PHYS UN1602 Recitation Week 3 Worksheet

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February 12-13, 2025

Problem 1

An infinitely long cylinder of radius R and total charge per unit length λ has a charge density that varies as a function of radius r as $\rho(r) = \frac{C}{r}$.

a) Calculate the coefficient C in terms of λ. To do this, set up and carry out an appropriate integral over the radius with the proper radial weighting.

b) Evaluate the electric field produced by the charged distribution for both $r < R$ and $r > R$.

c) Evaluate the electric potential (e.g. using $\Delta\phi = -\int \vec{E} \cdot d\vec{r}$) as a function of r, assuming $\phi(0) = 0$.

Solution.

a)

$$\lambda = \int_0^R \rho(r) 2\pi r dr = 2\pi \int_0^R C dr = 2\pi CR \implies C = \frac{\lambda}{2\pi R}$$

b) Since this system has cylindrical symmetry, the electric field only depends on the r coordinate. We thus apply Gauss' law over the surface of a cylinder of radius r and length L that is coaxial with the z axis:

$$\oint \vec{E} \cdot d\vec{A} = 2\pi L r E(r)$$

We then need to evaluate the amount of charge contained in the Gaussian cylinder for both $r < R$ and $r > R$. For $r < R$:

$$Q = \int_0^r dr' 2\pi r' \rho(r') = 2\pi L \int_0^r \frac{\lambda}{2\pi R} dr' = \frac{\lambda L r}{R}$$

For $r > R$, $Q = \lambda L$. Then, applying Gauss' law $\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$, we obtain:

$$\vec{E}(r) = \begin{cases} \frac{\lambda}{2\pi\varepsilon_0 R}\hat{r}, \; r < R \\ \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}, \; r > R \end{cases}$$

c) For $r < R$:

$$\phi(r) = -\int_0^r \frac{\lambda}{2\pi\varepsilon_0 R} dr' = -\frac{\lambda r}{2\pi\varepsilon_0 R}$$

so $\phi(r = R) = -\frac{\lambda}{2\pi\varepsilon_0}$. Then, for $r > R$:

$$\phi(r) = \phi(R) - \int_0^r \frac{\lambda}{2\pi\varepsilon_0 r'} dr' = -\frac{\lambda}{2\pi\varepsilon_0} - \frac{\lambda}{2\pi\varepsilon_0} \left[\ln(r')\right]_R^r = -\frac{\lambda}{2\pi\varepsilon_0} \left(1 + \ln\left(\frac{r}{R}\right)\right)$$

Problem 2

An electrically neutral atom consists of a nucleus that is a point particle with positive charge Q at the center of the atom and an electron of volume charge density modeled by:

$$\rho(r) = -\frac{\beta}{r^2} e^{-r/a}$$

for $r < a$ and $\rho(r) = 0$ for $r > a$.

a) Use Gauss' law to derive an expression for the electric field.

b) Graph the electric field as a function of radius.

Solution.

a) We first consider the negative charge enclosed, given by the volume charge density:

$$\begin{aligned} Q_{neg} &= \int \rho(r) dV = -\beta \int \frac{e^{-r/a}}{r^2} dV = -4\pi\beta \int_0^r \frac{e^{-r'/a}}{r'^2} r'^2 dr' = 4\pi\beta a \left[e^{-r'/a} \right]_0^r \\ &= 4\pi\beta a (e^{-r/a} - 1) \end{aligned}$$

The total charge enclosed is $Q + Q_{neg} = Q - 4\pi\beta a(1 - e^{-r/a})$. Then, applying Gauss' law:

$$\begin{aligned} E(4\pi r^2) &= \frac{Q - 4\pi\beta a(1 - e^{-r/a})}{\varepsilon_0} \\ \vec{E}(r) &= \frac{Q - 4\pi\beta a(1 - e^{-r/a})}{4\pi\varepsilon_0 r^2} \hat{r} \end{aligned}$$

b)