PHYS UN1601 Recitation Worksheet 6

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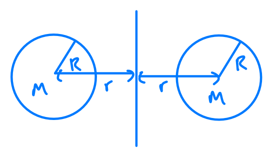
Problem 1

Find the shortest possible period of revolution of two identical gravitating spheres which are in circular orbit in free space about a point midway between them. Give your answer in terms of the density ρ of the spheres.



$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \Rightarrow M = \frac{4}{3}\pi\rho R^3$$

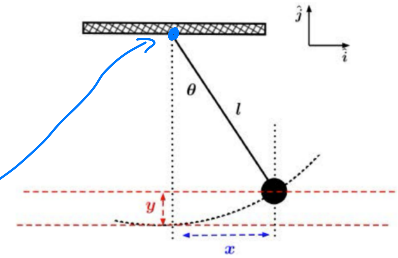
grav. force: $\frac{GM^{\cancel{2}}}{(2r)^2} = \frac{Mv^2}{r} = \cancel{M}\omega^2 r$

$$\frac{GM}{4r^2} = \omega^2 r \Rightarrow \omega = \sqrt{\frac{GM}{4r^3}}$$

period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4r^3}{GM}} = 4\pi\sqrt{\frac{r^3}{GM}}$

minimize period $\Rightarrow$ set $r = R \Rightarrow T = 4\pi\sqrt{\frac{R^3}{GM}} = 4\pi\sqrt{\frac{\cancel{R^3}}{G\frac{4}{3}\pi \cancel{R^3}\rho}} = \sqrt{\frac{12\pi}{G\rho}}$

Problem 2



An object of mass m is suspended from a massless rope of length l. It can swing such that the angle of the rope from the vertical, θ, is time-dependent. There are no drag forces acting on the mass. We will use Cartesian coordinates with the origin located at the position of the mass when it is hanging straight down. → reset origin

a) Make a free-body diagram for the mass when the rope is at an angle θ. Specify the constraint and the constraint force.

b) Write out Newton's second law ($\vec{F} = m\vec{a}$) in complete vector form, including all forces. Project onto $\hat{\imath}$ and $\hat{\jmath}$ to obtain expressions for $\ddot{x}$ and $\ddot{y}$ in terms of various parameters in the problem, including the rope's tension T and trigonometric functions of the angle θ.

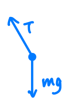
c) Express x and y in terms of θ. Substitute your results into the equations of motion for $\ddot{x}$ and $\ddot{y}$ from part b) to obtain what appear to be separable equations relating $\ddot{x}$ to x and $\ddot{y}$ to y.

d) The equations obtained in part c) are not truly separable because the tension on the rope T is not constant. It depends on θ, or equivalently on x and y. However, suppose we only consider the motion of the pendulum over small angles $\theta \ll 1$. Show that when keeping terms to the first power in θ, y becomes constant. Then show that, as a result, T also becomes constant.

e) Use your results from part d) and the small-angle approximation for $\sin\theta$ to obtain a simple harmonic oscillator (SHM) equation of motion for x. What is the angular frequency ω?

f) Find $x(t)$ assuming that at $t = 0$, $x = 0$ and $\dot{x} = v_0$.

a)



constraint is mass is distance ℓ from pivot
force is tension T

b) $m\vec{a} = \vec{T} + m\vec{g}$

$\Rightarrow$ $m\ddot{x} = -T\sin\theta$

$m\ddot{y} = T\cos\theta - mg$

c) $x = \ell\sin\theta$
$y = -\ell\cos\theta$ $\Rightarrow$ $m\ddot{x} = -\frac{T}{\ell}x$
$m\ddot{y} = -\frac{T}{\ell}y - mg$

d) $\cos\theta = 1 - \cancel{\frac{\theta^2}{2}} + \dots$

$\Rightarrow$ $y = -\ell\cos\theta = -\ell \Rightarrow \dot{y} = 0$

then $\ddot{y} = -\frac{T}{m\cancel{\ell}}\cancel{y} - mg = 0 \Rightarrow |T| = mg$ const.

e) $\sin\theta = \theta - \cancel{\frac{\theta^3}{6}} + \dots$

$\Rightarrow$ $\ddot{x} = -\frac{T}{m\ell}x = -\frac{\cancel{m}g}{\cancel{m}\ell}x = -\frac{g}{\ell}x$

$\Rightarrow$ $\ddot{x} = -\omega^2 x$, $\omega = \sqrt{\frac{g}{\ell}}$

f) general solution: $x(t) = A\sin(\omega t) + B\cos(\omega t)$

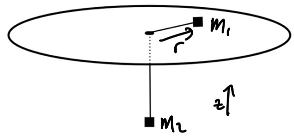
$\Rightarrow$ $x(0) = B\cos(\omega t) = 0 \Rightarrow B = 0$

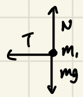
$\dot{x}(0) = A\omega\cos(\omega t) = A\omega \Rightarrow A = \frac{v_0}{\omega}$

$\Rightarrow$ $\boxed{x(t) = \frac{v_0}{\omega}\sin(\omega t)}$

demo problem

An object of mass m moves without friction on a horizontal table. It is connected to a (massless) rope that passes through a hole in the center of the table and hangs vertically. Another object, also of mass m, hangs from the rope. Find the conditions under which the hanging mass remains at constant vertical position, which also means that object on the table moves in a circle. Provide your answer in terms of an equation that expresses the necessary condition(s).





we know $z(m_2) = \text{const.} \Rightarrow \ddot{z}(m_2) = 0 \Rightarrow T = mg$

since m_1 moves in a circle, $\cancel{-m}\omega^2 r = -T = \cancel{-m}g$

$$\omega^2 = \frac{g}{r} \Rightarrow \boxed{\omega = \sqrt{\frac{g}{r}}}$$

this is the condition for m_2 to remain in constant vertical position