PHYS UN1602 Recitation Week 11 Worksheet

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Problem 1

Consider the two oppositely traveling electric-field waves:

$$\vec{E}_1 = E_0 \cos(kz - \omega t)\hat{x} \text{ and } \vec{E}_2 = E_0 \cos(kz + \omega t)\hat{x}$$

The sum of these two waves is the standing wave $2E_0 \cos(kz)\cos(\omega t)\hat{x}$.

a) Find the magnetic field associated with this standing electric wave by finding the $\vec{B}$ fields associated with each of the above traveling $\vec{E}$ fields, and then adding them.

b) Find the magnetic field by instead using Maxwell's equations to find the $\vec{B}$ field associated with the standing electric wave $2E_0 \cos(kz)\cos(\omega t)\hat{x}$.

Solution.

a) The traveling $\vec{B}$ fields must point in the $\pm\hat{y}$ directions because they must be perpendicular to both the associated $\vec{E}$ field and the direction of propagation, which is $\pm\hat{z}$. The magnitudes of the $\vec{B}$ fields are E_0/c. The signs are determined by the fact that $\vec{E} \times \vec{B}$ points in the direction of propagation. The two magnetic waves are therefore:

$$\vec{B}_1 = \frac{E_0}{c}\cos(kz - \omega t)\hat{y}$$

$$\vec{B}_2 = -\frac{E_0}{c}\cos(kz + \omega t)\hat{y}$$

The sum of these waves if $\vec{B} = \frac{2E_0}{c}\sin(kz)\sin(\omega t)\hat{y}$.

b) We use the Maxwell equation $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ to find $\vec{B}$. The curl of $\vec{E} = 2E_0 \cos(kz)\cos(\omega t)\hat{x}$ is:

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2E_0\cos(kz)\cos(\omega t) & 0 & 0 \end{vmatrix} = -2kE_0 \sin(kz)\cos(\omega t)\hat{y}$$

Setting this equal to $-\frac{\partial \vec{B}}{\partial t}$ gives (up to a constant):

$$\vec{B} = \frac{2kE_0}{\omega}\sin(kz)\sin(\omega t)\hat{y}$$

But we know that $\omega/k = c$, so we have the same formula for $\vec{B}$ as derived in part a).

Problem 2

We are familiar with the Lorentz boost in the following form (in the x direction):

$$\begin{aligned} ct' &= \gamma\,(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

where $\beta = v/c$. In this problem, we will write the Lorentz boost in a different form.

a) Define a spacetime vector for an event as:

$$\mathbf{X} = \begin{pmatrix} ct \\ x \end{pmatrix}$$

Write the Lorentz transformation in matrix form using this 2-vector, i.e., express the transformation as:

$$\mathbf{X}' = \mathbf{\Lambda X}$$

Give the 2×2 matrix $\mathbf{\Lambda}$ explicitly in terms of γ and β.

b) Recall that $\beta = v/c$. Taylor expand $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ for small β (i.e. $v \ll c$) and rewrite $\mathbf{\Lambda}$ for this case. Show that this recovers the Galilean transformation, except for a small correction to t.

c) Let's define a new variable called *rapidity*, ζ, such that:

$$\beta = \tanh\zeta \text{ and } \gamma = \cosh\zeta$$

Show that $\gamma\beta = \sinh\zeta$.

d) Using these identities, rewrite the Lorentz transformation matrix $\mathbf{\Lambda}$ in terms of hyperbolic functions $\cosh\zeta$ and $\sinh\zeta$.

e) Compare the matrix you obtained to a standard 2D rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

How are they similar? How are they different?

f) Show that under this transformation, the spacetime interval

$$s^2 = -(ct)^2 + x^2$$

is preserved, i.e. $s'^2 = s^2$. Compare this to the way distances are preserved under circular rotations in Euclidean space. What's the analogy?

Solution.

a) From the Lorentz transformation:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \end{aligned}$$

Writing this in matrix form gives:

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

b)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \frac{1}{2}\beta^2 \dots$$

$$\implies \mathbf{\Lambda} = \begin{pmatrix} 1+\frac{1}{2}\beta^2 & -\beta-\frac{1}{2}\beta^3 \\ -\beta-\frac{1}{2}\beta^3 & 1+\frac{1}{2}\beta^2 \end{pmatrix} \approx \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \mathbf{\Lambda} \begin{pmatrix} ct \\ x \end{pmatrix} \approx \begin{pmatrix} ct - \frac{v}{c}x \\ x - vt \end{pmatrix}$$

This is the same as a Galilean transformation except $t' = t - \frac{v}{c^2}x$ instead of t, but for $v \ll c$ this correction is virtually negligible.

c)

$$\gamma\beta = \cosh\zeta \tanh\zeta = \frac{\cosh\zeta \cdot \sinh\zeta}{\cosh\zeta} = \sinh\zeta$$

d)

$$\mathbf{\Lambda} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta \\ -\sinh\zeta & \cosh\zeta \end{pmatrix}$$

e) They are very similar except for two differences: replacing trigonometric functions with their hyperbolic equivalents, and the bottom entry having a minus sign in the Lorentz transform case (as opposed to the rotation matrix having it be positive), making the Lorentz transform matrix fully symmetric.

f)

$$\begin{aligned} s'^2 &= -\left(\cosh\zeta\, ct - \sinh\zeta\, x\right)^2 + \left(-\sinh\zeta\, ct + \cosh\zeta\, x\right)^2 \\ &= -\left(\cosh^2\zeta\, c^2t^2 - 2\cosh\zeta\sinh\zeta\, ct\, x + \sinh^2\zeta\, x^2\right) \\ &\quad + \left(\sinh^2\zeta\, c^2t^2 - 2\cosh\zeta\sinh\zeta\, ct\, x + \cosh^2\zeta\, x^2\right) \\ &= -\cosh^2\zeta\, c^2t^2 + 2\cosh\zeta\sinh\zeta\, ct\, x - \sinh^2\zeta\, x^2 \\ &\quad + \sinh^2\zeta\, c^2t^2 - 2\cosh\zeta\sinh\zeta\, ct\, x + \cosh^2\zeta\, x^2 \\ &= \left(-\cosh^2\zeta + \sinh^2\zeta\right) c^2t^2 + \left(\cosh^2\zeta - \sinh^2\zeta\right) x^2 \\ &= -c^2t^2 + x^2 = s^2 \end{aligned}$$

In Euclidean space, circular rotations preserve:

$$r^2 = x^2 + y^2$$

This is preserved under rotation matrices using cos and sin. Similarly, Lorentz boosts preserve:

$$s^2 = -(ct)^2 + x^2$$

but using cosh and sinh. We can hence think of Lorentz boosts as "rotations" in spacetime, preserving a different (hyperbolic) type of distance — the spacetime interval.