PHYS UN1601 Recitation Worksheet 3

TA: Nitya Nigam

September 25, 2024

Problem 1

Consider a particle subject to an acceleration of the form

$$a(t) = 0,\ t < 0$$
$$a(t) = a_0 e^{-t/\tau},\ t > 0$$

with $x(0) = x_0$ and $v(0) = v_0$.

a) Evaluate $v(t)$ for $t > 0$.

b) Evaluate $x(t)$ for $t > 0$.

a) $v(t) = \int_0^t a(t')\,dt' + v_0 = \int_0^t a_0 e^{-t'/\tau}\,dt' + v_0 = a_0\left[-\tau e^{-t'/\tau}\right]_0^t + v_0$

$= -a_0\tau\left(e^{-t/\tau} - 1\right) + v_0 = \boxed{a_0\tau\left(1 - e^{-t/\tau}\right) + v_0}$

b) $x(t) = \int_0^t v(t')\,dt' + x_0 = \int_0^t a_0\tau - a_0\tau e^{-t'/\tau} + v_0\ dt' + x_0 = \left[a_0\tau t' + \tau a_0\tau e^{-t'/\tau} + v_0 t'\right]_0^\tau + x_0$

$= \boxed{(a_0\tau + v_0)t + a_0\tau^2\left(e^{-t/\tau} - 1\right) + v_0 t + x_0}$

Problem 2

A particle has an acceleration given by

$$\ddot{x} = -\omega^2(x - \lambda)$$

a) What are the dimensions of ω and λ?

b) Suppose $x(0) = 0$ and $v(0) = \dot{x}(0) = v_0$. Find expressions for $x(t)$ and $v(t)$ for $t > 0$.

a) $[\ddot{x}] = m s^{-2}$, $[x] = m \Rightarrow \boxed{[\lambda] = m}$, $[\omega] = \left[\sqrt{\frac{\ddot{x}}{x-\lambda}}\right] = \sqrt{\frac{m s^{-2}}{m}} = \sqrt{s^{-2}} = \boxed{s^{-1} = [\omega]}$

b) change coordinates: $y = x - \lambda \Rightarrow \ddot{y} = \ddot{x} = -\omega^2 y \Rightarrow y(t) = A\cos(\omega t) + B\sin(\omega t)$; $\dot{y}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$

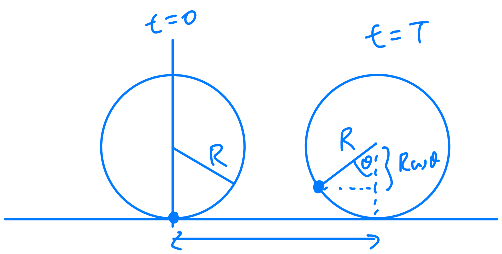
apply conditions: $x(0) = 0 \Rightarrow y(0) = -\lambda = A$; $\dot{y}(0) = \dot{x}(0) = v_0 = B\omega \Rightarrow B = \frac{v_0}{\omega}$

$\Rightarrow y(t) = -\lambda\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) \Rightarrow \boxed{x(t) = \lambda - \lambda\cos(\omega t) + \frac{v_0}{B}\sin(\omega t)}$

$\dot{y}(t) = \boxed{\lambda\omega\sin(\omega t) + v_0\cos(\omega t) = \dot{x}(t)}$

Problem 3

A tire rolls in a straight line without slipping. Its center moves with constant speed v. A small pebble lodged in the tread of the tire touches the road at time $t = 0$. Find the pebble's position, velocity and acceleration as functions of time.



$$vt = R\theta \implies \theta = \frac{vt}{R}$$

$$\dot\theta = \frac{v}{R}$$

$$\ddot\theta = 0$$

$$x = R\theta - R\sin\theta = vt - R\sin\left(\frac{vt}{R}\right)$$

$$y = R - R\cos\theta = R - R\cos\left(\frac{vt}{R}\right)$$

$$\dot x = R\dot\theta - R\dot\theta\cos\theta = v\left(1-\cos\left(\frac{vt}{R}\right)\right)$$

$$\dot y = R\dot\theta\sin\theta = v\sin\left(\frac{vt}{R}\right)$$

$$\ddot x = \cancel{R\ddot\theta} - \cancel{R\ddot\theta\cos\theta} + R\dot\theta^2\sin\theta = \frac{v^2}{R}\sin\left(\frac{vt}{R}\right)$$

$$\ddot y = \cancel{R\ddot\theta\sin\theta} + R\dot\theta^2\cos\theta = \frac{v^2}{R}\cos\left(\frac{vt}{R}\right)$$

Problem 4

A particle moves in the $x - y$ plane. Starting from $\vec{r} = r\hat{r}$ (the position vector is described entirely radially), do the following, showing all the steps in your evaluation of the derivatives.

a) Obtain the most general expression for the velocity using the polar unit vectors $\hat{r}$ and $\hat{\theta}$.

b) Obtain the most general expression for the acceleration assuming $\theta = \omega t$ and that r has an arbitrary time dependence. Express your results using the polar unit vectors.

c) Apply your results from parts a) and b) to calculate the velocity and acceleration, $\vec{v}(t)$ and $\vec{a}(t)$, for a particle moving with $\theta = \omega t$ and $r = r_0 e^{-t/\tau}$.

a) $\vec r = r\hat r \implies \dot{\vec r} = \vec v = \dot r\hat r + r\dot\theta\hat\theta$ (since $\dot{\hat r} = \dot\theta\hat\theta$)

b) $\vec a = \dot{\vec v} = \ddot r\hat r + \dot r\dot\theta\hat\theta + \dot r\dot\theta\hat\theta + r\ddot\theta\hat\theta + r\dot\theta(-\dot\theta\hat r)$ $\quad(\dot{\hat\theta} = -\dot\theta\hat r)$

using $\theta = \omega t$, $\dot\theta = \omega$, $\ddot\theta = 0$ $\;\to\; = \ddot r\hat r + 2\dot r\omega\hat\theta - r\omega^2\hat r = (\ddot r - r\omega^2)\hat r + 2\dot r\omega\hat\theta$

c) $r = r_0 e^{-t/\tau} \implies \dot r = -\frac{r_0}{\tau}e^{-t/\tau}$, $\ddot r = \frac{r_0}{\tau^2}e^{-t/\tau}$

$$\implies \vec v(t) = -\frac{r_0}{\tau}e^{-t/\tau}\hat r + r_0\omega e^{-t/\tau}\hat\theta$$

$$\vec a(t) = \left(\frac{r_0}{\tau^2}e^{-t/\tau} - r_0\omega^2 e^{-t/\tau}\right)\hat r - \frac{2\omega r_0}{\tau}e^{-t/\tau}\hat\theta$$

$$= r_0 e^{-t/\tau}\left[\left(\frac{1}{\tau^2} - \omega^2\right)\hat r - \frac{2\omega}{\tau}\hat\theta\right]$$