PHYS UN1602 Recitation Week 2 Worksheet

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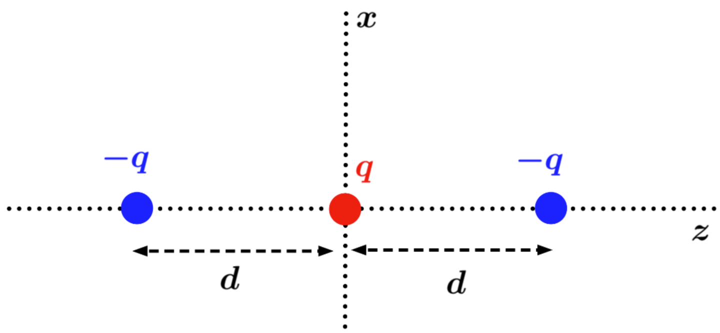
Problem 1

Three charged particles with charges $\{-q, q, -q\}$ are placed on the z axis at $z = \{-d, 0, d\}$, as shown in the diagram below.



a) List the symmetries associated with this arrangement of charges.

b) Calculate the electric field on the x axis, $\vec{E}(x, 0, 0)$.

c) Evaluate the electric potential on the x axis, $\phi(x, 0, 0)$.

d) Show that $E_x = -\frac{\partial \phi}{\partial x}$

e) What is $\vec{E}(x, 0, 0)$ in the limit $x \gg d$ to zeroth order in d/x?

f) What is $\phi(x, 0, 0)$ in the limit $x \gg d$ to zeroth order in d/x?

Solution.

a) Rotational/cylindrical symmetry about the z axis, inversion symmetry $z \to -z$, and an infinite set of inversion symmetries associated with rotations, including $x \to -x$ and $y \to -y$

b) We add up the contributions from the three charges:

$$\begin{aligned}\vec{E} &= k\left[-q\left(\frac{x\hat{i} + z\hat{k}}{\sqrt{x^2 + d^2}^3}\right) + \frac{qx\hat{i}}{|x|^3} - q\left(\frac{x\hat{i} - z\hat{k}}{\sqrt{x^2 + d^2}^3}\right)\right] \\ &= kq\left(\frac{-2x}{(x^2 + d^2)^{3/2}} + \frac{x}{|x|^3}\right)\hat{i}\end{aligned}$$

c) Recall that potential is defined as follows:

$$\phi(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|}$$

Applying the vectors $|\vec{r} - \vec{r}_i|$ from part b):

$$\phi = kq\left(\frac{-2x}{(x^2 + d^2)^{3/2}} + \frac{x}{|x|^3}\right)$$

d) Treating $|x|$ as $\sqrt{x^2}$ in the result from part c), we get:

$$-\frac{\partial\phi}{\partial x} = -kq\left(-\left(-\frac{1}{2}\right)\frac{2}{\sqrt{x^2+d^2}}\times 2x + \left(-\frac{1}{2}\right)\frac{1}{\sqrt{x^2}^3}\times 2x\right) \quad = kq\left(-\frac{2x}{(x^2+d^2)^{3/2}} + \frac{x}{|x|^3}\right)$$

e) To zeroth order in d/x:

$$\vec{E} = kq\left(-\frac{2x}{|x|^3} + \frac{x}{|x|^3}\right) = -\frac{kqx}{|x|^3}\hat{i}$$

f) Similarly:

$$\phi = kq\left(-\frac{2}{|x|} + \frac{1}{|x|}\right) = -\frac{kq}{|x|}$$

Problem 2

A spherical volume of radius a is filled with charge of uniform density ρ. We want to know the potential U of this sphere of charge. *(Hint: this is essentially the work required to assemble this sphere.)* Express your final result in terms of the total charge Q in the sphere.

a) Take a sphere of initial radius r and find the work required to add a point charge q to the sphere.

b) Write the charge of the sphere as a function of the radius.

c) Solve for total energy by building the sphere up layer by layer.

d) Why are we allowed to do what we did in step c)?

Solution.

a) Let the charge of the sphere at this time be $Q(r)$. Then the amount of energy needed to add a point charge q is:

$$U = \frac{kQ(r)q}{r}$$

b) Given the density of the sphere, we know:

$$Q(r) = \frac{4}{3}\pi\rho r^3$$

c) To calculate the total energy of the configuration, we integrate over the amount of charge added. First, using our result from part b), we notice that

$$dQ = 4\pi\rho r^2 dr$$

and from part a), we know:

$$dU = k\frac{Q(r)}{r}dQ = \frac{k}{r}\frac{4}{3}\pi\rho r^3 \times 4\pi\rho r^2 dr = \frac{16}{3}k\pi^2\rho^2 r^4 dr$$

Therefore:

$$U = \frac{16}{3}k\pi^2\rho^2\int_0^a r^4 dr = \frac{16}{15}k\pi^2\rho^2 a^5$$

We know that $Q(a) = \frac{4}{3}\pi\rho a^3 \implies \rho = \frac{3}{4\pi a^3}Q$. Hence:

$$U = \frac{16}{15}k\pi^2\frac{9}{16\pi^2 a^6 Q^2}a^5 = \frac{3}{5}\frac{Q^2}{a}$$