PHYS UN1601 Recitation Week 10

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Reduced mass

Consider two massive bodies that interact via a force. By Newton's second law, the force exerted by mass 2 on mass 1 is $\vec{F}_{12} = m_1\vec{a}_1$, and the forced exerted by mass 1 on mass 2 is $\vec{F}_{21} = m_2\vec{a}_2$. By Newton's third law, these forces must be equal and opposite:

$$\vec{F}_{12} = -\vec{F}_{21} \implies m_1\vec{a}_1 = -m_2\vec{a}_2 \implies \vec{a}_2 = -\frac{m_1}{m_2}\vec{a}_1$$

Then the relative acceleration between the two masses is:

$$\vec{a}_{rel} = \vec{a}_1 - \vec{a}_2 = \vec{a_1} + \frac{m_1}{m_2}\vec{a}_1 = \frac{m_1 + m_2}{m_1 m_2} m_1\vec{a}_1 = \frac{\vec{F}_{12}}{\mu}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the **reduced mass**. Using this new mass, the system can be described in terms of one force, one coordinate $\vec{x}_{rel}$ specifying the separation between the two masses, and one mass μ. This premise allows us to simplify two-body problems into one-body ones.

Gravitation

Consider two masses m_1 and m_2, with position vectors $\vec{r}_1$ and $\vec{r}_2$, interacting solely via gravity. The energy of this system is:

$$E = K + U = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - \frac{Gm_1m_2}{\vec{r}_1 - \vec{r}_2}$$

We can simplify this equation by rewriting it in terms of the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ and the center of mass coordinate $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$:

$$E = \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 - \frac{Gm_1m_2}{r}$$

where μ is the reduced mass introduced above. Note that since there are no external forces acting on the system, its center of mass moves at constant velocity. This means there must be an inertial frame where the COM velocity is zero, allowing us to drop the $\frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2$ in the above equation. This can simplify problem solving in a number of situations.

Two masses on a spring

Consider two masses m_1 and m_2 lying on a horizontal table attached to either end of a spring with spring constant k. We denote their displacements from equilibrium x_1 and x_2. Then by Newton's second law, we have:

$$\ddot{x}_1 = -\frac{k}{m_1}(x_1 - x_2)$$
$$\ddot{x}_2 = \frac{k}{m_2}(x_1 - x_2)$$

We introduce the relative coordinate $x = x_1 - x_2$, then subtract the second equation from the first to get:

$$\ddot{x}_1 - \ddot{x}_2 = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x$$
$$\implies \mu\ddot{x} = -kx$$

This implies the relative coordinate oscillates harmonically with angular frequency $\sqrt{\frac{k}{\mu}}$.