Hamiltonian Mechanics Revisited

Nitya Nigam

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1 Introduction

In my previous seminar talk, I introduced Lagrangian mechanics as a formalism within which to understand the deep connection between symmetries and conservation in physics. Emory expanded on this by discussing the notions of symmetry and conservation in the context of Hamiltonian mechanics, a more versatile framework within which to conduct theoretical physics. In this paper, I will formalize some of the notions of Hamiltonian mechanics in the mathematical language of algebra, lending greater insight into how symmetries are a deeply powerful tool for understanding physical systems.

2 From Lagrangian to Hamiltonian mechanics

2.1 Introducing some notation

For a system with N degrees of freedom, we define $\vec{v} = (v_1, \dots, v_N)$, where the v_i could be e.g. positions q_i, velocities $\dot{q}_i$, or momenta p_i. Correspondingly, for functions of $\vec{v}$, we write $\partial_{\vec{v}} f \equiv (\partial_{v_1} f, \dots \partial_{v_N} f)$. We abbreviate sums over indices with a dot: $\partial_{\vec{v}} f \cdot \partial_{\vec{w}} g = \sum_i \partial_{v_i} f \partial_{w_i} g$.

2.2 The Legendre transform

For an N-variable function $F(\vec{x})$, the Legendre transform $\tilde{F}(\vec{s})$ is defined as follows:

$$\tilde{F} \equiv \vec{s} \cdot \vec{x} - F, \, \vec{s} \equiv \partial_{\vec{x}} F$$

Note that the Legendre transform is an involution; i.e. $\tilde{F}(\tilde{F}(\vec{x})) = F(x)$. Another notable feature is that the matrix of second derivatives $\partial_{x_i} \partial_{x_j} F(\vec{x})$ and $\partial_{s_i} \partial_{s_j} \tilde{F}(\vec{s})$ are each other's inverse.

For a one-dimensional example, let us consider $F(x) = \frac{a}{2}x^2 + bx + c$ with $a > 0$. Then $s = \frac{dF}{dx} = ax + b$. Rearranging to get x as a function of s gives $x = \frac{1}{a}(s - b)$. The Legendre transform is thus

$$\tilde{F}(s) = sx - F = ax^2 + bx - (\frac{a}{2}x^2 + bx + c) = \frac{a}{2}x^2 - c = \frac{1}{2a}(s - b)^2 - c$$

Note that the slope of $\tilde{F}$ at s equals the original variable x: $\frac{d\tilde{F}}{ds} = \frac{1}{a}(s - b) = x$. This is true in general, and is a result of the transform being an involution. Legendre transforms arise as leading saddle point approximations of integrals with odd behavior, like sharply peaked or extremely oscillatory integrands. For example, in statistical mechanics, the free energy $F(\beta)$ (where β is the inverse temperature) is the Legendre transform of the etnropy $S(E)$, up to a factor of β.

2.3 From Lagrangian to Hamiltonian and back again

The Hamiltonian $H(\vec{q}, \vec{p})$ is just the Legendre transform of the Lagrangian $L(\vec{q}, \dot{\vec{q}})$, swapping the velocities $\dot{\vec{q}}$ with the momenta $\vec{p} = \partial_{\dot{\vec{q}}} L$ which are canonically conjugate to $\vec{q}$:

$$H = \vec{p} \cdot \dot{\vec{q}} - L, \; \vec{P} = \partial_{\dot{\vec{q}}} L \, ; \quad \dot{\vec{q}} = \partial_{\vec{p}} H, \; L = \vec{p} \cdot \dot{\vec{q}} - H$$

For example if $L = \frac{m}{2} \dot{\vec{q}}^2 - V(\vec{q})$, then $\vec{p} = \partial_{\dot{\vec{q}}} L = m\dot{\vec{q}}$, so the Hamiltonian is

$$H = \frac{\vec{p}^2}{2m} + V(\vec{q})$$

This is applicable to any coordinate system. For example, in spherical coordinates, if we have

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2) - V(r)$$

then $p_r = m\dot{r}$, $p_\theta = mr^2\dot{\theta}$, and $p_\phi = mr^2 \sin^2\theta \dot{\phi}$, so

