$$

Hence the force is:

$$F = -\frac{d}{dx}E = \frac{2\pi d Q^2}{L(l+x)^2}$$

b) Drawing the circuit, we see that the voltage across each capacitor (plate-slab system) is the same as that of the battery, V. For the capacitors, $C = \frac{Q}{V} = \frac{\sigma A}{V} \implies Q = \sigma A$. Thus:

$$E_1 = \frac{Q^2}{2C_1} = \frac{(\sigma_1 x L)^2 V}{2\sigma_1 x L} = \frac{1}{2}\sigma_1 x L V$$
$$E_2 = \frac{Q^2}{2C_2} = \frac{(\sigma_2 L(l-x))^2 V}{2\sigma_2 L(l-x)} = \frac{1}{2}\sigma_2 L(l-x)V$$
$$E_{cap} = E_1 + E_2 = \frac{1}{2}LV(\sigma_1 x + \sigma_2(l-x))F_{cap} = -\frac{dE_{cap}}{dx} = \frac{1}{2}LV(\sigma_1 - \sigma_2)$$

Now for the battery:

$$Q = \sigma_1(xL) + \sigma_2 L(l-x)$$
$$E_{bat} = QV = V(\sigma_1(xL) + \sigma_2 L(l-x))$$
$$F_{bat} = -\frac{dE_{bat}}{dx} = -LV(\sigma_1 - \sigma_2)$$

The total force is

$$F = \frac{1}{2}LV(\sigma_1 - \sigma_2) - LV(\sigma_1 - \sigma_2) = -\frac{1}{2}LV(\sigma_1 - \sigma_2)$$

c) The potential difference for the capacitors are $V_1 = 4\pi\sigma_1\frac{d}{2}$ and $V_2 = 4\pi\sigma_2 d$. Equating these yields $\sigma_1 = 2\sigma_2$. Plugging this into the charge in the battery:

$$\begin{aligned} Q &= \sigma_1(xL) + \sigma_2 L(l - x) \\ &= \sigma_1(Lx + \frac{1}{2}L(l - x)) \\ \implies \sigma_1 &= \frac{2Q}{L(l + x)} \end{aligned}$$

Then:

$$F = -\frac{1}{2}LV(\sigma_1 - \sigma_2) = -\frac{1}{4}\frac{2Q}{L(l + x)}L \cdot 2\pi\frac{2Q}{L(l + x)}d = \frac{2\pi dQ^2}{L(l + x)^2}$$

as obtained in part a).