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right) + V(r)$$

We can just as easily go in the other direction, starting with a given Hamiltonian $H(\vec{q}, \vec{p})$ and obtain the Lagrangian $L(\vec{q}, \dot{\vec{q}})$ by solving $\partial_{\dot{\vec{q}}} H$ and setting $L = \vec{p} \cdot \dot{\vec{q}} - H$. For example, the Hamiltonian for a free relativistic point particle is

$$H = \sqrt{\vec{p}^2 + m^2}$$

from which we get $\dot{\vec{q}} = \partial_{\vec{p}} H = \frac{\vec{p}}{\sqrt{\vec{p}^2 + m^2}} \implies \vec{p} = \frac{m\dot{\vec{q}}}{\sqrt{1 - \dot{\vec{q}}^2}} \implies L = -m\sqrt{1 - \dot{\vec{q}}^2}$

2.4 Hamilton's equations

We assume familiarity with Hamilton's equations:

$$\dot{\vec{q}} = \partial_{\vec{p}} H, \; \dot{\vec{p}} = -\partial_{\vec{q}} H$$

and how they follow from variations of the action S, since this has been discussed in several talks.

2.5 What if we can't get back?

While we derived the Hamiltonian formalism from the Lagrangian one, the Hamiltonian formalism is understood to be the more fundamental one. One reason for this is that the Hamiltonian formulation allows classical mechanics to emerge naturally from the operator formalism of quantum mechanics when we take $\hbar \to 0$, but the more pressing (and more mathematical) reason is that the Hamiltonian formalism allows us to consider a broader class of dynamical systems, since any function $H(\vec{q}, \vec{p})$ defines a Hamiltonian flow on the $2N$-dimensional space parametrized by $(\vec{q}, \vec{p})$ (which we call phase space), including Hamiltonians H for which the Legendre transform to L does not exist. One such (one-dimensional) example of a Hamiltonian and its equations is:

$$H = pq; \;\; \dot{q} = q, \; \dot{p} = p$$

While this Hamiltonian does not appear to describe the evolution of a physical system, is is a generator of a phase space flow that describes the action of a symmetry.

3 Symmetry revisited

3.1 Symmetries generated by Hamiltonian flow

Consider the Hamiltonian flow generated by the so-called "Hamiltonian" $H = p$. We introduce the notation $\mathring{v} = \frac{\partial v}{\partial a}$ to represent the derivative of a variable v with respect to the flow parameter (which in this case is a). This generates translations of $(q, p) \to (q + a, p)$:

$$\mathring{q} = \partial_p p = 1, \, \mathring{p} = -\partial_q p = 0 \implies q(a) = q(0) + a, \, p(a) = p(0)$$

The point we intend to show here is that the flow equations are viewed as the action of an infinitesimal symmetry on phase space; which in this case is the infinitesimal translation of q. Similarly, the solutions to these flow equations represent the action on phase space of finite symmetries, which are finite translations $q \to q + a$, $p \to p$.

We can similarly generate rotations in the 1-2 plane by

$$L_{12} \equiv q_1 p_2 - q_2 p_1$$

where the flow parameter is the rotation angle ϕ:

$$\mathring{q}_1 = \partial_{p_1} L_{12} = -q_2, \, \mathring{q}_2 = \partial_{p_2} L_{12} = q_1 \implies q_1(\phi) = \cos\phi \, q_1(0) - \sin\phi \, q_2(0)$$

$$\mathring{p}_1 = -\partial_{q_1} L_{12} = -p2, \, \mathring{p}_2 = -\partial_{q_2} L_{12} = p_1 \implies p_1(\phi) = \cos\phi \, p_1(0) - \sin\phi \, p_2(0)$$

This expresses the fact that an infinitesimal rotation generated by L_{12} acts as

$$\delta q_1 = -\varepsilon q_2, \; \delta q_2 = \varepsilon q_1, \; \delta p_1 = -\varepsilon p_2, \; \delta p_2 = \varepsilon p_1$$

and correspondingly, finite rotations act as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \; \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \to \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

We can similarly generate dilations (i.e. rescalings of $\vec{q}, \vec{p}$ by $e^{\pm\alpha}$) by $D \equiv \vec{q} \cdot \vec{p}$, but this will not be discussed explicitly for time's sake.

3.2 Poisson brackets

Let $A(\vec{q},\vec{p})$ be a function on phase space. Then its time derivative $\dot{A}$ can be expressed as a function of the phase space coordinates, as dictated by Hamilton's equations:

$$\dot{A} = \partial_{\vec{q}}A \cdot \dot{\vec{q}} + \partial_{\vec{p}}A \cdot \dot{\vec{p}} = \partial_{\vec{q}}A \cdot \partial_{\vec{p}}H - \partial_{\vec{p}}A \cdot \partial_{\vec{q}}H$$

Similarly, as we saw previously, a symmetry generated by a function G on phase space, e.g. the rotation generator $G = L_{ij} = q_i p_j - q_j p_i$, acts on phase space by a Hamiltonian flow generated by G, instead of H, with the flow parameter (or in this case an angle) parameterizing the symmetry, instead of time. Thus the derivative $\mathring{A}$ of A with respect to the flow parameter can be expressed as a function of the phase space coordinates in an analogous way:

$$\mathring{A} = \partial_{\vec{q}}A \cdot \mathring{\vec{q}} + \partial_{\vec{p}}A \cdot \mathring{\vec{p}} = \partial_{\vec{q}}A \cdot \partial_{\vec{p}}G - \partial_{\vec{p}}A \cdot \partial_{\vec{q}}G$$

This specific antisymmetric combination of derivatives with respect to $\vec{q}$ and $\vec{p}$ warrants the introduction of specific notation for this operation, namely the Poisson bracket:

$$\{A,B\} \equiv \partial_{\vec{q}}A \cdot \partial_{\vec{p}}B - \partial_{\vec{p}}A \cdot \partial_{\vec{q}}B$$

We can hence rewrite the above equations as

$$\dot{A} = \{A,H\} \text{ and } \mathring{A} = \{A,G\}$$

The Poisson brackets of the phase space coordinates themselves are

$$\{q_i,q_j\} = 0,\ \{p_i,p_j\} = 0,\ \{p_i,q_j\} = \delta_{ij}$$

3.2.1 Example: rotations

For $L_{ij} = q_i p_j - q_j p_i$, we have

$$\{q_1, L_{12}\} = \partial_{p_1}(q_1 p_2 - q_2 p_1) = -q_2,\ \{q_2, L_{12}\} = q_2,\ \text{and } \{q_k, L_{12}\} = 0 \text{ for } k \neq 1,2$$

More generally:

$$\{q_k, L_{ij}\} = \delta_{kj} q_i - \delta_{ki} q_j,\ \{p_k, L_{ij}\} = \delta_{kj} p_i - \delta_{ki} p_j$$

This generalizes to the following relations that may be familiar from quantum mechanics:

$$\{L_{12}, L_{23}\} = L_{31},\ \{L_{23}, L_{31}\} = L_{12},\ \{L_{31}, L_{12}\} = L_{23}$$

3.2.2 Properties of Poisson brackets

- Antisymmetry: $\{A,B\} = -\{B,A\}$
- Linearity: $\{A, B+C\} = \{A,B\} + \{A,C\}$
- Product rule: $\{A, BC\} = \{A,B\}C + B\{A,C\}$
- Jacobi identity: $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$

3.3 Symmetries and conservation laws

The Hamiltonian formulation, and the Poisson bracket in particular, makes Noether's theorem (the relationship between symmetries of the Hamiltonian and conservation laws) very clear. By definition, a symmetry generated by some function $G(\vec{q}, \vec{p})$ is a symmetry of the Hamiltonian $H(\vec{q}, \vec{p})$ if it leaves H unchanged:

$$\mathring{H} = \{H, G\} = 0$$

On the other hand, by definition, a function $G(\vec{q}, \vec{p})$ is a conserved quantity if time evolution generated by the Hamiltonian leaves G unchanged:

$$\dot{G} = \{G, H\} = 0 = -\{H, G\}$$

It is thus clear that generators of symmetries of the Hamiltonian are conserved, and conserved quantities generate symmetries of the Hamiltonian.

4 Conclusion

In revisiting Hamiltonian mechanics, we have seen how the formal structure of phase space, Hamilton's equations, and the Poisson bracket formalism collectively provide a powerful and flexible framework for understanding classical dynamics. By emphasizing symmetries through Hamiltonian flow and exploring their deep connection to conserved quantities via Noether's theorem, we gain not only mathematical elegance but also physical insight. This algebraic approach underlines the universality of Hamiltonian mechanics and its foundational role in both classical and quantum theories.

5 References

Landau and Lifshitz, Mechanics. 3rd ed., Butterworth-Heinemann, 1976.
Goldstein, Safko, and Poole, Classical Mechanics. 3rd ed., Pearson, 2001.